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UNIVERSAL R-MATRICES FOR THE QUANTUM
GROUP Uq(sl(N + 1, C)): THE ROOT OF UNITY
CASE

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Introduction

The aim of this paper is to construct a universal R-matrix for a certain quotient of the quantized universal enveloping algebra Uq(sl(N + 1, C)) in the sense of Drinfeld [2] and Jimbo [5][6] at roots of unity. The notion of universal R-matrix is due to Drinfeld. A universal R-matrix for a Hopf algebra A over C is an invertible element \( R \in A \otimes A \) with the following properties: (1) \( R \Delta(a) R^{-1} = \hat{\Delta}(a) \), for \( a \in A \), (2) \( (\Delta \otimes \text{id})(R) = R_{13} R_{23} \), \( (\text{id} \otimes \Delta)(R) = R_{13} R_{12} \). Here \( \Delta : A \rightarrow A \otimes A \) is the comultiplication, and \( \hat{\Delta} \) is the opposite comultiplication \( \hat{\Delta} = P \circ \Delta \) for the permutation \( P \) in \( A \otimes A \), \( P(a \otimes b) = b \otimes a \). The map \( \Delta \) is not in general symmetric in the sense that \( \hat{\Delta} \neq \Delta \), but from the property (1) of this universal R-matrix, there arises an \( A \)-module isomorphism \( V \otimes W \rightarrow W \otimes V \) for \( A \)-modules \( V \) and \( W \). It follows from two properties (1) and (2) that it satisfies the Yang-Baxter equation:

\[
R_{12} R_{13} R_{23} = R_{23} R_{12} R_{13},
\]

where \( R_{ij} \) is the embedding of \( R \) into the i-th and j-th factor of \( A \otimes A \).

In [14], Rosso gave an explicit formula of universal R-matrix for \( U_q(sl(N+1, C)) \) for generic \( q \), and in [15], he obtained a universal R-matrix for a quotient of \( U_q(sl(N+1, C)) \) when \( q \) is a primitive \( r \)-th root of unity for an integer \( r \) satisfying that \( r \geq N+1 \) and that \( r \) and \( N+1 \) are coprime. The result was independently obtained in [17]. In [23],[24],[25], and [26], Yamane introduced quasi-triangular Hopf algebras associated to complex simple Lie superalgebras of types A-G, and gave explicit formulas of their universal R-matrices, both in generic and non-generic cases. In particular, he got an explicit formula of a universal R-matrix for a quotient of \( U_q(sl(N+1, C)) \).

In the present paper, we give an explicit formula of a universal R-matrix for a quotient of \( U_q(sl(N+1, C)) \) for a primitive \( r \)-th root of unity, \( r \neq 1,2,4 \). Let \( E_i, F_i, \) and \( K_i, 1 \leq i \leq N \), be the generators of the Hopf algebra \( U_q(sl(N+1, C)) \). Let \( U^+ \) be the Hopf subalgebra \( U_q(sl(N+1, C)) \) generated by \( E_i, K_i, 1 \leq i \leq N \) and \( U^- \) the Hopf subalgebra generated by \( F_i, K_i, 1 \leq i \leq N \). The construction of the universal R-matrix
is based on the quantum double construction due to Drinfel'd [2]. An essential point of this construction is the existence of a non-degenerate pairing $U^+ \times U^- \to C$ compatible with the Hopf algebra structures of $U^+$ and $U^-$. Since a pairing naturally defined degenerates when $q$ is a root of unity, we consider, following Yamane [25], a certain quotient of $U_q(\mathfrak{sl}(N+1,C))$.

For $N \in \mathbb{N}$ and $1 < r \in \mathbb{N}$, we put $d = (r, N+1)$, $a = r \cdot (r, N+1)$. Let $\zeta$ be a primitive $r$-th root of unity with $(\zeta + \overline{\zeta})(\zeta - \overline{\zeta}) \neq 0$. We remark that $\zeta^{N+1}$ is a primitive $a$-th root of unity, and $\zeta^2$ is a primitive $r$-th root of unity. Let $(a_{ij})_{1 \leq i,j \leq N}$ be the Cartan matrix for $\mathfrak{sl}(N+1,C)$. In the present paper, we consider the Hopf algebra $U_\zeta$ which is a quotient Hopf algebra of $U_q(\mathfrak{sl}(N+1,C))$.

As an algebra $U_\zeta$ is generated by $E_i, F_i, K_i, K_i^{-1}, \Lambda = \Pi_{i=1}^{N} K_i^a$ for $1 \leq i \leq N$ with the relations:

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,
\]

\[
K_i E_j = \zeta^{(a_i a_j)} E_j K_i, \quad K_i F_j = \zeta^{- (a_i a_j)} F_j K_i,
\]

\[
[E_i F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{\zeta - \overline{\zeta}^{-1}},
\]

\[
E_i^2 E_j - (\zeta + \zeta^{-1}) E_i E_j E_i + E_i E_j^2 = 0 \quad (|i - j| = 1),
\]

\[
E_i E_j = E_j E_i \quad (|i - j| \geq 2),
\]

\[
F_i^2 F_j - (\zeta + \zeta^{-1}) F_i F_j F_i + F_i F_j^2 = 0 \quad (|i - j| = 1),
\]

\[
F_i F_j = F_j F_i \quad (|i - j| \geq 2),
\]

\[
E_i^\zeta = F_i^\zeta = 0,
\]

\[
K_i^\zeta = 1, \quad \Lambda^a = 1,
\]

where $(a_i a_j) = a_{ij}$ and for $1 \leq i \leq N+1$ and $X = E$ or $F$, the element $X_{ij}$ is inductively defined by

\[
X_{ij} = \begin{cases} X_i & \text{if } j = i + 1, \\ X_{i-1} X_{j-1} - \xi X_{j-1} X_{i-1} & \text{if } j > i + 1. \end{cases}
\]

Let $U^\zeta_+$ be the Hopf subalgebra of $U_\zeta$ generated by $E_i, K_i^\pm, 1 \leq i \leq N$, $U^\zeta_-$ the Hopf subalgebra of $U_\zeta$ generated by $F_i, K_i^\pm, 1 \leq i \leq N$, and $(U^\zeta_+)^\#$ the dual algebra of $U^\zeta_+$ with the opposite comultiplication. We construct a Hopf algebra isomorphism $\varphi: U^\zeta_- \to (U^\zeta_+)^\#$, and give an explicit formula of an orthonormal basis with respect to the pairing $\Phi$.

Applying the quantum double construction to the Hopf algebra $U^\zeta_+$, we see that the Hopf algebra isomorphism $\varphi$ induces a Hopf algebra epimorphism $\psi$ from the quantum double $D(U^\zeta_+)$ to the Hopf algebra $U_\zeta$. The image of the universal $R$ of $D(U^\zeta_+)$ under $\psi \otimes \psi$ is a universal $R$ of $U_\zeta$. 
As well-known, a universal $R$ can be used in producing tangle invariants obtained from the representations of the quantized universal enveloping algebras for classical simple Lie algebras (see for example [11][12][13][18][19]). As an application of our universal $R$, we can calculate some tangle invariants, which are essential in the construction of Witten's 3-manifold invariants [21].

For any positive integer $K$, let $P_+(K)$ be the set of the dominant integral weights $\lambda$ with $0 \leq (\lambda,\theta) \leq K$, where $\theta$ denotes the longest root. We consider the family of finite dimensional irreducible representations of $U_\xi$ whose highest weight $\lambda$ is contained in $P_+(K)$, in the case $\bar{r}=K+N+1$. For an oriented framed link $L$, we denote by $J(L)$ the tangle invariant obtained by using these irreducible representations. Using our explicit formula of universal $R$ for $U_\xi$ in the case $\bar{r}=K+N+1$, one can calculate $J(H_{\lambda\mu})$, where $H_{\lambda\mu}$ denotes Hopf link with two components assigned with $V_\lambda$ and $V_\mu$.

$$J(H_{\lambda\mu}) = \frac{\sum_{w \in \mathcal{W}} (\det w) q^{2(\lambda,\rho)+w(\mu+\rho)}}{\sum_{w \in \mathcal{W}} (\det w) q^{2(\rho,w(\rho))}}.$$ 

Here $\rho$ is half the sum of positive roots. Let $S=(S_{\lambda\mu})$ be the modular transformation $S$ matrix for characters of the integrable highest weight modules due to Kac and Peterson [7]. Using the equality $S_{\lambda\mu}=S_{00}J(H_{\lambda\mu})$, we show Verlinde's formula for the fusion algebra of type $A_N^{(1)}$. The fusion algebra is an associative commutative ring with basis labelled by $P_+(K)$ and the product $w_\lambda \cdot w_\mu$ of two basis elements can be written as a sum $\sum \Sigma N_{\lambda\mu}^{\nu} w_\nu$ with structure constants $N_{\lambda\mu}^{\nu} \in \mathbb{N}$ called the fusion rule. The modular transformation $S$-matrix and the fusion rules $N_{\lambda\mu}^{\nu}$ are related by Verlinde's formula [20]:

$$N_{\lambda\mu}^{\nu} = \sum_{\nu \in P_+(K)} \frac{S_{\lambda\nu} S_{\mu\nu} S_{\nu\nu}^{\nu\nu}}{S_{00}}.$$

The paper is organized as follows: In §1, we recall the quantum double construction due to Drinfel'd and define the Hopf algebra $U_\xi$. In §2, a universal $R$ for $U_\xi$ is obtained, applying the quantum double construction to the Hopf subalgebra $U_\xi^+$ of $U_\xi$. In §3, we state tangle operators derived from irreducible representations of $U_\xi$, and calculate some tangle invariants. As an application of the tangle invariants, we prove Verlinde's formula for the fusion algebra of type $A_N^{(1)}$.

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1. **Hopf algebra $U_\xi$ and quantum double construction**

In this section, we define the Hopf algebra and recall the quantum double
construction due to Drinfel'd [2].

Let $A$ be a Hopf algebra over $C$. A universal $R$-matrix for $A$ is an invertible element $R \in A \otimes A$ such that

\[
(1) \quad R \Delta(a) R^{-1} = \tilde{\Delta}(a) \quad \text{for} \quad a \in A,
\]

\[
(2) \quad (\Delta \otimes \text{id})(R) = R_{13} R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13} R_{12},
\]

where $\Delta$ is the comultiplication and $\tilde{\Delta} = P \circ \Delta$ for the permutation $P, P(a \otimes b) = b \otimes a$. Here $R_{12} = \Sigma_i a_i \otimes b_i \otimes 1$, $R_{13} = \Sigma_i a_i \otimes 1 \otimes b_i$, and $R_{23} = \Sigma_i 1 \otimes a_i \otimes b_i$, where the components of the universal $R$ are given by $R = \Sigma_i a_i \otimes b_i$. The pair $(A, R)$ is called a quasitriangular Hopf algebra.

