ON THE BOUNDEDNESS OF SOLUTIONS AND THE ATTRACTIVITY PROPERTIES FOR NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Errata Sheet

Page	Line	Error	Correction
4	20	Th. 3.16	Theorem 3.16
18	4	$\frac{r_2(t)y_2}{a(t)h(y)}$	$\frac{r_2(t)y^2}{a(t)g(y)}$
38	5	and so	for $s_2 \leqq t \leqq s_3$, and so
38	6	$\int_{s_3}^{t}$	$\int_{s_3}^{s_4}$
42	16	assumptions (A4)'	assumptions (A3)', (A4)'
43	19	assumptions (A4)'	assumptions (A3)', (A4)'
54	7	J.S.W. wong	J.S.W. Wong

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TABLE OF CONTENTS

Page

Introduction...1

1. Definitions and Lemmas....................................6

2. Boundedness...16

3. Attractivity..23

3.1. Attractivity for the equation $(a(t)x')' + b(t)f_1(x)g_1(x')x' + c(t)f_3(x)g_2(x')x = e(t,x,x')$........23

3.2. Uniform boundedness and attractivity for the equation $x'' + b(t)f_1(x)g_1(x')x' + c(t)f_2(x)g_2(x') = e(t,x,x')$........34

3.3. Uniform attractivity for the equation $x''+ b(t)f_1(x)g_1(x')x' + c(t)f_3(x)g_2(x')x = e(t,x,x')$.....47

Bibliography...52

Introduction

In this thesis I will consider the asymptotic behavior of solutions of the second order differential equations

(1.1) $(a(t)x')' + b(t)f_1(x)g_1(x')x' + c(t)f_2(x)g_2(x') = e(t,x,x')$

(1.2) $(a(t)x')' + h(t,x,x') + c(t)f(x)g(x') = e(t,x,x')$

where $a(t)$, $b(t)$, $c(t)$, $f_1(x)$, $g_1(x')$, $g_2(x')$ and $g(x')$ are all supposed positive, and $xf_2(x) > 0$ for $x \neq 0$. The problems treated are about the boundedness of solutions and the attractivity property of the origin for these equations.

To begin with, we shall review results on the boundedness problem for the equation (1.2). The conditions $\lim_{|x|\to\infty}\int_0^x f(u)du = \infty$ and $\lim_{|y|\to\infty}\int_0^y \frac{v}{g(v)}dv = \infty$ are generally assumed, for example in [2], [5], [6], [8], [9], [11], [19], [20], [21], [24] and so we will make our observations under these conditions. In [19], J.S.W. Wong and T.A.Burton considered the equation of the type $x'' + c(t)f(x)g(x') = 0$. They assumed that $c(t)$ is monotone and $\lim_{t\to\infty} c(t) = c > 0$. In later studies concerning equation (1.2), several types of conditions on $a(t)$ as well as $c(t)$ were considered. In [11], B.S.Lalli assumed that $a'(t) \geqq 0$ and $\lim_{t\to\infty} a(t) = a > 0$, and that $c'(t) \leqq 0$ and $\lim_{t\to\infty} c(t) = c > 0$. On the other hand, several authors tried to replace conditions on $a(t)$ by $\int_0^\infty |a'(t)|dt < \infty$. (See [5], [11] and [24].) Furthermore in [6], S.H.Chang used $0 < a_1 \leqq a(t)$ and $\int_0^\infty \frac{a'(t)_-}{a(t)}dt < \infty$ in place of these conditions where $a'(t)_- =$

$\max\{0,-a'(t)\}$. He showed also that if $a'(t) \geq 0$ and if $\lim_{t\to\infty} a(t) = a > 0$, then the condition "$c(t)$ is monotone and $\lim_{t\to\infty} c(t) = c > 0$" can be replaced by "$0 < c_1 \leq c(t)$ and $\int_0^\infty \frac{c'(t)_+}{c(t)}\,dt < \infty$" where $c'(t)_+ = \max\{0,c'(t)\}$. Similarly in [8], J.R.Graef and P.W.Spikes made their arguments under the consitions $0 < a(t) \leq a_2, \int_0^\infty \frac{a'(t)_-}{a(t)}\,dt < \infty$ and $\int_0^\infty \frac{c'(t)_-}{c(t)}\,dt < \infty$, and in [2] J.W.Baker assumed $\int_0^\infty \frac{c'(t)_+}{c(t)}\,dt < \infty$ and $\int_0^\infty \frac{c'(t)_-}{c(t)}\,dt < \infty$. This kind of conditions on $a(t)$ and $c(t)$ will also be imposed in our studies.

In the recent paper [9], Graef and Spikes discussed the boundedness of solutions of the equation (1.2) under some other conditions. Among them, a particular assumption was $|e(t,x,y)| \leq \frac{a(t)c'(t)}{Mc(t)}$ for some $M > 0$. This has an advantage that $\frac{a(t)c'(t)}{Mc(t)}$ may even diverge to ∞ as t tends to ∞, but still it contains some unsatisfactory limitations. One is that $c'(t)$ must be nonnegative, and the other is, when $c(t)$ is independent of t, the above inequality will imply that $e(t,x,y) \equiv 0$. So, in our studies on the boundedness of the solutions of (1.2), one of the assumptions on $e(t,x,y)$ will be $|e(t,x,y)| \leq \frac{a(t)|c'(t)|}{Mc(t)} + r_1(t) + r_2(t)|y|$ and $\int_0^\infty r_i(t)dt < \infty$ $(i=1,2)$, while we don't assume $c'(t) \geq 0$. Our result on the boundedness of solutions is Theorem 2.1, and it is an improvement of the result of Graef and Spikes.

Next we consider the attractivity property of the origin for the equation (1.1). D.W.Bushaw showed the global asymptotic stability of the zero solution of the system

$$(1.3) \qquad x' = y\ , \qquad y' = -f(x,y) - g(x)$$

where $xg(x) > 0$ for $x \neq 0$ and $yf(x,y) > 0$ for $y \neq 0$, under the extra condition that $\int_0^{\pm\infty} g(x)dx = \infty$. Without assuming this condition, T.A.Burton discussed the global asymptotic stability for the system

$$(1.4) \qquad x' = y\ , \qquad y' = -p(x)|y|\ y - g(x)$$

where $p(x) > 0$ and $0 < \alpha < 1$, and gave the following theorem.

Theorem ([3]). The zero solution of (1.4) is globally asymptotically stable if and only if $\int_0^{\pm\infty} [p(x) + |g(x)|]dx = \pm\infty$. In addition, Burton obtained in [4] an extension of this theorem for a more general system

$$(1.5) \qquad x' = y\ , \qquad y' = -f(x)h(y)y - g(x) + e(t)\ .$$

(See [7] also.) Furthermore for the system

$$(1.6) \qquad x' = y\ , \qquad y' = -f(x)h(y)y - g(x)k(y) + e(t)\ ,$$

J.W.Heidel proved in [10] that if $\int_0^{\pm\infty} [f(x) + |g(x)|]dx = \pm\infty$ and if the function $k(y)$ satisfies some conditions, then all solutions of (1.6) converge to the origin as t tends to ∞ .

In [1], J.W.Baker studied the convergence to zero of the solutions along with their derivatives of the nonautonomous second order differential equation

$$(a(t)x')' + \phi(t,x,x')x' + c(t)f(x) = e(t,x,x')$$

under the conditions that the functions $a(t)$ and $c(t)$ are monotone and tend to some positive constants as t tends to ∞. Also, M.Yamamoto and the author studied in [14], [15] and [23], the convergence to zero of the solutions along with their derivatives of the equation (1.1) without assuming these conditions. In our papers, we assumed that $0 < b_1 \leqq b(t) \leqq b_2$, $|e(t,x,y)| \leqq r_1(t) + r_2(t)|y|$ and the functions $\frac{a'(t)}{a(t)}$, $\frac{c'(t)}{c(t)}$, $r_1(t)$ and $r_2(t)$ are absolutely integrable on $[0,\infty)$. In this case, we note that, if we put $b^*(t) = \frac{b(t)}{a(t)}$, $c^*(t) = \frac{c(t)}{a(t)}$ and $e^*(t,x,y) = \frac{e(t,x,y)-a'(t)y}{a(t)}$ then the equation (1.1) is reduced to the equation

$$x'' + b^*(t)f_1(x)g_1(x')x' + c^*(t)f_2(x)g_2(x') = e^*(t,x,x')$$

which is easier to study. $b^*(t)$, $c^*(t)$ and $e^*(t,x,y)$ satisfy the same conditions as for $b(t)$, $c(t)$ and $e(t,x,y)$.

The main results of my study concerning the attractivity property of the origin for the equation (1.1) are Theorem 3.1, Theorem 3.15 and Theorem 3.16. In Theorem 3.1, a sufficient condition for the convergence to zero of all bounded solutions is given. In Theorem 3.15 a necessary and sufficient condition for the attractivity of the origin and in Th. 3.16 a necessary and sufficient condition for the uniform attractivity are given. These extend Burton's results for (1.5) to a wide class of equations (1.1). In the course, by generalizing Heidel's result, we proved Theorem 3.11. This is a consequence of Lemma 1.10 which is a modification of LaSalle's theorem in [12].

In this thesis, I will give accounts of my works concerning these problems. In the first section, I give basic definitions and fundamental lemmas which are used throughout. In section 2 I will discuss the boundedness of solutions, and in section 3 the attractivity.

1. Definitions and Lemmas

Consider a differential equation

$$(1.7) \qquad x' = f(t,x)$$

for $t \geqq 0$ and $x \in R^2$, where $f:[0,\infty) \times R^2 \to R^2$ is a continuous function. $x(t;t_0,x_0)$ will denote a solution of (1.7) through x_0 at $t = t_0$ and $\|.\|$ denote the Euclidean norm. Moreover Q_i means the i-th quadrant in R^2 ($i = 1,2,3,4$). Let U be a set in R^2. $\overline{U}$ is the closure of U and ∂U the boundary of U.

Definition 1.1. the solutions of (1.7) are uniformly bounded if for any $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that $\|x_0\| \leqq \alpha$ and $t_0 \geqq 0$ imply $\|x(t;t_0,x_0)\| \leqq \beta(\alpha)$ for $t \geqq t_0$.

Definition 1.2. The origin is globally uniformly attractive for (1.7) if for any $\alpha > 0$ and any $\varepsilon > 0$, there exists $T(\alpha,\varepsilon) > 0$ such that $\|x_0\| \leqq \alpha$ and $t_0 \geqq 0$ imply $\|x(t;t_0,x_0)\| < \varepsilon$ for $t \geqq t_0 + T(\alpha,\varepsilon)$.

Definition 1.3. Let the equation (1.7) have the zero solution. Then

(A) the zero solution of (1.7) is uniformly stable if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ and $t_0 \geqq 0$ imply $\|x(t;t_0,x_0)\| < \varepsilon$ for $t \geqq t_0$,

(B) in addition to (A), if the solutions are uniformly bounded and if the origin is globally uniformly attractive, then the zero solution of (1.7) is said to be globally uniform-asymptotically stable.

In what follows, we shall use the notations $a'(t)_+ = \max\{0, a'(t)\}$ and $a'(t)_- = \max\{0, -a'(t)\}$.

Lemma 1.4. Let $a(t)$ be a continuously differentiable, positive function defined on $[0,\infty)$.

(i) If $\int_0^\infty \frac{a'(t)_-}{a(t)}\,dt < \infty$, then there exists a constant $a_1 > 0$ such that $a_1 \leqq a(t)$ for $t \geqq 0$.

(ii) If $\int_0^\infty \frac{a'(t)_+}{a(t)}\,dt < \infty$, then there exists a constant $a_2 > 0$ such that $a(t) \leqq a_2$ for $t \geqq 0$.

Proof. (i) Put $a_1 = a(0)\exp[-\int_0^\infty \frac{a'(s)_-}{a(s)}ds]$. Then we have $a(t) = a(0)\exp[\int_0^t \frac{a'(s)}{a(s)}ds] \geqq a(0)\exp[-\int_0^t \frac{a'(s)_-}{a(s)}ds]$ $\geqq a(0)\exp[-\int_0^\infty \frac{a'(s)_-}{a(s)}ds] = a_1$ for $t \geqq 0$.

(ii) Put $a_2 = a(0)\exp[\int_0^\infty \frac{a'(s)_+}{a(s)}ds]$. Then we have $a(t) \leqq a(0)\exp[\int_0^t \frac{a'(s)_+}{a(s)}ds] \leqq a(0)\exp[\int_0^\infty \frac{a'(s)_+}{a(s)}ds] = a_2$ for $t \geqq 0$. Q.E.D.

