<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On special type of hereditary abelian categories</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Harada, Manabu</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 1967, 4(2), p. 243-256</td>
</tr>
<tr>
<td><strong>Version Type</strong></td>
<td>VoR</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/9065">https://doi.org/10.18910/9065</a></td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

Osaka University
ON SPECIAL TYPE OF HEREDITARY
ABELIAN CATEGORIES

MANABU HARADA

(Received July 20, 1967)

In the book of Mitchell [5] he has defined a category of a commutative
diagrams over an abelian category \( \mathfrak{A} \). Especially he has developed this idea to a
finite commutative diagrams and obtained many interesting results on global
dimension of this diagram. Among them he has shown in [5], p. 237, Corollary
10. 10 that if \( I \) is a linearly ordered set, then \( \text{gl dim } [I, \mathfrak{A}] = 1 + \text{gl dim } \mathfrak{A} \) for an
abelian category \( \mathfrak{A} \) with projectives. This is a generalization of Eilenberg,
Rosenberg and Zelinsky [1], Theorem 8.

On the other hand, the author has studied a semi-primary hereditary ring
and shown that it is a special type of generalized triangular matrix ring in [2].

In this note we shall generalize the notion of a generalized triangular matrix
ring to an abelian category of generalized commutative diagram \([I, \mathfrak{A}_i]\) over
abelian categories \( \mathfrak{A}_i \) and obtain the similar results in it to [2], Theorem 1,
where \( I \) is a finite linearly ordered set. The method in this note is quite similar
to [5], IX, §10 and different from that of [2]. Finally we shall show that if the
\( \mathfrak{A}_i \) are the abelian category of right \( R_i \)-modules, then \([I, \mathfrak{A}_i]\) is equivalent to a
generalized triangular matrix ring over \( R_i \) in [2], where \( R_i \) is a ring.

The author has shown many applications of generalized triangular matrix
ring to semi-primary rings with suitable conditions in [2], [3] and [4]. However
we do not study any applications of our results in this note and he hopes to
continue this work on some other day.

1. Abelian categories of generalized commutative diagrams

Let \( I = \{1, 2, \cdots, n\} \) be a linearly ordered set and \( \mathfrak{A}_i \) be abelian categories.
We consider additive covariant functors \( T_{ij} \) of \( \mathfrak{A}_i \) to \( \mathfrak{A}_j \) for \( i < j \). For objects
\( A_i \in \mathfrak{A}_i, A_j \in \mathfrak{A}_j \) we define an arrow \( D_{ij} : A_i \rightarrow A_j \) as follows:

\[
D_{ij} = d_{ij}T_{ij}, \quad \text{where } d_{ij} \text{ is a morphism in } \mathfrak{A}_j. \tag{1}
\]

Using those \( D_{ij} \) we can define a category \([I, \mathfrak{A}_i]\) of diagrams over \( \{\mathfrak{A}_i\}_{i \in I} \).
Namely, the objects of \([I, \mathfrak{A}_i]\) consist of sets \( \{A_i\}_{i \in I} \) with \( D_{ij}(A_i \in \mathfrak{A}_i) \) and the
morphism of \([I, \mathfrak{A}_i]\) consist of sets \( \{f_i\}_{i \in I}(f_i \in \mathfrak{A}_i) \) such that
(2) \[ d'_{ij}T_{ij}(f_i) = f_jd_{ij}, \]
where \( f_i: A_iA'_i \) and \( D_{ij} = d_{ij}T_{ij} \), \( D'_{ij} = d'_{ij}T_{ij} \) are arrows in \( A = (A_i) \) and \( A' = (A'_i) \), respectively.

Let \( f = (f_i)_{i \in I} \) be a morphism of \( A \) to \( A' \). Then we define a set \( (\text{Im} f_i), (\text{coker} f_i) - \ldots \) coincide with \( \text{Im} f, \text{coker} f - \ldots \) in \([I, \mathcal{A}], \) respectively, we shall call \([I, \mathcal{A}] \) a category induced naturally from \( \mathcal{A} \).

**Proposition 1.1.** Let \( I \) and \( \mathcal{A}, \) be as above. \([I, \mathcal{A}] \) is an abelian category induced naturally from \( \mathcal{A} \) if and only if \( T_{ij} \) is cokernel preserving.

**Proof.** We assume that \( T_{ij} \) is cokernel preserving. Let \( f = (f_i)_{i \in I}: (A_i) \to (A'_i) \) be a morphism in \( \mathcal{A} = [I, \mathcal{A}] \). Then we can easily see that \( (\ker f_i)_{i \in I} \) is \( \text{Ker} f \) in \( \mathcal{A} \) and \( (\text{coker} f_i)_{i \in I} \) is in \( \mathcal{A} \) since \( T_{ij} \) is cokernel preserving. Hence, we know from [1], p. 33, Theorem 20.1 that \( \mathcal{A} \) is an abelian category. Conversely, we assume \( \mathcal{A} \) is an abelian category as above. We may assume \( I = (1, 2) \).

Let \( f: A_1 \to C_1 \) be an epimorphism in \( \mathcal{A} \), and \( B_2 = \text{im} T(f) \), where \( T = T_{1,2} \). Put \( A = (A_1, T(A_1)) \), \( C = (C_1, T(C_1)) \) and \( f = (f, T(f)) \). Then \( \text{Im} f = (C_1, B_2) \), \( (f: A \to C) \). By the assumption \( f' \) and \( i \) are morphisms in \( \mathcal{A} \). Hence, there exists an arrow \( d: T(C_1) \to B_2 \) in \( \mathcal{A} \) such that \( dT \) is an arrow in \( \text{im} f \).