The so-called quantum double construction due to Drinfel'd allows us to produce quasitriangular Hopf algebras from Hopf algebras. It can be used to construct a univeasal $R$. The method can be sketched as follows. Let $A$ be a finite dimensional Hopf algebra and $A^*$ its dual with opposite comultiplication. Then, the quantum double $D(A)$ is isomorphic to $A \otimes A^*$ as a vector space, and it contains $A$ and $A^*$ as Hopf subalgebras via the natural embeddings, and the universal $R$ of $D(A)$ is the image of the canonical element of $A \otimes A^*$ i.e. $\Sigma \Sigma_i 1 \otimes 1 \otimes e_i$, if $\{e_i\}$ is a basis of $A$ and $\{e^i\}$ the dual basis in $A^*$ which is the dual space of $A$.

For $N \in \mathbb{N}$ and $1 < r \in N$, we put $d = (r, N+1)$, $a = \alpha^d$, and $n = \frac{r}{(r,2)}$.

Let $((\alpha_i, \alpha_j))_{1 \leq i, j \leq N}$ be the Cartan matrix of type $A_N$:

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & \ddots \\
\vdots & \ddots & \ddots \\
0 & \ddots & -1 & 2
\end{pmatrix}
\]

$((\alpha_i, \alpha_j)) = (2, -1, 0, \ldots, 2, -1)$.

Let $\zeta$ be a primitive $r$-th root of unity with $(\zeta + \bar{\zeta})(\zeta - \bar{\zeta}) \neq 0$. We remark that $\zeta^{N+1}$ is a primitive $a$-th root of unity, and that $\zeta^2$ is a primitive $\bar{r}$-th root of unity.

We define the Hopf algebra $U_\zeta$ which is a quotient Hopf algebra of $U_{\tilde{\zeta}}(\mathfrak{sl}(N+1, C))$.

The algebra $U_\zeta$ is generated by $E_i, F_i, K_i, K_i^{-1}, \Lambda = \Pi_{i=1}^N$ for $1 \leq i \leq N$ with the relations:

\[
K_i K_j - K_j K_i, K_i K_i^{-1} - K_i^{-1} K_i = 1,
\]

\[
K_i E_j = \zeta^{(\alpha_i, \alpha_j)} E_j K_i, K_i F_j = \zeta^{-(\alpha_i, \alpha_j)} F_j K_i
\]

\[
[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{\zeta - \bar{\zeta}^{-1}},
\]
where, for integers \( i \) and \( j \) with \( 1 \leq i < j \leq N + 1 \) and \( X = E \) or \( F \), the element \( X_{ij} \) is inductively defined by

\[
X_{ij} = \begin{cases} 
X_i & \text{if } j = i + 1, \\
X_{i-1}X_{j-1} - \zeta X_{j-1}X_{i-1} & \text{if } j > i + 1.
\end{cases}
\]

The algebra \( U_\zeta \) has a Hopf algebra structure with comultiplication \( \Delta \), counit \( \epsilon \), and antipode \( S \) given by

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]

\[
\Delta(K_i^\pm) = K_i^\pm \otimes K_i^\pm,
\]

\[
\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i^\pm) = 1,
\]

\[
S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i^\pm) = K_i^\mp.
\]

Let us show that the definitions of \( \Delta \) and \( S \) are compatible with (1.10). We prove some Lemmas.

We put

\[
[X, Y]_\zeta = XY - \zeta YX, \quad [X, Y]_\bar{\zeta} = XY - \bar{\zeta}^{-1} YX.
\]

**Lemma 1.1.** Let \( M \) be the C-algebra generated by \( A \) and \( B \) with the relations:

\[
A^2 B - (\zeta + \zeta^{-1})ABA + BA^2 = 0, \quad (1.12)
\]

\[
B^2 A - (\zeta + \zeta^{-1})BAB + AB^2 = 0. \quad (1.13)
\]

We put

\[
C = [A, B]_\zeta, \quad C' = [A, B]_{\bar{\zeta}}.
\]

Then it holds:

\[
C'' = C' + (1 - \zeta^{-2})\zeta^{\frac{\bar{B} - 1}{2}} A'B.'
\]
Proof. When \( C' = AB - \zeta^{-1}BA \), we have, for any positive integer \( n \),

\[
(C')^n = (\zeta^{-2}C + (1 - \zeta^{-2})AB)^n
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-2)(a-2) \cdot A^i C^{n+1-i} B^i,
\]

(1.14)

where

\[
\binom{n}{i} \zeta^{[n]...[n-i+1]} = \frac{[n]}{[i]...[1]}, \quad [n] = \frac{1 - \zeta^{-2n}}{1 - \zeta^{-2}}
\]

The equality is shown as follows. We have the following equalities for any non-negative integer \( n \):

\[
B^n AB = \zeta^{-n} AB = \zeta^{-n} AB^{n+1} - \zeta^{-n-2}[n]CB^n,
\]

(1.15)

\[
(1 - \zeta^{-2})[n] = 1 - \zeta^{-2n},
\]

(1.16)

\[
\binom{n}{i} \zeta^{(n-i+1)} = \binom{n+1}{i} \zeta^{i}.
\]

(1.17)

We show the equality (1.14) by induction on \( n \). We suppose that the equality (1.14) holds for \( n \), and then it follows from (1.15),(1.16),(1.17) that

\[
(C')^{n+1}
\]

\[
= (C')^{(\zeta^{-2}C + (1 - \zeta^{-2})AB)}
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-2)(a-2) \cdot A^i C^{n+1-i} B^i
\]

\[
+ \sum_{i=0}^{n} \binom{n}{i} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-1)(a-1) \cdot A^{i+1} C^{n-i} B^{i+1}
\]

\[
- \sum_{i=0}^{n} \binom{n}{i} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-2)(a-2) \cdot A^{i} C^{n+1-i} B^{i}
\]

\[
= \zeta^{-2(n+1)} C^{n+1} + (1 - \zeta^{-2})^n + 1 \cdot \zeta^{-\frac{(n+1)(n+2)}{2}} A^{n+1} B^{n+1}
\]

\[
+ \sum_{i=1}^{n} \left\{ \binom{n}{i-1} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-2)(a+1-i) \right\} A^i C^{n+1-i} B^i
\]

\[
+ \binom{n}{i} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-2)(a+1-i)(1 - (1 - \zeta^{-2})[i]) A^i C^{n+1-i} B^i
\]

\[
= \sum_{i=0}^{n+1} \binom{n+1}{i} (1 - \zeta^{-2})^{i} \zeta^{-\frac{(1-i)(n+1)}{2}} + (i-2)(a+1-i) A^i C^{n+1-i} B^i.
\]
So the equality (1.14) holds. As $\zeta^2$ is a primitive $r$-th root of unity, we obtain the claim, putting $n=r$ in the equality (1.14).

For $1 \leq i < j \leq N+1$, the elements $E_{ij}$ and $E'_{ij}$ are inductively defined by

$$
E_{ij} = \begin{cases} 
E_i & \text{if } j = i+1, \\
[E_{ij-1}, E_{j-1}]_\zeta & \text{if } j > i+1.
\end{cases}
$$

$$
E'_{ij} = \begin{cases} 
E_i & \text{if } j = i+1, \\
[E_{ij-1}, E_{j-1}]_\zeta & \text{if } j > i+1.
\end{cases}
$$

**Lemma 1.2.** (i) For $i < p < j$, we put $A = E_{ip}$ and $B = E'_{pj}$. Then these $A$ and $B$ satisfy the relations (1.12) and (1.13).

(ii) We have that $[E_{ip}, E'_{pj}]_\zeta = [E_{ip+1}, E'_{pj+1}]_\zeta$.

Proof. (i) We show by induction on $p$ that $[E_i, E_{ip}]_\zeta = 0$, for $p \geq i+2$. It follows from relation (1.6) that $[E_i, E_{i+2}]_\zeta = 0$. We suppose that $[E_i, E_{ip}]_\zeta = 0$. Then we obtain from the relation (1.7),

$$
[E_i, E_{ip+1}]_\zeta = [E_i, E_{ip}]_\zeta + [E_{ip}, E_{ip+1}]_\zeta = 0.
$$

Similarly, using the relation (1.7) and the equality

$$
[E'_{ip+1}, E_{j-1}]_\zeta = E_{ip+1}[E'_{ip}, E_{j-1}]_\zeta - \zeta[E_{ip}, E'_{ip+1}]_\zeta E_{ip+1},
$$

we obtain by induction on $p$ that $[E'_{ip}, E_{ip+1}]_\zeta = 0$ for $p \leq j-2$.

We put $X = [E_{ip}, E_{ip}]_\zeta$ and $Y = [[E_{ip}, E_{ip}]_\zeta]_\zeta$. Computing $[E_{ip}, [E_{i-1}X]]_\zeta$, $[X, E_{ip}]_\zeta$, $[[Y, E_{ip}]]_\zeta$, and $[E_{i-1}, Y]$, we prove that $E_{ip}$ and $E'_{pj}$ satisfy the relations (1.12) and (1.13). Noting that $[E_i, E_{ip}]_\zeta = 0$ and

$$
E_{i-1}[E_{ip}, E'_{pj}]_\zeta - (\zeta + \zeta^2)E_{i-1}[E_{ip}, E'_{pj}]_\zeta E_{i-1} - [E_{ip}, E'_{pj}]_\zeta E_{i-1} E_{i-1} = 0,
$$

it follows that

$$
[E_{i-1}, E_{i-1}, X]_\zeta = 
E_{i-1}^2 E_{ip}[E_{ip}, E'_{pj}]_\zeta - \zeta^2 E_{i-1}[E_{ip}, E'_{pj}]_\zeta E_{ip}
$$

$$
- \zeta^2 E_{i-1} E_{ip}[E_{ip}, E'_{pj}]_\zeta E_{i-1} + \zeta E_{i-1}[E_{ip}, E'_{pj}]_\zeta E_{i-1} E_{i-1}
$$

$$
- E_{i-1} E_{ip}[E_{ip}, E'_{pj}]_\zeta + \zeta E_{i-1}[E_{ip}, E'_{pj}]_\zeta E_{ip} E_{i-1}
$$

$$
+ \zeta^2 E_{ip}[E_{ip}, E'_{pj}]_\zeta E_{i-1} - \zeta [E_{ip}, E'_{pj}]_\zeta E_{ip} E_{i-1}^2
$$

$$
= (\zeta + \zeta^2)E_{i-1} E_{ip}[E_{ip}, E'_{pj}]_\zeta - E_{ip} E_{i-1}^2 [E_{ip}, E'_{pj}]_\zeta.
$$
Here we have used \[ [E_{i-1}p[E_{ip}E_{pj}],\zeta] = [E_{ip}[E_{ip}E_{pj}],\zeta] \zeta \]. So, when \( \zeta + \zeta \neq 0 \) and \( X = 0 \), it turns out that \( [E_{i-1}p[E_{ip}E_{pj}],\zeta] = 0 \). From the formula \( [E_{ip}E_{pj+1}],\zeta \) and the relation (1.7), we have

\[
[X,E_{j}] = \zeta[E_{ip}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} - \zeta[E_{ip}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} + \zeta^2[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} - \zeta^2E_{j}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} + \zeta^3E_{j}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} - \zeta^3E_{j}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} - \zeta^3E_{j}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j}.
\]

So, if \( [E_{ip}[E_{ip}E_{pj}],\zeta] = 0 \), then \( [E_{ip}[E_{ip}E_{pj}],\zeta] = 0 \). Thus the elements \( E_{ip} \) and \( E_{pj} \) satisfy the relations (1.12).

