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UNIVERSAL R -MATRICES FOR THE QUANTUM GROUP $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$: THE ROOT OF UNITY CASE

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Introduction

The aim of this paper is to construct a universal R -matrix for a certain quotient of the quantized universal enveloping algebra $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$ in the sense of Drinfel'd [2] and Jimbo [5][6] at roots of unity. The notion of universal R -matrix is due to Drinfel'd. A universal R -matrix for a Hopf algebra A over \mathbb{C} is an invertible element $R \in A \otimes A$ with the following properties: (1) $R\Delta(a)R^{-1} = \check{\Delta}(a)$, for $a \in A$, (2) $(\Delta \otimes id)(R) = R_{13}R_{23}$, $(id \otimes \Delta)(R) = R_{13}R_{12}$. Here $\Delta: A \rightarrow A \otimes A$ is the comultiplication, and $\check{\Delta}$ is the opposite comultiplication $\check{\Delta} = P \circ \Delta$ for the permutation P in $A \otimes A$, $P(a \otimes b) = b \otimes a$. The map Δ is not in general symmetric in the sense that $\check{\Delta} \neq \Delta$, but from the property (1) of this universal R -matrix, there arises an A -module isomorphism $V \otimes W \rightarrow W \otimes V$ for A -modules V and W . It follows from two properties (1) and (2) that it satisfies the Yang-Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where R_{ij} is the embedding of R into the i -th and j -th factor of $A \otimes A \otimes A$.

In [14], Rosso gave an explicit formula of universal R -matrix for $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$ for generic q , and in [15], he obtained a universal R -matrix for a quotient of $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$ when q is a primitive r -th root of unity for an integer r satisfying that $r \geq N+1$ and that r and $N+1$ are coprime. The result was independently obtained in [17]. In [23],[24],[25], and [26], Yamane introduced quasi-triangular Hopf algebras associated to complex simple Lie superalgebras of types A-G, and gave explicit formulas of their universal R -matrices, both in generic and non-generic cases. In particular, he got an explicit formula of a universal R -matrix for a quotient of $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$.

In the present paper, we give an explicit formula of a universal R -matrix for a quotient of $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$ for a primitive r -th root of q of unity, $r \neq 1, 2, 4$. Let E_i, F_i , and K_i , $1 \leq i \leq N$, be the generators of the Hopf algebra $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$. Let U^+ be the Hopf subalgebra $U_q(\mathfrak{sl}(N+1, \mathbb{C}))$ generated by E_i, K_i , $1 \leq i \leq N$ and U^- the Hopf subalgebra generated by F_i, K_i , $1 \leq i \leq N$. The construction of the universal R -matrix

is based on the quantum double construction due to Drinfel'd [2]. An essential point of this construction is the existence of a non-degenerate pairing $U^+ \times U^- \rightarrow \mathbb{C}$ compatible with the Hopf algebra structures of U^+ and U^- . Since a pairing naturally defined degenerates when q is a root of unity, we consider, following Yamane [25], a certain quotient of $U_q(sl(N+1, \mathbb{C}))$.

For $N \in \mathbb{N}$ and $1 < r \in \mathbb{N}$, we put $d = (r, N+1)$, $a = \frac{r}{d}$, $\bar{r} = \frac{r}{(r, 2)}$. Let ζ be a primitive r -th root of unity with $(\zeta + \bar{\zeta})(\zeta - \bar{\zeta}) \neq 0$. We remark that ζ^{N+1} is a primitive a -th root of unity, and ζ^2 is a primitive \bar{r} -th root of unity. Let $(a_{ij})_{1 \leq i, j \leq N}$ be the Cartan matrix for $sl(N+1, \mathbb{C})$. In the present paper, we consider the Hopf algebra U_ζ which is a quotient Hopf algebra of $U_\zeta(sl(N+1, \mathbb{C}))$.

As an algebra U_ζ is generated by $E_i, F_i, K_i, K_i^{-1}, \Lambda = \prod_{i=1}^N K_i$ for $1 \leq i \leq N$ with the relations:

$$\begin{aligned} K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ K_i E_j &= \zeta^{(a_i, \alpha_j)} E_j K_i, \quad K_i F_j = \zeta^{-(a_i, \alpha_j)} F_j K_i, \\ [E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{\zeta - \zeta^{-1}}, \\ E_i^2 E_j - (\zeta + \zeta^{-1}) E_i E_j E_i + E_i E_j^2 &= 0 \quad (|i-j|=1), \\ E_i E_j &= E_j E_i \quad (|i-j| \geq 2), \\ F_i^2 F_j - (\zeta + \zeta^{-1}) F_i F_j F_i + F_i F_j^2 &= 0 \quad (|i-j|=1), \\ F_i F_j &= F_j F_i \quad (|i-j| \geq 2), \\ E_i \bar{F}_j &= F_i \bar{E}_j = 0, \\ K_i^r &= 1, \quad \Lambda^a = 1, \end{aligned}$$

where $(\alpha_i, \alpha_j) = \alpha_{ij}$, and for $1 \leq i \leq j \leq N+1$ and $X = E$ or F , the element X_{ij} is inductively defined by

$$X_{ij} = \begin{cases} X_i & \text{if } j = i + 1, \\ X_{ij-1} X_{j-1} - \zeta X_{j-1} X_{ij-1} & \text{if } j > i + 1. \end{cases}$$

Let U_ζ^+ be the Hopf subalgebra of U_ζ generated by $E_i, K_i^\pm, 1 \leq i \leq N$, U_ζ^- the Hopf subalgebra of U_ζ generated by $F_i, K_i^\pm, 1 \leq i \leq N$, and $(U_\zeta^+)^o$ the dual algebra of U_ζ^+ with the opposite comultiplication. We construct a Hopf algebra isomorphism $\varphi: U_\zeta^- \rightarrow (U_\zeta^+)^o$, and give an explicit formula of an orthonormal basis with respect to the pairing Φ .

Applying the quantum double construction to the Hopf algebra U_ζ^+ , we see that the Hopf algebra isomorphism φ induces a Hopf algebra epimorphism ψ from the quantum double $D(U_\zeta^+)$ to the Hopf algebra U_ζ . The image of the universal R of $D(U_\zeta^+)$ under $\psi \otimes \psi$ is a universal R of U_ζ .

As well-known, a universal R can be used in producing tangle invariants obtained from the representations of the quantized universal enveloping algebras for classical simple Lie algebras (see for example [11][12][13][18][19]). As an application of our universal R , we can calculate some tangle invariants, which are essential in the construction of Witten's 3-manifold invariants [21].

For any positive integer K , let $P_+(K)$ be the set of the dominant integral weights λ with $0 \leq (\lambda, \theta) \leq K$, where θ denotes the longest root. We consider the family of finite dimensional irreducible representations of U_ζ whose highest weight λ is contained in $P_+(K)$, in the case $\bar{r} = K + N + 1$. For an oriented framed link L , we denote by $J(L)$ the tangle invariant obtained by using these irreducible representations. Using our explicit formula of universal R for U_ζ in the case $\bar{r} = K + N + 1$, one can calculate $J(H_{\lambda\mu})$, where $H_{\lambda\mu}$ denotes Hopf link with two components assigned with V_λ and V_μ :

$$J(H_{\lambda\mu}) = \frac{\sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(\mu + \rho))}}{\sum_{w \in W} (\det w) \bar{q}^{(\rho, w(\rho))}}.$$

Here ρ is half the sum of positive roots. Let $S = (S_{\lambda\mu})$ be the modular transformation S matrix for characters of the integrable highest weight modules due to Kac and Peterson [7]. Using the equality $S_{\lambda\mu} = S_{00} J(H_{\lambda\mu})$, we show Verlinde's formula for the fusion algebra of type $A_N^{(1)}$. The fusion algebra is an associative commutative ring with basis labelled by $P_+(K)$ and the product $w_\lambda \cdot w_\mu$ of two basis elements can be written as a sum $\sum N_{\lambda\mu}^\nu w_\nu$ with structure constants $N_{\lambda\mu}^\nu \in \mathbb{N}$ called the fusion rule. The modular transformation S -matrix and the fusion rules $N_{\lambda\mu}^\nu$'s are related by Verlinde's formula [20]:

$$N_{\lambda\mu}^\nu = \sum_{\varepsilon \in P_+(K)} \frac{S_{\lambda\varepsilon} S_{\mu\varepsilon} S_{\nu\varepsilon}^*}{S_{0\varepsilon}}.$$

The paper is organized as follows: In §1, we recall the quantum double construction due to Drinfel'd and define the Hopf algebra U_ζ . In §2, a universal R for U_ζ is obtained, applying the quantum double construction to the Hopf subalgebra U_ζ^+ of U_ζ . In §3, we state tangle operators derived from irreducible representations of U_ζ , and calculate some tangle invariants. As an application of the tangle invariants, we prove Verlinde's formula for the fusion algebra of type $A_N^{(1)}$.

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1. Hopf algebra U_ζ and quantum double construction

In this section, we define the Hopf algebra and recall the quantum double

$$E_i^2 E_j - (\zeta + \zeta^{-1}) E_i E_j E_i + E_i E_j^2 = 0 \quad (|i-j|=1), \tag{1.6}$$

$$E_i E_j = E_j E_i \quad (|i-j| \geq 2), \tag{1.7}$$

$$F_i^2 F_j - (\zeta + \zeta^{-1}) F_i F_j F_i + F_i F_j^2 = 0 \quad (|i-j|=1), \tag{1.8}$$

$$F_i F_j = F_j F_i \quad (|i-j| \geq 2), \tag{1.9}$$

$$E_{ij}^{\bar{r}} = F_{ij}^{\bar{r}} = 0, \tag{1.10}$$

$$K_i^r = 1, \Lambda^a = 1, \tag{1.11}$$

where, for integers i and j with $1 \leq i < j \leq N+1$ and $X = E$ or F , the element X_{ij} is inductively defined by

$$X_{ij} = \begin{cases} X_i & \text{if } j = i + 1, \\ X_{i,j-1} X_{j-1} - \zeta X_{j-1} X_{i,j-1} & \text{if } j > i + 1. \end{cases}$$

The algebra U_ζ has a Hopf algebra structure with comultiplication Δ , counit ε , and antipode S given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\Delta(K_i^\pm) = K_i^\pm \otimes K_i^\pm,$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i^\pm) = 1,$$

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i^\pm) = K_i^\mp.$$

Let us show that the definitions of Δ and S are compatible with (1.10). We prove some Lemmas.