Namely

\[
\begin{align*}
\text{T}(A_1) & \xrightarrow{T(f)} \text{T}(C_1) \\
\downarrow d_{12} & \quad \quad \quad \quad \downarrow d \\
\text{T}(A_1) & \xrightarrow{f'_2} B_2
\end{align*}
\]

is commutative, where \( if'_2 = T(f) \).

Therefore, \( f'_2 = dT(f) = df'f \). Since \( f'_2 \) is epimorphic \( di = I_{B_2} \). On the other hand, we obtain similarly from an morphism \( i \) that \( id = I_{T(C_1)} \). Hence, \( d \) is isomorphic and \( T \) is an epimorphic functor. Let \( A'_1 \xrightarrow{g} A_1 \xrightarrow{f} A_1/g(A'_1) \to 0 \) be exact and \( B'_2 = \text{im} T(g) \). Put \( A = (A'_1, B'_2) \), \( C = (A_1, T(A_1)) \), and \( f = (g, i) \), where \( T(g): T(A'_1) \to B'_2 \to T(A_1) \). From the assumption coker \( f = (A_1/g(A'_1), T(A_1)/B'_2) \). Hence there exists \( d: T(A_1/g(A'_1)) \to T(A_1)/B'_2 \) such that \( dT(f) = h \), where \( h = \text{coker} (B'_2 \i T(A_1)), (\text{cf.}(3)). \) Hence, \( \ker (f) \subseteq B'_2 \). \( B'_2 \subseteq \text{Ker} T(f) \) is clear, since \( fg = 0 \). Therefore, \( T \) is cokernel preserving.

From this proposition we always assume that \( T_{ij} \) is cokernel preserving.

We shall define functors \( T_i: \mathcal{A} \to \mathcal{A}_i \) and \( \tilde{S}_i: \mathcal{A}_i \to \mathcal{A} \) as follows:

Let \( A = (A_i)_{i \in I} \)

\[
\begin{align*}
T_i(A) & = A_i \\
T_j\tilde{S}_i(A_i) & = 0 \quad \text{for} \quad j < i,
\end{align*}
\]
HEREDITARY ABELIAN CATEGORIES

\[ T_j S(A_i) = \sum_{i < i_1 < \ldots < i_k < j} \bigoplus T_{i_{i_k}} T_{i_{i_{k-1}} \cdots i_k} \cdots T_{i_{i_1}}(A_i) \quad \text{for} \quad i < j, \]

with arrow \( D_{ik} = T_{jh} \) for \( j < k \).

Then we have a natural equivalence \( \eta: [S_i(A_i), D] \cong [A_i, T_i(D)] \) for any \( A_i \in \mathfrak{A}_i \) and \( D \in \mathfrak{A}_i \). Hence, we have from [5], p. 138, Coro. 7.4.

**Proposition 1.2.** We assume that each \( \mathfrak{A}_i \) has a projective class \( \Sigma_i \), and \( T_{ij} \) is cokernel preserving. Then \( \cap T_i^{-1}(\Sigma_i) \) is a projective class in \( \mathfrak{A}_i = [I, \mathfrak{A}_i] \), whose projectives are the objects of the form \( \bigoplus_{i \in I} S_i(P_i) \) and their retracts, where \( P_i \) is \( \Sigma_i \)-projective for all \( i \in I \).

2. Commutative diagrams with special arrows

In the previous section we study a general case of abelian categories of commutative diagrams. However, it is too general to discuss them. Hence, we shall consider the following conditions:

[I] \( T_{ij} \) is cokernel preserving.

[II] There exist natural transformations \( \psi_{ijk} : T_{jk} T_{ij} \to T_{ih} \) for any \( i < j < k \).

[III] For any \( i < j < k < l \) and \( N \) in \( A_i \)

\[
\begin{array}{ccc}
T_{hl} T_{jk} T_{ij} (N) & \xrightarrow{T_{hl} (\psi_{ijk})} & T_{hl} T_{ik} (N) \\
\downarrow \psi_{jhl} & & \downarrow \psi_{ihl} \\
T_{jl} T_{ij} (N) & \xrightarrow{\psi_{ijl}} & T_{il} (N)
\end{array}
\]

is commutative

[IV] For arrows \( d_{ij} : T_{ij}(A_i) \to A_j \) in \( \mathfrak{A}_i = [I, \mathfrak{A}_i] \)

\[
\begin{array}{ccc}
T_{jh} T_{ij} (A_i) & \xrightarrow{T_{jh} (d_{ij})} & T_{jh} (A_j) \\
\downarrow \phi_{ijh} & & \downarrow d_{ih} \\
T_{ih} (A_i) & \xrightarrow{d_{ik}} & A_k
\end{array}
\]

is commutative.

From now on we always assume I, II and for any arrows in \( \mathfrak{A}_i \), we require the condition IV.

We note that IV implies \( D_{jh} D_{ij}(A_i) \subseteq D_{ih}(A_i) \) for any \( A = (A_i)_{i \in I} \) in \( \mathfrak{A} \).

First we shall show that \( \mathfrak{A}_i \) is still an abelian category under the assumption I even if we require IV in \( \mathfrak{A}_i \).

**Proposition 2.1.** Let \( (\mathfrak{A}_i)_{i \in I} \) be abelian categories. We assume II. Then
$\mathcal{A} = [I, \mathcal{A}]$ requiring IV is abelian if and only if $I$ is satisfied.