From the equalities \( [E_{pj},E_{i-1}] = 0 \) and

\[
[E_{ip}E_{pj}],\zeta]E_{j}^2 - (\zeta + \zeta)E_{j}[E_{ip}E_{pj}],\zeta]E_{j} + E_{j}^2[E_{ip}E_{pj}],\zeta] = 0.
\]

it follows that

\[
[[Y,E_{j}],\zeta]E_{j} = \zeta[E_{ip}[E_{ip}E_{pj}],\zeta]E_{ip}E_{j} - \zeta^2E_{p}[E_{ip}E_{pj}],\zeta]E_{p}E_{j} - \zeta^3E_{p}[E_{ip}E_{pj}],\zeta]E_{p}E_{j} + \zeta^4E_{p}[E_{ip}E_{pj}],\zeta]E_{p}E_{j} - \zeta^5E_{p}[E_{ip}E_{pj}],\zeta]E_{p}E_{j} - \zeta^6E_{p}[E_{ip}E_{pj}],\zeta]E_{p}E_{j}.
\]
Here we have used the equality $[E_{ip}, E_{ip}' + 1]_\xi = [[E_{ip}, E_{ip}']_\xi, E_j]_\xi$.

So, if $\zeta + \zeta \neq 0$ and $[[E_{ip}, E_{ip}']_\xi, E_j]_\xi = 0$, then $[[E_{ip}, E_{ip}']_\xi, E_{ip+1}]_\xi = 0$. From the equality $[E_{i-1}, [E_{ip}, E_{ip}']_\xi]_\xi = [E_{i-1}, p(E_{ip}')_\xi]_\xi$, we have

$$[E_{i-1}, Y]_\xi = E_{i-1}[[E_{ip}, E_{ip}']_\xi, E_{ip+1} - \zeta E_{i-1} E_{ip+1} [E_{ip}, E_{ip}']_\xi - \zeta [[E_{ip}, E_{ip}']_\xi, E_{ip+1}]_\xi = [E_{i-1}, [E_{ip}, E_{ip}']_\xi]_\xi = [E_{i-1}, p(E_{ip}')_\xi]_\xi.$$ 

If $[[E_{ip}, E_{ip}']_\xi, E_{ip}]_\xi = 0$, then $[[E_{i-1}, p(E_{ip}')_\xi]_\xi, E_{ip}]_\xi = 0$. Thus the elements $E_{ip}$ and $E_{ip}'$ satisfy the relations (1.13).

(ii) Let us show that $[[E_{ip}, E_{ip}']_\xi]_\xi = [E_{ip+1}, E_{ip+1}]_\xi$. We have

$$[E_{ii+1}, E_{i+1}]_\xi = E_{i+1} E_{i+2} - \zeta E_{i+1} E_{i+2} - \zeta E_{i+2} E_{i+1} + E_{i+2} E_{i+1} E_i$$

$$= [E_{i+2}, E_{i+1}]_\xi.$$ 

We suppose that $[E_{ip}, E_{ip}']_\xi = [E_{ip+1}, E_{ip+1}]_\xi$. Then we obtain

$$[E_{ip}, E_{ip+1}]_\xi = [E_{ip}, E_{ip}']_\xi E_j - \zeta E_j [E_{ip}, E_{ip}']_\xi$$

$$= [E_{ip+1}, E_{ip+1}]_\xi E_j - \zeta E_j [E_{ip}, E_{ip+1}]_\xi$$

$$= [E_{ip+1}, E_{ip+1}]_\xi E_j - \zeta E_j [E_{ip}, E_{ip+1}]_\xi - \zeta (E_{ip+1} E_j - \zeta E_{ip+1}) E_{ip+1}$$

$$= [E_{ip+1}, E_{ip+1}]_\xi$$

and

$$[E_{i-1}, p(E_{ip})]_\xi$$

$$= E_{i-1} [E_{ip}, E_{ip}']_\xi - \zeta [E_{ip}, E_{ip}']_\xi E_{i-1}$$

$$= E_{i-1} [E_{ip+1}, E_{ip+1}]_\xi - \zeta [E_{ip+1}, E_{ip+1}]_\xi E_{i-1}$$

$$= (E_{i-1} E_{ip+1} - \zeta E_{ip+1} E_{i-1}) E_{ip+1} - \zeta E_{ip+1} E_{i-1} E_{ip+1} - \zeta E_{ip+1} E_{i-1}$$

$$= [E_{i-1}, p(E_{ip+1})]_\xi.$$ 

So the claim holds.

By Lemma 1.1 and Lemma 1.2, we have the equality
Lemma 1.3. We have the formula
\[ E_{ij}^\tilde{r} = E_{ij}^\tilde{r} + \sum_{i<p_1<\cdots<p_s<j} ((1-\zeta^{-2})^s \zeta^{-\tilde{r}^2-1} E_{ip_1} E_{p_s j}. \]

Proof. From the equality stated just before the lemma repeatedly, we have that
\[ E_{ij}^\tilde{r} = E_{ij}^\tilde{r} + \sum_{k=i+1}^{j-1} (1-\zeta^{-2})^s \zeta^{-\tilde{r}^2-1} E_{ik} E_{kj}. \]

By induction on \( j-i \), we get the claim.

By Lemma 1.3, we obtain \( E_{ij}^\tilde{r} = 0 \) and similarly, \( F_{ij}^\tilde{r} = 0 \).

Now we prove that the definition of the coproduct \( \Delta \) is compatible with the relation (1.10). We can prove the following formula
\[ \Delta(E_{ij}) = E_{ij} \otimes 1 + (1-\zeta^2) \sum_{i<k<j} K_{ik} E_{kj} \otimes E_{ik} + K_{ij} \otimes E_{ij}. \]

where \( K_{ij} = K_i \cdots K_{j-1} \). We put
\[ u_1 = E_{ij} \otimes 1, \]
\[ u_2 = K_{ii+1} E_{i+1, j} \otimes E_{ii+1}, \]
\[ \vdots \]
\[ u_{j-i-1} = K_{ij-1} E_{ij-1} \otimes E_{ij-1}, \]
\[ u_{j-i+1} = K_{ij} \otimes E_{ij}. \]

It follows that if \( k > l \), then \( u_k u_l = \zeta^2 u_l u_k \). As we can write that \( \Delta(E_{ij}) = u_1 + (1-\zeta^2)(u_2 + \cdots + u_{j-i}) + u_{j-i+1} \), we have
\[ \Delta(E_{ij})^m = \sum_{m_1 + \cdots + m_{j-i-1} = m} \phi_{m_1}(\zeta^2) \cdots \phi_{m_{j-i-1}}(\zeta^2) (1-\zeta^2)^{m_2 + \cdots + m_{j-i}} u_1^{m_1} \cdots u_{j-i+1}^{m_{j-i+1}}, \]

where \( \phi_m(\zeta^2) = (1-\zeta^2)(1-\zeta^4) \cdots (1-\zeta^{2m}) \) (see [14]). Putting \( m = \tilde{r} \), we can obtain the equality \( \Delta(E_{ij})^\tilde{r} = 0 \).

By induction, it follows that \( S(E_{ij}) = -K_{ij}^{-1} E_{ij} \) and \( S(F_{ij}) = -\zeta^{2(j-i-1)} F_{ij} \). We
recall that $E_{ij}^i = F_{ij}^i = 0$ and so one can obtain that $S(E_{ij}) = S(F_{ij}) = 0$.

2. A construction of a universal $R$-matrix for $U_\zeta$

In this section, we construct a universal $R$-matrix for $U_\zeta$, using the quantum double construction due to Drinfel'd [2]. Our method is similar to that of the construction of the universal $R$-matrix in [23] and [26].

Let $U_\zeta^+$ be the Hopf subalgebra of $U_\zeta$ generated by $E_i, K_i^\pm, 1 \leq i \leq N$ and $U_\zeta^-$ the Hopf subalgebra of $U_\zeta$ generated by $F_i, K_i, 1 \leq i \leq N$ and $(U_\zeta^+)^\epsilon$ be the dual algebra of $U_\zeta^+$ with the opposite comultiplication.

First we fix some notations. Let $\{x_i|1 \leq i \leq N\}$ be the system of simple roots and $\Pi_+$ the set of positive roots $x_1 + \cdots + x_{j-1}$ with $1 \leq i < j \leq N+1$ of $sl(N+1, C)$. We denote by $Q = \bigoplus \mathbb{Z} x_i$ the root lattice and let $\langle ., . \rangle: Q \times Q \rightarrow \mathbb{Z}$ be the pairing defined by $\langle x_i, x_j \rangle = \delta_{ij}$ where $(x_i)^0 = 0$ is the Cartan matrix of type $A_N$.

We shall put on the set $\{E_{ij}|1 \leq i < j \leq N+1\}$ a total order $<$ defining $E_{kl} < E_{ij}$ if $k < i$, or $k = i$ and $l < j$. We also denote $E_{ij}$ by $E_x$ for $x \in \Pi_+$ if $x = x_i + \cdots + x_{j-1}$. The following notation will be used in describing a $C$-basis of $U_\zeta^+$:

$$I = \{ (m_a)_{a \in \Pi_+} | 0 \leq m_a < r \},$$

$$J = \{ (v_i)_{1 \leq i \leq N} | 0 \leq v_p < r, p = 1, \ldots, N-1, 0 \leq v_N < a \},$$

$$P = \{ v | v = \sum_{i=1}^{N} v_i x_i, (v_i)_{1 \leq i \leq N} \in J \}.$$

Moreover, we denote by $\Pi_{a \in \Pi_+} E_x^{m_a}$ for $\{m_a\}_{a \in I}$ ordered monomials of the $E_x$s according to the total order defined above, $E_{12} E_{13} \cdots E_{NN}^{m_N}$, and for $v = \sum_{i=1}^{N} v_i x_i$ with $(v_i)_{1 \leq i \leq N} \in J$, set $K_v = \Pi_{1}^{N} K_i^{v_i}$. In a way similar to Lemma 4.2 in [22], we can derive a system of generators of $U_\zeta^+$.

**Proposition 2.1.** The algebra $U_\zeta^+$ is generated by $\{ \Pi_{a \in \Pi_+} E_x^{m_a} K_v | (m_a)_{a \in I}, (v_i)_{1 \leq i \leq N} \in J \}$ as a $C$-vector space.