Lemma 1.5. Let $\phi(t)$ and $r(t)$ be nonnegative, continuous functions defined on $[0,\infty)$ and let $\lambda > 0$. If

$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\phi(s)ds < \lambda$ and if $\lim_{t\to\infty} e^{-\varepsilon t}\int_0^t e^{\varepsilon s}r(s)ds = 0$ for any

$\varepsilon > 0$, then every nonnegative function $u(t)$ which satisfies the differential inequality

$$(1.8) \qquad u' \leqq (-\lambda + \phi(t))u + r(t) \quad \text{for} \quad t \geqq t_0$$

converges to zero as t tends to ∞ .

Proof. Let $u(t)$ be a nonnegative solution of (1.8) with $u(t_0) = u_0$. Since $\exp[-\lambda t + \int_0^t \phi(s)ds]$ is a fundamental solution of the differential equation $u' = (-\lambda + \phi(t))u$, we obtain the following estimate according to the well known comparison theorem,

$$u(t) \leqq u_0 \exp[-\lambda(t-t_0) + \int_{t_0}^{t} \phi(s)ds]$$
$$+ \int_{t_0}^{t} r(s)\exp[-\lambda(t-s) + \int_{s}^{t} \phi(u)du]ds$$

for $t \geqq t_0$. From the assumption on $\phi(t)$, there exist $\varepsilon \in (0,\alpha)$ and $T > t_0$ such that

$$(1.9) \qquad \frac{1}{v}\int_{s}^{s+v} \phi(u)du < \lambda-\varepsilon \quad \text{for} \quad s \geqq T \quad \text{and} \quad v \geqq T.$$

Hence $\int_T^t \phi(u)du < (\lambda-\varepsilon)(t-T)$ for any $t \geqq 2T$. This implies that for $t_0 \leqq s \leqq T$, $t \geqq 2T$,

$$-\lambda(t-s) + \int_{s}^{t} \phi(u)du \leqq -\lambda(t-T) + \int_{s}^{T} \phi(u)du + \int_{T}^{t} \phi(u)du$$
$$\leqq -\varepsilon(t-T) + \int_{t_0}^{T} \phi(u)du$$

and so $\int_{t_0}^{T} r(s)\exp[-\lambda(t-s) + \int_{s}^{t} \phi(u)du]ds$

$$\leqq e^{-\varepsilon(t-T)}\int_{t_0}^{T} r(s)\exp[\int_{t_0}^{T}\phi(u)du]ds \quad \text{for} \quad t \geqq 2T.$$

The right-hand term of the above last inequality tends to zero as $t \to \infty$. Similarly we have

$$\exp[-\lambda(t-t_0) + \int_{t_0}^{t}\phi(u)du] \leqq \exp[-\varepsilon(t-T) + \int_{t_0}^{T}\phi(u)du]$$

and the right-hand term tends to zero as $t \to \infty$. It follows from (1.9) that

$$\int_{s}^{t}\phi(u)du < (\lambda-\varepsilon)(t-s) \quad \text{for} \quad T \leqq s \leqq t-T,\ t \geqq 2T,$$

hence $\quad -\lambda(t-s) + \int_{s}^{t}\phi(u)du < -\varepsilon(t-s) \quad$ for $\quad T \leqq s \leqq t-T,\ t \geqq 2T.$

This implies that

$$\int_{T}^{t-T} r(s)\exp[-\lambda(t-s) + \int_{s}^{t}\phi(u)du]ds \leqq e^{-\varepsilon t}\int_{T}^{t} e^{\varepsilon s}r(s)ds$$

for $t \geqq 2T$. On the other hand, since

$\int_{t-T}^{t}\phi(u)du < (\lambda-\varepsilon)T$ for $t \geqq 2T$, we obtain

$$\begin{aligned}
&\int_{t-T}^{t} r(s)\exp[-\lambda(t-s) + \int_{s}^{t}\phi(u)du]ds \\
&\leqq \int_{T}^{t} r(s)\exp[-\lambda(t-s) + \int_{t-T}^{t}\phi(u)du]ds \\
&\leqq e^{(\lambda-\varepsilon)T}\, e^{-\lambda t}\int_{T}^{t} e^{\lambda s}r(s)ds \quad \text{for} \quad t \geqq 2T.
\end{aligned}$$

These estimates show from the assumption on $r(t)$ that $\int_{T}^{t} r(s)\exp[-\lambda(t-s) + \int_{s}^{t}\phi(u)du]ds$ tends to zero as $t \to \infty$. Thus we conclude that

$$u_0\exp[-\lambda(t-t_0) + \int_{t_0}^{t}\phi(u)du]$$

$$+ \left(\int_{t_0}^{T} + \int_{T}^{t}\right) r(s)\exp[-\lambda(t-s) + \int_{s}^{t} \phi(u)du]ds$$

tends to zero as $t \to \infty$. This completes the proof of Lemma 1.5. Q.E.D.

Remark 1.6. The following proposition was given by N.Onuchic. (See [13], [22].)
"Let $\phi(t)$ be a nonnegative, continuous function defined on $[0,\infty)$. Then the zero solution $u(t) \equiv 0$ of the differential equation $u' = [-\lambda + \phi(t)]u$ is globally asymptotically stable, if and only if the function $\phi(t)$ satisfies
$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_{t}^{t+v} \phi(s)ds < \lambda$."

Lemma 1.7. Let $r(t)$ be a nonnegative, continuous function defined on $[0,\infty)$. If $\int_0^\infty r(t)dt < \infty$, then $\lim_{t\to\infty} e^{-\varepsilon t}\int_0^t e^{\varepsilon s}r(s)ds = 0$ for any $\varepsilon > 0$.

Proof. For any positive number η, there exists $T_1 > 0$ such that $\int_{T_1}^{\infty} r(s)ds < \eta$. For $t > T_1$, we have $e^{-\varepsilon t}\int_0^t e^{\varepsilon s}r(s)ds$ $\leqq e^{-\varepsilon t}\int_0^{T_1} e^{\varepsilon s}r(s)ds + \int_{T_1}^{t} r(s)ds < e^{-\varepsilon t}\int_0^{T_1} e^{\varepsilon s}r(s)ds + \eta$.
Therefore there exists $T_2 > T_1$ such that $e^{-\varepsilon t}\int_0^t e^{\varepsilon s}r(s)ds < 2\eta$ for $t \geqq T_2$. This completes the proof of Lemma 1.7. Q.E.D.

Lemma 1.8. Let $r(t)$ be a nonnegative, continuous function. If $\lim_{t\to\infty} e^{-\varepsilon t}\int_0^t e^{\varepsilon s}r(s)ds = 0$ for some $\varepsilon > 0$, then

$$\lim_{(t,v)\to(\infty,\infty)} \sup \frac{1}{v}\int_t^{t+v} r(s)ds = 0.$$

Proof. Using the integration by parts formula, we have

$$\frac{1}{v}\int_t^{t+v} r(s)ds = \frac{1}{v}\int_t^{t+v} e^{-\varepsilon s} e^{\varepsilon s} r(s)ds$$
$$= \frac{1}{v} e^{-\varepsilon(t+v)}\int_t^{t+v} e^{\varepsilon u} r(u)du + \frac{\varepsilon}{v}\int_t^{t+v} e^{-\varepsilon s}\{\int_t^{s} e^{\varepsilon u} r(u)du\}ds$$
$$\leqq \frac{1}{v}\int_0^{t+v} e^{-\varepsilon(t+v-u)} r(u)du + \frac{\varepsilon}{v}\int_t^{t+v}\{\int_0^{s} e^{-\varepsilon(s-u)} r(u)du\}ds.$$

For any positive number η, there exists $T > 0$ such that $\int_0^s e^{-\varepsilon(s-u)}r(u)du < \eta$ for $s \geqq T$. This shows that for $t \geqq T$ and $v > 0$,

$$\int_0^{t+v} e^{-\varepsilon(t+v-u)} r(u)du < \eta$$

and $\int_t^{t+v} ds \int_0^s e^{-\varepsilon(s-u)} r(u)du < \int_t^{t+v} \eta ds = \eta v$,

hence $\frac{1}{v}\int_t^{t+v} r(s)ds < (1+\varepsilon)\eta$ for $t \geqq T$, $v \geqq 1$ which yields

$$\sup_{\substack{t\geqq T\\ v\geqq 1}} \frac{1}{v}\int_t^{t+v} r(s)ds \leqq (1+\varepsilon)\eta.$$

Therefore we conclude that $\lim_{(t,v)\to(\infty,\infty)} \sup \frac{1}{v}\int_t^{t+v} r(s)ds = 0.$ Q.E.D.

Remark 1.9. Let $r(t)$ be a nonnegative, continuous function. Then the following conditions are equivalent.

(i) $\lim_{t\to\infty} e^{-\varepsilon t}\int_0^t e^{\varepsilon s} r(s)ds = 0$ for any $\varepsilon > 0$.

(ii) $\lim_{t\to\infty} e^{-\varepsilon_0 t}\int_0^t e^{\varepsilon_0 s} r(s)ds = 0$ for some $\varepsilon_0 > 0$.

(iii) $\lim_{t\to\infty} \int_t^{t+1} r(s)ds = 0$.

(See [22], [23].)

Lemma 1.10. Let $f(t,x)$ be a continuous function from $[0,\infty) \times R^2$ into R^2 and let $x(t)$ be a solution of $x' = f(t,x)$ which is defined on $[t_0,\infty)$ and remains in a compact set D of R^2 for $t \geqq t_0$. Assume that there exist a positive constant N and a nonnegative, continuous function $r(t)$ such that $\|f(t,x)\| \leqq N + r(t)$ for all $t \geqq 0$ and all $x \in D$, and $\int_0^\infty r(t)dt < \infty$. If there exists a nonnegative function $V(t,x)$ on $[0,\infty) \times R^2$ such that

(i) $V(t,x)$ is continuously differentiable on $[0,\infty) \times R^2$,

(ii) $V'(t,x) \leqq -W(x) + \phi(t)$ for $t \geqq 0$ and $x \in D$, where $V'(t,x) = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{2} \frac{\partial V(t,x)}{\partial x_i} f_i(t,x)$ and $W(x)$ is a nonnegative, continuous function on D and $\phi(t) \geqq 0$, $\int_0^\infty \phi(t)dt < \infty$,

then $x(t)$ approaches $E = \{x \in D | W(x) = 0\}$ as t tends to ∞.

Proof. This lemma is an extension of Theorem 1(a) in [12:LaSalle] and the proof of this lemma is analogous to that of Theorem 1(a). Since $x(t)$ remains in the compact set D in the future, there exist a point $x^* \in D$ and an increasing sequence $\{t_n\}$ such that t_n tends to ∞ and $x(t_n)$ converges to x^* as $n \to \infty$. Now suppose that $x(t)$ does not approach E as t tends to ∞. Then we may assume, without any loss of generality, that $x^* \notin E$. This implies $W(x^*) > 0$, hence there exist a positive constant δ and a 2ε-neighborhood $U_{2\varepsilon}(x^*)$ of x^* such that $W(x) \geqq \delta$ for all x in $\overline{U_{2\varepsilon}(x^*)} \cap D$. On the other hand, since $\frac{dV(t,x(t))}{dt} \leqq - W(x(t)) + \phi(t)$ for $t \geqq t_0$, it follows that $0 \leqq V(t,x(t)) \leqq V(t_0,x(t_0)) - \int_{t_0}^{t} W(x(s))ds + \int_{t_0}^{t} \phi(s)ds$ for $t \geqq t_0$. Our assertion about $\phi(t)$ shows that $0 \leqq \int_{t_0}^{t} W(x(s))ds \leqq V(t_0,x(t_0)) + \int_0^\infty \phi(s)ds$ for $t \geqq t_0$ and so $\int_{t_0}^{t} W(x(s))ds$ is bounded for all $t \geqq t_0$. Thus $x(t)$ can not remain in $\overline{U_{2\varepsilon}(x^*)} \cap D$ for an infinite length of time. However $x(t)$ passes through $U_{2\varepsilon}(x^*)$ an infinite number of times, because $x(t_n)$ converges to x^* as $n \to \infty$. Hence there exist increasing sequences $\{t_n'\}$ and $\{s_n'\}$ such that $t_n' < s_n' < t_{n+1}'$, $\lim_{n\to\infty} t_n' = \infty$, $x(t_n') \in \partial U_{2\varepsilon}(x^*)$, $x(s_n') \in \partial U_\varepsilon(x^*)$ and $x(t) \in U_{2\varepsilon}(x^*)$ for $t_n' < t < s_n'$. Then $\sum_{n=1}^{\infty}(s_n'-t_n')$ is convergent. This implies that $s_n'-t_n'$ converges to zero as $n \to \infty$, and hence there exists a natural number n_0 such that $s_n'-t_n' < \frac{\varepsilon}{2N}$ for all $n \geqq n_0$. On the other hand, it follows from the assumption on $f(t,x)$ that

$$\varepsilon \leqq \| x(s_n') - x(t_n') \| \leqq \int_{t_n'}^{s_n'} \| f(s,x(s)) \| ds$$
$$\leqq N(s_n'-t_n') + \int_{t_n'}^{s_n'} r(s)ds$$

and so $\int_{t_n'}^{s_n'} r(s)ds > \frac{\varepsilon}{2}$ for $n \geqq n_0$, which contradicts $\int_0^\infty r(s)ds < \infty$. Therefore $x(t)$ approaches E as t tends to ∞. Q.E.D.