Proof. Let $f = (f_i): (A_i) \rightarrow (A'_i)$ in $\mathcal{A}$. We consider a diagram

\[
\begin{array}{ccccccccc}
\text{T}_{jk}T_{ij}(A_i) & \xrightarrow{T(d_{ij})} & \text{T}_{jk}(A_j) & \xrightarrow{d_{jk}} & \text{T}(f_j) \\
\downarrow \phi & & \downarrow \phi & & \downarrow d_{jk} \\
\text{T}_{lk}(A_l) & \xrightarrow{T(f_l)} & \text{T}_{lk}(A'_l) & \xrightarrow{d'_{lk}} & A'_k
\end{array}
\]

We only prove from Proposition 1.1 that for any morphism $g = (g_i), (\ker g_i)_{i \in I}$ (coker $g_i)_{i \in I}$ satisfy IV. Put $A_i = \ker g_i$ and $f_i = $ inclusion morphism in the above. Then all squares except the rear in (5) are commutative from II, IV and (2). Since $f_k$ is monomorphic, the rear one is commutative. Which shows $(\ker g_i)_{i \in I}$ satisfies IV. Similarly if $A_i = (\text{coker } g_i)$ and $f_i$ epimorphism of cokernel, then $(\text{coker } g_i)$ satisfies IV, since $T_{jk}T_{ij}(f_i)$ is epimorphic from I.

Next, we shall define functors similarly to $\tilde{S}_i$. For $A_i \in \mathcal{A}_i$, we put

\[(6) \quad S_i(A_i) = (0, 0, \ldots, A_i, T_{ii+1}(A_i), \ldots, T_{in}(A_i)) \text{ with arrows}
\]

\[
D_{ik} = 0 \quad \text{for} \quad t < i \\
D_{ik} = T_{ik} \quad \text{for} \quad k > i \\
D_{jk} = \phi_{ijk}T_{jk} \quad \text{for} \quad k > j > i.
\]

If $T_{ij}$'s satisfy III, then $S_i(A_i)$ is an object in $[I, \mathcal{A}_i]$ requiring IV. Furthermore, we can prove easily $[S_i(A_i), D] = [A_i, T_i(D)]$ for $D \in [I, \mathcal{A}_i]$. Hence, we have similarly to Proposition 1.2

**Proposition 1.2'.** We assume that each $\mathcal{A}_i$ has a projective class $\varepsilon_i$ and $I \sim III$ are satisfied. Then $\mathcal{A} = [I, \mathcal{A}_i]$ requiring IV has a projective class $\bigcap T_i^{-1}(\varepsilon_i)$ whose projectives are the objects of the form $\bigoplus_{i \in I} S_i(P_i)$ and their retracts, where $P_i$ is $\varepsilon_i$-projective for all $i \in I$.

In the rest of the paper we always assume that $[I, \mathcal{A}_i]$ is an abelian category.
of the commutative diagrams whose arrows are required IV and that I~III are satisfied.

**Proposition 2.2.** \((D_{kl}D_{kj})D_{ij} = D_{kl}(D_{kj}D_{ij})\) for \(i < j < k < l\).

Proof. \((D_{kl}D_{kj})D_{ij}(A) = d_{ji} \phi_{bjl}(T_{hi}T_{jh})(d_{ij})T_{hj}T_{ji}(A)\) (naturality of \(\phi\))

\[= d_{ii} \phi_{ij}T_{hi}T_{hi}T_{ji}(A) \quad (\text{IV})\]

\[= d_{ii} \phi_{ij}T_{hi}T_{hi}T_{ji}(A) \quad (\text{III})\]

\[= d_{ii} T_{hi}T_{hi}T_{ji}(A) \quad (\text{IV})\]

\[= d_{ii} T_{hi}T_{hi}T_{ji}(A) \quad (\text{IV})\]

\[= D_{kl}(D_{kj}D_{ij})(A) \quad \text{for any} \quad A \in \mathcal{A}.\]

**Theorem 2.3.** (cf. [1], p. 234, Lemma 9.3) Let \(I = I_1 \cup I_2\) and \(I_1 = \{1, 2, \ldots, i-1\}, I_2 = \{i, \ldots, n\}\). Then \(\mathcal{A}\) is isomorphic to \(\mathcal{A}' = \{(1, 2), [I, \mathcal{A}_1], [I_2, \mathcal{A}_2]\}\) with a suitable functor \(T_{iz} : [I_1, \mathcal{A}_1] \rightarrow [I_2, \mathcal{A}_2]\).

Proof. First we define a functor \(T_{iz}\). Let \(A_i = (A_i)_i = I_.\) For any \(k \geq i\) we consider a diagram \(D_k = \{T_{ih}(A_i), T_{ih}T_{ij}A_i\}\) for \(l < l' < i < k\) with arrows \(T_{ih}T_{ij}A_i \rightarrow T_{ih}(A_i)\) and \(T_{ih}T_{ij}A_i \rightarrow T_{ih}(A_i)\). \(A_k\) has a colimit \(A_k \in \mathcal{A}_k\) by [1], p. 46, Coro. 2.5. Put \(A_k = (A_i, \ldots, A_n)\). We shall show that \(A_k\) is in \([I_2, \mathcal{A}_2]\). We have to define \(D_{kk'}\) for \(i \leq k < k'\). Consider a diagram

\[
\begin{array}{c}
T_{kk'}T_{ih}(A_i) \xrightarrow{\phi_{kk'}} T_{kk'}(A_i) \\
\downarrow T(\psi) \\
T_{kk'}T_{ii'}T_{i'i'}(A_i) \xrightarrow{\phi_{kk'}} T_{kk'}(A_i) \\
\downarrow T(\psi) \\
T_{kk'}T_{ii'}T_{i'i'}(A_i) \xrightarrow{\phi_{kk'}} T_{kk'}(A_i) \\
\end{array}
\]