**Proof.** Using the relations (1.3),(1.4) and (1.11), any element $x$ of $U_\zeta^+$ can be written as a $C$-linear combination of the elements $E_{i_1} \cdots E_{i_m} K_v$ with $1 \leq i_k \leq N$ and $0 \leq v_i < r$. Let $L$ be the subalgebra generated by $K_v$ for $0 \leq i \leq N$. We remark that $L$ is generated by $\{ K_v | v \in P \}$ as a $C$-vector space. In fact, it follows, from the relations $\Pi_{i=1}^{N} K_i^a = 1$, $K^{a_{\epsilon}} = 1$, that $K_v = \Pi_{i=1}^{N} K_i^{v_i}$. So we can write $K_v^a$ for $a = b \leq r - 1$ as a product of elements in $\{ K_v | v \in J \}$. Let $P = \{ (i_1, j_1), \ldots, (i_k, j_k) | (i_p, j_p) \in N \times N, 1 \leq i_p < j_p < N+1 \} \cup \{ \phi \}$. For $\Sigma = (i_1, j_1), \ldots, (i_k, j_k) \in P$, we put $E_{\Sigma} = E_{i_1 j_1} \cdots E_{i_k j_k}$. We define a map $\eta: P_N \rightarrow \mathbb{Z}$ given by
\[ \eta(\Sigma) = i_1(j_1 - i_1) + \cdots + i_k(j_k - i_k) \text{ for } \Sigma \in P_N, \quad \eta(\phi) = 0. \]

We consider the subspace \( W_m \) generated by \( \{ E_{\Sigma} | \eta(\Sigma) \leq m \} \). A sequence \( \Sigma = ((i_1,j_1), \ldots, (i_k,j_k)) \in P_N \) is called increasing if \( (i_1,j_1) \leq (i_2,j_2) \leq \cdots \leq (i_k,j_k) \). In particular, \( \phi \) is increasing. From [22], for a pair \((s,t) \prec (x,y)\), we can show

\[ E_{xy}E_{st} = \zeta^{\delta_{sx} - \delta_{xt} - \delta_{ys} + \delta_{yt}}E_{xy} + \sum_{\eta(\Sigma) < \eta((x,y);(s,t))} c_\Sigma E_{i_1j_1} \cdots E_{i_kj_k} \tag{*} \]

for some \( c_\Sigma \in C \). By induction on \( m \), we can show that for any \( m \), any element in \( W_m \) is written as a \( C \)-linear combination of the elements in the set \( \{ E_{\Sigma} | \eta(\Sigma) \leq m, \Sigma \text{ is increasing} \} \) (see [22]).

We give a triangular decomposition of \( \tilde{U}_\zeta \), using a way similar to one in [22].

Let us prepare some notations.

- \( \tilde{U}_\zeta \) is the algebra over \( C \) generated by \( E_i, F_i, K_i^\pm, 1 \leq i \leq N \) with relations (1.3), (1.4), (1.5).
- \( N_+ \) (resp. \( N_- \)) is the subalgebra of \( U_\zeta \) (resp. \( \tilde{U}_\zeta \)) generated by \( E_i \), \( 1 \leq i \leq N \) along with 1.
- \( N_- \) (resp. \( N_+ \)) is the subalgebra of \( U_\zeta \) (resp. \( \tilde{U}_\zeta \)) generated by \( F_i \), \( 1 \leq i \leq N \) along with 1.
- \( T \) (resp. \( \tilde{T} \)) is the subalgebra of \( U_\zeta \) (resp. \( \tilde{U}_\zeta \)) generated by \( K_i^\pm, 1 \leq i \leq N \) along with 1.
- \( \phi^+, \phi^-_i, 1 \leq i \neq j \leq N \) are the elements of \( \tilde{U}_\zeta \), defined
  \[
  \phi^+_i = \begin{cases} 
  E_iE_j - E_jE_i & \text{if } |i-j| \geq 2, \\
  E_i^2E_j - (\zeta + \zeta^{-1})E_iE_jE_i + E_iE_j^2 & \text{if } |i-j| = 1,
  \end{cases}
  \]
  \[
  \phi^-_i = \begin{cases} 
  F_iF_j - F_jF_i & \text{if } |i-j| \geq 2, \\
  F_i^2F_j - (\zeta + \zeta^{-1})F_iF_jF_i + F_iF_j^2 & \text{if } |i-j| = 1.
  \end{cases}
  \]

- \( J_+ \) (resp. \( J_- \)) is the two sided ideal of \( \tilde{N}_+ \) (resp. \( \tilde{N}_- \)) generated by \( \phi^+_i \), \( 1 \leq i \neq j \leq N \), \( E_i^p \), \( 1 \leq i < j \leq N + 1 \) (resp. \( \phi^-_i \), \( 1 \leq i \neq j \leq N \), \( F_i^p \), \( 1 \leq i < j \leq N + 1 \)).
- \( J_0 \) is the two sided ideal of \( \tilde{T} \) generated by \( K_i^\pm - 1, 1 \leq i \leq N, \Lambda^a - 1 \).
- \( J \) is the two sided ideal of \( \tilde{U}_\zeta \) generated by \( \phi^+_i \), \( 1 \leq i \neq j \leq N \), \( E_i^p \), \( 1 \leq i < j \leq N + 1 \), \( \phi^-_i \), \( 1 \leq i \neq j \leq N \), \( F_i^p \), \( 1 \leq i < j \leq N + 1 \), \( K_i^\pm - 1, 1 \leq i \leq N, \Lambda^a - 1 \).

We investigate the structure of \( \tilde{U}_\zeta \) as a vector space, in a way similar to the proof in Lemma 2.1 and 2.2 in [22].

Let \( \mathcal{A}_+ \) (resp. \( \mathcal{A}_- \)) be the free associative \( C \)-algebra with 1 generators \( e_i \), \( 1 \leq i \leq N \) (resp. \( f_i \), \( 1 \leq i \leq N \)). Let \( C[k_1^\pm, \ldots, k_N^\pm] \) be the \( C \)-algebra of Laurent polynomials in indeterminates \( k_1, \ldots, k_N \). Let \( \mathcal{M} = \mathcal{A}_- \otimes_c C[k_1^\pm, \ldots, k_N^\pm] \otimes_c \mathcal{A}_+ \). The elements \( f_{i_1} \cdots f_{i_n} k_{j_1}^\pm \cdots k_{j_r}^\pm \varepsilon_{j_1} \cdots \varepsilon_{j_r} v_1, \ldots, v_n \in \mathcal{M}, 1 \leq i_1, \ldots, i_n, j_1, \ldots, j_r \leq N \), form an \( C \)-basis.
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$\mathcal{M}$ has a left $U_\zeta$-module structure defined by

$$K_p f_{i_1} \cdots f_{i_k} k^{v_1}_1 \cdots k^{v_k}_N e_{j_1} \cdots e_{j_h} = \zeta^{-\sum_{i=1}^k v_i} f_{i_1} \cdots f_{i_k} k^{v_1}_1 \cdots k^{v_k}_N e_{j_1} \cdots e_{j_h},$$

$$F_p f_{i_1} \cdots f_{i_k} k^{x_1}_1 \cdots k^{x_k}_N e_{j_1} \cdots e_{j_h} = \zeta^{\sum_{i=1}^k x_i} f_{i_1} \cdots f_{i_k} k^{x_1}_1 \cdots k^{x_k}_N e_{j_1} \cdots e_{j_h},$$

where $\zeta$ is a Laurent polynomial ring in the variables $K^\pm$.

By this fact, it follows that the elements $F_{i_1} \cdots F_{i_k} K^{v_1}_1 \cdots K^{v_N}_k E_{j_1} \cdots E_{j_h}$, $v_1, \ldots, v_N \in \mathbb{Z}, 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_h \leq N$, form a basis of $\tilde{U}_\zeta$. We have $U_\zeta = U_\zeta / S_\zeta$ as a vector space, $\tilde{N}_+ (\text{resp. } \tilde{N}_-)$ is a free algebra in the variables $E_i$ (resp. $F_i$), and $\tilde{T}$ is the Laurent polynomial ring in the variables $K^\pm$. We have $\tilde{N}_+ = \tilde{N}_- \otimes \tilde{T} \otimes \tilde{N}_+$ as a vector space, and $(\tilde{T} / \mathcal{S}_0)$ as an algebra over $\mathbb{C}$. It is proved in the following way, which is analogous to the proof of Proposition 2.3 in [22]. It suffices to prove:

$$\mathcal{S} = \tilde{N}_- \tilde{T} \mathcal{S}_+ + \tilde{N}_- \mathcal{S}_0 \tilde{N}_+ + \mathcal{S}_- \tilde{T} \tilde{N}_+.$$

To prove it, we show that $\tilde{N}_- \tilde{T} \mathcal{S}_+, \tilde{N}_- \mathcal{S}_0 \tilde{N}_+$, and $\mathcal{S}_- \tilde{T} \tilde{N}_+$ are ideals of $U_\zeta$. Firstly, we consider $\mathcal{S}_- \tilde{T} \tilde{N}_+$. The argument for $\tilde{N}_- \tilde{T} \mathcal{S}_+$ is analogous. Let $Y = \tilde{N}_- \tilde{T} \mathcal{S}_+$. It is clear that $K^\pm_i Y \subset Y$, $Y K^\pm_i \subset Y$, $F_i Y \subset Y$, and $Y E_i \subset Y$. Let us show that $E_i Y \subset Y$. We define the two $\mathbb{C}$-linear maps $E_i^\pm : \tilde{N}_- \rightarrow \tilde{N}_-$ by
$E_i^\pm(F_{i_1} \cdots F_{i_0}) = \sum_{i_0 = \pm} \zeta^\pm a_u F_{i_1} \cdots F_{i_u} \cdots F_{i_0}$,

where $a_u = (\alpha_{i_0} \alpha_{i_1} + \cdots + \alpha_{i_u})$, so that

$$
E_i \cdot F_{i_1} \cdots F_{i_u} K_1^{i_1} \cdots K_N^{i_u} E_{j_1} \cdots E_{j_t} \\
= \zeta^{-(\alpha_{i_0} \alpha_{i_1} + \cdots + \alpha_{i_u} + 1)N} F_{i_1} \cdots F_{i_u} K_1^{i_1} \cdots K_N^{i_u} E_{i_0} E_{j_1} \cdots E_{j_t} \\
+ \frac{1}{\zeta - \zeta^{-1}} \sum_{i_0 = \pm} (E_i^\pm(F_{i_1} \cdots F_{i_u})K_1^{i_1} \cdots K_1^{i_u+1} \cdots K_N^{i_u} E_{j_1} \cdots E_{j_t} \\
- E_i^\pm F_{i_1} \cdots F_{i_u})K_1^{i_1} \cdots K_1^{i_u-1} \cdots K_N^{i_u} E_{j_1} \cdots E_{j_t}).
$$

We can show

$$
E_i^\pm(F_{i_1} \cdots F_{i_p} F_{j_1} \cdots F_{j_x}) \in \mathcal{I}_-
$$

(see Proposition 2.3 in [22]). Moreover we have

$$
E_p F_{i_1} \cdots F_{i_x} F_{i+1} F_{i_1} \cdots F_{i_x+1} K_1^{i_1} \cdots K_N^{i_x} E_{j_1} \cdots E_{j_t} \\
= F_{i_1} \cdots F_{i_x} E_p F_{j_1} F_{i_1} \cdots F_{i_x+1} K_1^{i_1} \cdots K_N^{i_x} E_{j_1} \cdots E_{j_t} \\
+ \frac{1}{\zeta - \zeta^{-1}} \sum_{i_0 = p} (E_p^+(F_{i_1} \cdots F_{i_x})K_1^{i_1} \cdots K_N^{i_x} E_{j_1} \cdots E_{j_t} \\
- E_p^+(F_{i_1} \cdots F_{i_x})K_1^{i_1} \cdots K_N^{i_x} E_{j_1} \cdots E_{j_t}),
$$

for $1 \leq p \leq N$ and $1 \leq i < j \leq N + 1$.