We put

$$[X, Y]_\zeta = XY - \zeta YX, \quad [X, Y]_{\bar{\zeta}} = XY - \zeta^{-1} YX.$$

Lemma 1.1. *Let M be the \mathbb{C} -algebra generated by A and B with the relations:*

$$A^2 B - (\zeta + \zeta^{-1}) A B A + B A^2 = 0, \tag{1.12}$$

$$B^2 A - (\zeta + \zeta^{-1}) B A B + A B^2 = 0. \tag{1.13}$$

We put

$$C = [A, B]_\zeta, \quad C' = [A, B]_{\bar{\zeta}}.$$

Then it holds:

$$C'^{\bar{r}} = C^{\bar{r}} + (1 - \zeta^{-2})^{\bar{r}} \zeta^{-\frac{\bar{r}(\bar{r}-1)}{2}} A^{\bar{r}} B^{\bar{r}}.$$

Proof. When $C' = AB - \zeta^{-1}BA$, we have, for any positive integer n ,

$$\begin{aligned} (C')^n &= (\zeta^{-2}C + (1 - \zeta^{-2})AB)^n \\ &= \sum_{i=0}^n \binom{n}{i} (1 - \zeta^{-2})^i \zeta^{-\frac{i(i-1)}{2} + (i-2)(n-i)} A^i C^{n-i} B^i, \end{aligned} \tag{1.14}$$

where

$$\binom{n}{i}_\zeta = \frac{[n] \cdots [n-i+1]}{[i] \cdots [1]}, \quad [n] = \frac{1 - \zeta^{-2n}}{1 - \zeta^{-2}}$$

The equality is shown as follows. We have the following equalities for any non-negative integer n :

$$B^n A B = \zeta^{-n} A B = \zeta^{-n} A B^{n+1} - \zeta^{n-2} [n] C B^n, \tag{1.15}$$

$$(1 - \zeta^{-2}) [n] = 1 - \zeta^{-2n}, \tag{1.16}$$

$$\binom{n}{i-1}_\zeta + \binom{n}{i}_\zeta \zeta^{-2i} = \binom{n+1}{i}_\zeta. \tag{1.17}$$

We show the equality (1.14) by induction on n . We suppose that the equality (1.14) holds for n , and then it follows from (1.15),(1.16),(1.17) that

$$\begin{aligned} (C')^{n+1} &= (C')^n (\zeta^{-2}C + (1 - \zeta^{-2})AB) \\ &= \sum_{i=0}^n \binom{n}{i}_\zeta (1 - \zeta^{-2})^i \zeta^{-\frac{i(i-1)}{2} + (i-2)(n-i) + i-2} A^i C^{n+1-i} B^i \\ &\quad + \sum_{i=0}^n \binom{n}{i}_\zeta (1 - \zeta^{-2})^{i+1} \zeta^{-\frac{i(i-1)}{2} + (i-1)(n-i)} A^{i+1} C^{n-i} B^{i+1} \\ &\quad - \sum_{i=0}^n \binom{n}{i}_\zeta (1 - \zeta^{-2})^{i+1} \zeta^{-\frac{i(i-1)}{2} + (i-2)(n-i+1)} [i] A^i C^{n+1-i} B^i \\ &= \zeta^{-2(n+1)} C^{n+1} + (1 - \zeta^{-2})^{n+1} \zeta^{-\frac{n(n+1)}{2}} A^{n+1} B^{n+1} \\ &\quad + \sum_{i=1}^n \left\{ \binom{n}{i-1}_\zeta (1 - \zeta^{-2}) \zeta^{-\frac{i(i-1)}{2} + (i-2)(n+1-i)} \right. \\ &\quad \left. + \binom{n}{i}_\zeta (1 - \zeta^{-2})^i \zeta^{-\frac{i(i-1)}{2} + (i-2)(n+1-i)} (1 - (1 - \zeta^{-2}) [i]) \right\} A^i C^{n+1-i} B^i \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i}_\zeta (1 - \zeta^{-2})^i \zeta^{-\frac{i(i-1)}{2} + (i-2)(n+1-i)} A^i C^{n+1-i} B^i. \end{aligned}$$

So the equality (1.14) holds. As ζ^2 is a primitive \bar{r} -th root of unity, we obtain the claim, putting $n = \bar{r}$ in the equality (1.14).

For $1 \leq i < j \leq N+1$, the elements E_{ij} and E'_{ij} are inductively defined by

$$E_{ij} = \begin{cases} E_i & \text{if } j = i + 1, \\ [E_{ij-1}, E_{j-1}]_{\zeta} & \text{if } j > i + 1. \end{cases}$$

$$E'_{ij} = \begin{cases} E_i & \text{if } j = i + 1, \\ [E'_{ij-1}, E_{j-1}]_{\bar{\zeta}} & \text{if } j > i + 1. \end{cases}$$

Lemma 1.2. (i) For $i < p < j$, we put $A = E_{ip}$ and $B = E'_{pj}$. Then these A and B satisfy the relations (1.12) and (1.13).

(ii) We have that $[E_{ip}, E'_{pj}]_{\zeta} = [E_{ip+1}, E'_{p+1j}]_{\bar{\zeta}}$.

Proof. (i) We show by induction on p that $[E_i, E_{ip}]_{\bar{\zeta}} = 0$, for $p \geq i + 2$. It follows from relation (1.6) that $[E_i, E_{i+2}]_{\bar{\zeta}} = 0$. We suppose that $[E_i, E_{ip}]_{\bar{\zeta}} = 0$. Then we obtain from the relation (1.7),

$$[E_i, E_{ip+1}]_{\bar{\zeta}} = [E_i, E_{ip}]_{\bar{\zeta}} E_p - \zeta E_p [E_i, E_{ip}]_{\bar{\zeta}} = 0.$$

Similarly, using the relation (1.7) and the equality

$$[E'_{p-1j}, E_{j-1}]_{\zeta} = E_{p-1} [E'_{pj}, E_{j-1}]_{\zeta} - \bar{\zeta} [E'_{pj}, E_{j-1}]_{\bar{\zeta}} E_{p-1},$$

we obtain by induction on p that $[E'_{pj}, E_{j-1}]_{\zeta} = 0$ for $p \leq j - 2$.

We put $X = [E_{ip}, [E_{ip}, E'_{pj}]_{\zeta}]_{\bar{\zeta}}$ and $Y = [[E_{ip}, E'_{pj}]_{\zeta}, E'_{pj}]_{\bar{\zeta}}$. Computing $[E_{i-1}, [E_{i-1}, X]_{\zeta^2}]$, $[X, E_j]_{\bar{\zeta}}$, $[[Y, E_j]_{\bar{\zeta}^2}, E_j]$ and $[E_{i-1}, Y]$, we prove that E_{ip} and E'_{pj} satisfy the relations (1.12) and (1.13). Noting that $[E_i, E_{ip}]_{\bar{\zeta}} = 0$ and

$$E_{i-1}^2 [E_{ip}, E'_{pj}]_{\zeta} - (\zeta + \bar{\zeta}) E_{i-1} [E_{ip}, E'_{pj}]_{\zeta} E_{i-1} - [E_{ip}, E'_{pj}]_{\zeta} E_{i-1}^2 = 0,$$

it follows that

$$\begin{aligned} & [E_{i-1}, [E_{i-1}, X]_{\zeta^2}] \\ &= E_{i-1}^2 E_{ip} [E_{ip}, E'_{pj}]_{\zeta} - \bar{\zeta} E_{i-1}^2 [E_{ip}, E'_{pj}]_{\zeta} E_{ip} \\ & \quad - \zeta^2 E_{i-1} E_{ip} [E_{ip}, E'_{pj}]_{\zeta} E_{i-1} + \zeta E_{i-1} [E_{ip}, E'_{pj}]_{\zeta} E_{ip} E_{i-1} \\ & \quad - E_{i-1} E_{ip} [E_{ip}, E'_{pj}]_{\zeta} + \bar{\zeta} E_{i-1} [E_{ip}, E'_{pj}]_{\zeta} E_{ip} E_{i-1} \\ & \quad + \zeta^2 E_{ip} [E_{ip}, E'_{pj}]_{\zeta} E_{i-1}^2 - \zeta [E_{ip}, E'_{pj}]_{\zeta} E_{ip} E_{i-1}^2 \\ &= (\zeta + \bar{\zeta}) E_{i-1} E_{ip} E_{i-1} [E_{ip}, E'_{pj}]_{\zeta} - E_{ip} E_{i-1}^2 [E_{ip}, E'_{pj}]_{\zeta} \end{aligned}$$

$$\begin{aligned}
 & -\bar{\zeta}\{(\zeta + \bar{\zeta})E_{i-1}[E_{ip}, E'_{pj}]_{\zeta}E_{i-1} - [E_{ip}, E'_{pj}]_{\zeta}E_{i-1}^2\}E_{ip} \\
 & -(\zeta^2 + 1)E_{i-1}E_{ip}[E_{ip}, E'_{pj}]_{\zeta}E_{i-1} + (\zeta + \bar{\zeta})E_{i-1}[E_{ip}, E'_{pj}]_{\zeta}E_{ip}E_{i-1} \\
 & + \zeta^2E_{ip}\{(\zeta + \bar{\zeta})E_{i-1}[E_{ip}, E'_{pj}]_{\zeta}E_{i-1} - E_{i-1}^2[E_{ip}, E'_{pj}]_{\zeta}\} \\
 & - \zeta[E_{ip}, E'_{pj}]_{\zeta}\{(\zeta + \bar{\zeta})E_{i-1}E_{ip}E_{i-1} - E_{i-1}^2E_{ip}\} \\
 = & (\zeta + \bar{\zeta})E_{i-1}E_{ip}[E_{i-1}, [E_{ip}, E'_{pj}]_{\zeta}]_{\zeta} + (\zeta + \bar{\zeta})[E_{i-1}, [E_{ip}, E'_{pj}]_{\zeta}]_{\zeta}E_{ip}E_{i-1} \\
 & - \zeta(\zeta + \bar{\zeta})E_{ip}E_{i-1}[E_{i-1}, [E_{ip}, E'_{pj}]_{\zeta}]_{\zeta} - \bar{\zeta}(\zeta + \bar{\zeta})[E_{i-1}, [E_{ip}, E'_{pj}]_{\zeta}]_{\zeta}E_{i-1}E_{ip} \\
 = & (\zeta + \bar{\zeta})[E_{i-1p}, [E_{i-1p}, E'_{pj}]_{\zeta}]_{\bar{\zeta}}.
 \end{aligned}$$

Here we have used $[E_{i-1p}, E'_{pj}]_{\zeta} = [E_{i-1}, [E_{ip}, E'_{pj}]_{\zeta}]_{\zeta}$. So, when $\zeta + \bar{\zeta} \neq 0$ and $X=0$, it turns out that $[E_{i-1p}, [E_{i-1p}, E'_{pj}]_{\zeta}]_{\bar{\zeta}} = 0$. From the formula $[E_{ip}, E'_{pj+1}]_{\zeta} = [[E_{ip}, E'_{pj}]_{\zeta}, E_j]_{\bar{\zeta}}$ and the relation (1.7), we have

$$\begin{aligned}
 [X, E_j]_{\bar{\zeta}} &= E_{ip}[E_{ip}, E'_{pj}]_{\zeta}E_j - \bar{\zeta}[E_{ip}, E'_{pj}]_{\zeta}E_{ip}E_j \\
 & \quad - \bar{\zeta}E_jE_{ip}[E_{ip}, E'_{pj}]_{\zeta} + \bar{\zeta}^2[E_{ip}, E'_{pj}]_{\zeta}E_{ip} \\
 &= E_{ip}[[E_{ip}, E'_{pj}]_{\zeta}, E_j]_{\bar{\zeta}} - \bar{\zeta}[[E_{ip}, E'_{pj}]_{\zeta}, E_j]_{\bar{\zeta}}E_{ip} \\
 &= [E_{ip}, [E_{ip}, E'_{pj+1}]_{\zeta}]_{\bar{\zeta}}.
 \end{aligned}$$

So, if $[E_{ip}, [E_{ip}, E'_{pj}]_{\zeta}]_{\bar{\zeta}} = 0$, then $[E_{ip}, [E_{ip}, E'_{pj+1}]_{\zeta}]_{\bar{\zeta}} = 0$. Thus the elements E_{ip} and E'_{pj} satisfy the relations (1.12).