Lemma 1.11. Let $\phi_1(t)$ and $\phi_2(t)$ be nonnegative, continuous functions defined on $[0,\infty)$ and let $u(t)$ be a nonnegative solution defined on $[t_0,\infty)$ of the differential inequality $u' \leqq [-\lambda+\phi_1(t)]u + \phi_2(t)$ for some $\lambda > 0$. If $\int_0^\infty \phi_i(t)dt < \infty$ ($i = 1,2$), then for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $u(t) \leqq \varepsilon + K\{u(t_0)e^{-\lambda(t-t_0)} + e^{-\lambda(t-T)}\}$ for $t \geqq t_0$, where K is a positive constant which is independent of $(t_0,u(t_0))$.

Proof. Since $\exp[-\lambda t + \int_0^t \phi_1(s)ds]$ is a fundamental solution of the equation $u' = [-\lambda+\phi_1(t)]u$, it follows from a comparison theorem and the variation of constant formula that for $t \geqq t_0$,

$$u(t) \leqq u(t_0)\exp[-\lambda(t-t_0) + \int_{t_0}^t \phi_1(s)ds]$$
$$+ \int_{t_0}^t \phi_2(s)\exp\{-\lambda(t-s) + \int_s^t \phi_1(\tau)d\tau]ds \;.$$

Hence

$$u(t) \leqq \{u(t_0)\exp[-\lambda(t-t_0)]+\int_{t_0}^t \phi_2(s)\exp[-\lambda(t-s)]ds\}\exp[\int_0^\infty \phi_1(s)ds]$$

Let $\varepsilon > 0$. Then there exists a time $T > 0$ such that

$$\int_T^\infty \phi_2(s)ds < \varepsilon \exp[-\int_0^\infty \phi_1(s)ds] \;.$$

If $T > t_0$, then

$$\int_{t_0}^t \phi_2(s)\exp[-\lambda(t-s)]ds = (\int_{t_0}^T + \int_T^t)\phi_2(s)\exp[-\lambda(t-s)]ds$$
$$\leqq \int_{t_0}^T \phi_2(s)\exp[-\lambda(t-T)]ds + \int_T^t \phi_2(s)ds$$

$$\leqq \left(\int_0^{\infty}\phi_2(s)ds\right)\exp[-\lambda(t-T)] + \int_T^t \phi_2(s)ds$$

for $t \geqq t_0$. On the other hand, if $t_0 \geqq T$, then

$$\int_{t_0}^t \phi_2(s)\exp[-\lambda(t-s)]ds \leqq \int_T^t \phi_2(s)ds \quad \text{for} \quad t \geqq t_0.$$

Thus we conclude by putting $K = \left(1+\int_0^{\infty}\phi_2(s)ds\right)\exp\left[\int_0^{\infty}\phi_1(s)ds\right]$ that

$$u(t) \leqq K\{u(t_0)\exp[-\lambda(t-t_0)] + \exp[-\lambda(t-T)]\} + \varepsilon \quad \text{for} \quad t \geqq t_0.$$

Q.E.D.

2. Boundedness

Here we consider the equation

$$(2.1)\quad (a(t)x')' + h(t,x,x') + c(t)f(x)g(x') = e(t,x,x') .$$

This equation is equivalent to the system

$$(2.2)\quad \begin{aligned} x' &= y \\ y' &= \frac{1}{a(t)}\{-a'(t)y - h(t,x,y) - c(t)f(x)g(y) + e(t,x,y)\} . \end{aligned}$$

For the boundedness of solutions of (2.1), we make the following assumptions about the equation (2.1):

(B1) The functions $a(t)$ and $c(t)$ are continuously differentiable, and $a(t) > 0$ and $c(t) > 0$ for all $t \geqq 0$.

(B2) The functions $f(x)$ and $g(y)$ are continuous, and $g(y) > 0$. The functions $h(t,x,y)$ and $e(t,x,y)$ are also continuous.

(B3) $\int_0^\infty \frac{|a'(t)|}{a(t)}\,dt < \infty$ and $\int_0^\infty \frac{c'(t)_-}{c(t)}\,dt < \infty$.

(B4) $\int_0^{\pm\infty} f(x)dx = \infty$.

(B5) There exist positive constants M and k such that

$$\frac{y^2}{g(y)} \leqq M\int_0^y \frac{v}{g(v)}\,dv \quad \text{for} \quad |y| \geqq k .$$

(B6) $yh(t,x,y) \geqq 0$ for all $t \geqq 0$ and all real x and y .

(B7) There exist nonnegative, continuous functions $r_1(t)$ and $r_2(t)$ such that $\int_0^\infty r_i(t)dt < \infty$ ($i = 1,2$) and

$$|e(t,x,y)| \leqq \frac{a(t)|c'(t)|}{Mc(t)} + r_1(t) + r_2(t)|y| .$$

Theorem 2.1. Suppose the assumptions (B1) ~ (B7). Then every solution $x(t)$ of (2.1) is bounded. If in addition to the above assumptions, the following hold:

(B3)' $\int_0^{\infty} \frac{|c'(t)|}{c(t)} dt < \infty$,

(B8) $\int_0^{\pm\infty} \frac{y}{g(y)} dy = \infty$,

then every solution $(x(t),y(t))$ of (2.2) is bounded.

Proof. Put $F(x) = \int_0^x f(u)du$. From the assumption (B4) we can find a positive number F_0 such that $F(x) + F_0 \geqq 0$ for all x . Moreover, the assumption (B5) implies that there exists $m \geqq 0$ such that

(2.3) $\frac{|y|}{g(y)} \leqq m + MG(y)$ for all real y ,

where $G(y) = \int_0^y \frac{v}{g(v)} dv$. Let

$$V_1(t,x,y) = [\frac{c(t)}{a(t)} \{F(x)+F_0\} + G(y) + \frac{m}{M}]\exp[2\int_0^t \frac{c'(s)_-}{c(s)}ds]$$

and let $(x(t),y(t))$ be a solution of (2.2) through (x_0,y_0) at $t = t_0$. Put $V_1(t) = V_1(t,x(t),y(t))$ and differentiate $V_1(t)$. Then we have

$$V_1'(t)\exp[-2\int_0^t \frac{c'(s)_-}{c(s)}ds]$$

$$= \{\frac{c'(t)}{a(t)} - \frac{a'(t)c(t)}{a(t)^2}\}\{F(x) + F_0\} + \frac{c(t)}{a(t)}f(x)x' + \frac{yy'}{g(y)}$$

$$+ \frac{2c'(t)_-}{a(t)}\{F(x) + F_0\} + \frac{2c'(t)_-}{c(t)}\{G(y) + \frac{m}{M}\}$$

$$= \{\frac{|c'(t)|}{c(t)} - \frac{a'(t)}{a(t)}\} \frac{c(t)}{a(t)} \{F(x) + F_0\} + \frac{2c'(t)_-}{c(t)}\{G(y) + \frac{m}{M}\}$$

$$- \frac{a'(t)y^2}{a(t)g(y)} - \frac{yh(t,x,y)}{a(t)g(y)} + \frac{ye(t,x,y)}{a(t)g(y)} \quad .$$

Here the assumption (B7) implies that

$$\frac{|ye(t,x,y)|}{a(t)g(y)} \leqq \{\frac{|c'(t)|}{Mc(t)} + \frac{r_1(t)}{a(t)}\}\frac{|y|}{g(y)} + \frac{r_2(t)y_2}{a(t)h(y)}$$

which yields from (B5) and (2.3)

$$\frac{|ye(t,x,y)|}{a(t)g(y)} \leqq \{\frac{|c'(t)|}{c(t)} + M\frac{r_1(t)}{a(t)} + M\frac{r_2(t)}{a(t)}\}\{G(y) + \frac{m}{M}\} .$$

Furthermore it follows from (B5) that

$$\frac{|a'(t)|y^2}{a(t)g(y)} \leqq M\frac{|a'(t)|}{a(t)} \{G(y) + \frac{m}{M}\} .$$

From the above facts and the assumption (B6), we obtain

$$V_1'(t)\exp[-2\int_0^t \frac{c'(s)_-}{c(s)}ds]$$

$$\leqq \{\frac{|c'(t)|}{c(t)} + \frac{|a'(t)|}{a(t)}\}\frac{c(t)}{a(t)}\{F(x) + F_0\} + \{\frac{2c'(t)_-}{c(t)} + M\frac{|a'(t)|}{a(t)}$$

$$+ \frac{|c'(t)|}{c(t)} + M\frac{r_1(t)}{a(t)} + M\frac{r_2(t)}{a(t)}\}\{G(y) + \frac{m}{M}\}$$

$$\leqq \{\frac{|c'(t)|}{c(t)} + (1+M)\frac{|a'(t)|}{a(t)} + \frac{2c'(t)_-}{c(t)} + M\frac{r_1(t)}{a(t)} + M\frac{r_2(t)}{a(t)}\}$$

$$\times [\frac{c(t)}{a(t)}\{F(x) + F_0\} + G(y) + \frac{m}{M}] .$$

The assumption (B3) implies from Lemma 1.4 that $0 \leqq a_1 \leqq a(t) \leqq a_2$ and $\frac{1}{a(t)} \leqq \frac{1}{a_1}$. Therefore for all $t \geqq t_0$,

$$V_1'(t) \leqq [\frac{c'(t)}{c(t)} + L_1\{\frac{|a'(t)|}{a(t)} + \frac{c'(t)_-}{c(t)} + r_1(t) + r_2(t)\}]V_1(t) ,$$

where $L_1 = \max\{1+M, 4, \frac{M}{a_1}\}$. Integrating $\frac{V_1'(t)}{V_1(t)}$ from t_0 to t, we obtain from the assumptions (B3) and (B7) that

$$V_1(t) \leqq V_1(t_0)\exp[\int_{t_0}^{t}\{\frac{c'(s)}{c(s)}+L_1(\frac{|a'(s)|}{a(s)}+\frac{c'(s)_-}{c(s)}+r_1(s)+r_2(s))\}ds]$$

$$\leqq V_1(t_0)\frac{c(t)}{c(t_0)}\exp[L_1\int_0^{\infty}\{\frac{|a'(s)|}{a(s)}+\frac{c'(s)_-}{c(s)}+r_1(s)+r_2(s)\}ds]$$

$$= L_2 c(t)$$

for $t \geqq t_0$, where $L_2 > 0$. From the definition of $V_1(t,x,y)$,

$$F(x(t)) \leqq \frac{a(t)}{c(t)}V_1(t)\exp[-2\int_0^{t}\frac{c'(s)_-}{c(s)}ds]$$

$$\leqq a_2 L_2$$

for $t \geqq t_0$. This implies from the assumption (B4) that the solution $x(t)$ of (2.1) is bounded for $t \geqq t_0$. Analogously, we have

$$G(y(t)) \leqq V_1(t)\exp[-2\int_0^{t}\frac{c'(s)_-}{c(s)}ds]$$

$$\leqq L_2 c(t)$$

for $t \geqq t_0$ and so if the assumptions (B3)' and (B8) hold, then $y(t)$ is bounded for $t \geqq t_0$. The proof of Theorem 2.1 is now completed. Q.E.D.

Corollary 2.2. Suppose the assumptions (B1), (B2), (B4), (B6) and the following:

(B3)" $a'(t) \geqq 0$, $a(t) \leqq a_2$ for some constant $a_2 > 0$ and $\int_0^{\infty}\frac{c'(t)_-}{c(t)}dt < \infty$, (B5)' There exist constants $m \geqq 0$ and $M > 0$ such that $\frac{|y|}{g(y)} \leqq m + M\int_0^y \frac{v}{g(v)}dv$ for all real y,

(B7)' There exists a nonnegative, continuous function $r_1(t)$ such that $\int_0^\infty r_1(t)dt < \infty$ and $|e(t,x,y)| \leqq \frac{a(t)|c'(t)|}{Mc(t)} + r_1(t)$.

Then every solution $x(t)$ of (2.1) is bounded. If in addition, the assumptions (B3)' and (B8) hold, then every solution $(x(t),y(t))$ of (2.2) is bounded.