Let us show that $[E_p F_{ij}] = 0$, for $1 \leq p \leq N$ and $1 \leq i < j \leq N + 1$.

If $i < p < j - 1$, then we can obtain

$$
E_p F_{ij} = E_p(F_{ip} - F_{pj} - \zeta F_p F_{ip}) \\
= F_{ip} E_p F_{pj} - \zeta E_p F_{pj} F_{ip} \\
= F_{ip} E_p + \zeta F_p K_p^{-1} F_{p+1} F_{ij} \kappa - \zeta^2 K_p^{-1} F_{p+1} F_{ip} \\
= F_{ip} E_p,
$$

using the equality $E_p F_{pj} = F_{pj} E_p + \zeta K_p^{-1} F_{p+1} F_{ij}$ and so it follows that $[E_p F_{ij}] = 0$.

We consider the case $p = i$. We have

$$
E_i F_{ij} = (E_i F_{i+1} E_i + \zeta F_{i+1} E_i F_{i+1}) F_{ij}^{-1} \\
= (F_{ij} E_i + \zeta K_i^{-1} F_{i+1}) F_{ij}^{-1}.
$$
Here we used the equality $F_{ij} + F_{i+1,j} = \zeta^{-1}F_{i+1,j}F_{ij}$. By induction, we can obtain that

$$E_i F_{ij} = F_{ij} E_i + \zeta(1 + \zeta^{-2} + \cdots + \zeta^{-2(r-1)}) = F^r_{ij} E_i.$$ 

Similarly, we can prove that $[E_{j-1} F_{ij}] = 0$.

Thus, we obtain that $E_i Y = Y$.

Nextly, we consider $\mathcal{N}_{-\mathcal{A}_o} \mathcal{N}_+$. It suffices to prove that for $X = E$ or $F$ and $1 \leq i, j \leq N$,

$$[X_{ij}, K^r_{j} - 1] = 0 \quad \text{and} \quad [X_{ij}, \Lambda^a - 1] = 0.$$ 

Let us show the formulas for $E_i$. Indeed, we have

$$E_i K^r_{j} = \zeta^r K^r_{j} E_i = K^r_{j} E_i$$

and

$$E_i \left( \prod_{j=1}^{N} K^r_{j} \right)^a = \zeta^{a_1(a_2 + a_3)} \left( \prod_{j=1}^{N} K^r_{j} \right)^a E_i$$ 

$$= \zeta^{a_1(a_1 + \Lambda + a_2)} E_i$$ 

$$= \zeta^{a_1(N-\Lambda + a_2)} \Lambda^a E_i$$ 

$$= \Lambda^a E_i.$$ 

Similarly, we can prove the formulas $[F_i, K^r_{j} - 1] = 0$ and $[F_i, \Lambda^a - 1] = 0$.

The following map $\varphi: U^- \to (U^+)^\sigma$ plays an important role.

**Proposition 2.2.** There is a Hopf algebra homomorphism $\varphi: U^- \to (U^+)^\sigma$ such that for $X = E_{i_1} \cdots E_{i_m} K_{v_0}$

$$\varphi(F_i)(X) = \begin{cases} b & \text{if } X = E_i K_{v_0}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varphi(K_{i \pm})(X) = \begin{cases} \zeta^\mp(a_1, e) & \text{if } X = K_{v_0}, \\ 0 & \text{otherwise}, \end{cases}$$

where $b = -\frac{1}{1 - \zeta^{-1}}$.

**Proof.** We put $\varphi(F_i) = \xi_i, \varphi(K_{i \pm}) = \eta_i \pm$ and $\eta_{ij} = \eta_i \cdots \eta_{j-1}$. We define $\xi_{ij}$ inductively by
We remark that if \( \{i_1, \ldots, i_m\} \neq \{j_1, \ldots, j_n\} \), then
\[
\xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}(E_{j_1}E_{j_2} \cdots E_{j_n}) = 0. \tag{**}
\]

Let us prove the fact by induction on \( m \). We assume that it holds for \( m-1 \). Then we have
\[
\begin{align*}
\xi_{i_1} \xi_{i_2} \cdots \xi_{i_m}(E_{j_1}E_{j_2} \cdots E_{j_n}) &= \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(\xi_{i_m}(\Delta(E_{j_1}) \cdots \Delta(E_{j_n}))) \\
&= \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(\sum_{1 \leq p \leq n} E_{j_1} \cdots E_{j_{p-1}} K_{j_p} E_{j_{p+1}} \cdots E_{j_n} E_{j_p}) \\
&= \sum_{1 \leq p \leq n} \delta_{i_{m-1}, j_p} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(E_{j_1} \cdots \hat{E}_{j_p} \cdots E_{j_n} E_{j_p}).
\end{align*}
\]

By the hypothesis of induction, if \( \{i_1, \ldots, i_{m-1}\} \neq \{j_1, \ldots, j_{p-1} j_p + 1, \ldots, j_n\} \), then
\[
\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(E_{j_1} \cdots \hat{E}_{j_p} \cdots E_{j_n}) = 0.
\]

We consider the pair \((i'j')\) satisfying that \((i'j') < (ij)\) and that there is no pair \((i''j'')\) with \((i'j') < (i''j'') < (ij)\). It follows that
\[
\begin{align*}
\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(E_{i_1} \cdots E_{i_j}) &= 0, \tag{1} \\
\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(E_{i_1} E_{i_3} \cdots E_{i_j}) &= 0 \tag{2}.
\end{align*}
\]

In fact, \(\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(E_{i_1} \cdots E_{i_j})\) and \(\xi_{i_1} \xi_{i_2} \cdots \xi_{i_{m-1}}(E_{i_1} E_{i_3} \cdots E_{i_j})\) are \(C\)-linear combinations of the elements in (**).

We note that
\[
\Delta(E_{ij}) = E_{ij} \otimes 1 + (1 - \zeta^2) \sum_{i < k < j} K_{ik} E_{kj} \otimes E_{ik} + K_{ij} \otimes E_{ij},
\]
\[
\Delta(\xi_{ij}) = \xi_{ij} \otimes \eta_{ij}^{-1} + (1 - \zeta^2) \sum_{i < k < j} \xi_{ik} \otimes \eta_{ik}^{-1} \xi_{kj} + 1 \otimes \xi_{ij}.
\]

From these facts, it follows that if \( m_x > n_x \) and for any \( \beta \) with \( E_x < E_{\beta} \), \( m_\beta = 0 \) or \( n_\beta = 0 \), then
\[
\left( \prod_{x \in \Pi_+} \xi_x^{n_x} \eta_x \right) \left( \prod_{x \in \Pi_+} E_x^{m_x} K_v \right) = Y_{\xi_x^{n_x} \otimes \eta_x}(X E_x^{m_x} \otimes 1)(K_v \otimes K_v)
\]
universal elements for quantum group $U_q(sl(N+1,C))$,

$$= Y\zeta_{\alpha x}^{n_x}(X E_x^{m_x} K_x) \eta_w(K_v)$$

$$= Y\zeta_{\alpha x}^{n_x}(X E_x^{m_x}) \eta_w(K_v),$$

where $X = \Pi_x E_x E_x^{m_x}$ and $Y = \Pi_x E_x^{\xi x}$. By the equality $K_{ij} E_{ij} = \zeta E_{ij} K_{ij}$, we obtain

$$Y\zeta_{\alpha x}^{n_x}(X E_x^{m_x})$$

$$= Y\zeta_{\alpha x}^{n_x-1} \otimes \zeta((X \otimes 1) \Delta(E_x)^{m_x})$$

$$= Y\zeta_{\alpha x}^{n_x-1}(X E_x^{m_x-1}) \zeta(E_x)[m_x]$$

$$= \prod_{\alpha \in \Pi_+} \delta_{m_a n_a x}(E_x)^{m_w}[m_x],$$

where $[m] = \frac{\zeta^m-1}{\zeta-1}$. Here we have used the formula (\ast). Similarly, for $m_a < n_a$, the similar equality holds. Thus, we compute

$$\left( \prod_{\alpha \in \Pi_+} \zeta_{\alpha x}^{n_x} \eta_w \right) \left( \prod_{\alpha \in \Pi_+} E_x^{m_x} K_x \right) = \prod_{\alpha \in \Pi_+} \delta_{m_a n_a x}(E_x)^{m_w}[m_x] \zeta^{(v,w)}.$$

It follows that any element $\Pi_{\alpha \in \Pi_+} \zeta_{\alpha x}^{n_x} \eta_w$ is zero on $\phi_{ij}$, $1 \leq i \neq j \leq N$, $E_x^{ij}$, $1 \leq i < j \leq N+1$, $K_{ij}$, $1 \leq i \leq N$, and $\Lambda^{a-1}$, and from the triangular decomposition of $U_q$, $\phi$ is well-defined.