From the equalities $[E'_{pj}, E_{i-1}] = 0$ and

$$[E_{ip}, E'_{pj}]_{\zeta}E_j^2 - (\zeta + \bar{\zeta})E_j[E_{ip}, E'_{pj}]_{\zeta}E_j + E_j^2[E_{ip}, E'_{pj}]_{\zeta} = 0.$$

it follows that

$$\begin{aligned}
 & [[Y, E_j]_{\bar{\zeta}^2}, E_j] \\
 = & [E_{ip}, E'_{pj}]_{\zeta}E'_{pj}E_j^2 - \bar{\zeta}E'_{pj}[E_{ip}, E'_{pj}]_{\zeta}E_j^2 \\
 & - \bar{\zeta}^2E_j[E_{ip}, E'_{pj}]_{\zeta}E'_{pj}E_j + \zeta^3E_j\bar{E}'_{pj}[E_{ip}, E'_{pj}]_{\zeta}E_j \\
 & - E_j[E_{ip}, E'_{pj}]_{\zeta}E'_{pj}E_j + \bar{\zeta}E_jE'_{pj}[E_{ip}, E'_{pj}]_{\zeta}E_j \\
 & + \bar{\zeta}^2E_j^2[E_{ip}, E'_{pj}]_{\zeta}E'_{pj} - \bar{\zeta}^3E_j^2E'_{pj}[E_{ip}, E'_{pj}]_{\zeta} \\
 = & [E_{ip}, E'_{pj}]_{\zeta}\{(\zeta + \bar{\zeta})E_jE'_{pj}E_j - E_j^2E'_{pj}\} \\
 & - \bar{\zeta}E'_{pj}\{(\zeta + \bar{\zeta})E_j[E_{ip}, E'_{pj}]_{\zeta}E_j - E_j^2[E_{ip}, E'_{pj}]_{\zeta}\} \\
 & - (\bar{\zeta}^2 + 1)E_j[E_{ip}, E'_{pj}]_{\zeta}E'_{pj}E_j + (\bar{\zeta}^3 + \bar{\zeta})E_jE'_{pj}[E_{ip}, E'_{pj}]_{\zeta}E_j \\
 & + \bar{\zeta}^2\{(\zeta + \bar{\zeta})E_j[E_{ip}, E'_{pj}]_{\zeta}E_j - [E_{ip}, E'_{pj}]_{\zeta}E_j^2\}E'_{pj} \\
 & - \bar{\zeta}^3\{(\zeta + \bar{\zeta})E_jE'_{pj}E_j[E_{ip}, E'_{pj}]_{\zeta} - E'_{pj}E_j^2[E_{ip}, E'_{pj}]_{\zeta}\}
 \end{aligned}$$

$$\begin{aligned} &= (\zeta + \bar{\zeta})[[E_{i_p}, E'_{p_j}]_{\zeta}, E_j]_{\bar{\zeta}} E'_p E_j + \bar{\zeta}^2 (\zeta + \bar{\zeta}) E_j E'_p [[E_{i_p}, E'_{p_j}]_{\zeta}, E_j]_{\bar{\zeta}} \\ &\quad - \bar{\zeta} (\zeta + \bar{\zeta}) [[E_{i_p}, E'_{p_j}]_{\zeta}, E_j]_{\bar{\zeta}} E_j E'_p - \bar{\zeta} (\zeta + \bar{\zeta}) E'_p E_j [[E_{i_p}, E'_{p_j}]_{\zeta}, E_j]_{\bar{\zeta}} \\ &= (\zeta + \bar{\zeta}) [[E_{i_p}, E'_{p_j+1}]_{\zeta}, E'_{p_j+1}]_{\bar{\zeta}}. \end{aligned}$$

Here we have used the equality $[E_{i_p}, E'_{p_j+1}]_{\zeta} = [[E_{i_p}, E'_{p_j}]_{\zeta}, E_j]_{\bar{\zeta}}$.

So, if $\zeta + \bar{\zeta} \neq 0$ and $[[E_{i_p}, E'_{p_j}]_{\zeta}, E'_{p_j}]_{\bar{\zeta}} = 0$, then $[[E_{i_p}, E'_{p_j+1}]_{\zeta}, E_{p_j+1}]_{\bar{\zeta}} = 0$. From the equality $[E_{i-1}, [E_{i_p}, E'_{p_j}]_{\zeta}]_{\bar{\zeta}} = [E_{i-1, p}, E'_{p_j}]_{\bar{\zeta}}$, we have

$$\begin{aligned} [E_{i-1}, Y]_{\zeta} &= E_{i-1} [E_{i_p}, E'_{p_j}]_{\zeta} E'_p - \bar{\zeta} E_{i-1} E'_p [E_{i_p}, E'_{p_j}]_{\zeta} \\ &\quad - \zeta [E_{i_p}, E'_{p_j}]_{\zeta} E'_p E_{i-1} + E'_p [E_{i_p}, E'_{p_j}]_{\zeta} E_{i-1} \\ &= [E_{i-1}, [E_{i_p}, E'_{p_j}]_{\zeta}]_{\bar{\zeta}} E'_p - \bar{\zeta} E'_p [E_{i-1}, [E_{i_p}, E'_{p_j}]_{\zeta}]_{\bar{\zeta}} \\ &= [[E_{i-1, p}, E'_{p_j}]_{\zeta}, E'_{p_j}]_{\bar{\zeta}}. \end{aligned}$$

If $[[E_{i_p}, E'_{p_j}]_{\zeta}, E'_{p_j}]_{\bar{\zeta}} = 0$, then $[[E_{i-1, p}, E'_{p_j}]_{\zeta}, E'_{p_j}]_{\bar{\zeta}} = 0$. Thus the elements E_{i_p} and E'_{p_j} satisfy the relations (1.13).

(ii) Let us show that $[E_{i_p}, E'_{p_j}]_{\zeta} = [E_{i_p+1}, E'_{p+1}]_{\bar{\zeta}}$.

We have

$$\begin{aligned} [E_{ii+1}, E'_{i+1+3}]_{\zeta} &= E_i E_{i+1} E_{i+2} - \zeta E_{i+1} E_i E_{i+2} - \bar{\zeta} E_{i+2} E_i E_{i+1} + E_{i+2} E_{i+1} E_i \\ &= [E_{ii+2}, E_{i+2}]_{\bar{\zeta}}. \end{aligned}$$

We suppose that $[E_{i_p}, E'_{p_j}]_{\zeta} = [E_{i_p+1}, E'_{p+1}]_{\bar{\zeta}}$. Then we obtain

$$\begin{aligned} [E_{i_p}, E'_{p_j+1}]_{\zeta} &= [E_{i_p}, E'_{p_j}]_{\zeta} E_j - \bar{\zeta} E_j [E_{i_p}, E'_{p_j}]_{\zeta} \\ &= [E_{i_p+1}, E'_{p+1}]_{\bar{\zeta}} E_j - \bar{\zeta} E_j [E_{i_p+1}, E'_{p+1}]_{\bar{\zeta}} \\ &= E_{i_p+1} (E'_{p+1} E_j - \bar{\zeta} E_j E'_{p+1}) - \bar{\zeta} (E'_{p+1} E_j - \bar{\zeta} E_j E'_{p+1}) E_{i_p+1} \\ &= [E_{i_p+1}, E'_{p+1+1}]_{\bar{\zeta}} \end{aligned}$$

and

$$\begin{aligned} [E_{i-1, p}, E'_{p_j}]_{\zeta} &= E_{i-1} [E_{i_p}, E'_{p_j}]_{\zeta} - \zeta [E_{i_p}, E'_{p_j}]_{\zeta} E_{i-1} \\ &= E_{i-1} [E_{i_p+1}, E'_{p+1}]_{\bar{\zeta}} - \zeta [E_{i_p+1}, E'_{p+1}]_{\bar{\zeta}} E_{i-1} \\ &= (E_{i-1} E_{i_p+1} - \zeta E_{i_p+1} E_{i-1}) E'_{p+1} - \bar{\zeta} E'_{p+1} (E_{i-1} E_{i_p+1} - \zeta E_{i_p+1} E_{i-1}) \\ &= [E_{i-1, p+1}, E'_{p+1}]_{\bar{\zeta}}. \end{aligned}$$

So the claim holds.

By Lemma 1.1 and Lemma 1.2, we have the equality

$$\begin{aligned} E'_{ij}{}^{\bar{r}} &= ([E_{ii+1}, E'_{i+1j}]_{\zeta})^{\bar{r}} \\ &= ([E_{ii+1}, E'_{i+1j}]_{\zeta})^{\bar{r}} + (1 - \zeta^{-2})^{\bar{r}} \zeta^{-\frac{\bar{r}(\bar{r}-1)}{2}} E_{ii+1}{}^{\bar{r}} E'_{i+1j}{}^{\bar{r}} \\ &= ([E_{ii+2}, E'_{i+2j}]_{\zeta})^{\bar{r}} + (1 - \zeta^{-2})^{\bar{r}} \zeta^{-\frac{\bar{r}(\bar{r}-1)}{2}} E_{ii+1}{}^{\bar{r}} E'_{i+1j}{}^{\bar{r}}. \end{aligned}$$

Lemma 1.3. *We have the formula*

$$E'_{ij}{}^{\bar{r}} = E_{ij}{}^{\bar{r}} + \sum_{i < p_1 < \dots < p_s < j} ((1 - \zeta^{-2})^{\bar{r}} \zeta^{-\frac{\bar{r}(\bar{r}-1)}{2}} E_{ip_1}{}^{\bar{r}} \dots E_{p_s j}{}^{\bar{r}}).$$

Proof. From the equality stated just before the lemma repeatedly, we have that

$$E'_{ij}{}^{\bar{r}} = E_{ij}{}^{\bar{r}} + \sum_{k=i+1}^{j-1} (1 - \zeta^{-2})^{\bar{r}} \zeta^{-\frac{\bar{r}(\bar{r}-1)}{2}} E_{ik}{}^{\bar{r}} E'_{kj}{}^{\bar{r}}.$$

By induction on $j-i$, we get the claim.

By Lemma 1.3, we obtain $E'_{ij}{}^{\bar{r}} = 0$ and similarly, $F'_{ij}{}^{\bar{r}} = 0$.

Now we prove that the definition of the coproduct Δ is compatible with the relation (1.10). We can prove the following formula

$$\Delta(E_{ij}) = E_{ij} \otimes 1 + (1 - \zeta^2) \sum_{i < k < j} K_{ik} E_{kj} \otimes E_{ik} + K_{ij} \otimes E_{ij},$$

where $K_{ij} = K_i \dots K_{j-1}$. We put

$$\begin{aligned} u_1 &= E_{ij} \otimes 1, \\ u_2 &= K_{ii+1} E_{i+1j} \otimes E_{ii+1}, \\ &\vdots \\ u_{j-i} &= K_{ij-1} E_{j-1j} \otimes E_{ij-1}, \\ u_{j-i+1} &= K_{ij} \otimes E_{ij}. \end{aligned}$$

It follows that if $k > l$, then $u_k u_l = \zeta^2 u_l u_k$. As we can write that $\Delta(E_{ij}) = u_1 + (1 - \zeta^2)(u_2 + \dots + u_{j-i}) + u_{j-i+1}$, we have

$$\begin{aligned} \Delta(E_{ij})^m &= \sum_{m_1 + \dots + m_{j-i+1} = m} \frac{\phi_m(\zeta^2)}{\phi_{m_1}(\zeta^2) \dots \phi_{m_{j-i+1}}(\zeta^2)} \\ &\quad (1 - \zeta^2)^{m_2 + \dots + m_{j-i}} u_1^{m_1} \dots u_{j-i+1}^{m_{j-i+1}}, \end{aligned}$$

where $\phi_m(\zeta^2) = (1 - \zeta^2)(1 - \zeta^4) \dots (1 - \zeta^{2m})$ (see [14]). Putting $m = \bar{r}$, we can obtain the equality $\Delta(E_{ij})^{\bar{r}} = 0$.

By induction, it follows that $S(E_{ij}) = -K_{ij}^{-1} E'_{ij}$ and $S(F_{ij}) = -\zeta^{2(j-i-1)} F'_{ij}$. We

recall that $E'_{ij}{}^{\bar{r}} = \bar{F}'_{ij}{}^{\bar{r}} = 0$ and so one can obtain that $S(E_{ij})^{\bar{r}} = S(F_{ij})^{\bar{r}} = 0$.

2. A construction of a universal R -matrix for U_ζ

In this section, we construct a universal R -matrix for U_ζ , using the quantum double construction due to Drinfel'd [2]. Our method is similar to that of the construction of the universal R -matrix in [23] and [26].

Let U_ζ^+ be the Hopf subalgebra of U_ζ generated by $E_i, K_i^\pm, 1 \leq i \leq N$ and U_ζ^- the Hopf subalgebra of U_ζ generated by $F_i, K_i, 1 \leq i \leq N$ and $(U_\zeta^+)^o$ be the dual algebra of U_ζ^+ with the opposite comultiplication.

First we fix some notations. Let $\{\alpha_i | 1 \leq i \leq N\}$ be the system of simple roots and Π_+ the set of positive roots $\alpha_i + \dots + \alpha_{j-1}$ with $1 \leq i < j \leq N+1$ of $sl(N+1, \mathbb{C})$. We denote by $Q = \bigoplus \mathbb{Z}\alpha_i$ the root lattice and let $(,): Q \times Q \rightarrow \mathbb{Z}$ be the pairing defined by $(\alpha_i, \alpha_j) = a_{ij}$, where $(a_{ij})_{1 \leq i, j \leq N}$ is the Cartan matrix of type A_N .