Proof. Using the same function $V_1(t,x,y)$ as in the proof of Theorem 2.1, we have

$$V_1'(t)\exp[-2\int_0^t \frac{c'(s)_-}{c(s)} ds]$$

$$\leqq \frac{|c'(t)|}{a(t)}\{F(x)+F_0\} + \frac{2c'(t)_-}{c(t)}\{G(y)+\frac{m}{M}\} + \{\frac{|c'(t)|}{Mc(t)}+\frac{r_1(t)}{a(t)}\}\frac{|y|}{g(y)}$$

because of $a'(t) \geqq 0$.

Hence $V_1'(t) \leqq [\frac{c'(t)}{c(t)} + L_1\{\frac{c'(t)_-}{c(t)} + r_1(t)\}]V_1(t)$. Thus we have the same conclusion as that of Theorem 2.1. Q.E.D.

Remark 2.3. Here we consider what function $g(y)$ satisfies the condition (B5) or (B5)'. First, the following are easily shown :

(i) It is necessary for (B5) that $g(y) \geqq \frac{k^M}{MG(\pm k)}|y|^{2-M}$ for $y \gtrless \pm k$,

(ii) It is necessary for (B5)' that $g(y) \geqq \frac{1}{m}|y|e^{-M|y|}$ for all y .

For example, if $g(y)$ is a positive constant, then the condition (B5) holds for $M = 2$ and for an arbitrary $k \geqq 0$. Moreover, the following classes of functions satisfy the condition (B5) :

(2.4) $A|y|^{\alpha} \leqq g(y) \leqq B|y|^{\alpha}$ for $|y| \geqq k$, where $B \geqq A > 0$ and $\alpha \leqq 2$,

(2.5) $A|y|^{\alpha} \leqq g(y) \leqq B|y|^{\beta}$ for $|y| \geqq k$, where $A > 0$, $B > 0$, $\alpha \geqq 2$ and $\beta > \alpha$,

(2.6) $A\, e^{\alpha|y|} \leqq g(y) \leqq B\, e^{\beta|y|}$ for $|y| \geqq k$, where $A > 0$, $B > 0$ and $\beta \geqq \alpha > 0$.

On the other hand, $g(y) = e^{-|y|}$ satisfies (B5)' but not (B5). Of course, any function which satisfies (B5) does also (B5)'.

Example 2.4. Consider the equation

$$(2.7) \quad x'' + \frac{x'}{4(t+1)} + \frac{x}{8(t+1)^2} = 0 ,$$

where the latter of the assumption (B3) in Theorem 2.1 is not satisfied, since $\int_0^\infty \frac{c'(t)}{c(t)}\, dt = 2\int_0^\infty \frac{1}{t+1}\, dt = \infty$. The other conditions in Theorem 2.1 are all fulfilled. The function $x(t) = \sqrt{t+1}$ is a solution of (2.7) which is unbounded.

Example 2.5. Consider the equation

$$(2.8) \quad x'' + 4t^2 x = 6t \cos t^2 ,$$

where $e(t) = 6t \cos t^2$ is not absolutely integrable on $[0,\infty)$. Hence the assumption (B7) in Theorem 2.1 is not valid. The other conditions in Theorem 2.1 are all fulfilled. Then $x(t) = t \sin t^2$ is an unbounded solution of (2.8).

Example 2.6. Consider the equation

$$(2.9) \quad (a(t)x')' + c(t)x = e(t) ,$$

where $a(t) = (t+7)^2\log(t+7)$, $c(t) = \frac{1}{4}(t+7)$ and $e(t) = \frac{1}{4}(t+11)\log(t+7) + 1$. Then $x(t) = \log(t+7)$ is an unbounded solution of (2.9). The function $e(t)$ satisfies the estimate

$$|e(t)| \leqq \frac{a(t)|c'(t)|}{2c(t)} ,$$

while the former of the assumption (B3) in Theorem 2.1 does not hold, since $a(t) \to \infty$ as $t \to \infty$. The other conditions in Theorem 2.1 are all satisfied.

Example 2.7. Consider the equation

$$(2.10) \quad x'' + x' + x = (2+t)x' .$$

Then $x(t) = t + 1$ is an unbounded solution of this equation, where $r_2(t) = 2 + t$ is not integrable on $[0,\infty)$.

3. Attractivity

3.1. Attractivity for the equation $(a(t)x')' + b(t)f_1(x)g_1(x')x' + c(t)f_3(x)g_2(x')x = e(t,x,x')$.

We shall here discuss the convergence to zero of bounded solutions of the equation

$$(3.1)\quad (a(t)x')' + b(t)f_1(x)g_1(x')x' + c(t)f_2(x)g_2(x') = e(t,x,x')$$

with $xf_2(x) > 0$ for $x \neq 0$ and $\lim_{x\to 0}\frac{f_2(x)}{x} > 0$. Put $f_3(x) = \frac{f_2(x)}{x}$ for $x \neq 0$ and $f_3(0) = \lim_{x\to 0}\frac{f_2(x)}{x}$. Then we see that $f_2(x) = f_3(x)x$ and $f_3(x) > 0$ for all x. Hence in what follows, we consider the equation

$$(3.2)\quad (a(t)x')' + b(t)f_1(x)g_1(x')x' + c(t)f_3(x)g_2(x')x = e(t,x,x')$$

which is equivalent to the system

$$(3.3)\quad \begin{aligned} x' &= y \\ y' &= \frac{1}{a(t)}\{-a'(t)y - b(t)f_1(x)g_1(y)y - c(t)f_3(x)g_2(y)x + e(t,x,y)\}. \end{aligned}$$

It is convenient to use the functions

$$F_1(x) = \int_0^x f_1(u)du \quad , \quad F_3(x) = \int_0^x uf_3(u)du \; ,$$

$$G_1(y) = \int_0^y \frac{1}{g_1(v)}dv \quad , \quad G_2(y) = \int_0^y \frac{v}{g_2(v)}dv$$

and $G_L(y) = L\, G_2(y) - \frac{1}{2}\{G_1(y)\}^2$. We first give a sufficient condition for the convergence to the origin (0,0) of all bounded solutions of (3.3).

Theorem 3.1. Suppose the following assumptions.

(A1) The functions $a(t)$ and $c(t)$ are continuously differentiable and the function $b(t)$ is continuous, and also $a(t) > 0$, $b(t) > 0$, $c(t) > 0$ for all $t \geqq 0$.

(A2) There exist positive constants a_1, a_2, b_1, b_2, c_1 and c_2 such that $a_1 \leqq a(t) \leqq a_2$, $b_1 \leqq b(t) \leqq b_2$ and $c_1 \leqq c(t) \leqq c_2$ for $t \geqq 0$.

(A3) The functions $f_1(x)$, $f_3(x)$, $g_1(y)$ and $g_2(y)$ are all continuous and positive on R^1 , and also the function $e(t,x,y)$ is continuous on $[0,\infty) \times R^2$.

(A4) There exist nonnegative continuous functions $r_1(t)$ and $r_2(t)$ such that $|e(t,x,y)| \leqq r_1(t) + r_2(t)(|x|+|y|)$ for $t \geqq 0$ and for any real x and y .

(A5) $\lim_{t\to\infty} e^{-\varepsilon t}\int_0^t e^{\varepsilon s} r_1(s)ds = 0$ for some $\varepsilon > 0$.

(A6) $\lim_{(t,v)\to(\infty,\infty)} \sup \frac{1}{v}\int_t^{t+v}\{\frac{|a'(s)|}{a(s)} + \frac{|c'(s)|}{c(s)} + r_2(s)\}ds = 0$.

Then every bounded solution of (3.3) converges to the origin (0,0) as $t \to \infty$.

Proof. Let $(x(t), y(t))$ be a bounded solution of (3.3) which is defined on $[t_0, \infty)$. Then there exists a positive constant K such that $|x(t)| + |y(t)| \leqq K$ for $t \geqq t_0$. According to the assumptions imposed on $f_1(x)$, $f_3(x)$, $g_1(y)$ and $g_2(y)$, there exist positive constants $d_1, d_2, \ldots, d_8$ such that

$$(3.4) \quad \begin{aligned} &d_1 \leqq f_1(x) \leqq d_2 \;, \quad d_3 \leqq f_3(x) \leqq d_4 \;, \quad d_5 \leqq g_1(y) \leqq d_6 \;, \\ &d_7 \leqq g_2(y) \leqq d_8 \end{aligned}$$

for any (x,y) satisfying $|x| + |y| \leqq K$. Therefore it follows from the above relations that

$$d_1|x| \leqq |F_1(x)| \leqq d_2|x| \;, \quad \frac{1}{2}d_3x^2 \leqq F_3(x) \leqq \frac{1}{2}d_4x^2 \;,$$

$$(3.5) \quad |G_1(y)| \leqq \frac{1}{d_5}|y| \;, \quad \frac{1}{2d_8}y^2 \leqq G_2(y) \leqq \frac{1}{2d_7}y^2$$

$$\text{and} \quad G_L(y) \geqq \frac{1}{2}\left(\frac{L}{d_8} - \frac{1}{d_5^2}\right)y^2 \;.$$

Let $V_2(t,x,y) = \frac{a(t)}{2}\{F_1(x) + g_1(y)\}^2 + Lc(t)F_3(x) + a(t)G_L(y)$. Then according to (A2) and (3.5)

$$\begin{aligned} V_2(t,x,y) &\geqq Lc(t)F_3(x) + a(t)G_L(y) \\ &\geqq \frac{1}{2}c_1d_3Lx^2 + \frac{a_1}{2}\left(\frac{L}{d_8} - \frac{1}{d_5^2}\right)y^2 \end{aligned}$$

and

$$\begin{aligned} V_2(t,x,y) &= \frac{a(t)}{2}\{F_1(x)^2 + 2F_1(x)G_1(y)\} + Lc(t)F_3(x) + La(t)G_2(y) \\ &\leqq \frac{a_2}{2}\{d_2^2x^2 + \frac{2d_2}{d_5}|xy|\} + \frac{1}{2}c_2d_4Lx^2 + \frac{a_2L}{2d_7}y^2 \\ &\leqq \frac{1}{2}\left(a_2d_2^2 + \frac{a_2d_2}{d_5} + c_2d_4L\right)x^2 + \frac{a_2}{2}\left(\frac{d_2}{d_5} + \frac{L}{d_7}\right)y^2 \end{aligned}$$

for $t \geqq 0$ and for $|x| + |y| \leqq K$. By choosing L large enough, we have $G_L(y) \geqq 0$ and

$$(3.6) \quad d_9(x^2 + y^2) \leqq V_2(t,x,y) \leqq d_{10}(x^2 + y^2)$$

for $t \geqq 0$ and $|x| + |y| \leqq K$, where d_9 and d_{10} are positive constants. Put $V_2(t) = V_2(t,x(t),y(t))$ and differentiate $V_2(t)$ with respect to t. Then we obtain

$$V_2'(t) = \frac{a'(t)}{2}\{F_1(x)+G_1(y)\}^2 + a(t)\{F_1(x)+G_1(y)\}\{f_1(x)x'+\frac{1}{g_1(y)}y'\}$$

$$+ Lc'(t)F_3(x) + Lc(t)xf_3(x)x' + a'(t)G_L(y)$$

$$+ a(t)\{\frac{Ly}{g_2(y)} - G_1(y)\frac{1}{g_1(y)}\}y'$$

$$= a'(t)[\frac{1}{2}\{F_1(x)+G_1(y)\}^2+G_L(y)] + a(t)f_1(x)\{F_1(x)y+yG_1(y)\}$$

$$- a'(t)\frac{F_1(x)y}{g_1(y)} - b(t)f_1(x)F_1(x)y - c(t)\frac{f_3(x)g_2(y)xF_1(x)}{g_1(y)}$$

$$+ Lc'(t)F_3(x) - La'(t)\frac{y^2}{g_2(y)} - Lb(t)\frac{f_1(x)g_1(y)y^2}{g_2(y)}$$

$$+ \{\frac{F_1(x)}{g_1(y)} + \frac{Ly}{g_2(y)}\}e(t,x,y) ,$$

hence from (A4)

$$V_2'(t) \leqq |a'(t)|[\frac{1}{2}\{F_1(x) + G_1(y)\}^2 + G_L(y)]$$

$$+ a(t)f_1(x)\{|F_1(x)y| + yG_1(y)\} + |a'(t)|\frac{|F_1(x)y|}{g_1(y)}$$

$$+ b(t)f_1(x)|F_1(x)y| - c(t)\frac{f_3(x)g_2(y)xF_1(x)}{g_1(y)}$$

$$+ L|c'(t)|F_3(x) + L|a'(t)|\frac{y^2}{g_2(y)} - Lb(t)\frac{f_1(x)g_1(y)y^2}{g_2(y)}$$