The upper and lower squares are commutative by III and naturality of \(\psi\), respectively. Then (7) implies that these exist compatible morphism: \(\{T_{kk'}(D_k)\} \rightarrow A_{k'}\). Since \(T_{kk'}\) is colimit preserving by [5], p. 55. Proposition 6.4, we have a unique morphism \(d_{kk'} : T_{kk'}(A_k) \rightarrow A_{k'}\). Hence we can define \(D_{kk'} = d_{kk'}T_{kk'}\). Next we show that those \(D_{kk'}\) satisfy IV. For \(i \leq k < k' < k''\) we have a diagram
All squares except bottom are commutative by III and the definitions $d_{kk}$, $d_{hh}$ and $d_{kk'}$. On the other hand, it is clear that $T_k: T_{kk'}(D_k) \rightarrow T_{kk'}(D_k')$. Since $T_k: T_{kk'}$ is colimit preserving, we have a unique morphism $\Phi: T_k: T_{kk'}(D_k) \rightarrow T_{kk'}(D_k')$ such that $\Phi = \Phi TT(\alpha)$. Therefore, the bottom square is also commutative, which means II. Thus we have shown that $T_{12}$ is a functor. Let $(A_1, A_2)$ be in $\mathfrak{H}'$, where $A_1 = (A_i)_{i \in I_1}$ and $A_2 = (B_j)_{j \in I_2}$. From the definition of $T_{12}$ we have a morphism:

$$T_{jk}(A_j) \rightarrow A_k \xrightarrow{d_k} B_k \text{ for } j \in I_1, k \in I_2, \text{ where } (d_i)_{i \in I_1}: T_{12}(A_1) \rightarrow A_2.$$ We put

$$D'_{jk} = d_k \alpha_k T_{jk} \text{ for } j < i < k \text{ and } D'_{st} = D_{st} \text{ for } s, t \in I_1 \text{ or } T_2.$$

We shall show that $D'_{ij}$ satisfy IV. Take $j < h < k$. If $j \in I_2$ or $k \in I_1$, then it is obvious. We assume $j \in I_1$ and $h, k \in I_2$. Then we have

$$T_{hk} T_{jk}(A_j) \xrightarrow{\alpha_k} T_{kk}(A_k) \xrightarrow{T(d_k)} T_{hh}(B_h)$$

where $d'_{hh}$ is a given morphism in $A_2$. The left side is commutative by the definition of $T_{12}$ and so is the right side, since $h, k \in I_2$. Hence, the out side square means IV. We can easily see by the definition of {$D_k$} that IV is satisfied for $j, h \in I_1$ and $k \in I_2$. Hence, $T(A_1, A_2) = (A_1, \ldots, A_{i-1}, B_i, \ldots, B_n)$ is an object in $\mathfrak{H}$. Conversely, for $A = (A_1, \ldots, A_n)$ we put $S(A) = ((A_i, \ldots, A_{i-1}), (A_i, \ldots, A_n))$. Then it is clear that $S(A) \in \mathfrak{H}'$ and $TS = I_{\mathfrak{H}'}, ST = I_{\mathfrak{H}}$. This
shows that $T_{12}$ is cokernel preserving by Proposition 1.1.

3. Hereditary categories

In this section, we always assume that I~IV are satisfied and every $\mathcal{A}_i$ has projectives and hence $\mathcal{A}=[I, \mathcal{A}_i]$ has projectives by Proposition 1.2'.

If every object in an abelian category $\mathcal{B}$ is projective, we call $\mathcal{B}$ a semi-simple category, which is equivalent to a fact $\text{gl dim } \mathcal{B}=0$. If $\text{gl dim } \mathcal{B} \leq 1$ we call $\mathcal{B}$ hereditary.

**Proposition 3.1.** ([5], p. 235, Coro. 10.3). We assume that $\mathcal{A}_i$ has projectives and that $T_{ij}$ is projective preserving. Let $D=(D_i)_{i \in I}$ be an object in $[I, \mathcal{A}_i]$ and $m=\max (\text{hd } D_i)$, $n=$ the number of elements of $I$. Then $\text{hd } D \leq n+m-1$.

Since $T_{ij}$ is projective preserving, we can prove it similarly to [1], p. 235.

**Corollary.** Let $I=(1,2)$ and $T_{12}$ be projective preserving. Then

$$\max (\text{gl dim } \mathcal{A}_1, \text{gl dim } \mathcal{A}_2) \leq \text{gl dim } [(1, 2), \mathcal{A}_1, \mathcal{A}_2] \leq \max (\text{gl dim } \mathcal{A}_i)+1.$$

**Proof.** The right side inequality is clear from Proposition 3.1. Let $A$ be an object in $\mathcal{A}_1$. It is clear that $\text{hd}(A, 0) \leq \text{hd } A$. Since $T_{12}$ is projective preserving, we have similarly $\text{hd}(0, A') \leq \text{hd } A'$ for $A' \in \mathcal{A}_2$.

**Lemma 3.2.** Let $\mathcal{A}=[(1, 2), \mathcal{A}_1, \mathcal{A}_2]$. If $\text{gl dim } \mathcal{A} \leq 1$, then $T_{12}$ is projective preserving.