We shall put on the set $\{E_{ij} | 1 \leq i < j \leq N+1\}$ a total order $<$ defining $E_{kl} < E_{ij}$ if $k < i$, or $k = i$ and $l < j$. We also denote E_{ij} by E_α for $\alpha \in \Pi_+$ if $\alpha = \alpha_i + \dots + \alpha_{j-1}$. The following notation will be used in describing a \mathbb{C} -basis of U_ζ^+ :

$$I = \{(m_\alpha)_{\alpha \in \Pi_+} | 0 \leq m_\alpha < \bar{r}\},$$

$$J = \{(v_i)_{1 \leq i \leq N} | 0 \leq v_p < r, p = 1, \dots, N-1, 0 \leq v_N < a\},$$

$$P = \{v | v = \sum_{i=1}^N v_i \alpha_i, (v_i) \in J\}.$$

Moreover, we denote by $\Pi_{\alpha \in \Pi_+} E_\alpha^{m_\alpha}$ for $(m_\alpha) \in I$ ordered monomials of the E_α 's according to the total order defined above, $E_{12}^{m_{12}} E_{13}^{m_{13}} \dots E_{NN+1}^{m_{N+1}}$, and for $v = \sum_{i=1}^N v_i \alpha_i$ with $(v_i)_{1 \leq i \leq N} \in J$, set $K_v = \prod_{i=1}^N K_i^{v_i}$. In a way similar to Lemma 4.2 in [22], we can derive a system of generators of U_ζ^+ .

Proposition 2.1. *The algebra U_ζ^+ is generated by $\{\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v | (m_\alpha) \in I, (v_i) \in J\}$ as a \mathbb{C} -vector space.*

Proof. Using the relations (1.3), (1.4) and (1.11), any element x of U_ζ^+ can be written as a \mathbb{C} -linear combination of the elements $E_{i_1} \dots E_{i_m} K_v$ with $1 \leq i_k \leq N$ and $0 \leq v_i < r$. Let L be the subalgebra generated by $K_i, 0 \leq i \leq N$. We remark that L is generated by $\{K_\lambda | \lambda \in P\}$ as a \mathbb{C} -vector space. In fact, it follows, from the relations $(\prod_{i=1}^N K_i)^a = 1, K_N^{ad} = 1$, that $K_N^a = (\prod_{i=1}^{N-1} K_i)^a$. So we can write K_N^b for $a \leq b \leq r-1$ as a product of elements in $\{K_v | (v_i) \in J\}$. Let $P_N = \{((i_1, j_1), \dots, (i_k, j_k)) | (i_p, j_p) \in N \times N, 1 \leq i_p < j_p \leq N+1\} \cup \{\phi\}$. For $\Sigma = ((i_1, j_1), \dots, (i_k, j_k)) \in P_N$, we put $E_\Sigma = E_{i_1 j_1} \dots E_{i_k j_k}$. We define a map $\eta: P_N \rightarrow \mathbb{Z}$ given by

$$\eta(\Sigma) = i_1(j_1 - i_1) + \dots + i_k(j_k - i_k) \text{ for } \Sigma \in P_N, \eta(\phi) = 0.$$

We consider the subspace W_m generated by $\{E_\Sigma | \eta(\Sigma) \leq m\}$. A sequence $\Sigma = ((i_1, j_1), \dots, (i_k, j_k)) \in P_N$ is called increasing if $(i_1, j_1) \leq (i_2, j_2) \leq \dots \leq (i_k, j_k)$. In particular, ϕ is increasing. From [22], for a pair $(s, t) < (x, y)$, we can show

$$E_{xy}E_{st} = \zeta^{\delta_{xs} - \delta_{xt} - \delta_{ys} + \delta_{yt}} E_{st}E_{xy} + \sum_{\substack{\eta(\Sigma) < \eta((s,t),(x,y)) \\ \Sigma = ((i_1, j_1), \dots, (i_n, j_n))}} c_\Sigma E_{i_1 j_1} \dots E_{i_n j_n} \tag{*}$$

for some $c_\Sigma \in C$. By induction on m , we can show that for any m , any element in W_m is written as a C -linear combination of the elements in the set $\{E_\Sigma | \eta(\Sigma) \leq m, \Sigma \text{ is increasing}\}$ (see [22]).

We give a triangular decomposition of U_ζ using a way similar to one in [22].

Let us prepare some notations.

- \tilde{U}_ζ is the algebra over C generated by $E_i, F_i, K_i^\pm, 1 \leq i \leq N$ with relations (1.3), (1.4), (1.5).
- \mathcal{N}_+ (resp. $\tilde{\mathcal{N}}_+$) is the subalgebra of U_ζ (resp. \tilde{U}_ζ) generated by $E_i, 1 \leq i \leq N$ along with 1.
- \mathcal{N}_- (resp. $\tilde{\mathcal{N}}_-$) is the subalgebra of U_ζ (resp. \tilde{U}_ζ) generated by $F_i, 1 \leq i \leq N$ along with 1.
- T (resp. \tilde{T}) is the subalgebra of U_ζ (resp. \tilde{U}_ζ) generated by $K_i^\pm, 1 \leq i \leq N$ along with 1.
- $\phi_{ij}^+, \phi_{ij}^-, 1 \leq i \neq j \leq N$ are the elements of \tilde{U}_ζ , defined

$$\phi_{ij}^+ = \begin{cases} E_i E_j - E_j E_i & \text{if } |i-j| \geq 2, \\ E_i^2 E_j - (\zeta + \zeta^{-1}) E_i E_j E_i + E_i E_j^2 & \text{if } |i-j| = 1, \end{cases}$$

$$\phi_{ij}^- = \begin{cases} F_i F_j - F_j F_i & \text{if } |i-j| \geq 2, \\ F_i^2 F_j - (\zeta + \zeta^{-1}) F_i F_j F_i + F_i F_j^2 & \text{if } |i-j| = 1. \end{cases}$$

- \mathcal{I}_+ (resp. \mathcal{I}_-) is the two sided ideal of $\tilde{\mathcal{N}}_+$ (resp. $\tilde{\mathcal{N}}_-$) generated by $\phi_{ij}^+, 1 \leq i \neq j \leq N, E_{ij}^\pm, 1 \leq i < j \leq N+1$ (resp. $\phi_{ij}^-, 1 \leq i \neq j \leq N, F_{ij}^\pm, 1 \leq i < j \leq N+1$).
- \mathcal{I}_0 is the two sided ideal of \tilde{T} generated by $K_i^r - 1, 1 \leq i \leq N, \Lambda^a - 1$.
- \mathcal{I} is the two sided ideal of \tilde{U}_ζ generated by $\phi_{ij}^+, 1 \leq i \neq j \leq N, E_{ij}^\pm, 1 \leq i < j \leq N+1, \phi_{ij}^-, 1 \leq i \neq j \leq N, F_{ij}^\pm, 1 \leq i < j \leq N+1, K_i^r - 1, 1 \leq i \leq N, \Lambda^a - 1$.

We investigate the structure of \tilde{U}_ζ as a vector space, in a way similar to the proof in Lemma 2.1 and 2.2 in [22].

Let \mathcal{X}_+ (resp. \mathcal{X}_-) be the free associative C -algebra with 1 generators $e_i, 1 \leq i \leq N$ (resp. $f_i, 1 \leq i \leq N$). Let $C[k_1^\pm, \dots, k_N^\pm]$ be the C -algebra of Laurent polynomials in indeterminates k_1, \dots, k_N . Let $\mathcal{M} = \mathcal{X}_- \otimes_C C[k_1^\pm, \dots, k_N^\pm] \otimes_C \mathcal{X}_+$. The elements $f_{i_1} \dots f_{i_s} k_1^{v_1} \dots k_N^{v_N} e_{j_1} \dots e_{j_t}, v_1, \dots, v_N \in \mathbf{Z}, 1 \leq i_1, \dots, i_s, j_1, \dots, j_t \leq N$, form an C -basis

of \mathcal{M} .

\mathcal{M} has a left U_ζ -module structure defined by

$$\begin{aligned} &K_p \cdot f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t} \\ &= \zeta^{-(\alpha_p, \alpha_{i_1} + \cdots + \alpha_{i_s})} f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_p^{v_p+1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t}, \\ &F_p \cdot f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t} \\ &= f_p f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t}, \\ &E_p \cdot f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t} \\ &= \zeta^{-(\alpha_p, v_1 \alpha_1 + \cdots + v_N \alpha_N)} f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_N^{v_N} e_p e_{j_1} \cdots e_{j_t} \\ &+ \frac{1}{\zeta - \zeta_{i_u=p}} \sum \{ \zeta^{-(\alpha_p, \alpha_{i_u+1} + \cdots + \alpha_{i_s})} f_{i_1} \cdots \hat{f}_{i_u} \cdots f_{i_s} k_1^{v_1} \cdots k_p^{v_p+1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t} \\ &- \zeta^{-(\alpha_p, \alpha_{i_u+1} + \cdots + \alpha_{i_s})} f_{i_1} \cdots \hat{f}_{i_u} \cdots f_{i_s} k_1^{v_1} \cdots k_p^{v_p-1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t} \}, \end{aligned}$$

where \hat{f}_{i_u} means that f_{i_u} is omitted.

By this fact, it follows that the elements $F_{i_1} \cdots F_{i_s} K_1^{v_1} \cdots K_N^{v_N} E_{j_1} \cdots E_{j_t}$, $v_1, \dots, v_N \in \mathbb{Z}$, $1 \leq i_1, \dots, i_s, j_1, \dots, j_t \leq N$, form a basis of \tilde{U}_ζ . In fact, we have the left \tilde{U}_ζ -module isomorphism $\tau: \tilde{U}_\zeta \rightarrow \mathcal{M}$ defined by

$$\begin{aligned} \tau(F_{i_1} \cdots F_{i_s} K_1^{v_1} \cdots K_N^{v_N} E_{j_1} \cdots E_{j_t}) &= F_{i_1} \cdots F_{i_s} K_1^{v_1} \cdots K_N^{v_N} E_{j_1} \cdots E_{j_t} \cdot (1 \otimes 1 \otimes 1) \\ &= f_{i_1} \cdots f_{i_s} k_1^{v_1} \cdots k_N^{v_N} e_{j_1} \cdots e_{j_t}. \end{aligned}$$

So we have $\tilde{U}_\zeta \cong \tilde{\mathcal{N}}_- \otimes \tilde{T} \otimes \tilde{\mathcal{N}}_+$ as a vector space, $\tilde{\mathcal{N}}_+$ (resp. $\tilde{\mathcal{N}}_-$) is a free algebra in the variables E_i (resp. F_i), and \tilde{T} is the Laurent polynomial ring in the variables K_i^\pm .

We have $U_\zeta \cong \tilde{U}_\zeta / \mathcal{I}$ as an algebra over \mathbb{C} .