$$+ \{r_1(t) + r_2(t)(|x|+|y|)\}\{\frac{|F_1(x)|}{g_1(y)} + \frac{L|y|}{g_2(y)}\}$$

and

$$V_2'(t) \leqq \frac{|a'(t)|}{a(t)}[\frac{a(t)}{2}\{F_1(x) + G_1(y)\}^2 + a(t)G_L(y)]$$

$$+ \frac{|c'(t)|}{c(t)}Lc(t)F_3(x) + |a'(t)|\{\frac{|F_1(x)y|}{g_1(y)} + \frac{Ly^2}{g_2(y)}\}$$

$$+ r_2(t)(|x|+|y|)\{\frac{|F_1(x)|}{g_1(y)} + \frac{L|y|}{g_2(y)}\}$$

$$+ r_1(t)\{\frac{|F_1(x)|}{g_1(y)} + \frac{L|y|}{g_2(y)}\} + \{a(t) + b(t)\}f_1(x)|F_1(x)y|$$

$$+ a(t)f_1(x)yG_1(y) - c(t)\frac{f_3(x)g_2(y)}{g_1(y)}xF_1(x)$$

$$- Lb(t)\frac{f_1(x)g_1(y)}{g_2(y)}y^2$$

for $t \geqq t_0$. This implies from (A2), (3.4) and the definition of $V_2(t,x,y)$ that

$$V_2'(t) \leqq \{\frac{|a'(t)|}{a(t)} + \frac{|c'(t)|}{c(t)}\}V_2(t) + \frac{a_2|a'(t)|}{a(t)}\{\frac{|F_1(x)y|}{d_5} + \frac{Ly^2}{d_7}\}$$

$$+ r_2(t)(|x|+|y|)\{\frac{|F_1(x)|}{d_5} + \frac{L|y|}{d_7}\} + r_1(t)\{\frac{|F_1(x)|}{d_5} + \frac{L|y|}{d_7}\}$$

$$+ (a_2+b_2)d_2|F_1(x)y| + a_2d_2yG_1(y) - \frac{c_1d_3d_7}{d_6}xF_1(x)$$

$$- \frac{Lb_1d_1d_5}{d_8}y^2$$

for $t \geqq t_0$. On the other hand, it follows from (3.5) that $|F_1(x)y| \leqq d_2|xy| \leqq \frac{1}{2}d_2(x^2+y^2)$ and so

$$\frac{|F_1(x)y|}{d_5} + \frac{Ly^2}{d_7} \leqq \frac{d_2}{2d_5}(x^2+y^2) + \frac{L}{d_7}y^2$$

$$\leqq (\frac{d_2}{2d_5} + \frac{L}{d_7})(x^2+y^2) .$$

We have also

$$\frac{|F_1(x)|}{d_5} + \frac{L|y|}{d_7} \leqq \frac{d_2}{d_5}|x| + \frac{L}{d_7}|y| \leqq (\frac{d_2}{d_5} + \frac{L}{d_7})(|x|+|y|) .$$

This implies that

$$(|x|+|y|)\{\frac{|F_1(x)|}{d_5} + \frac{L|y|}{d_7}\} \leqq (\frac{d_2}{d_5} + \frac{L}{d_7})(|x|+|y|)^2$$

$$\leqq 2(\frac{d_2}{d_5} + \frac{L}{d_7})(x^2+y^2) .$$

An analogous estimation shows the inequality

$$(a_2+b_2)d_2|F_1(x)y| + a_2d_2yG_1(y) - \frac{c_1d_3d_7}{d_6}xF_1(x) - \frac{Lb_1d_1d_5}{d_8}y^2$$

$$\leqq (a_2+b_2)d_2^2|xy| + \frac{a_2d_2}{d_5}y^2 - \frac{c_1d_1d_3d_7}{d_6}x^2 - \frac{Lb_1d_1d_5}{d_8}y^2$$

and hence by choosing L large enough,

$$(a_2+b_2)d_2|F_1(x)y| + a_2d_2yG_1(y) - \frac{c_1d_3d_7}{d_6}xF_1(x) - \frac{Lb_1d_1d_5}{d_8}y^2$$

$$\leqq - d_{11}(x^2+y^2)$$

where d_{11} is a suitably chosen positive constant.
Furthermore, we get the following estimate

$$\frac{|F_1(x)|}{d_5} + \frac{L|y|}{d_7} \leqq (\frac{d_2}{d_5} + \frac{L}{d_7})K \quad \text{for} \quad |x|+|y| \leqq K .$$

Thus we obtain from (3.6)

$$V_2'(t) \leqq [- \frac{d_{11}}{d_{10}} + L_3\{\frac{|a'(t)|}{a(t)} + \frac{|c'(t)|}{c(t)} + r_2(t)\}]V_2(t) + L_4r_1(t)$$

for $t \geqq t_0$, where L_3 and L_4 are properly chosen positive constants. Now Lemma 1.5 shows that $V_2(t)$ converges to zero as $t \to \infty$. This completes the proof of Theorem 3.1, since $x(t)^2 + y(t)^2 \leqq \frac{1}{d_9}V_2(t)$. Q.E.D.

Remark 3.2. Let $p(t)$ be a continuously differentiable function and let $0 < p_1 \leqq p(t) \leqq p_2$ for some constants p_1 and p_2. Then $\frac{1}{v}\int_t^{t+v} \frac{\pm p'(s)}{p(s)} ds = \pm \frac{1}{v} \log \frac{p(t+v)}{p(t)}$ and so $\frac{1}{v}\log\frac{p_1}{p_2} \leqq \frac{1}{v}\int_t^{t+v}\frac{\pm p'(s)}{p(s)}ds \leqq \frac{1}{v}\log\frac{p_2}{p_1}$ for any t and $v > 0$. Hence we have $\lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{\pm p'(s)}{p(s)}ds = 0$. Since $p'(t)_+ = p'(t) + p'(t)_-$, we obtain

$$0 \leqq \lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_+}{p(s)}ds \leqq \lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_-}{p(s)}ds \leqq \infty .$$

Moreover since $p'(t)_- = p'(t)_+ - p'(t)$, we obtain

$$\lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_-}{p(s)}ds \leqq \lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_+}{p(s)}ds .$$

These show that

$$\lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_+}{p(s)}ds = \lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_-}{p(s)}ds .$$

Therefore we conclude that $\lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{|p'(s)|}{p(s)}ds = 0$ if and only if $\lim\sup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\frac{p'(s)_-}{p(s)}ds = 0$. (See [23, Theorem 2].)

Theorem 3.3. Suppose the assumptions (A1) ~ (A4) and the following.

(A7) $\int_0^\infty \frac{|a'(t)|}{a(t)}dt < \infty$ and $\int_0^\infty \frac{|c'(t)|}{c(t)}dt < \infty$.

(A8) There exists a positive constant M such that

$$\frac{y^2}{g_2(y)} \leqq M\int_0^y \frac{v}{g_2(v)}dv \quad \text{for any } y .$$

(A9) $\displaystyle\int_0^{\pm\infty} \frac{y}{g_2(y)}dy = \infty$.

(A10) $\displaystyle\int_0^{\infty} r_i(t)dt < \infty$ ($i = 1,2$).

If each of $f_3(x)$ and $g_2(y)$ has a positive lower bound, then every solution of (3.3) converges to the origin (0,0) as t tends to ∞ .

Proof. To show the boundedness of solutions of (3.3), let

$$V_3(t,x,y) = \frac{c(t)}{a(t)}F_3(x) + G_2(y) + 1$$

and let $(x(t),y(t))$ be a solution of (3.3) through (x_0,y_0) at $t = t_0$. Differentiating $V_3(t) = V_3(t,x(t),y(t))$ with respect to t, we have

$$\begin{aligned} V_3'(t) &= \frac{c'(t)}{a(t)}F_3(x) - \frac{c(t)a'(t)}{a(t)^2}F_3(x) + \frac{c(t)}{a(t)}xf_3(x)x' + \frac{yy'}{g_2(y)} \\ &= \frac{c'(t)}{a(t)}F_3(x) - \frac{c(t)a'(t)}{a(t)^2}F_3(x) - \frac{a'(t)y^2}{a(t)g_2(y)} - \frac{b(t)f_1(x)g_1(y)y^2}{a(t)g_2(y)} \\ &\quad + \frac{ye(t,x,y)}{a(t)g_2(y)} \end{aligned}$$

for $t \geqq t_0$. Also from (A4)

$$V_3' \leqq \left(\frac{|a'(t)|}{a(t)}+\frac{|c'(t)|}{c(t)}\right)\frac{c(t)}{a(t)}F_3(x) + \frac{|a'(t)|y^2+r_1(t)|y|+r_2(t)|xy|+r_2(t)y^2}{a(t)g_2(y)} .$$

Here, by our assertion about $f_3(x)$ and $g_2(y)$, there exists a positive constant d such that $f_3(x) > d$ and $g_2(y) > d$. Then $F_3(x) \geqq \frac{1}{2}dx^2$, hence $|x| \leqq \sqrt{\frac{2F_3(x)}{d}}$ for any x. Since $\frac{|y|}{g_2(y)} \leqq \sqrt{\frac{1}{g_2(y)}MG_2(y)}$, it follows that $\frac{|y|}{g_2(y)} \leqq \sqrt{\frac{M}{d}\,\frac{G_2(y)+1}{2}}$ and also $\sqrt{\frac{c(t)}{a(t)}}\frac{|xy|}{g_2(y)} \leqq \sqrt{\frac{2Mc(t)}{d^2 a(t)}F_3(x)G_2(y)} \leqq \frac{1}{d}\sqrt{\frac{M}{2}}\{\frac{c(t)}{a(t)}F_3(x)+G_2(y)\}$

for any x and any y. Thus we conclude from (A2) and (A8) that

$$V_3'(t) \leqq L_5\{\frac{|a'(t)|}{a(t)} + \frac{|c'(t)|}{c(t)} + r_1(t) + r_2(t)\}V_3(t)$$

for $t \geqq t_0$, where L_5 is a positive constant. Therefore it is easily shown by (A7), (A9) and (A10) that the solution $(x(t),y(t))$ is bounded for $t \geqq t_0$, because the existence of the constant d implies $\lim_{|x|\to\infty} F_3(x) = \infty$.

Next, (A7) and (A10) imply (A5) and (A6) by applying Lemma 1.7 and Lemma 1.8. Hence the conclusion of Theorem 3.3 is an immediate consequence of Theorem 3.1. Q.E.D.

Corollary 3.4. Suppose the same assumptions as in Theorem 3.3 except for (A8), (A9) and the existence of a positive lower bound of $g_2(y)$. If $g_2(y)$ has an upper bound and if $\frac{|y|}{g_2(y)} \leqq M\sqrt{G_2(y)}$ for any y and for some $M > 0$, then every solution of (3.3) converges to the origin (0,0) as t tends to ∞ .

The proof of this corollary is analogous to that of Theorem 3.3 and so we omit it.

Example 3.5. Consider the equation

(3.7) $ax'' + bx' + cx = 0$,

where a, b and c are positive constants. This is in the case that the equation (3.2) is autonomous and $f_1(x) = f_3(x) = g_1(x')$

$=g_2(x') \equiv 1$ and $e(t,x,x') \equiv 0$. Then the condition that a, b and c are positive means the well known Routh-Hurwitz's criterion for the asymptotic stability of the zero solution.

Example 3.6. Consider the equation

$$(3.8) \qquad x'' + x' + x = 1 + \cos t .$$

In this case $r_1(t) = 1 + \cos t$ does not satisfy (A5). The function $x(t) = 1 + \sin t$ is a bounded solution of (3.8) which does not converge to zero as t tends to ∞.

Example 3.7. Consider the equation

$$(3.9) \qquad x'' + x' + \frac{2}{1+t}x = \frac{1}{1+t}(1 + \sqrt{x^2+x'^2}) .$$

Then $x(t) = 1$ is a bounded solution of (3.9) which does not converge to zero. In this equation, it is obvious that $c(t) = \frac{2}{1+t}$ does not satisfy (A2).

Example 3.8. Consider the equation

$$(3.10) \qquad x'' + e^t x' + x = 2e^{-t} ,$$

where $b(t) = e^t$ tends to ∞ as $t \to \infty$. Hence (A2) is not valid. The solution $x(t) = 1 + e^{-t}$ of (3.10) does not converge to zero as $t \to \infty$.

Example 3.9. Consider the equation

(3.11) $$x'' + \frac{1}{(1+t)^2}x' + x = \frac{1+\cos t}{(1+t)^2} + \frac{6}{(1+t)^4} - \frac{2}{(1+t)^5} .$$

Since $b(t) = \frac{1}{(1+t)^2}$ tends to zero as $t \to \infty$, (A2) is not valid again. This equation also has a solution which does not converge to zero as $t \to \infty$, for example $x(t) = \sin t + \frac{1}{(1+t)^2}$.