Moreover, the elements $\xi_{ij}$ and $\eta_i^{\pm}$ satisfy the following relations:

1. $\eta_i \eta_j = \eta_j \eta_i$, $\eta_i^{-1} \eta_i = \eta_i^{-1} = \varepsilon_i$, (2.1)
2. $\eta_i \xi_j = \varepsilon^{-1}(\xi_{ij}) \xi_j \eta_i$, (2.2)
3. $\xi_i \xi_j = (\xi + \xi^{-1}) \xi_i \xi_j + \xi_i \xi_j = 0$ (for $i = j = 1$), (2.3)
4. $\xi_i \xi_j = \xi_j \xi_i$ (for $i = j \geq 2$), (2.4)
5. $\xi_i^{1} = 0$, (2.5)
6. $\eta_i^{1} = \varepsilon_i \left( \prod_{j=1}^{N} \eta_i^{a} \right)^a = \varepsilon_i$, (2.6)
7. $\Delta(\xi_i) = \xi_i \otimes \eta_i^{-1} + 1 \otimes \xi_i \Delta(\eta_i^{1}) = \eta_i^{1} \otimes \eta_i^{1}$, (2.7)
8. $e(\xi_i) = 0$, $e(\eta_i^{1}) = 1$, (2.8)
9. $S(\xi_i) = -\xi_i \eta_i$, $S(\eta_i^{1}) = \eta_i^{1}$. (2.9)

One can prove these formulas by easy computations. In the following, we show only the formulas (2.2), (2.5), (2.6) and (2.7). For (2.2), $\eta_i \xi_i$ is non-zero only on $E_x K_x$.
where its value is \( \zeta^{(\Lambda, \omega)} b \zeta^{(\Lambda, \omega)} \) and \( \xi_{i} \) is non-zero only on \( E_{j} K_{w} \) where its value is \( b \zeta^{(\Lambda, \omega)} \). For (2.5), it follows from the above equality that \( \zeta^{(\Lambda, \omega)} (\Pi_{a \in \Pi,} E_{a}^{m_{a}} K_{w}) = 0 \). For (2.6), \( (\Pi_{j=1}^{N} \eta_{i})^{a} \) is non-zero only on \( K_{v} \). We have that for \( 1 \leq p \leq N \), \( (\Pi_{j=1}^{N} \eta_{i})^{a}(K_{p}) = 1 \). In fact, by the definition of \( \eta_{i} \), we have
\[
\left( \prod_{j=1}^{N} \eta_{i} \right)^{a}(K_{p}) = \zeta^{(\Lambda, \omega)}(\sum_{j=1}^{N} \eta_{i}) = \zeta^{(N, \omega)(N+1)} = 1.
\]
For (2.7), \( \Delta(\xi_{i}) \) is non-zero only on \( E_{i} K_{v} \otimes K_{w} \) and \( K_{v} \otimes E_{i} K_{w} \) where their values are respectively \( b \zeta^{(\Lambda, \omega)} \) and \( b \). On the other hand, \( \xi_{i} \otimes \eta_{i} \) is non-zero only on \( E_{i} K_{v} \otimes K_{w} \) where its value is \( b \zeta^{(\Lambda, \omega)} \) and \( \eta_{i}^{-1} \otimes \xi_{i} \) is non-zero only on \( K_{v} \otimes E_{i} K_{w} \) where its value is \( b \). The map \( \phi \) is a Hopf algebra homomorphism.

**Proposition 2.3.** We define \( \Phi: U_{i}^{+} \times U_{i}^{-} \rightarrow \mathbb{C} \) by \( \Phi(x, y) = \phi(y)(x) \) for \( (x, y) \in U_{i}^{+} \times U_{i}^{-} \). Then \( \Phi \) is non-degenerate. Moreover, \( \{ \Pi_{a \in \Pi,} E_{a}^{m_{a}} K_{v}(m_{a}) \in I, (v_{i}) \in J \} \) in proposition 2.1 is a \( \mathbb{C} \)-basis of \( U_{i}^{+} \) and the Hopf algebra homomorphism \( \phi \) is an isomorphism.

**Proof.** By the discussion in the proof of Proposition 2.2, it follows that
\[
\Phi \left( \prod_{a \in \Pi} E_{a}^{m_{a}} K_{v} \prod_{a \in \Pi} F_{a}^{n_{a}} K_{w} \right) = \prod_{a \in \Pi} \delta_{m_{a}, n_{a}} \zeta^{(E_{a})^{m_{a}}[m_{a}]_{\zeta}^{(\ell, w)}},
\]
where \( [m] = \frac{\ell_{m} - 1}{2} \), and \( [m]! = [m][m-1] \cdots [1] \).

For \( v, w \in P \), we put
\[
h_{v-w} = \sum_{\mu \in P} \zeta^{(\mu, v-w)}
\]
and
\[
u-w = x_{1} \alpha_{1} + \cdots + x_{N-1} \alpha_{N-1} x_{N} (x_{i}) = (v_{i}) - (w_{i}), (v_{i}) \in J.
\]
We have that
\[
h_{v-w} = \sum_{\substack{0 \leq u_{1}, \ldots, u_{N-1} \leq r-1 \leq r-1 \leq a-1 \leq a-1}} \zeta^{(\ell_{a})^{u_{1}+2x_{1}-x_{1}+1}} \left( \prod_{i=1}^{N-1} \sum_{u_{i}=0}^{r_{u_{i}}(-x_{i-1}+2x_{i}-x_{i})} \right) = \left( \prod_{i=1}^{N-1} \sum_{u_{i}=0}^{r_{u_{i}}(-x_{i-1}+2x_{i}-x_{i})} \right) \zeta^{(\ell_{a})^{u_{1}+2x_{1}-x_{1}+1}}.
\]
We assume \( h_{v-w} \neq 0 \). Then \( \Pi_{i=1}^{N-1} \sum_{u_{i}=0}^{r_{u_{i}}(-x_{i-1}+2x_{i}-x_{i})} \neq 0 \). Hence we have that
\[-x_{i-1}+2x_{i}-x_{i+1} \equiv 0 \pmod{r}, 2 \leq i \leq N-1 \text{ and } x_{2} \equiv 2x_{1} \pmod{r}.
\]
So, it follows that
\[ x_{i+1} = 2x_i - x_{i-1} = 2ix_1 - (i-1)x_1 \equiv (i+1)x_1 \pmod{r}. \]

Thus we obtain \( x_i \equiv ix_1 \pmod{r}, 1 \leq i \leq N \). From the equality

\[
\sum_{u_N=0}^{a-1} \xi^u x_{N-1} + 2x_N = \sum_{u_N=0}^{a-1} \xi^u x_{N+1} + x_1 \neq 0,
\]

we obtain that \( x_1 \equiv 0 \pmod{a} \), noting that \( \xi^a \) is a primitive \( a \)-th root unity. While \( x_N \equiv N x_1 \pmod{r} \) and \( ad = r \), we have that \( x_N \equiv 0 \pmod{a} \). As \( |x_N| < a \), it follows that \( x_N = 0 \). From the formulas \( x_i \equiv ix_1 \pmod{r} \) and \( x_i = 0 \pmod{a} \), we have that \( -x_{N-1} + 2x_N = (N+1)x_1 = 0 \pmod{r} \) and so \( x_{N-1} = 2x_N \pmod{r} \). From the equality \( x_{i-1} = 2x_i - x_{i+1} \pmod{r} \), by induction, we have that \( x_i \equiv (N-i+1)x_N = 0 \pmod{r} \). As \( |x_i| < r \) for \( 1 \leq i \leq N-1 \), we obtain that \( x_i = 0 \) for \( 1 \leq i \leq N-1 \). Thus we obtain that \( h_{v-w} \neq 0 \) if and only if \( v = w \). Let \( L = |J| \), and then

\[
\Phi \left( \frac{1}{L} \sum_{e \in \Pi^+} \xi^{(v,w)} K_{w} K_{v} \right) = \delta_{vw}.
\]

For \( m = (m_{\alpha})_{\alpha \in \Pi^+} \), we put

\[
c_m = \prod_{\alpha \in \Pi^+} \left( \frac{1}{\xi - \xi^{-1}(-\xi)^{ht(\alpha)-1}} \right)^{m_{\alpha}} [m_*]!
\]

where \( c_m \) is non-zero. From the above discussion,

\[
\left\{ \frac{1}{L} \sum_{e \in \Pi^+} \xi^{(v,w)} \prod_{\alpha \in \Pi^+} E_{\alpha}^{m_{\alpha}} K_{v} \right\}_{(m_{\alpha})_{\alpha \in \Pi^+}} = \left\{ \prod_{\alpha \in \Pi^+} F_{\alpha}^{m_{\alpha}} K_{w} \right\}_{(m_{\alpha})_{\alpha \in \Pi^+}}
\]

is a basis for \( U_{\xi^+} \) and \( U_{\xi^-} \), and they are orthonormal for the pairing \( \Phi \). Thus \( \Phi \) is non-degenerate and \( \left\{ \prod_{\alpha \in \Pi^+} E_{\alpha}^{m_{\alpha}} K_{v} \right\}_{(m_{\alpha})_{\alpha \in \Pi^+}} \) is a \( \mathcal{C} \)-basis of \( U_{\xi^+} \), by Proposition 2.1. From the definition of \( \Phi \), the homomorphism \( \varphi \) is an isomorphism.

Now we apply the quantum double construction to the Hopf algebra \( U_{\xi^+} \). By the definition of the multiplication of the quantum double, one can derive the following Lemma.

**Lemma 2.4.** Let \( e_i = E_i \otimes 1 \), \( k_i^+ = K_i^+ \otimes 1 \), \( f_i = 1 \otimes \varphi(F_i) \), and \( h_i^\pm = 1 \otimes \varphi(K_i^\pm) \) in the quantum double \( D(U_{\xi^+}) \). These elements satisfy the following commutation relations:

1. \( k_i h_j = h_j k_i k_i^{-1} h_i = k_i^{-1} h_i = 1 \)
2. \( h_i e_j = \xi^{(t_i, s_j)} e_j h_i k_i f_j = \xi^{-(t_i, s_j)} f_j k_i \)
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(3) \[ [e_i f_j] = \delta_{i j} \frac{k_i - h_i^{-1}}{\zeta - \zeta^{-1}}. \] (2.13)

Proof. For (2.13), we have

\[ f j [\xi_j] = S(\xi_j) (e_i f_j) = f_j [\xi_j] S(\xi_j) (1) + S(1) (f_j) [\xi_j] = \delta_{ij} \frac{k_i}{\zeta - \zeta^{-1}} + e_i f_j - \delta_{ij} \frac{k_i}{\zeta - \zeta^{-1}}, \]

where \[ \xi_j = \varphi(F_j) \] and \( \eta_i = \varphi(K_i) \). The other relations are also immediately obtained.

The Hopf algebra structure on \( D(U_\zeta^+) \) induces the one on \( U_\zeta^+ \).

Proposition 2.5. Let us define a map \( \psi : D(U_\zeta^+) \to U_\zeta^+ \) by \( \psi(x \otimes y) = x \varphi^{-1}(y) \) \( \psi(x \otimes y) = x \varphi^{-1}(y) \) for \( x \otimes y \in U_\zeta^+ \otimes (U_\zeta^+)^* \cong D(U_\zeta^+) \). Then the map \( \psi \) is a Hopf algebra epimorphism.