We obtain a triangular decomposition of U_ζ . It follows that $U_\zeta \cong \mathcal{N}_- \otimes T \otimes \mathcal{N}_+$ as a vector space, $\mathcal{N}_\pm \cong \tilde{\mathcal{N}}_\pm / \mathcal{I}_\pm$ and $T \cong \tilde{T} / \mathcal{I}_0$ as an algebra over \mathbb{C} . It is proved in the following way, which is analogous to the proof of Proposition 2.3 in [22]. It suffices to prove:

$$\mathcal{I} = \tilde{\mathcal{N}}_- \tilde{T} \mathcal{I}_+ + \tilde{\mathcal{N}}_- \mathcal{I}_0 \tilde{\mathcal{N}}_+ + \mathcal{I}_- \tilde{T} \tilde{\mathcal{N}}_+.$$

To prove it, we show that $\tilde{\mathcal{N}}_- \tilde{T} \mathcal{I}_+$, $\tilde{\mathcal{N}}_- \mathcal{I}_0 \tilde{\mathcal{N}}_+$, and $\mathcal{I}_- \tilde{T} \tilde{\mathcal{N}}_+$ are ideals of U_ζ . Firstly, we consider $\mathcal{I}_- \tilde{T} \tilde{\mathcal{N}}_+$. The argument for $\tilde{\mathcal{N}}_- \tilde{T} \mathcal{I}_+$ is analogous. Let $Y = \tilde{\mathcal{N}}_- \tilde{T} \mathcal{I}_+$. It is clear that $K_i^\pm Y \subset Y$, $Y K_i^\pm \subset Y$, $F_i Y \subset Y$, $Y F_i \subset Y$, $Y E_i \subset Y$. Let us show that $E_i Y \subset Y$. We define the two \mathbb{C} -linear maps $E_i^\pm: \tilde{\mathcal{N}}_- \rightarrow \tilde{\mathcal{N}}_-$ by

$$E_i^\pm(F_{i_1} \cdots F_{i_s}) = \sum_{i_u=i} \zeta^{\pm a_u} F_{i_1} \cdots F_{i_u} \cdots F_{i_s},$$

where $a_u = (\alpha_{i_s} \alpha_{i_{u+1}} + \cdots + \alpha_{i_s})$, so that

$$\begin{aligned} & E_i \cdot F_{i_1} \cdots F_{i_s} K_1^{l_1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \\ &= \zeta^{-(\alpha_{i_1} l_1 \alpha_1 + \cdots + l_N \alpha_N)} F_{i_1} \cdots F_{i_s} K_1^{l_1} \cdots K_N^{l_N} E_i E_{j_1} \cdots E_{j_t} \\ &+ \frac{1}{\zeta - \zeta^{-1}} \sum_{i_u=i} \{ E_i^-(F_{i_1} \cdots F_{i_s}) K_1^{l_1} \cdots K_i^{l_i+1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \\ &- E_i^+(F_{i_1} \cdots F_{i_s}) K_1^{l_1} \cdots K_i^{l_i-1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \}. \end{aligned}$$

We can show

$$E_i^\pm(F_{i_1} \cdots F_{i_p} \phi_{lm}^- F_{i_s} \cdots F_{i_{s+1}}) \in \mathcal{I}_-$$

(see Proposition 2.3 in [22]). Moreover we have

$$\begin{aligned} & E_p F_{i_1} \cdots F_{i_s} F_{ij}^\bar{F} F_{i_k} \cdots F_{i_{k+1}} K_1^{l_1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \\ &= F_{i_1} \cdots F_{i_s} E_p F_{ij}^\bar{F} F_{i_k} \cdots F_{i_{k+1}} K_1^{l_1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \\ &+ \frac{1}{\zeta - \zeta^{-1}} \sum_{i_u=p} \{ E_p^-(F_{i_1} \cdots F_{i_s}) K_p F_{ij}^\bar{F} F_{i_k} \cdots F_{i_{k+1}} K_1^{l_1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \\ &- E_p^+(F_{i_1} \cdots F_{i_s}) K_p^{-1} F_{ij}^\bar{F} F_{i_k} \cdots F_{i_{k+1}} K_1^{l_1} \cdots K_N^{l_N} E_{j_1} \cdots E_{j_t} \}, \end{aligned}$$

for $1 \leq p \leq N$ and $1 \leq i < j \leq N+1$.

Let us show that $[E_p, F_{ij}^\bar{F}] = 0$, for $1 \leq p \leq N$ and $1 \leq i < j \leq N+1$.

If $i < p < j-1$, then we can obtain

$$\begin{aligned} E_p F_{ij} &= E_p(F_{ip} - F_{pj} - \zeta F_{pj} F_{ip}) \\ &= F_{ip} E_p F_{pj} - \zeta E_p F_{pj} F_{ip} \\ &= F_{ij} E_p + \zeta F_{ip} K_p^{-1} F_{p+1j} - \zeta^2 K_p^{-1} F_{p+1j} F_{ip} \\ &= F_{ij} E_p, \end{aligned}$$

using the equality $E_p F_{pj} = F_{pj} E_p + \zeta K_p^{-1} F_{p+1j}$ and so it follows that $[E_p, F_{ij}^\bar{F}] = 0$.

We consider the case $p=i$. We have

$$\begin{aligned} E_i F_{ij}^\bar{F} &= (E_i F_i F_{i+1j} - \zeta F_{i+1j} E_i F_i) F_{ij}^\bar{F}^{-1} \\ &= (F_{ij} E_i + \zeta K_i^{-1} F_{i+1j}) F_{ij}^\bar{F}^{-1} \end{aligned}$$

$$= F_{ij}(F_{ij}E_iF_{ij}^{\bar{r}-2} + \zeta K_i^{-1}F_{i+1j}F_{ij}^{\bar{r}-2}) + \zeta K_i^{-1}F_{i+1j}F_{ij}^{\bar{r}-1}.$$

Here we used the equality $F_{ij} + F_{i+1j} = \zeta^{-1}F_{i+1j}F_{ij}$. By induction, we can obtain that

$$E_iF_{ij}^{\bar{r}} = F_{ij}^{\bar{r}}E_i + \zeta(1 + \zeta^{-2} + \dots + \zeta^{-2(\bar{r}-1)}) = F_{ij}^{\bar{r}}E_i.$$

Similarly, we can prove that $[E_{j-1}, F_{ij}^{\bar{r}}] = 0$.

Thus, we obtain that $E_iY \subset Y$.

Nextly, we consider $\tilde{\mathcal{N}}_- \mathcal{S}_0 \tilde{\mathcal{N}}_+$. It suffices to prove that for $X = E$ or F and $1 \leq i, j \leq N$,

$$[X_i, K_j^r - 1] = 0 \text{ and } [X_i, \Lambda^a - 1] = 0.$$

Let us show the formulas for E_i . Indeed, we have

$$E_iK_j^r = \zeta^r K_j^r E_i = K_j^r E_i$$

and

$$\begin{aligned} E_i \left(\prod_{j=1}^N K_j^r \right)^a &= \zeta^{a(\alpha_i, \sum_j \alpha_j)} \left(\prod_{j=1}^N K_j^r \right)^a E_i \\ &= \zeta^{\delta_i N a(N+1)} \Lambda^a E_i \\ &= \zeta^{\delta_i N^r \frac{N+1}{d}} \Lambda^a E_i \\ &= \Lambda^a E_i. \end{aligned}$$

Similarly, we can prove the formulas $[F_i, K_j^r - 1] = 0$ and $[F_i, \Lambda^a - 1] = 0$.

The following map $\varphi: U_\zeta^- \rightarrow (U_\zeta^+)^0$ plays an important role.

Proposition 2.2. *There is a Hopf algebra homomorphism $\varphi: U_\zeta^- \rightarrow (U_\zeta^+)^0$ such that for $X = E_{i_1} \cdots E_{i_m} K_v$,*

$$\begin{aligned} \varphi(F_i)(X) &= \begin{cases} b & \text{if } X = E_i K_v, \\ 0 & \text{otherwise,} \end{cases} \\ \varphi(K_i^\pm)(X) &= \begin{cases} \zeta^{\mp(\alpha_i, v)} & \text{if } X = K_v, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $b = -\frac{1}{\zeta - \zeta^{-1}}$.

Proof. We put $\varphi(F_i) = \xi_i, \varphi(K_i^\pm) = \eta_i^\pm$ and $\eta_{ij} = \eta_i \cdots \eta_{j-1}$. We define ξ_{ij} inductively by

$$\xi_{ij} = \begin{cases} \xi_i & \text{if } j = i + 1, \\ \xi_{i_{j-1}}\xi_{i_{j-1}} - \zeta\xi_{i_{j-1}}\xi_{i_{j-1}} & \text{if } j > i + 1. \end{cases}$$

We remark that if $\{i_1, \dots, i_m\} \neq \{j_1, \dots, j_n\}$, then

$$\xi_{i_1}\xi_{i_2}\dots\xi_{i_m}(E_{j_1}E_{j_2}\dots E_{j_n}) = 0. \tag{**}$$

Let us prove the fact by induction on m . We assume that it holds for $m - 1$. Then we have

$$\begin{aligned} & \xi_{i_1}\xi_{i_2}\dots\xi_{i_m}(E_{j_1}E_{j_2}\dots E_{j_n}) \\ &= \xi_{i_1}\dots\xi_{i_{m-1}} \otimes \xi_{i_m}(\Delta(E_{j_1})\dots\Delta(E_{j_n})) \\ &= \xi_{i_1}\xi_{i_{m-1}} \otimes \xi_{i_m} \left(\sum_{1 \leq p \leq n} E_{j_1}\dots E_{j_{p-1}}K_{j_p}E_{j_{p+1}}\dots E_{j_n}E_{j_p} \right) \\ &= \sum_{1 \leq p \leq n} \delta_{i_m j_p} \zeta^{(\alpha_{j_{p+1}} + \dots + \alpha_{j_n, \alpha_{j_p}})} \xi_{i_1}\xi_{i_2}\dots\xi_{i_{m-1}}(E_{j_1}\dots\hat{E}_{j_p}\dots E_{j_n})\xi_{i_m}(E_{j_p}). \end{aligned}$$

By the hypothesis of induction, if $\{i_1, \dots, i_{m-1}\} \neq \{j_1, \dots, j_{p-1}, j_{p+1}, \dots, j_n\}$, then

$$\xi_{i_1}\xi_{i_2}\dots\xi_{i_{m-1}}(E_{j_1}\dots\hat{E}_{j_p}\dots E_{j_n}) = 0.$$

We consider the pair $(i'j')$ satisfying that $(i'j') < (ij)$ and that there is no pair $(i''j'')$ with $(i'j') < (i''j'') < (ij)$. It follows that

$$\xi_{12}^{m_{12}}\xi_{13}^{m_{13}}\dots\xi_{i'j'}^{m_{i'j'}}(E_{ij}) = 0, \tag{1}$$

$$\xi_{ij}(E_{12}^{m_{12}}E_{13}^{m_{13}}\dots E_{i'j'}^{m_{i'j'}}) = 0. \tag{2}$$

In fact, $\xi_{12}^{m_{12}}\xi_{13}^{m_{13}}\dots\xi_{i'j'}^{m_{i'j'}}(E_{ij})$ and $\xi_{ij}(E_{12}^{m_{12}}E_{13}^{m_{13}}\dots E_{i'j'}^{m_{i'j'}})$ are \mathbb{C} -linear combinations of the elements in (**).

We note that

$$\Delta(E_{ij}) = E_{ij} \otimes 1 + (1 - \zeta^2) \sum_{i < k < j} K_{ik}E_{kj} \otimes E_{ik} + K_{ij} \otimes E_{ij},$$

$$\Delta(\xi_{ij}) = \xi_{ij} \otimes \eta_{ij}^{-1} + (1 - \zeta^2) \sum_{i < k < j} \xi_{ik} \otimes \eta_{ik}^{-1} \zeta_{kj} + 1 \otimes \xi_{ij}.$$

From these facts, it follows that if $m_\alpha > n_\alpha$ and for any β with $E_\alpha < E_\beta$, $m_\beta = 0$ or $n_\beta = 0$, then

$$\left(\prod_{\alpha \in \Pi_+} \xi_\alpha^{n_\alpha} \eta_\omega \right) \left(\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_\nu \right) = Y \xi_\alpha^{n_\alpha} \otimes \eta_\omega ((X E_\alpha^{m_\alpha} \otimes 1)(K_\nu \otimes K_\nu))$$

$$\begin{aligned} &= Y \zeta_\alpha^{n_\alpha} \otimes (X E_\alpha^{m_\alpha} K_\nu) \eta_w(K_\nu) \\ &= Y \zeta_\alpha^{n_\alpha} (X E_\alpha^{m_\alpha}) \eta_w(K_\nu), \end{aligned}$$

where $X = \prod_{E_\beta < E_\alpha} E_\beta^{m_\beta}$ and $Y = \prod_{E_\beta < E_\alpha} \zeta_\beta^{n_\beta}$. By the equality $K_{ij} E_{ij} = \zeta^2 E_{ij} K_{ij}$, we obtain

$$\begin{aligned} &Y \zeta_\alpha^{n_\alpha} (X E_\alpha^{m_\alpha}) \\ &= Y \zeta_\alpha^{n_\alpha - 1} \otimes \zeta_\alpha ((X \otimes 1) \Delta(E_\alpha)^{m_\alpha}) \\ &= Y \zeta_\alpha^{n_\alpha - 1} (X E_\alpha^{m_\alpha - 1}) \zeta_\alpha(E_\alpha) [m_\alpha] \\ &= \prod_{\alpha \in \Pi_+} \delta_{m_\alpha n_\alpha} \zeta_\alpha(E_\alpha)^{m_\alpha} [m_\alpha]!, \end{aligned}$$

where $[m] = \frac{\zeta^{2m} - 1}{\zeta^2 - 1}$. Here we have used the formula (*). Similarly, for $m_\alpha < n_\alpha$, the similar equality holds. Thus, we compute

$$\left(\prod_{\alpha \in \Pi_+} \zeta_\alpha^{n_\alpha} \eta_w \right) \left(\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_\nu \right) = \prod_{\alpha \in \Pi_+} \delta_{m_\alpha n_\alpha} \zeta_\alpha(E_\alpha)^{m_\alpha} [m_\alpha]! \zeta^{\bar{c}(v, w)}.$$

It follows that any element $\prod_{\alpha \in \Pi_+} \zeta_\alpha^{n_\alpha} \eta_w$ is zero on ϕ_{ij} , $1 \leq i \neq j \leq N$, E_{ij}^r , $1 \leq i < j \leq N+1$, $K_i^r - 1$, $1 \leq i \leq N$, and $\Lambda^a - 1$, and from the triangular decomposition of U_ζ , φ is well-defined.