3.2. Uniform boundedness and attractivity for the equation $x'' + b(t)f_1(x)g_1(x')x' + c(t)f_2(x)g_2(x') = e(t,x,x')$.

We shall consider the equation

$$x'' + b(t)f_1(x)g_1(x')x' + c(t)f_2(x)g_2(x') = e(t,x,x') \tag{3.12}$$

which is equivalent to the system

$$\begin{aligned} x' &= y \\ y' &= -\,b(t)f_1(x)g_1(y)y - c(t)f_2(x)g_2(y) + e(t,x,y) \,. \end{aligned} \tag{3.13}$$

In the previous section, we assumed that $f_2(x) = f_3(x)x$, $f_3(x) > 0$. Particularly, we considered the equations with $f_2(x) = x$ in several examples. But we can not apply Theorem 3.1 to the equation (3.12) with $f_2(x) = x^3$. In this section, we give a theorem which can be applied to such an equation.

In [4], T.A.Burton considered the system

$$x' = y \,, \quad y' = -\,f(x)h(y)y - g(x) + e(t) \,,$$

where $xg(x) > 0$ for $x \neq 0$. Under some assumptions, he has given a necessary and sufficient condition for the boundedness of all solutions of this system. That is $\int_0^{\pm\infty} [f(x) + |g(x)|]dx = \pm\infty$. He has also shown that, if $f(0) > 0$ and $f(x) \geqq 0$, then this condition is a necessary and sufficient condition for the convergence to the origin of all solutions. Our result is an extension of Burton's that. (See [16], [17].)

Theorem 3.10. Suppose the assumptions (A8), (A9), (A10) and the following.

(A1)' The functions $b(t)$ and $c(t)$ are continuous and positive for $t \geqq 0$, and also $c(t)$ is continuously differentiable.

(A3)' The functions $f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(y)$ are all continuous on R^1 and $xf_2(x) > 0$ for $x \neq 0$, $f_1(x) > 0$, $g_1(y) > 0$ and $g_2(y) > 0$. Also, the function $e(t,x,y)$ is continuous on $[0,\infty) \times R^2$.

(A4)' There exist nonnegative, continuous functions $r_1(t)$ and $r_2(t)$ such that $|e(t,x,y)| \leqq r_1(t) + r_2(t)|y|^{\ell}$ for any $t \geqq 0$, x and y, where $0 < \ell \leqq 1$.

(A11) $\int_0^{\infty} \frac{|c'(t)|}{c(t)} dt < \infty$ and there exist positive constants b_1 and b_2 such that $b_1 \leqq b(t) \leqq b_2$ for $t \geqq 0$.

(A12) $\int_0^{\pm\infty} \{f_1(x) + |f_2(x)|\} dx = \pm\infty$.

Then the solutions of (3.13) are uniformly bounded.

Proof. It follows from (A3)' and (A8) that

$$\frac{|y|^{1+\ell'}}{g_2(y)} \leqq m + MG_2(y) \quad \text{for any } y \text{ and } 0 \leqq \ell' \leqq 1, \tag{3.14}$$

where $m > 0$. Let $V_4(t,x,y) = c(t)F_2(x) + G_2(y) + \frac{m}{M}$ in which $F_2(x) = \int_0^x f_2(u)du$. Let $\alpha > 1$ and let $x_0^2 + y_0^2 \leqq \alpha^2$.

For any solution $(x(t),y(t))$ of (3.13) through (x_0,y_0) at $t = t_0$, differentiating $V_4(t) = V_4(t,x(t),y(t))$ with respect to t, we have

$$V_4'(t) = c'(t)F_2(x) + c(t)f_2(x)y + \frac{y}{g_2(y)}\{ - b(t)f_1(x)g_1(y)y - c(t)f_2(x)g_2(y) + e(t,x,y)\}$$

$$= c'(t)F_2(x) - \frac{b(t)f_1(x)g_1(y)y^2}{g_2(y)} + \frac{ye(t,x,y)}{g_2(y)}$$

$$\leqq |c'(t)|F_2(x) + \frac{|ye(t,x,y)|}{g_2(y)}$$

for $t \geqq t_0$. Here (A4)' and (3.14) imply that

$$\frac{|ye(t,x,y)|}{g_2(y)} \leqq \frac{r_1(t)|y|+r_2(t)|y|^{1+\ell}}{g_2(y)} \leqq \{r_1(t)+r_2(t)\}\{m+MG_2(y)\} .$$

Hence $V_4'(t) \leqq |c'(t)|F_2(x) + M\{r_1(t)+r_2(t)\}\{\frac{m}{M}+G_2(y)\}$

$$\leqq \{\frac{|c'(t)|}{c(t)} + Mr_1(t) + Mr_2(t)\}V_4(t) .$$

Integrating $\frac{V_4'(t)}{V_4(t)}$ from t_0 to t, we have

$$V_4(t) \leqq V_4(t_0)\exp[\int_{t_0}^{t}\{\frac{|c'(s)|}{c(s)} + Mr_1(s) + Mr_2(s)\}ds] .$$

Put $L_6 = \exp[\int_0^\infty\{\frac{|c'(s)|}{c(s)} + Mr_1(s) + Mr_2(s)\}ds]$. Then $V_4(t) \leqq L_6V_4(t_0)$ and so $G_2(y(t)) \leqq L_6V_4(t_0)$ for $t_0 \leqq t < t_1$, whenever the solution $(x(t),y(t))$ is defined on $[t_0,t_1)$. From (A9), $G_2(y)$ tends to ∞ as $|y| \to \infty$. Therefore there exists $\beta' = \beta'(\alpha) > \alpha$ such that $G_2(y) \leqq L_6 \sup_{x^2+y^2\leqq\alpha^2}\{c_2F_2(x) + G_2(y) + \frac{m}{M}\}$ implies $|y| \leqq \beta'$, where β' depends only on α but is independent of (t_0,x_0,y_0) whenever $x_0^2 + y_0^2 \leqq \alpha^2$.

Thus $|y(t;t_0,x_0,y_0)|$ has an upper bound β' . This implies that the solution $(x(t),y(t))$ is defined in the future, since $x'(t) = y(t)$.

Now we shall consider four cases.

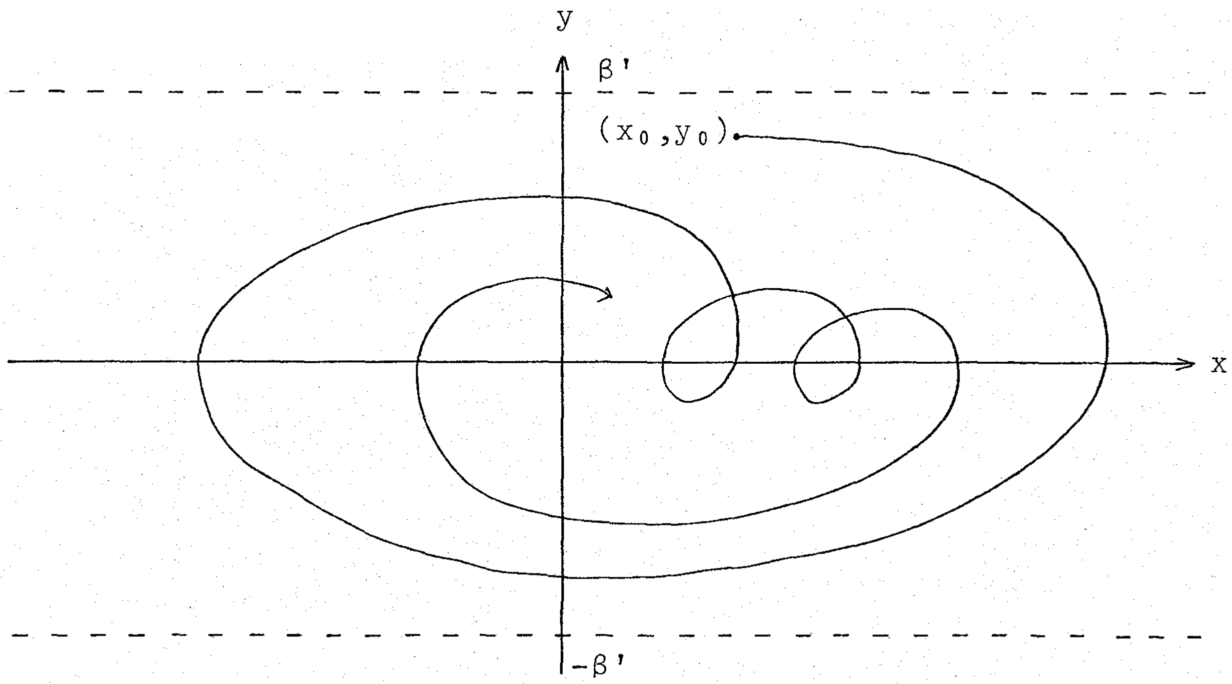
Case I: $F_2(x)$ tends to ∞ as $|x| \to \infty$. It follows from Lemma 1.4 that $F_2(x(t)) \leqq c_1^{-1}L_6V_4(t_0)$ for $t \geqq t_0$, and so there exists $\beta'' = \beta''(\alpha) > \alpha$ such that $F(x) \leqq c_1^{-1}L_6 \sup_{x^2+y^2\leqq\alpha^2}\{c_2F_2(x) + G_2(y) + \frac{m}{M}\}$ implies $|x| \leqq \beta''$. Hence $|x(t;t_0,x_0,y_0)| \leqq \beta''$ for $t \geqq t_0$. Put $\beta(\alpha) = \sqrt{(\beta')^2 + (\beta'')^2}$. Then $x(t)^2 + y(t)^2 \leqq \beta^2$ for $t \geqq t_0$ and β depends only on α but is independent of (t_0,x_0,y_0), whenever $x_0{}^2 + y_0{}^2 \leqq \alpha^2$. Thus the solutions of (3.13) are uniformly bounded.

Case II: $F_1(x)$ tends to $\pm\infty$ as $x \to \pm\infty$. Since $|y(t)| \leqq \beta'$ for $t \geqq t_0$, we define $V_5(x,y) = b_1F_1(x) + G_1(y)$ for any x and $|y| \leqq \beta'$. Differentiating $V_5(t) = V_5(x(t),y(t))$ with respect to t, we have from (A4)' that

$$V_5'(t) = b_1f_1(x)y+\frac{1}{g_1(y)}\{-b(t)f_1(x)g_1(y)y-c(t)f_2(x)g_2(y)+e(t,x,y)\}$$

$$\leqq \{b_1-b(t)\}f_1(x)y - \frac{c(t)f_2(x)g_2(y)}{g_1(y)} + \frac{r_1(t)+r_2(t)|y|^{\ell}}{g_1(y)}$$

for $t \geqq t_0$. Suppose that $x(t) > 0$ for $t_1 < t < t_2 \leqq \infty$. Let (s_1,s_2) be any subinterval of (t_1,t_2) in which $y(t) > 0$. Put $g_0 = \sup_{|y|\leqq\beta'} \frac{1 + |y|^{\ell}}{g_1(y)}$. Then $V_5'(t) \leqq g_0(r_1(t) + r_2(t))$ for $s_1 < t < s_2$, hence $V_5(t) \leqq V_5(s_1) + g_0\int_{s_1}^{t}(r_1(s) + r_2(s))ds$ for $s_1 \leqq t < s_2$.