**Proof.** Let $P_1$ be projective in $\mathcal{A}_1$. Then $(P_1, T_{12}(P_1))$ is projective in $\mathcal{A}$ by Proposition 1.2. Let $0 \rightarrow T_{12}(P_1) \rightarrow Q$ be an exact sequence in $\mathcal{A}_1$ with $Q$ projective. Then $(0, 0) \rightarrow (P_1, 0) \rightarrow (P_1, T_{12}(P_1)) \rightarrow (0, Q)$ is exact in $\mathcal{A}$. Since $\text{gl dim } \mathcal{A}_i \leq 1$, $(0, T_{12}(P_1))$ is projective in $\mathcal{A}((0, T_{12}(P_1)) \subset (P_1, T_{12}(P_1))$. Hence, $T(P_1) \rightarrow Q$ is retract and $T_{12}(P_1)$ is projective in $\mathcal{A}_2$.

Similarly to the category of modules we have

**Lemma 3.3.** Let $A$ be an abelian category. If $A \oplus B = A' \oplus C$ and $A \supset A'$, then $A'=A \oplus A'$, $A''=A \cap C$ and $C=A' \oplus C'$.

**Lemma 3.4.** Let $I=(1,2)$ and $\mathcal{A}=[I, \mathcal{A}_i]$. If $T_{12}$ is projective preserving, then every projective object $A$ in $\mathcal{A}$ is of a form $(P_1, T_{12}(P_1) \oplus P_2)$ and the arrow $d_{12}$ in $A$ is monomorphic, where $P_1$ is projective in $\mathcal{A}_1$.

**Proof.** Since $A=(A_1, A_2)$ is a retraction of an object of a form $P=(P_1, T_{12}(P_1) \oplus P_2)$ with $P_1$ projective in $\mathcal{A}_1$. Hence, $0 \rightarrow A \rightarrow P$ splits. Let $P_1=A_1 \oplus Q_1$. Then $T_{12}(P_1)=T_{12}(A_1) \oplus T_{12}(Q_1)$ and $A_2$ is a coretract of $T_{12}(A_1) \oplus T_{12}(Q_1) \oplus P_2$. Furthermore, $T_{12}(A_1) \xrightarrow{d_{12}} A_2 \rightarrow T_{12}(P_1) \oplus P_2=T_{12}(A_1) \rightarrow T_{12}(P_1) \oplus P_2$ and the right side is monomorphic. Hence, $d_{12}$ is monomorphic. Thus we
may assume $T_{13}(A_i) = A_i \subset T_{13}(P_1) \oplus P_2$. Therefore, $A_2 = T_{13}(A_i) \oplus A_2$ by Lemma 3.3. Since $P_1$ is projective and $T_{13}$ is projective preserving, $T_{13}(P_1) \oplus P_2$ is projective in $\mathcal{A}$. Hence, $A_2'$ is projective by Lemma 3.3.

**Lemma 3.5.** Let $\mathcal{A}_1$, $\mathcal{A}_2$ be hereditary and $T_{12}$ projective preserving. If $T_{13}(P_3)$ is a coretract of $T_{13}(P_1)$ for any projective objects $P_1 \supset P_2$ in $\mathcal{A}_1$, then $\mathcal{A} = [(1, 2), \mathcal{A}_1, \mathcal{A}_2]$ is hereditary.

**Proof.** Let $(A_1, A_2)$ be any object in $\mathcal{A}$ and $0 \rightarrow (A_1, A_2) \rightarrow P$ be exact, where $P \mathcal{A}$-projective. Then $P = (P_1, T_{13}(P_1) \oplus P_2)$ with $P_1$ projective by Lemma 3.4. Put $\text{ker} f = (K_1, K_2)$. Since $\mathcal{A}_1$ is hereditary, $K_1$ is projective. Hence, $T_{13}(K_1)$ is a coretract of $T_{13}(P_1)$ by the assumption. Hence, $K_2 = T_{13}(K_1) \oplus K_2'$ by Lemma 3.3. Since $K_2$ is projective, $(K_1, K_2')$ is $\mathcal{A}_2$-projective.

**Theorem 3.6.** Let $I = (1, 2, \ldots, n)$ be a linearly ordered set, $\mathcal{A}_i$ abelian categories with projectives. Let $\mathcal{A} = [I, \mathcal{A}_i]$ be the abelian category of commutative diagrams over $\mathcal{A}_i$ with functors $T_{ij}$ satisfying $I \sim IV$. If $\mathcal{A}$ is hereditary, then we have:

i) Every projective object of $\mathcal{A}$ is of a form $\bigoplus_{i \in I} S_i(P_i)$, where $P_i$ is projective in $\mathcal{A}_i$.

ii) $T_{ij}$ is projective preserving for any $i < j$.

iii) $T_{ij}(P_i)$ is a coretract of $T_{ij}(P_i)$ for any projective objects $P_1 \supset P_2$ in $\mathcal{A}_i$.

iv) If $P = (P_{i_1}, P_{i_2}, \ldots, P_{i_k}) = \mathcal{A}(i_1, i_2, \ldots, i_k)$ is hereditary for any $i_1 < i_2 < \cdots < i_k$.

v) If $P = (P_{i_{\ell}})_{i \in I}$ is projective in $\mathcal{A}$, then every $d_{ij}$ in $P$ is a coretract. $T_{ij} = \mathcal{A}(i_1, i_2, \ldots, i_k)$-projective.