Moreover, the elements ζ_i and η_i^\pm satisfy the following relations:

$$(1) \quad \eta_i \eta_j = \eta_j \eta_i, \eta_i^{-1} \eta_i = \eta_i \eta_i^{-1} = \varepsilon, \tag{2.1}$$

$$(2) \quad \eta_i \zeta_j = \zeta^{-(\alpha_i, \alpha_j)} \zeta_j \eta_i, \tag{2.2}$$

$$(3) \quad \zeta_i^2 \zeta_j - (\zeta + \zeta^{-1}) \zeta_i \zeta_j \zeta_i + \zeta_i \zeta_j^2 = 0 \quad (|i - j| = 1), \tag{2.3}$$

$$(4) \quad \zeta_i \zeta_j = \zeta_j \zeta_i \quad (|i - j| \geq 2), \tag{2.4}$$

$$(5) \quad \zeta_{ij}^r = 0, \tag{2.5}$$

$$(6) \quad \eta_i^r = \varepsilon, \left(\prod_{j=1}^N \eta_i^j \right)^a = \varepsilon, \tag{2.6}$$

$$(7) \quad \Delta(\zeta_i) = \zeta_i \otimes \eta_i^{-1} + 1 \otimes \zeta_i, \Delta(\eta_i^\pm) = \eta_i^\pm \otimes \eta_i^\pm, \tag{2.7}$$

$$(8) \quad \varepsilon(\zeta_i) = 0, \varepsilon(\eta_i^\pm) = 1, \tag{2.8}$$

$$(9) \quad S(\zeta_i) = -\zeta_i \eta_i, S(\eta_i^\pm) = \eta_i^\mp. \tag{2.9}$$

One can prove these formulas by easy computations. In the following, we show only the formulas (2.2), (2.5), (2.6) and (2.7). For (2.2), $\eta_i \zeta_j$ is non-zero only on $E_j K_\nu$

where its value is $\bar{\zeta}^{(\alpha_i, \alpha_j)} b \bar{\zeta}^{(\alpha_i, v)}$ and $\xi_j \eta_i$ is non-zero only on $E_j K_v$ where its value is $b \bar{\zeta}^{(\alpha_i, v)}$. For (2.5), it follows from the above equality that $\xi_{ij} \bar{\zeta}^{(\alpha_i, v)} (\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v) = 0$. For (2.6), $(\prod_{j=1}^N \eta_i^j)^a$ is non-zero only on K_v . We have that for $1 \leq p \leq N$, $(\prod_{j=1}^N \eta_i^j)^a (K_p) = 1$. In fact, by the definition of η_i , we have

$$\left(\prod_{j=1}^N \eta_i^j \right)^a (K_p) = \bar{\zeta}^{a(\sum_{j \neq p} \alpha_j, \alpha_p)} = \bar{\zeta}^{\partial_{N,p} a(N+1)} = 1.$$

For (2.7), $\Delta(\xi_i)$ is non-zero only on $E_i K_v \otimes K_w$ and $K_v \otimes E_i K_w$, where their values are respectively $b \bar{\zeta}^{(\alpha_i, w)}$ and b . On the other hand, $\xi_i \otimes \eta_i$ is non-zero only on $E_i K_v \otimes K_w$, where its value is $b \bar{\zeta}^{(\alpha_i, w)}$, and $\eta_i^{-1} \otimes \xi_i$ is non-zero only on $K_v \otimes E_i K_w$, where its value is b . The map φ is a Hopf algebra homomorphism.

Proposition 2.3. *We define $\Phi: U_\zeta^+ \times U_\zeta^- \rightarrow \mathcal{C}$ by $\Phi(x, y) = \varphi(y)(x)$ for $(x, y) \in U_\zeta^+ \times U_\zeta^-$. Then Φ is non-degenerate. Moreover, $\{\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v \mid (m_\alpha) \in I, (v_i) \in J\}$ in proposition 2.1 is a \mathcal{C} -basis of U_ζ^+ and the Hopf algebra homomorphism φ is an isomorphism.*

Proof. By the discussion in the proof of Proposition 2.2, it follows that

$$\Phi \left(\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v, \prod_{\alpha \in \Pi_+} F_\alpha^{n_\alpha} K_w \right) = \prod_{\alpha \in \Pi_+} \delta_{m_\alpha n_\alpha} \bar{\zeta}_\alpha (E_\alpha)^{m_\alpha} [m_\alpha]! \bar{\zeta}^{(v, w)},$$

where $[m] = \frac{\zeta^{2m-1}}{\zeta^{2-1}}$, and $[m]! = [m][m-1] \cdots [1]$.

For $v, w \in P$, we put

$$h_{v-w} = \sum_{\mu \in P} \zeta^{(\mu, v-w)}$$

and

$$v-w = x_1 \alpha_1 + \cdots + x_{N-1} \alpha_{N-1} x_N \alpha_N \quad ((x_i) = (v_i) - (w_i), (v_i), (w_i) \in J).$$

We have that

$$\begin{aligned} h_{v-w} &= \sum_{\substack{0 \leq u_1, \dots, u_{N-1} \leq r-1 \\ 0 \leq u_N \leq a-1}} \zeta^{\sum_{i=1}^N u_i (-x_{i-1} + 2x_i - x_{i+1})} \\ &= \left(\prod_{i=1}^{N-1} \sum_{u_i=0}^{r-1} \zeta^{u_i (-x_{i-1} + 2x_i - x_{i+1})} \right) \sum_{u_N=0}^{a-1} \zeta^{u_N (-x_{N-1} + 2x_N)}. \end{aligned}$$

We assume $h_{v-w} \neq 0$. Then $\prod_{i=1}^{N-1} \sum_{u_i=0}^{r-1} \zeta^{u_i (-x_{i-1} + 2x_i - x_{i+1})} \neq 0$. Hence we have that $-x_{i-1} + 2x_i - x_{i+1} \equiv 0 \pmod{r}$, $2 \leq i \leq N-1$ and $x_2 \equiv 2x_1 \pmod{r}$. So, it follows that

$$x_{i+1} \equiv 2x_i - x_{i-1} \equiv 2ix_1 - (i-1)x_1 \equiv (i+1)x_1 \pmod{r}.$$

Thus we obtain $x_i \equiv ix_1 \pmod{r}$, $1 \leq i \leq N$. From the equality

$$\sum_{u_N=0}^{a-1} \zeta^{u_N(-x_{N-1} + 2x_N)} = \sum_{u_N=0}^{a-1} \zeta^{u_N(N+1)x_1} = \sum_{u_N=0}^{a-1} \zeta^{nu_N x_1} \neq 0,$$

we obtain that $x_1 \equiv 0 \pmod{a}$, noting that ζ^n is a primitive a -th root unity. While $x_N \equiv Nx_1 \pmod{r}$ and $ad=r$, we have that $x_N \equiv 0 \pmod{a}$. As $|x_N| < a$, it follows that $x_N=0$. From the formulas $x_i \equiv ix_1 \pmod{r}$ and $x_1 \equiv 0 \pmod{a}$, we have that $-x_{N-1} + 2x_N \equiv (N+1)x_1 \equiv 0 \pmod{r}$ and so $x_{N-1} \equiv 2x_N \pmod{r}$. From the equality $x_{i-1} \equiv 2x_i - x_{i+1} \pmod{r}$, by induction, we have that $x_i \equiv (N-i+1)x_N=0 \pmod{r}$. As $|x_i| < r$ for $1 \leq i \leq N-1$, we obtain that $x_i=0$ for $1 \leq i \leq N-1$. Thus we obtain that $h_{v-w} \neq 0$ if and only if $v=w$. Let $L=|J|$, and then

$$\Phi \left(\frac{1}{L} \sum_{(u_i) \in J} \zeta^{(v,u)} K_u K_w \right) = \delta_{vw}.$$

For $m=(m_\alpha)_{\alpha \in \Pi_+}$, we put

$$c_m = \prod_{\alpha \in \Pi_+} (\xi_\alpha(E_\alpha))^{m_\alpha} [m_\alpha]! = \prod_{\alpha \in \Pi_+} \left(-\frac{1}{\zeta - \zeta^{-1}} (-\zeta)^{\text{ht}(\alpha) - 1} \right)^{m_\alpha} [m_\alpha]!,$$

where c_m is non-zero. From the above discussion,

$$\left\{ \frac{1}{L} \sum_{(m_\alpha) \in I, (v_i) \in J} \zeta^{(v,u)} \prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v \right\}, \left\{ \prod_{\alpha \in \Pi_+} F_\alpha^{n_\alpha} K_w \right\}_{(n_\alpha) \in I, (w_i) \in J}$$

is a basis for U_ζ^+ and U_ζ^- , and they are orthonormal for the pairing Φ . Thus Φ is non-degenerate and $\{\prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v\}_{(m_\alpha) \in I, (v_i) \in J}$ is a \mathbb{C} -basis of U_ζ^+ , by Proposition 2.1. From the definition of Φ , the homomorphism φ is an isomorphism.

Now we apply the quantum double construction to the Hopf algebra U_ζ^+ . By the definition of the multiplication of the quantum double, one can derive the following Lemma.

Lemma 2.4. *Let $e_i = E_i \otimes 1$, $k_i^\pm = K_i^\pm \otimes 1$, $f_i = 1 \otimes \varphi(F_i)$, and $h_i^\pm = 1 \otimes \varphi(K_i^\pm)$ in the quantum double $D(U_\zeta^+)$. These elements satisfy the following commutation relations:*

$$(1) \quad k_i h_j = h_j k_i, k_i h_i^{-1} = k_i^{-1} h_i = 1, \tag{2.11}$$

$$(2) \quad h_i e_j = \zeta^{(\alpha_i, \alpha_j)} e_j h_i, k_i f_j = \zeta^{-(\alpha_i, \alpha_j)} f_j k_i, \tag{2.12}$$

$$(3) [e_i, f_j] = \delta_{ij} \frac{k_i - h_i^{-1}}{\zeta - \zeta^{-1}}. \tag{2.13}$$

Proof. For (2.13), we have

$$\begin{aligned} f_j e_i &= S(\xi_j)(E_i) \cdot h_j^{-1} \cdot \eta_j^{-1}(1) + S(1)(K_i) \cdot e_i f_j \cdot \eta_j^{-1}(1) + S(1)(K_i) \cdot k_j \cdot \xi_j(E_i) \\ &= \delta_{ij} \frac{h_i^{-1}}{\zeta - \zeta^{-1}} + e_i f_j - \delta_{ij} \frac{k_i}{\zeta - \zeta^{-1}}, \end{aligned}$$

where $\xi_j = \varphi(F_j)$ and $\eta_i = \varphi(K_i)$. The other relations are also immediately obtained.

The Hopf algebra structure on $D(U_\zeta^+)$ induces the one on U_ζ .