Let (s_3, s_4) be another subinterval such that $y(t) \leqq 0$ for $s_2 \leqq t \leqq s_3$ and $y(t) > 0$ for $s_3 < t < s_4$. As long as $y(t) \leqq 0$, $x(t)$ is monotonically decreasing in t. Hence $x(t) \leqq x(s_2)$ for $s_2 \leqq t \leqq s_3$ which implies $V_5(t) = b_1F_1(x(t)) + G_1(y(t)) \leqq b_1F_1(x(s_2)) = V_5(s_2)$ and so

$$V_5(t) \leqq V_5(s_1) + g_0\left(\int_{s_1}^{s_2} + \int_{s_3}^{t}\right)(r_1(s) + r_2(s))ds$$

for $s_2 \leqq t < s_4$. Thus we obtain that

$$V_5(t) \leqq b_1F_1(x(t_1)) + G_1(\beta') + g_0\int_0^\infty (r_1(s) + r_2(s))ds$$

for $t_1 \leqq t < t_2$. If $t_1 = t_0$ and $x_0 \geqq 0$, then

$$V_5(t) \leqq b_1F_1(\alpha) + G_1(\beta') + g_0\int_0^\infty (r_1(s) + r_2(s))ds$$

for $t_0 \leqq t < t_2$. On the other hand, if $x(t_1) = 0$, then $F_1(x(t_1)) = 0$, hence $V_5(t) \leqq G_1(\beta') + g_0\int_0^\infty (r_1(s) + r_2(s))ds$ for $t_1 \leqq t < t_2$. Note that the upper bound of $V_5(t)$ is independent of t_1 and t_2 even if the curve $\{(x(t),y(t))\,|\,t \geq t_0\}$ spirals about the origin. Therefore $b_1F_1(x(t)) \leq b_1F_1(\alpha) + G_1(\beta') - G_1(-\beta') + g_0\int_0^\infty (r_1(s) + r_2(s))ds$ for all $t \geq t_0$, because $b_1F_1(x) = V_5(x,y) - G_1(y)$. Since $F_1(x)$ tends to ∞ as $x \to \infty$, the above estimate shows that there exists a constant $\overline{x}_\alpha > \alpha$ such that $x(t) \leq \overline{x}_\alpha$ for $t \geq t_0$. Similarly, the existence of a lower bound $\underline{x}_\alpha$ of $x(t)$ follows by using $V_5(x,y)$ for $x < 0$ and $|y| \leq \beta'$. Here we note that $\overline{x}_\alpha$ and $\underline{x}_\alpha$ are independent of (t_0,x_0,y_0). Let $\beta(\alpha) = \sqrt{(\beta')^2+\overline{x}_\alpha{}^2+\underline{x}_\alpha{}^2}$ Then $x(t)^2 + y(t)^2 \leq \beta(\alpha)^2$ for $t \geq t_0$. This implies that the solutions of (3.13) are uniformly bounded.

In the case III: $\lim_{x\to\infty} F_1(x) = \infty$ and $\lim_{x\to-\infty} F_2(x) = \infty$, use the function

$$V_6(t,x,y) = \begin{cases} V_5(x,y) & \text{for } t \geq 0,\ x \geq 0,\ |y| \leq \beta' \\ V_4(t,x,y) & \text{for } t \geq 0,\ x < 0,\ |y| \leq \beta'. \end{cases}$$

Then we can show that the solutions of (3.13) are uniformly bounded. Moreover in the case IV: $\lim_{x\to\infty} F_2(x) = \infty$ and $\lim_{x\to-\infty} F_1(x) = -\infty$, use the function

$$V_7(t,x,y) = \begin{cases} V_4(t,x,y) & \text{for } t \geq 0,\ x \geq 0,\ |y| \leq \beta' \\ V_5(x,y) & \text{for } t \geq 0,\ x < 0,\ |y| \leq \beta'. \end{cases}$$

Then we have the same conclusion as proved above. The proof of Theorem 3.10 is now completed. Q.E.D.

Theorem 3.11. Suppose the assumptions (A3)', (A4)', (A8) ~ (A10), (A12) and the following.

(A1)" The functions $b(t)$ and $c(t)$ are continuously differentiable and positive for all $t \geqq 0$.

(A11)' $\int_0^\infty \frac{|b'(t)|}{b(t)}dt < \infty$ and $\int_0^\infty \frac{|c'(t)|}{c(t)}dt < \infty$.

Then every solution of (3.13) converges to the origin (0,0) as t tends to ∞.

Proof. Let $(x(t),y(t))$ be a solution of (3.13) through (x_0,y_0) at $t = t_0$. Since Theorem 3.10 implies that $(x(t),y(t))$ is bounded for $t \geqq t_0$, there exists a compact set $D = \{(x,y) | x^2 + y^2 \leqq \beta^2 , \beta > 0\}$ such that $(x(t),y(t))$ remains in D for $t \geqq t_0$. Now let

$$V_8(t,x,y) = \frac{1}{2}\{b(t)F_1(x) + G_1(y)\}^2 + Lc(t)F_2(x) + G_L(y) .$$

Differentiate $V_8(t) = V_8(t,x(t),y(t))$ with respect to t. Then we have

$$V_8'(t) = \{b(t)F_1(x) + G_1(y)\}\{b'(t)F_1(x) + b(t)f_1(x)x' + \frac{y'}{g_1(y)}\}$$
$$+ Lc'(t)F_2(x) + Lc(t)f_2(x)x' + L\frac{yy'}{g_2(y)} - G_1(y)\frac{y'}{g_1(y)}$$
$$= b'(t)\{b(t)F_1(x)^2 + F_1(x)G_1(y)\} + b(t)f_1(x)yG_1(y)$$
$$- \frac{b(t)c(t)g_2(y)F_1(x)f_2(x)}{g_1(y)} + Lc'(t)F_2(x)$$
$$- \frac{Lb(t)f_1(x)g_1(y)y^2}{g_2(y)} + \{\frac{b(t)F_1(x)}{g_1(y)} + \frac{Ly}{g_2(y)}\}e(t,x,y)$$

which implies from (A4)' that

$$V_8' \leqq \frac{|b'(t)|}{b(t)}\{b(t)^2F_1(x)^2 + b(t)|F_1(x)G_1(y)|\} + \frac{|c'(t)|}{c(t)}Lc(t)F_2(x)$$

$$+ r_1(t)\{\frac{b(t)|F_1(x)|}{g_1(y)} + \frac{L|y|}{g_2(y)}\} + r_2(t)\{\frac{b(t)|F_1(x)y|}{g_1(y)} + \frac{Ly^2}{g_2(y)}\}$$

$$- \frac{b(t)c(t)g_2(y)}{g_1(y)}F_1(x)f_2(x) - \frac{Lb(t)f_1(x)g_1(y)}{g_2(y)}y^2$$

$$+ b(t)f_1(x)yG_1(y)$$

for $t \geqq t_0$. By Lemma 1.4, the assumption (A11)' implies that $b(t)$ and $c(t)$ are bounded for $t \geqq 0$. Recall that the solution $(x(t),y(t))$ is bounded. Then by choosing L large enough, we see the existence of positive constants L_7 and L_8 such that

$$V_8'(t) \leqq - L_7\{F_1(x)f_2(x)+y^2\} + L_8\{\frac{|b'(t)|}{b(t)} + \frac{|c'(t)|}{c(t)} + r_1(t)+r_2(t)\}$$

for $t \geqq t_0$. Now put $E = \{(x,y) \in D | F_1(x)f_2(x) + y^2 = 0\}$. Since $xF_1(x) > 0$ and $xf_2(x) > 0$ for $x \neq 0$, E consists only one point $(0,0)$. Therefore we conclude from Lemma 1.10 that $(x(t),y(t))$ converges to $(0,0)$ as t tends to ∞ . Thus the proof is completed. Q.E.D.

Corollary 3.12. Suppose the same assumptions as in Theorem 3.11. If in addition, $a(t) > 0$ and $\int_0^\infty \frac{|a'(t)|}{a(t)}dt < \infty$, then every solution of the system

$$(3.15)\qquad \begin{aligned} x' &= y \\ y' &= \frac{1}{a(t)}\{-a'(t)y-b(t)f_1(x)g_1(y)y-c(t)f_2(x)g_2(y)+e(t,x,y)\} \end{aligned}$$

converges to the origin $(0,0)$ as t tends to ∞ .

Note that $b^*(t) = \frac{b(t)}{a(t)}$, $c^*(t) = \frac{c(t)}{a(t)}$, $r_1{}^*(t) = \frac{r_1(t)}{a(t)}$ and $r_2{}^*(t) = \frac{|a'(t)|}{a(t)} + \frac{r_2(t)}{a(t)}$ fulfil the assumptions in Theorem 3.11. This corollary is an immediate consequence of Theorem 3.11.

In [18], D.W.Willett and J.S.W.Wong discussed the boundedness of solutions and the global asymptotic stability of the zero solution for the autonomous system (3.16) $x' = y$, $y' = -f(x,y)-g(x)$, under the conditions "In $\{(x,y)|x^2+y^2 > \rho_1{}^2\}$, $yf(x,y) \geqq 0$ and the solutions of (3.16) are unique" and "$xg(x) > 0$ for $|x| > \rho_2$". They showed that the zero solution is globally asymptotically stable if and only if every trajectory which starts in the first or third quadrant and is eventually in some neighborhood of ∞ intersects the x-axis, under the condition that $\rho_1 = \rho_2 = 0$ and the set $\{(x,y)|yf(x,y) = 0\}$ contains no nontrivial trajectories of (3.16).

Lemma 3.13. Suppose the assumptions (A4)' and (A10), and suppose that $b(t)$ and $c(t)$ are continuous and $b(t) > 0$, $c(t) \geqq c_1$ for some $c_1 > 0$. Then for every bounded solution $(x(t),y(t))$ of (3.13) with $(x(t_0),y(t_0)) \in Q_1 \cup Q_3$, the curve $\{(x(t),y(t))|t \geqq t_0\}$ intersects the x-axis.

Proof. Let $(x(t),y(t))$ be a bounded solution of (3.13) through (x_0,y_0) at $t = t_0$, where $x_0 > 0$ and $y_0 > 0$.

Suppose that $(x(t),y(t))$ remains in Q_1 for all $t \geqq t_0$. From (3.13), we have $x(t) \geqq x_0$ and

$$y'(t) \leqq -c(t)f_2(x(t))g_2(y(t)) + |e(t,x(t),y(t))|$$

for $t \geqq t_0$. Since $x(t)$ and $y(t)$ are bounded, it follows from (A4)' that

$$y'(t) \leqq -A + r_1(t) + Br_2(t) \quad \text{for some } A > 0 \text{ and } B > 0 .$$

This implies

$$y(t) \leqq y_0 - A(t-t_0) + \int_0^\infty r_1(s)ds + B\int_0^\infty r_2(s)ds$$

for $t \geqq t_0$. Hence $y(t)$ tends to $-\infty$ as $t \to \infty$, which is a contradiction. Thus we conclude that the curve $\{(x(t),y(t))|t \geqq t_0\}$ intersects the x-axis at a finite time $t_1 > t_0$. The proof for the case of $x_0 < 0$ and $y_0 < 0$ is similar to the above. Q.E.D.

Under some assumptions, Theorem 3.10 and Lemma 3.13 show that, if $\int_0^{\pm\infty} \{f_1(x) + |f_2(x)|\}dx = \pm\infty$, then for every solution of (3.13) with $(x(t_0),y(t_0)) \in Q_1 \cup Q_3$, the curve $\{(x(t),y(t))|t \geqq t_0\}$ intersects the x-axis. The following theorem shows that the converse is also valid under more relaxed assumptions.

Theorem 3.14. Suppose the assumptions (A4)' and (A10), and suppose that $b(t)$ and $c(t)$ are bounded from above by b_2 and c_2 respectively. If for any solution $(x(t),y(t))$ of (3.13) with $(x(t_0),y(t_0)) \in Q_1 \cup Q_3$, the curve $\{(x(t),y(t)) | t \geqq t_0\}$ intersects the x-axis, then $\int_0^{\pm\infty} \{f_1(x) + |f_2(x)|\}dx = \pm\infty$.

Proof. We shall prove $\int_0^{\infty} \{f_1(x) + f_2(x)\}dx = \infty$. Let $V_9(y) = \int_0^y \frac{1}{1+|v|}dv$. Since $V_9(y)$ tends to ∞ as $y \to \infty$, there exists $y_0 > 1$ such that

$$V_9(y_0) > V_9(1) + 1 + \int_0^{\infty} \{r_1(t) + r_2(t)\}dt .$$

Now suppose $\int_0^{\infty} \{f_1(x) + f_2(x)\}dx < \infty$. Then taking $g^* = \sup_{1 \leqq y \leqq y_0} \{g_1(y) + g_2(y)\}$, we choose $x_0 > 0$ so large that $(b_2 + c_2)g^* \int_{x_0}^{\infty} \{f_1(x) + f_2(x)\}dx < 1$. Let $(x(t),y(t))$ be a solution of (3.13) through (x_0,y_0) at $t = t_0$. Then the curve $\{(x(t),y(t)) | t \geqq t_0\}$ must intersect the x-axis at a finite time. Hence we can find the first time $t_2 > t_0$ satisfying $y(t_2) = 1$ and the last time $t_1 < t_2$ satisfying $y(t_1) = y_0$. Then of course $x'(t) = y(t) \geqq 1$ for $t_0 \leqq t \leqq t_2$ and so $x(t) \geqq x_0$ for $t_0 \leqq t \leqq t_2$. Differentiating $v(t) = V_9(y(t))$, we obtain from (A4)'

$$v'(t) = \frac{1}{1+|y|}\{-b(t)f_1(x)g_1(y)y-c(t)f_2(x)g_2(y)+e(t,x,y)\}$$

$$\geqq \frac{1}{1+|y|}\{-b(t)f_1(x)g_1(y)y-c(t)f_2(x)g_2(y)-r_1(t)-r_2(t)|y|\}$$

for $t_1 \leqq t \leqq t_2$. Since $\frac{1}{1+|y(t)|} < 1$ and $\frac{|y(t)|}{1+|y(t)|} < 1$,

$$v'(t) \geqq -b(t)f_1(x)g_1(y)y - c(t)f_2(x)g_2(y) - r_1(t) - r_2(t)$$

for $t_1 \leqq t \leqq t_2$, hence

$$v'(t) \geqq -b_2 g^* f_1(x)x' - c_2 g^* f_2(x)x' - r_1(t) - r_2(t)$$

for $t_1 \leqq t \leqq t_2$. Integrate $v'(t)$ from t_1 to t_2 . Then we have

$$v(t_2) \geqq v(t_1) - b_2 g^* \int_{t_1}^{t_2} f_1(x)x'dt - c_2 g^* \int_{t_1}^{t_2} f_2(x)x'dt$$
$$-\int_{t_1}^{t_2} \{r_1(t) + r_2(t)\}dt$$

and

$$v(t_2) \geqq V_9(y_0)-(b_2+c_2)g^* \int_{x_0}^{x(t_2)} \{f_1(x)+f_2(x)\}dx-\int_{t_1}^{t_2} \{r_1(t)+r_2(t)\}dt.$$

Therefore $v(t_2) \geqq V_9(y_0) - 1 - \int_0^\infty \{r_1(t) + r_2(t)\}dt$. This implies $v(t_2) > V_9(1)$, which contradicts $y(t_2) = 1$. Thus we conclude that $\int_0^\infty \{f_1(x) + f_2(x)\}dx = \infty$. It is analogously proved that

$$\int_0^{-\infty} \{f_1(x) - f_2(x)\}dx = -\infty . \qquad \text{Q.E.D.}$$

Theorem 3.15. Suppose the assumptions (A1)", (A3)', (A4)' (A8) ~ (A10) and (A11)'. Then the following statements are equivalent.