**Proof.** We shall prove the theorem by the induction on the number $n$ of element of $I$. We obtain $\mathcal{A} \approx [(1, 2), \mathcal{A}_1, \mathcal{A}(I-1)] = \Lambda'$ from Theorem 3.2. Then $\mathcal{A}(I-1)$ is hereditary by Lemma 3.2 and Corollary to Proposition 3.1. Furthermore, $T_{12}$ in $\mathcal{A}$ is projective preserving.

i) Let $P = (P_{i_{\ell}})_{i \in I}$ be projective in $\mathcal{A}$. Then $P = (P_1, T_{13}(P_1) \oplus P_2)$ by Lemma 3.4, where $P_1$ is projective in $\mathcal{A}(I-1)$. We obtain, by the definition of $T_{13}$, that $T_{13}(P_1) = (T_{13}(P_1))_{i \in I-1}$. Hence, $P = \bigoplus_{i \in I} S_i(P_i)$ by the induction hypothesis.

ii) Every component of projective object in $\mathcal{A}(I-1)$ is projective by the induction. Hence, $T_{13}(P_i)$ is projective in $\mathcal{A}_i$. iii) Let $P_1 \supset P_2$ be projective in $\mathcal{A}_i$. Put $A = (P_1/P_2, 0, \ldots, 0)$. Then we have an exact sequence $0 \rightarrow A \rightarrow (P_1, T_{13}(P_1))$. Since $\mathcal{A}$ is hereditary, its kernel $(P_2, T_{13}(P_1))$ is projective. Therefore, $T_{13}(P_2)$ is a coretract from i). iv) We may show that $\mathcal{A}(I-1-i)$ is hereditary for any $i$. $\mathcal{A} \approx [I, i, I_i, \mathcal{A}_i, \mathcal{A}_2]$, where $I_i = (1, \ldots, i-1), I_i = (i+1, \ldots, n)$, $\mathcal{A}_i = \mathcal{A}(I_i)$ and $\mathcal{A}_2 = \mathcal{A}(I_2)$. From Lemma 3.2 $T_{13}$ is projective preserving and hence $\mathcal{A}(I-i)$ is hereditary from iii) and Lemma 3.5 and the definition of $T_{13}$. v) Since $P = (P_1, T_{13}(P_1) \oplus P_2), d_{ij}: T_{13}(P_1) \rightarrow P_i$ is a coretract.
\( P \approx (P'_1, P_2, P'_3) \), where \( P'_1 = (P_j)_{j \in I_1} \) and \( P'_3 = (P_j)_{j \in I_3} \). Then it is clear from i) and induction that \((P'_1, P'_3)\) is \( \mathbb{A}(\ell-i)\)-projective.

Next we shall study a condition of every projective objects in \( \mathbb{A} \) being of a form \( \oplus S_i(P_i) \), when \( T_{ij} \) is projective preserving.

**Lemma 3.7.** Let \( \mathbb{A} \) and \( \mathbb{A}_i \) be as above and \( T_{ij} \) projective preserving. If we have

\[
(*) \quad T_{ij}(P_i) = T_{i+1}T_{i+1}(P_i) \oplus T_{i+2}(K^{i+2}(P_i)) \oplus \cdots \oplus T_{j-1}(K^{j-1}(P_i)) \oplus K^j(P_i)
\]

for any projective object \( P_i \) in \( \mathbb{A}_i \) for all \( i \), then every object \( A = (A_i)_{i \in I} \) in \( \mathbb{A} \) is of a form \( \oplus S_i(Q_i) \) whenever \( A \) is subobject of \( P = (Q_i)_{i \in I} \) and \( A_i \) is a coretract of \( Q'_i \) for all \( i \), where \( K^j(P_i) \) is an object in \( \mathbb{A}_i \), \( Q_i \) and \( Q'_i \) are \( \mathbb{A}_i \)-projective, and the equality in \( (*) \) is given by taking suitable transformation from the right side to the left in \( (*) \).

**Proof.** We may assume \( P = \oplus S_i(P_i) \) and \( P_i \) is \( \mathbb{A}_i \)-projective. Put \( P = (P_i)_{i \in I} \). From the assumption \( P_i = A_i \oplus Q_i \). We shall show the following fact by the induction on \( i \).

i) \quad A_i = T_{i+1}(A_i) \oplus T_{i+1}(K^{i+1}(Q_i)) \oplus \cdots \oplus T_{i-1}(K^{i-1}(Q_i)) \oplus K^i

ii) \quad K^i \oplus Q_i = P_i \oplus \oplus (Q_i) \oplus \oplus (Q_i) \oplus \cdots \oplus K^i(Q_i) \oplus T_{i-1}(Q_i) \oplus \cdots

and this is a coretract of \( P_i \), where \( K^i(Q_i) \) is the object in \( (*) \) for projective \( Q_i \) and the equalities are considered in \( P_i \) by suitable imbedding mappings. If \( i = 1, 2, i \) and ii) are clear (see the proof of Lemma 3.4). We assume i) and ii) are true for \( k < i \). Using this assumption we first show for \( 2 < j < i - 1 \) that

iii) \quad P_i = T_{i+1}(A_i) \oplus T_{i+1}(Q_i) \oplus \cdots \oplus T_{i+1}(Q_i) \oplus K^i(P_i)

\[
\oplus T_{i+1}(P_i) \oplus T_{i+1}(Q_i) \oplus \cdots \oplus T_{i+1}(Q_i) \oplus K^i(P_i)
\]

\[
\oplus T_{i+1}(P_i) \oplus T_{i+1}(K^{i+1}(Q_i)) \oplus \cdots \oplus T_{i+1}(K^{i+1}(Q_i)) \oplus K^i(P_i)
\]

\[
\oplus \cdots \cdots \cdots
\]

\[
\oplus T_{i-1}(P_i) \oplus T_{i-1}(K^{i-1}(Q_i)) \oplus \cdots \oplus T_{i-1}(K^{i-1}(Q_i)) \oplus K^i(P_i)
\]