Proposition 2.5. *Let us define a map $\psi : D(U_\zeta^+) \rightarrow U_\zeta$ by $\psi(x \otimes y) = x\varphi^{-1}(y)$ for $x \otimes y \in U_\zeta^+ \otimes (U_\zeta^+)^o \cong D(U_\zeta^+)$. Then the map ψ is a Hopf algebra epimorphism.*

Proof. Comparing Lemma 2.4 with the commutation relations between E_i, F_i and $K_i, 1 \leq i \leq N$, one can easily show that ψ is an algebra homomorphism. From the fact that φ^{-1} is a Hopf algebra isomorphism, due to the Hopf algebra structure of $D(U_\zeta^+)$, it follows that ψ is a Hopf algebra homomorphism. The surjectivity of ψ follows from the fact that any element $X_1 \cdots X_p, X_i \in \{E_i, F_i, K_i^\pm | 1 \leq i \leq N\}$ is written as a \mathbb{C} -linear combination of the elements $X_+ Y_-, X_+ \in U_\zeta^+, Y_- \in U_\zeta^-$, using the relations (1.4) and (1.5).

Now, we obtain an explicit formula for a universal R of U_ζ , as the image of the universal R of $D(U_\zeta^+)$ under $\psi \otimes \psi$.

Theorem 2.6. *A universal R -matrix for U_ζ is given by*

$$R = \frac{1}{L} \sum_{\substack{(m_\alpha) \in I \\ (v_i), (w_i) \in J}} \frac{1}{c_m} \zeta^{(v,w)} \prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_v \otimes \prod_{\alpha \in \Pi_+} F_\alpha^{m_\alpha} K_w, \tag{2.14}$$

where

$$I = \{(m_\alpha)_{\alpha \in \Pi_+} | 0 \leq m_\alpha < \bar{r}\},$$

$$J = \{(v_i)_{1 \leq i \leq N} | 0 \leq v_p < r_p p = 1, \dots, N-1, 0 \leq v_N < a\},$$

$$L = r^{N-1} a,$$

$$c_m = \prod_{\alpha \in \Pi_+} \left(-\frac{1}{\zeta - \zeta^{-1}} (-\zeta)^{\text{ht}(\alpha) - 1} \right)^{m_\alpha} [m_\alpha]! \quad \text{for } m = (m_\alpha)_{\alpha \in \Pi_+}.$$

Proof. Since the universal R of $D(U_\zeta^+)$ satisfies (1.1) and (1.2), and ψ is a Hopf algebra epimorphism, R also satisfies (1.1) and (1.2).

3. Results from the universal R -matrix for U_ζ

We recall how one can obtain tangle operators from representations of the quasitriangular Hopf algebra (U_ζ, R) , where R is the universal R -matrix for U_ζ in the previous section [13].

For non negative integers k and l , a (k, l) -tangle T is a smooth 1-manifold in $\mathbb{R}^2 \times [0, 1]$ such that its boundary $\partial T = \{(i, 0, 0) | 1 \leq i \leq k\} \cup \{(j, 0, 1) | 1 \leq j \leq l\}$. We put $\partial T_+ = \{(i, 0, 0) | 0 \leq i \leq k\}$ and $\partial T_- = \{(j, 0, 1) | 1 \leq j \leq l\}$. All tangles are assumed to be oriented.

It is well-known that every tangle diagram can be reconstructed from the elementary diagrams in Fig.3.1, using the composition \circ (when defined) and the tensor product \otimes in the Fig.3.2.

A coloring of a tangle T is defined to be an assignment of a U_ζ -module to each component of T . According to a coloring, we assign U_ζ -modules T_\pm to ∂T_\pm as follows: if an arc S of T has a color V , then to each boundary point in $\mathbb{R}^2 \times \{0, 1\}$ associate V if the orientation is downwards and associate V^* if it is upwards. Then the U_ζ -module T_+ (resp. T_-) is the tensor product from left to right of the U_ζ -modules associated to ∂T_+ (resp. ∂T_-). By convention, $T_\pm = \mathbb{C}$ if T is a link.

In this paper, we consider the following family of irreducible representations of U_ζ with $\bar{r} = K + N + 1$ for a positive integer K . Let $\alpha_1, \dots, \alpha_N$ be the simple roots of $sl(N+1, \mathbb{C})$ and we put

$$P_+(K) = \{ \lambda \in \mathfrak{h}^* | (\lambda, \alpha_i) \in \mathbb{Z}, 0 \leq (\lambda, \alpha_i), i = 1, \dots, N, 0 \leq (\lambda, \theta) \leq K \},$$

where θ is the longest root, \mathfrak{h} is the Cartan subalgebra of $sl(N+1, \mathbb{C})$. Let $\lambda_1, \dots, \lambda_N$ be the fundamental dominant integral weight: each λ_i satisfies $(\lambda_i, \alpha_j) = \delta_{ij}$ for any α_j . We see that $\lambda = \sum_{i=1}^N m_i \lambda_i$ for integers m_1, \dots, m_N . For each $\lambda \in P_+(K)$, there exists an irreducible highest weight module V_λ of U_ζ with highest weight λ and

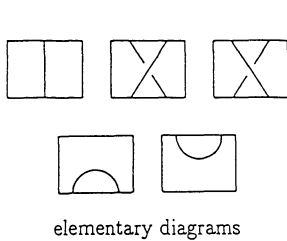


Fig. 3.1

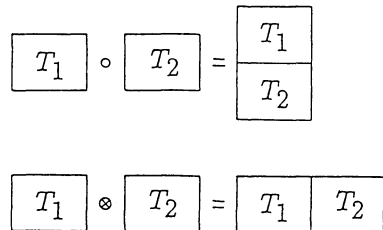


Fig. 3.2

highest weight vector e_λ such that

$$\mathcal{N}_+ e_\lambda = 0, \quad V_\lambda = \mathcal{N}_- e_\lambda, \quad K_\nu e_\lambda = \zeta^{(\lambda, \nu)} e_\lambda.$$

Here \mathcal{N}_+ is the subalgebra of U_ζ generated by $E_i, 1 \leq i \leq N$ and \mathcal{N}_- is the subalgebra of U_ζ generated by $F_i, 1 \leq i \leq N$.

Let T be a colored tangle such that each color of a component of T is contained in the set $\{V_\lambda | \lambda \in P_+(K)\}$. When S_1, \dots, S_n are the components of T , a coloring of T can be viewed as the map $\{1, \dots, n\} \rightarrow P_+(K)$. As is shown in [13], there exists a U_ζ -linear map $F_T: T_- \rightarrow T_+$ such that it satisfies $F_{T \circ T'} = F_T \circ F_{T'}$ and $F_{T \otimes T'} = F_T \otimes F_{T'}$, and for elementary diagrams,

$$F_\downarrow = \text{id}_{V_\lambda}, \quad F_\uparrow = \text{id}_{V_{\lambda^*}}$$

$$F_{\times}(x \otimes y) = \sum_k \beta_k y \otimes \alpha_k x, \quad \text{where } R = \sum_k \alpha_k \otimes \beta_k,$$

$$F_{\times}(x \otimes y) = \sum_k \beta'_k y \otimes \alpha'_k x, \quad \text{where } R^{-1} = \sum_k \alpha'_k \otimes \beta'_k,$$

$$F_{\cap}(f \otimes x) = f(x), \quad F_{\cap}(x \otimes f) = f(K_\rho^{-1} x),$$

$$F_{\cup}(1) = \sum_i e_i \otimes e^i, \quad F_{\cup}(1) = \sum_i e^i \otimes K_\rho e_i, \quad (\text{for any basis } \{e_i\}),$$

where $K_\rho = \prod_{\alpha \in \Pi_+} K_\alpha$. If L is a colored oriented link with coloring ν , F_L is a scalar map. We denote this scalar by $J(L, \nu)$.

In the following proposition, using the explicit formula (2.14) of the universal R for U_ζ , we shall compute two values, which are essential in the construction of 3-manifold invariants. We put $q = \zeta^2$.

Proposition 3.1. (1) *Let $H_{\lambda, \mu}$ be a colored Hopf link such that the colors of the two components are V_λ and V_μ drawn in Fig.3.3. Then we have*

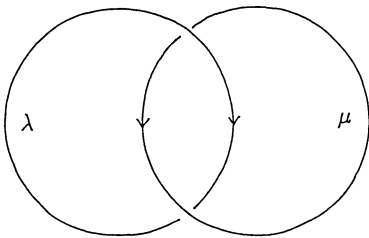


Fig. 3.3

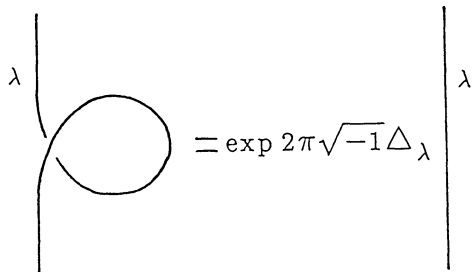


Fig. 3.4

$$J(H_{\lambda\mu}) = \frac{\sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(u + \rho))}}{\sum_{w \in W} (\det w) \bar{q}^{(\rho, w(\rho))}}, \quad (3.1)$$

where ρ is half the sum of positive roots and W is the Weyl group.

(2) Let T be a colored (1,1)-tangle such that the one component has a color V_λ in Fig.3.4. Then F_T is the multiplication by $\exp 2\pi\sqrt{-1}\Delta_\lambda$, where $\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2r}$.

Proof. (1) We consider the colored (1,1)-tangle in Fig.3.5. Since V_λ is irreducible, F_T is a scalar map. We denote this scalar by $b_{\lambda\mu}$. To compute $b_{\lambda\mu}$, it is enough to evaluate $F_T(e_\lambda)$ for the highest weight vector e_λ . If $R = \sum_k \alpha_k \otimes \beta_k$, then we see $R^{-1} = (\text{id} \otimes S)(R)$. From the definitions of tangle operators, one can obtain

$$F_T(e_\lambda) = b_{\lambda\mu} e_\lambda = \sum_{k,l} S(\beta_k) \alpha_l \text{Tr}_\mu(K_\rho^{-1} \alpha_k S(\beta_l)) e_\lambda.$$

By the formula (2.14), one has

$$b_{\lambda\mu} e_\lambda = \frac{1}{L^2} \sum_{\substack{(m_\alpha)(n_\alpha) \in I \\ (v_i)(u_i)(w_i)(u'_i) \in J}} S\left(\prod_{\alpha \in \Pi_+} F_\alpha^{n_\alpha} K_w\right) \left(\frac{1}{c_m} \zeta^{(u,v)} \prod_{\alpha \in \Pi_+} E_\alpha^{m_\alpha} K_u\right) \\ \text{Tr}_\mu\left(K_\rho^{-1} \frac{1}{c_n} \zeta^{(u',w)} \prod_{\alpha \in \Pi_+} E_\alpha^{n_\alpha} K_u \cdot S\left(\prod_{\alpha \in \Pi_+} F_\alpha^{m_\alpha} K_v\right)\right) e_\lambda.$$

Since e_λ is the highest weight vector, the only terms with $m_\alpha = n_\alpha = 0$ for any $\alpha \in \Pi_+$ are non zero. Thus one can get

$$b_{\lambda\mu} e_\lambda = \frac{1}{L^2} \sum_{(v_i)(u_i)(w_i)(u'_i) \in J} K_w^{-1} \zeta^{(v,u)} K_u \text{Tr}_\mu(K_\rho^{-1} \zeta^{(w,u')} K_u \cdot K_v^{-1}) e_\lambda \\ = \frac{1}{L^2} \sum \zeta^{(w,\lambda)} \zeta^{(v,u)} \bar{\zeta}^{(u,\lambda)} \text{Tr}_\mu(K_\rho^{-1} \zeta^{(w,u')} K_u \cdot K_v^{-1}) e_\lambda.$$

Noting that $\sum_{(u_i) \in J} \zeta^{(u,\lambda-v)} \neq 0$ if and only if $\lambda = v$, we can compute

$$b_{\lambda\mu} e_\lambda = \frac{1}{L} \sum \zeta^{(w,\lambda)} \text{Tr}_\mu(K_\rho^{-1} \zeta^{(w,u')} K_u \cdot K_v^{-1}) e_\lambda \\ = \sum_{\mu_s} \zeta^{(\mu_s, \lambda)} \zeta^{(2\rho, \mu_s)} \zeta^{(\mu_s, \lambda)} e_\lambda \\ = \sum_{\mu_s} \bar{q}^{(\lambda + \rho, \mu_s)} e_\lambda,$$

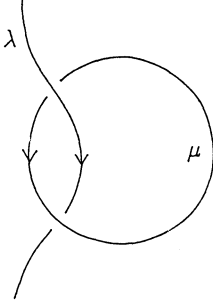


Fig. 3.5

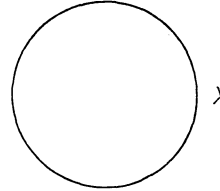


Fig. 3.6

where $\{\mu_s\}$ is the set of weights of V_μ with multiplicity and $\zeta^2 = q$. It follows from the character formula of Weyl (see for example [10]) that

$$b_{\lambda\mu} = \frac{\sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(\mu + \rho))}}{\sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(\rho))}}$$

Let L_λ be a colored unknot with a color λ in Fig.3.6. Then we see

$$J(L_\lambda) = \frac{\sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(\rho))}}{\sum_{w \in W} (\det w) \bar{q}^{(\rho, w(\rho))}}$$

which is called the quantum dimension of V_λ and we write it by $\dim_q V_\lambda$. Since $J(H_{\lambda\mu}) = b_{\lambda\mu} \dim_q V_\lambda$ according to [13, Lemma 2.6], the formula (3.1) holds.