(i) For any solution $(x(t),y(t))$ of (3.13) with $(x(t_0),y(t_0)) \in Q_1 \cup Q_3$, the curve $\{(x(t),y(t)) | t \geqq t_0\}$ intersects the x-axis at a finite time.

(ii) $\int_0^{\pm\infty} \{f_1(x) + |f_2(x)|\}dx = \pm\infty$.

(iii) the solutions of (3.13) are uniformly bounded.

(iv) Every solution of (3.13) converges to the origin as t tends to ∞ .

This theorem is an immediate consequence of Theorem 3.10, Theorem 3.11, Lemma 3.13 and Theorem 3.14.

3.3. Uniform attractivity for the equation

$$x'' + b(t)f_1(x)g_1(x')x' + c(t)\ f_3(x)g_2(x')x = e(t,x,x') .$$

In this section we shall consider the uniform attractivity for the system

$$(3.17)\quad \begin{aligned} x' &= y \\ y' &= -b(t)f_1(x)g_1(y)y - c(t)f_3(x)g_2(y)x + e(t,x,y) . \end{aligned}$$

Theorem 3.16. Suppose the assumptions (A1)', (A3), (A4)', (A8) ~ (A11). Then the following statements are equivalent.

(i) For any solution $(x(t),y(t))$ of (3.17) with $(x(t_0),y(t_0)) \in Q_1 \cup Q_3$, the curve $\{(x(t),y(t))\,|\,t \geqq t_0\}$ intersects the x-axis at a finite time.

(ii) $\displaystyle\int_0^{\pm\infty}\{f_1(x) + |x|f_3(x)\}dx = \pm\infty$.

(iii) The solutions of (3.17) are uniformly bounded.

(iv) The origin is globally uniformly attractive for (3.17).

Proof. The implication (i) ⇒ (ii) ⇒ (iii) is a consequence of Theorem 3.10 and Theorem 3.14. Therefore by Lemma 3.13, we need only show that the implication (iii) ⇒ (iv) is valid. Suppose $\alpha > 0$ and $x_0{}^2 + y_0{}^2 \leqq \alpha^2$. Let $(x(t),y(t))$ be a solution of (3.17) through (x_0,y_0) at $t = t_0$. Then from (iii), there exists a positive constant $\beta = \beta(\alpha)$ such that $x(t)^2 + y(t)^2 \leqq \beta^2$ for $t \geqq t_0$. Now we choose positive constants d_1 , $d_2, \dots, d_8$ such that $d_1 \leqq f_1(x) \leqq d_2$, $d_3 \leqq f_3(x) \leqq d_4$, $d_5 \leqq g_1(y) \leqq d_6$ and $d_7 \leqq g_2(y) \leqq d_8$ for $|x| \leqq \beta$ and $|y| \leqq \beta$. Let

$$V_{10}(t,x,y) = \frac{1}{2}[F_1(x) + G_1(y)]^2 + Lc(t)F_3(x) + G_L(y) .$$

Then $V_{10}(t,x,y) \geqq \frac{1}{2}Lc_1d_3x^2 + \frac{1}{2}(\frac{L}{d_8} - \frac{1}{d_5^2})y^2$ and $V_{10}(t,x,y) \leqq \frac{1}{2}(d_2^2 + Lc_2d_4 + \frac{d_2}{d_5})x^2 + \frac{1}{2}(\frac{d_2}{d_5} + \frac{L}{d_7})y^2$ for all $t \geqq 0$ and for $x^2 + y^2 \leqq \beta^2$. Therefore there exist positive constants $\mu_1 = \mu_1(\alpha)$ and $\mu_2 = \mu_2(\alpha)$ such that

$$\mu_1(x^2 + y^2) \leqq V_{10}(t,x,y) \leqq \mu_2(x^2 + y^2) \tag{3.18}$$

for $t \geqq 0$, $x^2 + y^2 \leqq \beta^2$. Next differentiating $V_{10}(t) = V_{10}(t,x(t),y(t))$ with respect to t, we have

$$\begin{aligned} V_{10}'(t) &= [F_1(x) + G_1(y)][f_1(x)x' + \frac{y'}{g_1(y)}] + Lc'(t)F_3(x) \\ &\quad + Lc(t)xf_3(x)x' + \frac{Lyy'}{g_2(y)} - G_1(y)\frac{y'}{g_1(y)} \\ &= F_1(x)f_1(x)y + G_1(y)f_1(x)y + Lc'(t)F_3(x) + Lc(t)xf_3(x)y \\ &\quad +\{ \frac{F_1(x)}{g_1(y)} + \frac{Ly}{g_2(y)}\}\{-b(t)f_1(x)g_1(y)y-c(t)f_3(x)g_2(y)x+e(t,x,y)\} \\ &= Lc'(t)F_3(x) + \{1 - b(t)\}F_1(x)f_1(x)y + f_1(x)G_1(y)y \\ &\quad - \frac{c(t)F_1(x)f_3(x)g_2(y)x}{g_1(y)} - \frac{Lb(t)f_1(x)g_1(y)y^2}{g_2(y)} \\ &\quad + \{\frac{F_1(y)}{g_1(y)} + \frac{Ly}{g_2(y)}\} e(t,x,y) . \end{aligned}$$

Here $\{1 - b(t)\}F_1(x)f_1(x)y \leqq (1 + b_2)d_2^2|xy|$ and $f_1(x)G_1(y)y \leqq$

$\frac{d_2}{d_5} y^2$. On the other hand, $\frac{c(t)F_1(x)f_3(x)g_2(y)x}{g_1(y)} \geqq \frac{c_1 d_1 d_3 d_7}{d_6} x^2$ and $\frac{Lb(t)f_1(x)g_1(y)y^2}{g_2(y)} \geqq \frac{Lb_1 d_1 d_5}{d_8} y^2$. Furthermore (A4)' implies

$$\{\frac{F_1(x)}{g_1(y)} + \frac{Ly}{g_2(y)}\}e(t,x,y) \leqq (\frac{d_2|x|}{d_5} + \frac{L|x|}{d_7})\{r_1(t) + r_2(t)|y|^{\ell}\}$$

and so

$$\{\frac{F_1(x)}{g_1(y)} + \frac{Ly}{g_2(y)}\}e(t,x,y) \leqq (\frac{d_2}{d_5} + \frac{L}{d_7})\beta(1 + \beta^{\ell})\{r_1(t) + r_2(t)\} ,$$

because of $|x| \leqq \beta$ and $|y| \leqq \beta$. Thus by choosing L large enough

$$V_{10}'(t) \leqq L|c'(t)|F_3(x) - d_9(x^2 + y^2) + d_{10}\{r_1(t) + r_2(t)\}$$

for $t \geqq t_0$, where positive constants d_9 and d_{10} are independent of (t_0,x_0,y_0) . Now (3.18) implies

$$V_{10}'(t) \leqq [-\frac{d_9}{\mu_2} + \frac{|c'(t)|}{c(t)}]V_{10}(t) + d_{10}\{r_1(t) + r_2(t)\}$$

for $t \geqq t_0$. From Lemma 1.11, for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that

$$V_{10}(t) \leqq \varepsilon + K\{V_{10}(t_0)\exp[-\frac{d_9}{\mu_2}(t-t_0)] + \exp[-\frac{d_9}{\mu_2}(t-T)]\}$$

for $t \geqq t_0$. Let $\tau(t) = \mathrm{Min}\{t-t_0, t-T\}$. Since $V_{10}(t_0) \leqq \mu_2(x_0^2 + y_0^2) \leqq \mu_2\alpha^2$, $V_{10}(t) \leqq \varepsilon + K(\mu_2\alpha^2 + 1)\exp[-\frac{d_9}{\mu_2}\tau(t)]$. Choose $T^* = T^*(\alpha,\varepsilon)$ so that $T^* > \frac{\mu_2}{d_9}\log\frac{K(\mu_2\alpha^2+1)}{\varepsilon} + T$. Then we obtain $\tau(t) > \frac{\mu_2}{d_9}\log\frac{K(\mu_2\alpha^2+1)}{\varepsilon}$ for $t \geqq t_0 + T^*$. Hence $V_{10}(t) < 2\varepsilon$ for $t \geqq t_0 + T^*$. Thus we conclude from (3.18) that $x(t)^2 + y(t)^2 < \frac{2}{\mu_1}\varepsilon$ for $t \geqq t_0 + T^*$. This completes the proof of Theorem 3.16. Q.E.D.

In Theorem 3.16, we have not asked whether the system (3.17) has the zero solution $(x(t),y(t)) \equiv (0,0)$. But if $|e(t,x,y)| \leqq r(t)|y|$, then the system (3.17) has the zero solution. In this case the following theorem holds.

Theorem 3.17. Suppose the assumptions (A1)', (A3), (A8), (A9), (A11) and suppose that $|e(t,x,y)| \leqq r(t)|y|$ and $r(t)$ is a nonnegative, continuous function satisfying $\int_0^\infty r(t)dt < \infty$. Then the zero solution of (3.17) is globally uniform-asymptotically stable if and only if $\int_0^{\pm\infty} \{f_1(x) + |x|f_3(x)\}dx = \pm\infty$.

Proof. We need only show that if $\int_0^{\pm\infty} \{f_1(x) + |x|f_3(x)\}dx = \pm\infty$, then the zero solution of (3.17) is uniformly stable. For any solution $(x(t),y(t))$ of (3.17) through (x_0,y_0) at $t = t_0$ satisfying $x_0{}^2 + y_0{}^2 \leqq 1$, it follows from Theorem 3.10 that $x(t)^2 + y(t)^2 \leqq \beta(1)^2$ for $t \geqq t_0$. Then using the same function $V_{10}(t,x,y)$ as in the proof of Theorem 3.16, we have

$$\mu_1(x^2 + y^2) \leqq V_{10}(t,x,y) \leqq \mu_2(x^2 + y^2)$$

for $t \geqq 0$ and for $x^2 + y^2 \leqq \beta(1)^2$. Also, we have for the derivative, of $V_{10}(t,x,y)$ along the solution of (3.17),

$$(V_{10})'_{(3.17)}(t,x,y) \leqq [-\frac{d_9}{\mu_2} + \frac{|c'(t)|}{c(t)}]V_{10}(t,x,y)+\{\frac{d_2|x|}{d_5} +\frac{L|y|}{d_7}\}r(t)|y|.$$

Furthermore

$$\{\frac{d_2|x|}{d_5} + \frac{L|y|}{d_7}\}r(t)|y| = r(t)\{\frac{d_2|xy|}{d_5} + \frac{Ly^2}{d_7}\} \leqq d_{11}r(t)(x^2 + y^2)$$

for some $d_{11} > 0$. Thus we obtain

$$(V_{10})'_{(3.17)}(t,x,y) \leqq \left[\frac{|c'(t)|}{c(t)} + \frac{d_{11}}{\mu_1} r(t)\right] V_{10}(t,x,y)$$

for any $t \geqq 0$. This implies that the zero solution is uniformly stable, according to the well known K.T.Persidski's theorem. Q.E.D.

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