Now \( P_i = T_{i+1}(P_i) \oplus T_{i+1}(P_i) \oplus \cdots \oplus T_{i+1}(P_i) \oplus P_i \)

\[
= T_{i+1}(P_i) \oplus T_{i+1}(P_i) \oplus (P_i' = T_{i+1}(P_i) \oplus \cdots) \oplus P_i
\]

\[
= T_{i+1}(A_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(P_i) \oplus P_i
\]

\[
= T_{i+1}(A_i) \oplus (T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(P_i) \oplus P_i
\]

\[
= T_{i+1}(A_i) \oplus (T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(P_i) \oplus P_i
\]

\[
= T_{i+1}(A_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(P_i) \oplus P_i
\]

\[
= T_{i+1}(A_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(P_i) \oplus P_i
\]

\[
= T_{i+1}(A_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(Q_i) \oplus T_{i+1}(P_i) \oplus P_i
\]
\[ \oplus T_{ij}(P_i \oplus K^i(Q_i) \oplus K^i(Q_i)) \oplus \cdots \]
\[ \oplus T_{i-1}(P_{i-1} \oplus K^{i-1}(Q_i) \oplus K^{i-1}(Q_j)) \]
\[ \oplus P_i \oplus \mathcal{K}^i(Q_i) \oplus K^i(Q_i). \]

This is a case of \( j=2 \) in iii). We assume iii) is true for \( k \leq j \). Since \( j+1 < i \), we obtain from ii) and taking
\[ T_{j+1}(K^{j+1}) \oplus T_{j+1}(Q_{j+1}) = T_{j+1}(P_{j+1} \oplus K^{j+1}(Q_i) \oplus K^{j+1}(Q_j) \oplus \cdots \oplus K^{j+1}(Q_{j-1}) \]
\[ \oplus T_{j+1}(Q_j). \]

On the other hand,
\[ T_{j+1}(Q_{j+1}) = T_{j+2}T_{j+1+j+2}(Q_{j+1}) \oplus T_{j+1}(K^{j+2}(Q_{j+1})) \oplus \cdots \]
\[ \oplus T_{i-1}(K^{i-1}(Q_{i+1})) \oplus K^i(Q_i). \]

Since \( Q_{j+1} \) is a coretract of \( P_{j+1} \) and \( T_{j+1}(P_{j+1}) \) is a coretract of \( P_i \) by the following Lemma 3.8, we may regard the above objects on the both sides as sub objects in \( P_i \). Hence, we obtain
\[ P_i = T_{i}(A_i) \oplus T_{i}(K^i) \oplus \cdots \oplus T_{i}(K^{i-1})(Q_i) \oplus \cdots \]
\[ \oplus K^{i-1}(Q_{i-1}) \oplus T_{i-1}(K^{i-2})(Q_{i-2}) \oplus \cdots \]
\[ = \{ T_{i}(A_i) \oplus T_{i}(K^i) \oplus \cdots \oplus T_{i-1}(K^{i-1}) \oplus \cdots \oplus K^{i-1}(Q_{i-1}) \}. \]

Thus we obtain from i) and ii)
\[ P_i = T_{i}(A_i) \oplus T_{i}(K^i) \oplus \cdots \oplus T_{i-1}(K^{i-1})(Q_i) \oplus \cdots \]
\[ \oplus K^{i-1}(Q_{i-1}) \oplus T_{i-1}(K^{i-2})(Q_{i-2}) \oplus \cdots \]
\[ = \{ T_{i}(A_i) \oplus T_{i}(K^i) \oplus \cdots \oplus T_{i-1}(K^{i-1}) \oplus \cdots \oplus K^{i-1}(Q_{i-1}) \}. \]

Since \( A_i \supseteq K^i \) and \( A_i \supseteq T_{i}(A_i) \oplus T_{i}(K^i) \oplus \cdots \oplus T_{i-1}(K^{i-1}) = A_i' \), we obtain
\[ A_i = A_i' \oplus K^i \] and \( Q_i \) in \( \mathcal{A}_i \) such that
\[ K^i \oplus Q_i = P_i \oplus K^i(Q_i) \oplus \cdots \oplus K^i(Q_{i-1}) \oplus T_{i-1}(Q_{i-1}), \]
and hence, \( K^i \oplus Q_i \) is a coretract of \( P_i \). Therefore, \( A = \bigoplus_{i \in I} S_i(K^i) \oplus S_i(A_i) \). Since \( T_{ij} \) is projective preserving, each \( K^i \) is \( \mathcal{A}_i \)-projective.

**Lemma 3.8.** Let \( \mathcal{A} \) and \( \mathcal{A}_i \) and \( T_{ij} \) be as above. We assume that \( T_{ij} \) satisfies the condition (\( \ast \)). Then \( T_{ij}(P_i) \) is a coretract of \( P_j \) for any projective object \( P_i = (P_i)_{i \in I} \).

Proof. We may assume \( P = \bigoplus_{i \in I} S_i(Q_i) \) by Lemma 3.3, where \( Q_i \) is \( \mathcal{A}_i \)-pro-
jective. Then $P_i = \sum_{k=1}^{l_i} T_{i,k}Q_k \oplus Q_i$. We shall show under the assumption of Lemma 3.8 that $T_{ij}T_{ij}(P_i) \xrightarrow{\phi_{ij}} T_{ij}(P_i)$ is a coretract. Let $t = l - i$. If $t < 2$, then the fact is clear from (*). We assume it for $t < k$ and $k = l - i$. $T_{ij}T_{ij}(P_i) = T_{ij}T_{i+1}(R_i) \oplus T_{ij}(T_{i+2}(K_{ij}Q_i) \oplus \cdots \oplus T_{j-1}(K_{ij}Q_i) + \sum_{k=1}^{l_i} T_{i,k}Q_k) + \sum_{k=1}^{l_i} T_{i,k}Q_k$.