(2) As the representation V_λ is irreducible, the tangle operator F_T is a scalar map. We denote this scalar by v_λ . To compute v_λ , it is enough to evaluate $F_T(e_\lambda)$ for the highest weight vector e_λ of V_λ . When $R = \sum \alpha_k \otimes \beta_k$, one can see

$$F_T(e_\lambda) = \alpha_k K_\rho e^\lambda (\beta_k e_\lambda) e_\lambda.$$

From computations similar to the one made in the proof of (1), it follows that

$$\begin{aligned} v_\lambda e_\lambda &= \frac{1}{L_{(v_i)(w_i) \in J}} \sum \zeta^{(v,w)} K_v K_\rho e^\lambda (K_w e_\lambda) e_\lambda \\ &= \frac{1}{L_{(v_i)(w_i) \in J}} \sum \zeta^{(v,w)} \bar{\zeta}^{(v,\lambda)} \bar{\zeta}^{(\lambda, 2\rho)} e^\lambda (\bar{\zeta}^{(\lambda,w)} e_\lambda) e_\lambda \\ &= \bar{\zeta}^{(\lambda,\lambda)} \bar{\zeta}^{(2\rho,\lambda)} e_\lambda \\ &= q^{\frac{1}{2}(\lambda,\lambda + 2\rho)} e_\lambda \end{aligned}$$

Thus the claim holds.

Let $S=(S_{\lambda\mu})$ be the so-called S -matrix due to Kac [7], which is given by

$$S_{\lambda\mu} = \frac{\sqrt{-1}^{N(N+1)/2}}{\sqrt{(N+1)^{r^N}}} \sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(\mu + \rho))}. \tag{3.2}$$

Comparing (3.1) with (3.2), one easily sees that $S_{\lambda\mu} = S_{00} b_{\lambda\mu}$.

By the discussion in [9], for any closed oriented connected 3-manifold M ,

$$Z_r(M) = C^\sigma \sum_{v \in \text{col}(L)} S_{0v(1)} \cdots S_{0v(n)} J(L, v)$$

is a topological invariant of M , where $C = (\exp 2\pi\sqrt{-1} \frac{c}{24})^{-3}$, $c = \frac{K \dim sl(N+1, \mathbb{C})}{r}$, L is a framed link with n components such that M is obtained by Dehn surgery of S^3 along L , σ is the signature of the linking matrix of L , and $\text{col}(L)$ means the set of colorings of L .

We denote by $\text{Rep}(sl(N+1, \mathbb{C}))$ the representation ring of $sl(N+1, \mathbb{C})$. It is well-known that the representations of $sl(N+1, \mathbb{C})$ with fundamental weight λ_i , $1 \leq i \leq N$, generate $\text{Rep}(sl(N+1, \mathbb{C}))$. We put $\partial P_+(K) = P_+(K+1) \setminus P_+(K)$. Let I_K be the ideal of $\text{Rep}(sl(N+1, \mathbb{C}))$ generated by the representations W_λ , $\lambda \in \partial P_+(K)$. We put $R_K = \text{Rep}(sl(N+1, \mathbb{C})) / I_K$.

In [4], Goodman-Wenzl showed that the algebra R_K is a free \mathbb{Z} -module with basis w_λ corresponding to $\lambda \in P_+(K)$ and that

$$w_\lambda \cdot w_\mu = \sum N_{\lambda\mu}^v w_v,$$

for non-negative integers $N_{\lambda\mu}^v$, which are called the fusion rule.

In $\text{Rep}(U_\zeta)$, the irreducible representation V_λ , $\lambda \in P_+(K)$, can be written as a formal sum of monomials in the fundamental representations V_{λ_i} , $1 \leq i \leq N$ such that the monomials are in the span of $\{V_\omega | \omega \in P_+(K)\}$. This follows from the induction on the lexicographic order of Young diagrams, applying Littlewood-Richardson rule to the decomposition of the tensor products of V_λ and V_{λ_i} . Using the formal expressions, we can obtain the decomposition $V_\lambda \otimes V_\mu = \sum_{v \in P_+(K)} n_{\lambda\mu}^v V_v + Z_{\lambda\mu}$, for λ, μ , where $n_{\lambda\mu}^v$ are integers and $Z_{\lambda\mu}$ is contained in the ideal generated by the irreducible representations V_ω for $\omega \in \partial P_+(K)$. Since in decomposing tensor products of the fundamental representations and V_λ , $\lambda \in P_+(K)$, we can apply Littlewood-Richardson rule, in a way similar to the proof in Lemma 3.1 in [4], we get $n_{\lambda\mu}^v = N_{\lambda\mu}^v$. It follows that for $\lambda, v \in P_+(K)$,

$$V_\lambda \otimes V_\mu = \sum_{v \in P_+(K)} N_{\lambda\mu}^v V_v + Z_{\lambda\mu}. \tag{3.3}$$

We recall that the quantum dimension means the trace of the representation matrix

of K_ρ and denote the quantum dimension of U_ζ module by $\dim_q V$. One can extend the definition of the quantum dimension to a \mathbb{C} -linear map from $Rep(U_\zeta)$ to \mathbb{C} . As the quantum dimension of V_ω , for $\omega \in P_+(K)$, is equal to 0 from the equality $[\bar{r}] = 0$ (also see [3]), that of the tensor product of V_ω and any representation of U_ζ is also equal to 0. From these two facts, the extended quantum dimension of $Z_{\lambda\mu}$ is 0.

REMARK. It is shown in [1] that for λ, μ , we have a decomposition

$$V_\lambda \otimes V_\mu = \oplus (M_{\lambda\nu}^\nu \otimes V_\nu) \oplus Z_{\lambda\mu}$$

where the dimension of \mathbb{C} -module $M_{\lambda\mu}^\nu$ is equal to $N_{\lambda\mu}^\nu$ and the quantum dimension of $Z_{\lambda\mu}$ is 0. Although, we don't need the fact.

As is shown in [13] for $sl(2, \mathbb{C})$ by Reshetikhin and Turaev, we extend $Z_r(M)$ to $Z_r(M, T)$ for M which contains a colored framed link L . Let T be a colored framed link in S^3 and we suppose that M is obtained by Dehn surgery on L . Then we think of $T \cup L$ as a framed link in S^3 , and we put

$$Z_r(M, T) = C^\sigma \sum_{v \in col(L)} S_{0v(1)} \cdots S_{0v(n)} J(L \cup T, v).$$

From the above observation, one can get Verlinde's formula for the fusion algebra R_K with the fusion rule due to Goodman-Wenzl.

Proposition 3.2. *The S-matrix $(S_{\lambda\mu})_{\lambda, \mu \in P_+(K)}$ and the fusion rule $N_{\lambda\mu}^\nu$ satisfy Verlinde's formula:*

$$N_{\lambda\mu}^\nu = \sum_{\varepsilon \in P_+(K)} \frac{S_{\lambda\varepsilon} S_{\mu\varepsilon} S_{\nu\varepsilon}^*}{S_{0\varepsilon}}$$

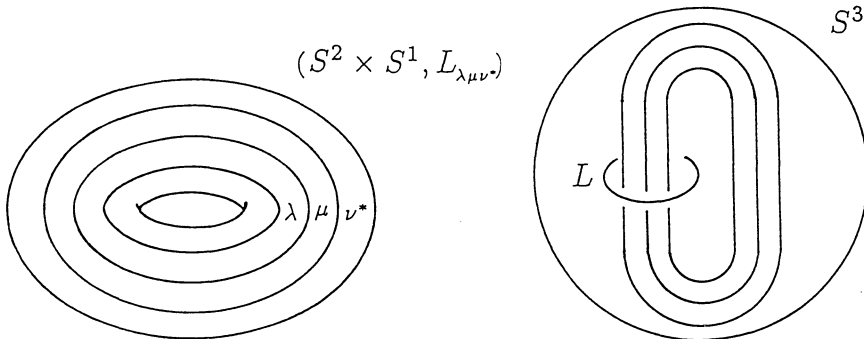


Fig. 3.7

Fig. 3.8

where for $\lambda, \mu \in P_+(K)$,

$$S_{\lambda\mu} = \frac{\sqrt{-1}^{N(N+1)/2}}{\sqrt{(N+1)!^{2N}}} \sum_{w \in W} (\det w) \bar{q}^{(\lambda + \rho, w(\mu + \rho))}.$$

Proof. Let us consider $S^2 \times S^1$ containing the 3-component link $L_{\lambda\mu\nu^*}$ with colors λ, μ, ν^* drawn in Fig.3.7, where for the longest element w_0 in the Weyl group, $\lambda^* = -w_0(\lambda)$. Let L be an unknotted circle with the zero framing which links $L_{\lambda\mu\nu^*}$ drawn in Fig.3.8. By the Dehn surgery on S^3 along the circle L , one can obtain $(S^2 \times S^1, L_{\lambda\mu\nu^*})$. In a way similar to the proof in [16, §3], we prove the assertion, evaluating $Z_r(S^2 \times S^1, L_{\lambda\mu\nu^*})$ in two ways.

We note that for $\lambda \in \partial P_+(K)$, V_λ is irreducible and the quantum dimension $\dim_q V_\lambda = 0$, and that a colored link with a component assigned with the tensor product of V_ω , $\omega \in \partial P_+(K)$ and the fundamental representations can be regarded as a colored link with a component assigned V_ω , $\omega \in \partial P_+(K)$. Then, by the formula (3.3) and the unitarity of the S-matrix $(S_{\lambda\mu})$ [7], we can compute

$$\begin{aligned} Z_r(S^2 \times S^1, L_{\lambda\mu\nu^*}) &= \sum_{\varepsilon \in P_+(K)} S_{\varepsilon 0} \left(\sum_{\varepsilon' \in P_+(K)} \frac{S_{\varepsilon' \varepsilon} S_{\varepsilon' \nu^*} S_{\varepsilon 0}}{S_{\varepsilon 0} S_{0 0}} N_{\lambda\mu}^{\varepsilon'} \right) \\ &= \sum_{\varepsilon' \in P_+(K)} N_{\lambda\mu}^{\varepsilon'} \left(\frac{1}{S_{0 0}} \delta_{\varepsilon' \nu} \right) \\ &= \frac{1}{S_{0 0} N_{\lambda\mu}^\nu}. \end{aligned}$$

On the other hand, a link $L_{\lambda\mu\nu^*} \cup L$ can be regarded as the result of connecting 3 Hopf links in a way analogous to the proof in [16], and so we can directly compute from Proposition 3.1 (1)

$$Z_r(S^2 \times S^1, L_{\lambda\mu\nu^*}) = \frac{1}{S_{0 0}} \sum_{\varepsilon} \frac{S_{\lambda \varepsilon} S_{\mu \varepsilon} S_{\nu \varepsilon}^*}{S_{0 \varepsilon}}.$$

Thus the claim follows from the comparison of these two evaluations.

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