Proof. Comparing Lemma 2.4 with the commutation relations between \( E_i, F_i \) and \( K_i, 1 \leq i \leq N \), one can easily show that \( \psi \) is an algebra homomorphism. From the fact that \( \varphi^{-1} \) is a Hopf algebra isomorphism, due to the Hopf algebra structure of \( D(U_\zeta^+) \), it follows that \( \psi \) is a Hopf algebra homomorphism. The surjectivity of \( \psi \) follows from the fact that any element \( X_1 \cdots X_p, X_i \in \{ E_i, F_i, K_i \}, 1 \leq i \leq N \) is written as a \( C \)-linear combination of the elements \( X_\zeta Y_\zeta, X_\zeta, Y_\zeta \in U_\zeta^+ \), \( Y_\zeta \in U_\zeta^- \), using the relations (1.4) and (1.5).

Now, we obtain an explicit formula for a universal \( R \) of \( U_\zeta \), as the image of the universal \( R \) of \( D(U_\zeta^+) \) under \( \psi \otimes \psi \).

Theorem 2.6. A universal \( R \)-matrix for \( U_\zeta \) is given by

\[ R = \frac{1}{[c_m]} \sum_{(m_a)_{a \in \Pi^+}, 0 \leq m_a < \bar{r}} \frac{1}{c_m} \prod_{a \in \Pi^+} E_a^{m_a} K_a \otimes \prod_{a \in \Pi^+} F_a^{m_a} K_a, \] (2.14)

where

\[ I = \{(m_a)_{a \in \Pi^+}, 0 \leq m_a < \bar{r} \}, \]
\[ J = \{(v_i)_{1 \leq i \leq N}, 0 \leq v_p < r_p = 1, \cdots, N - 1, 0 \leq v_N < a \}, \]
\[ L = r^{-1}a, \]
\[ c_m = \prod_{a \in \Pi^+} \left( -\frac{1}{\zeta - \zeta^{-1}}(\zeta^{-1} - 1) \right)^{m_a} \] for \( m = (m_a)_{a \in \Pi^+} \).
Proof. Since the universal $R$ of $D(U_q)$ satisfies (1.1) and (1.2), and $\psi$ is a Hopf algebra epimorphism, $R$ also satisfies (1.1) and (1.2).

3. Results from the universal $R$-matrix for $U_q$

We recall how one can obtain tangle operators from representations of the quasitriangulra Hopf algebra $(U_q, R)$, where $R$ is the universal $R$-matrix for $U_q$ in the previous section [13].

For non negative integers $k$ and $l$, a $(k,l)$-tangle $T$ is a smooth 1-manifold in $R^2 \times [0,1]$ such that its boundary $\partial T = \{(i,0,0)|1 \leq i \leq k\} \cup \{(j,0,1)|1 \leq j \leq l\}$. We put $\partial T_+ = \{(i,0,0)|0 \leq i \leq k\}$ and $\partial T_- = \{(j,0,1)|1 \leq j \leq l\}$. All tangles are assumed to be oriented.

It is well-known that every tangle diagram can be reconstructed from the elementary diagrams in Fig.3.1, using the composition $\circ$ (when defined) and the tensor product $\otimes$ in the Fig.3.2.

A coloring of a tangle $T$ is defined to be an assignment of a $U_q$-module to each component of $T$. According to a coloring, we assign $U_q$-modules $T_\pm$ to $\partial T_\pm$ as follows: if an arc $S$ of $T$ has a color $V$, then to each boundary point in $R^2 \times \{0,1\}$ associate $V$ if the orientation is downwards and associate $V^*$ if it is upwards. Then the $U_q$-module $T_+$ (resp. $T_-$) is the tensor product from left to right of the $U_q$-modules associated to $\partial T_+$ (resp. $\partial T_-$). By convention, $T_\pm = \mathbb{C}$ if $T$ is a link.

In this paper, we consider the following family of irreducible representations of $U_q$ with $\bar{r} = K + N + 1$ for a positive integer $K$. Let $\alpha_1, \ldots, \alpha_N$ be the simple roots of $\mathfrak{sl}(N + 1, \mathbb{C})$ and we put

$$P_+(K) = \{ \lambda \in \mathfrak{h}^*| (\lambda, \alpha_i) \in \mathbb{Z}, 0 \leq (\lambda, \alpha_i), i = 1, \ldots, N, 0 \leq (\lambda, \theta) \leq K \},$$

where $\theta$ is the longest root, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}(N + 1, \mathbb{C})$. Let $\lambda_1, \ldots, \lambda_N$ be the fundamental dominant integral weight: each $\lambda_i$ satisfies $(\lambda_i, \alpha_j) = \delta_{ij}$ for any $\alpha_j$. We see that $\lambda = \sum_{i=1}^N m_i \lambda_i$ for integers $m_1, \ldots, m_N$. For each $\lambda \in P_+(K)$, there exists an irreducible highest weight module $V_\lambda$ of $U_q$ with highest weight $\lambda$ and

![elementary diagrams](Fig. 3.1)

![elementary diagrams](Fig. 3.2)
highest weight vector $e_\lambda$ such that
\[ N_+ e_\lambda = 0, \quad V_\lambda = N_- e_\lambda, \quad K \rho e_\lambda = ^{\rho(\lambda,\nu)} e_\lambda. \]
Here $N_+$ is the subalgebra of $U_\zeta$ generated by $E_i$, $1 \leq i \leq N$ and $N_-$ is the subalgebra of $U_\zeta$ generated by $F_i$, $1 \leq i \leq N$.

Let $T$ be a colored tangle such that each color of a component of $T$ is contained in the set \( \{ V_\lambda | \lambda \in P_+(K) \} \). When $S_1, \cdots, S_n$ are the components of $T$, a coloring of $T$ can be viewed as the map \( \{1, \cdots, n\} \to P_+(K) \). As is shown in [13], there exists a $U_\zeta$-linear map $F_T: T_- \to T_+$ such that it satisfies $F_{T'T'} = F_T \circ F_{T'}$ and $F_{T \otimes T} = F_T \otimes F_T$, and for elementary diagrams,
\[ F_{\downarrow} = \text{id}_{V_\lambda}, \quad F_{\downarrow} = \text{id}_{V_\lambda}, \]
\[ F_{\chi}(x \otimes y) = \sum_k \beta_k y \otimes \alpha_k x, \quad \text{where} \quad R = \sum_k x_k \otimes \beta_k, \]
\[ F_{\chi}(x \otimes y) = \sum_k \beta_k y \otimes \alpha_k x, \quad \text{where} \quad R^{-1} = \sum_k x_k \otimes \beta_k, \]
\[ F_{\chi}(f \otimes x) = f(x), \quad F_{\chi}(x \otimes f) = f(K_{\rho}^{-1} x), \]
where $K_\rho = \Pi_{\rho \in P_+} K_\rho$. If $L$ is a colored oriented link with coloring $\nu$, $F_L$ is a scalar map. We denote this scalar by $J(L, \nu)$.

In the following proposition, using the explicit formula (2.14) of the universal $R$ for $U_\zeta$, we shall compute two values, which are essential in the construction of 3-manifold invariants. We put $q = \xi^2$.

**Proposition 3.1.** (1) Let $H_{\lambda \mu}$ be a colored Hopf link such that the colors of the two components are $V_\lambda$ and $V_\mu$ drawn in Fig. 3.3. Then we have
\[ \lambda \quad \text{Fig. 3.3} \]
\[ \mu \quad \text{Fig. 3.4} \]
\[ = \exp 2\pi \sqrt{-1} \Delta \lambda \]
where $\rho$ is half the sum of positive roots and $W$ is the Weyl group.

(2) Let $T$ be a colored $(1,1)$-tangle such that the one component has a color $V_\lambda$ in Fig.3.4. Then $F_T$ is the multiplication by $\exp 2\pi i/\Delta_\lambda$, where $\Delta_\lambda = (\lambda, \lambda + 2\rho)$.

Proof. (1) We consider the colored $(1,1)$-tangle in Fig.3.5. Since $V_\lambda$ is irreducible, $F_T$ is a scalar map. We denote this scalar by $b_{\lambda\mu}$. To compute $b_{\lambda\mu}$, it is enough to evaluate $F_T(e_\lambda)$ for the highest weight vector $e_\lambda$. If $R = \Sigma a_k \otimes \beta_k$, then we see $R^{-1} = (\text{id} \otimes S)(R)$. From the definitions of tangle operators, one can obtain

$$F_T(e_\lambda) = b_{\lambda\mu} e_\lambda = \sum_{k,l} S(\beta_k) \alpha_k \text{Tr}_\mu (K_\rho^{-1} \alpha_k S(\beta_l)) e_\lambda.$$ 

By the formula (2.14), one has

$$b_{\lambda\mu} e_\lambda = \frac{1}{L^2} \sum_{(m_a)(n_a) \in J} S \left( \prod_{a \in \Pi^+} F_a^{n_a} K_w \right) \left( \prod_{a \in \Pi^+} F_a^{m_a} K_u \right) \text{Tr}_\mu \left( K_\rho^{-1} \prod_{a \in \Pi^+} E_a^{n_a} K_u \cdot \left( \prod_{a \in \Pi^+} F_a^{m_a} K_v \right) \right) e_\lambda.$$

Since $e_\lambda$ is the highest weight vector, the only terms with $m_a = n_a = 0$ for any $\alpha \in \Pi^+$ are non zero. Thus one can get

$$b_{\lambda\mu} e_\lambda = \frac{1}{L^2} \sum_{(m_a)(n_a) \in J} K_w^{-1} r(\nu, \omega) K_u \text{Tr}_\mu (K_\rho^{-1} r(\nu, \omega) K_v K_v^{-1}) e_\lambda$$

$$= \frac{1}{L^2} \sum_{(m_a)(n_a) \in J} r(\nu, \omega) r(\nu, \lambda) \text{Tr}_\mu (K_\rho^{-1} r(\nu, \omega) K_v K_v^{-1}) e_\lambda.$$

Noting that $\Sigma_{(u,\nu) \in J} r(\nu, \nu - \nu) \neq 0$ if and only if $\nu = \nu$, we can compute

$$b_{\lambda\mu} e_\lambda = \frac{1}{L} \sum_{(m_a)(n_a) \in J} r(\nu, \lambda) \text{Tr}_\mu (K_\rho^{-1} r(\nu, \omega) K_v K_v^{-1}) e_\lambda$$

$$= \sum_{m_a} r(\nu, \lambda) r(\nu, \nu) e_\lambda$$

$$= \sum_{m_a} \delta(\lambda + \rho, \nu) e_\lambda.$$
where \{\mu_j\} is the set of weights of \( V_\mu \) with multiplicity and \( \xi^2 = q \). It follows from the character formula of Weyl (see for example [10]) that

\[
\mathbf{b}_\lambda = \frac{\sum_{w \in W} (\det w) q^{(\lambda, \mu + \rho + w(\mu + \rho))}}{\sum_{w \in W} (\det w) q^{(\lambda + \rho, w(\rho))}}.
\]

Let \( L_\lambda \) be a colored unknot with a color \( \lambda \) in Fig. 3.6. Then we see

\[
J(L_\lambda) = \frac{\sum_{w \in W} (\det w) q^{(\lambda + \rho, w(\rho))}}{\sum_{w \in W} (\det w) q^{(\rho, w(\rho))}},
\]

which is called the quantum dimension of \( V_\lambda \) and we write it by \( \dim_q V_\lambda \). Since \( J(H_{\lambda \mu}) = \mathbf{b}_\lambda \dim_q V_\lambda \) according to [13, Lemma 2.6], the formula (3.1) holds.

(2) As the representation \( V_\lambda \) is irreducible, the tangle operator \( F_\tau \) is a scalar map. We denote this scalar by \( v_\lambda \). To compute \( v_\lambda \), it is enough to evaluate \( F_\tau(e_\lambda) \) for the highest weight vector \( e_\lambda \) of \( V_\lambda \). When \( R = \sum_{k} \otimes_\beta_k \), one can see

\[
F_\tau(e_\lambda) = \alpha_\lambda K_\rho e^\lambda(\beta_\lambda e_\lambda) e_\lambda.
\]

From computations similar to the one made in the proof of (1), it follows that

\[
v_\lambda e_\lambda = \frac{1}{L(v)} \sum_{(v,w) \in J} \xi^{(v,w)} K_\rho e_\lambda(\xi K_\rho e_\lambda) e_\lambda
\]

\[
= \frac{1}{L(v)} \sum_{(v,w) \in J} \xi^{(v,w)} \xi^{(v,\lambda)} \xi^{(\lambda + \rho, \lambda)} e_\lambda(\xi \xi e_\lambda)
\]

\[
= \xi^{(\lambda, \lambda)} \xi^{(\lambda + \rho, \lambda)} e_\lambda
\]

\[
= q^{(\lambda, \lambda + 2\rho)} e_\lambda
\]

Thus the claim holds.
Let $S = (S_{\lambda \mu})$ be the so-called $S$-matrix due to Kac [7], which is given by

$$S_{\lambda \mu} = \sqrt{-\frac{1}{N(N+1)/2}} \sum_{w \in W} (\det w) \overline{q}^{\lambda + \rho, w(\mu + \rho)}.$$  \hfill (3.2)

Comparing (3.1) with (3.2), one easily sees that $S_{\lambda \mu} = S_{\lambda 0} b_{\lambda \mu}$.

By the discussion in [9], for any closed oriented connected 3-manifold $M$,

$$Z_\sigma(M) = C^{\sigma} \sum_{\text{col}(L)} S_{0v(L)} \cdots S_{0v(n)} I(L, v)$$

is a topological invariant of $M$, where $C = (\exp 2\pi \sqrt{-1})^{-\frac{3}{2}}$, $c = \frac{K \dim \mathfrak{sl}(N+1, \mathbb{C})}{L}$, $L$ is a framed link with $n$ components such that $M$ is obtained by Dehn surgery of $S^3$ along $L$, $\sigma$ is the signature of the linking matrix of $L$, and $\text{col}(L)$ means the set of colorings of $L$.

We denote by $\text{Rep}(\mathfrak{sl}(N+1, \mathbb{C}))$ the representation ring of $\mathfrak{sl}(N+1, \mathbb{C})$. It is well-known that the representations of $\mathfrak{sl}(N+1, \mathbb{C})$ with fundamental weight $\lambda_i$, $1 \leq i \leq N$, generate $\text{Rep}(\mathfrak{sl}(N+1, \mathbb{C}))$. We put $\partial P_+(K) = P_+(K+1) \setminus P_+(K)$. Let $I_K$ be the ideal of $\text{Rep}(\mathfrak{sl}(N+1, \mathbb{C}))$ generated by the representations $W_{\lambda}$, $\lambda \in \partial P_+(K)$. We put $R_K = \text{Rep}(\mathfrak{sl}(N+1, \mathbb{C}))/I_K$.

In [4], Goodman-Wenzl showed that the algebra $R_K$ is a free $\mathbb{Z}$-module with basis $w_{\lambda}$ corresponding to $\lambda \in P_+(K)$ and that

$$w_{\lambda} \cdot w_{\mu} = \sum_{\nu} N_{\lambda \mu}^{\nu} w_{\nu},$$

for non-negative integers $N_{\lambda \mu}^{\nu}$, which are called the fusion rule.

In $\text{Rep}(U_q)$, the irreducible representation $V_{\lambda}$, $\lambda \in P_+(K)$, can be written as a formal sum of monomials in the fundamental representations $V_{\lambda_i}$, $1 \leq i \leq N$ such that the monomials are in the span of $\{V_{\omega} \mid \omega \in P_+(K)\}$. This follows from the induction on the lexicographic order of Young diagrams, applying Littlewood-Richardson rule to the decomposition of the tensor products of $V_{\lambda}$ and $V_{\lambda_i}$. Using the formal expressions, we can obtain the decomposition $V_{\lambda} \otimes V_{\mu} = \sum_{\nu} n_{\lambda \mu}^{\nu} V_{\nu} + Z_{\lambda \mu}$ for $\lambda$, $\mu$, where $n_{\lambda \mu}^{\nu}$ are integers and $Z_{\lambda \mu}$ is contained in the ideal generated by the irreducible representations $V_{\omega}$ for $\omega \in \partial P_+(K)$. Since in decomposing tensor products of the fundamental representations and $V_{\lambda}$, $\lambda \in P_+(K)$, we can apply Littlewood-Richardson rule, in a way similar to the proof in Lemma 3.1 in [4], we get $n_{\lambda \mu}^{\nu} = N_{\lambda \mu}^{\nu}$. It follows that for $\lambda, \nu \in P_+(K)$,

$$V_{\lambda} \otimes V_{\mu} = \sum_{\nu \in P_+(K)} N_{\lambda \mu}^{\nu} V_{\nu} + Z_{\lambda \mu}. \hfill (3.3)$$

We recall that the quantum dimension means the trace of the representation matrix.
of $K_\omega$ and denote the quantum dimension of $U_\zeta$ module by $\dim_q V$. One can extend the definition of the quantum dimension to a $C$-linear map from $\text{Rep}(U_\zeta)$ to $C$. As the quantum dimension of $V_\omega$ for $\omega \in P_+(K)$, is equal to 0 from the equality $[\bar{f}] = 0$ (also see [3]), that of the tensor product of $V_\omega$ and any representation of $U_\zeta$ is also equal to 0. From these two facts, the extended quantum dimension of $Z_{\lambda\mu}$ is 0.

**Remark.** It is shown in [1] that for $\lambda$, $\mu$, we have a decomposition

$$V_\lambda \otimes V_\mu = \bigoplus (M_{\lambda\mu} \otimes V_\lambda) \oplus Z_{\lambda\mu},$$

where the dimension of $C$-module $M_{\lambda\mu}$ is equal to $N_{\lambda\mu}$ and the quantum dimension of $Z_{\lambda\mu}$ is 0. Although, we don't need the fact.

As is shown in [13] for $sl(2,C)$ by Reshetikhin and Turaev, we extend $Z_\zeta(M)$ to $Z_\zeta(M,T)$ for $M$ which contains a colored framed link $L$. Let $T$ be a colored framed link in $S^3$ and we suppose that $M$ is obtained by Dehn surgery on $L$. Then we think of $T \cup L$ as a framed link in $S^3$, and we put

$$Z_\zeta(M,T) = C^* \sum_{\text{vec}(L)} S_{0v(1)} \cdots S_{0v(n)} I(L \cup T,v).$$

From the above observation, one can get Verlinde's formula for the fusion algebra $R_\zeta$ with the fusion rule due to Goodman-Wenzl.

**Proposition 3.2.** The $S$-matrix $(S_{\lambda\mu})_{\lambda,\mu \in P_+(K)}$ and the fusion rule $N^*_{\lambda\mu}$ satisfy Verlinde's formula:

$$N^*_{\lambda\mu} = \sum_{\nu \in P_+(K)} \frac{S_{\lambda\nu} S_{\mu\nu} S^*_{\nu\nu}}{S_{0\nu}},$$

![Fig. 3.7](image1.png)  
![Fig. 3.8](image2.png)
where for \( \lambda, \mu \in P_+(K) \),

\[
S_{\lambda \mu} = \frac{-1^{(N+1)/2}}{\sqrt{(N+1)^2}} \sum_{w \in W} (\det w) \hat{q}^{(\lambda, w(\mu, \mu))}.
\]

Proof. Let us consider \( S^2 \times S^1 \) containing the 3-component link \( L_{\lambda \mu \nu} \) with colors \( \lambda, \mu, \nu \) as drawn in Fig.3.7, where for the longest element \( w_0 \) in the Weyl group, \( \lambda^* = -w_0(\lambda) \). Let \( L \) be an unknotted circle with the zero framing which links \( L_{\lambda \mu \nu} \) as drawn in Fig.3.8. By the Dehn surgery on \( S^3 \) along the circle \( L \), one can obtain \( (S^2 \times S^1, L_{\lambda \mu \nu}) \). In a way similar to the proof in [16, §3], we prove the assertion, evaluating \( Z_r(S^2 \times S^1, L_{\lambda \mu \nu}) \) in two ways.

We note that for \( \lambda \in \partial P_+(K) \), \( V_\lambda \) is irreducible and the quantum dimension \( \dim_q V_\lambda = 0 \), and that a colored link with a component assigned with the tensor product of \( V_{\omega} \), \( \omega \in \partial P_+(K) \) and the fundamental representations can be regarded as a colored link with a component assigned \( V_{\omega} \), \( \omega \in \partial P_+(K) \). Then, by the formula (3.3) and the unitarity of the S-matrix \( (S_{\lambda \mu}) \) [7], we can compute

\[
Z_r(S^2 \times S^1, L_{\lambda \mu \nu}) = \sum_{\epsilon \in P_+(K)} S_{\epsilon 0}\left( \sum_{\epsilon' \in P_+(K)} \frac{S_{\epsilon \epsilon'} S_{\epsilon' \nu}}{S_{\epsilon 0} S_{00}} N_{\lambda \mu}^{\epsilon'} \right)
\]

\[
= \sum_{\epsilon' \in P_+(K)} N_{\lambda \mu}^{\epsilon'} \left( \frac{1}{S_{00}} \delta_{\epsilon' \nu} \right)
\]

\[
= \frac{1}{S_{00} N_{\lambda \mu}^{\epsilon}}.
\]

On the other hand, a link \( L_{\lambda \mu \nu} \cup L \) can be regarded as the result of connecting 3 Hopf links in a way analogous to the proof in [16], and so we can directly compute from Proposition 3.1 (1)

\[
Z_r(S^2 \times S^1, L_{\lambda \mu \nu}) = \frac{1}{S_{00}} \sum_{\epsilon} S_{\lambda \mu}^\epsilon S_{\mu \nu}^\epsilon S_{\lambda \nu}^\epsilon.
\]

Thus the claim follows from the comparison of these two evaluations.

References