Hence, we obtain $\phi_{ij}$ is a coretract from the assumption III, naturality of $\phi$ and induction hypothesis. From those facts we can easily prove Lemma 3.8.

**Lemma 3.9.** Let $\mathcal{A}$ and $\mathcal{B}$ be as above, and $I'$ a subset of $I$. Then there exist functors $M: [I', \mathcal{A}] \to [I, \mathcal{B}]$, $F: [I, \mathcal{A}] \to [I', \mathcal{A}]$ such that $FM = I[I', \mathcal{A}]$, where $F$ is the restriction functor.

**Proof.** We may assume $I = I' \cup \{i\}$ by the induction. Let $I' = \{j \in I, j < i\}$ $I'' = \{j \in I, j > i\}$ and $A = (A_j)_{j \in I'}$. If $I = \phi$, we put $A = 0$. We assume $I = \phi$.

We consider a family $D_i = \{T_{ik}(A_k), T_{ik}T_{ik}(A_k) \xrightarrow{\phi_{kk'}} T_{ik}(A_k)\}$ and $T_{kk'}(A_k) \xrightarrow{\phi_{kk'}} T_{kk'}(A_k)$ for $k < k' < i$. Put $A_i$ is a colimit of $D_i$. Then we have defined arrows $D_{ik}$ and $D_{ik}$ for $k \in I_1, l \in I_2$ from (7). It is easily seen from the definition of colimit that those $D_{ik}$ satisfy IV. Then $M(A) = (A_k)_{k \in I}$ is a desired functor.

**Remark.** We note that if $A = (A_k)$ is a coretract of $B = (B_k)_{k \in I'}$, then $M(A)$ is a coretract of $M(B)$, (cf. [5], p. 47, Coro. 2.10).

**Proposition 3.10.** Let $\mathcal{A}_i$ be abelian categories with projective class $\mathcal{E}_i$, and $\mathcal{B}(I) = [I, \mathcal{A}_I]$. We assume $T_{ij}$ is projective preserving. Then every projective object $P = (P_i)_{i \in I}$ in $\mathcal{B}(I')$ is of a form $\bigoplus_{i \in I'} S_i(Q_i)$ with $Q_i$ projective in $\mathcal{A}_i$ for any subset $I'$ of $I$ and $(P_j)_{j \in I'}$ is $\mathcal{B}(I')$-projective for any subset $I''$ of $I'$ if and only if (*) is satisfied.

**Proof.** “only if”. Let $P_i$ be projective in $\mathcal{A}_i$. Then $S_i(P_i)$ is $\mathcal{A}_i$-projective, and hence, $P' = (T_{ii+1}(P_i), \cdots, T_{im}(P_i))$ is $\mathcal{B}(I')$-projective. Therefore, the fact $P' = \bigoplus_{i \in I'} S_i(Q_i)$ from the assumption is equivalent to (*). “if”. Let $P' = (P'_{i})_{i \in I'}$ be projective in $\mathcal{B}(I')$. Then $P'$ is a retract of $\bigoplus_{i \in I'} S_i(P_i)$, where $P_i$ is $\mathcal{A}_i$-projective and $S_i$ is functor: $\mathcal{A}_i \to \mathcal{B}(I')$ in (6). Let $M$ be a functor in Lemma 3.9. Then $M(\bigoplus_{i \in I'} S_i(P_i)) = \bigoplus_{i \in I'} S_i(P_i)$ from the construction of $M$, and $M(P')$ is its retract from the above remark. Hence, $M(P')$ is $\mathcal{A}_i$-projective.
Therefore, $M(P') = \bigoplus_{i \in I} S_i(Q_i)$ with $Q_i$ projective in $\mathcal{A}_i$ by Lemma 3.7. Let $I' = \{i_1, \ldots, i_t\}$. We shall show $A_{ik} = (T_{ik''}i_{ik''}(Q_{ik''}))_h = \sum_{k''} S_{ik''}(P'_{ik''})$, where $T_{ik''}i_{ik''} = \mathcal{A}_i$ and $P'_{ik''}$ is $\mathcal{A}_i$-projective. We obtain from Lemma 3.7 that $T_{ik''}i_{ik''}(Q_{ik''}) = T_{ik''}i_{ik''}(Q_{ik''}) \oplus P'_{ik''}$ and $T_{ik''}i_{ik''}(Q_{ik''}) = T_{ik''}i_{ik''}(Q_{ik''}) \oplus P'_{ik''}$. Hence,

$$T_{ik''}(Q_{ik''}) = T_{ik''}i_{ik''}i_{ik''}(Q_{ik''}) \oplus P'_{ik''}$$

from III. Repeating this argument we have $A_{ik''} = \sum_{k''} S_{ik''}(P'_{ik''})$. Therefore,

$$P = \sum_{k''} A_{ik''} = \sum_{k''} S_{ik''}(P'_{ik''})$$

This completes the proof.

Proposition 3.11. Let $\mathcal{A}$ and $\mathcal{A}_i$ be as above. We assume $T_{ij}$ is projective preserving and satisfies (*), then for $D = (D_i)_{i=1}^n$ in $\mathcal{A}$

$$hd D \leq \max (hd D_i) + 1$$

Proof. Put $n = \max (hd D_i)$. Let $0 \to D \to P_0 \to \cdots \to P_n \to 0$ be a projective resolution of $D$ and $K_n = ker d_n$. Since $n \geq hd D_i$, every component of im $d_n$ is projective. Hence, $K_n$ is $\mathcal{A}$-projective by Lemma 3.7.

Corollary. Let $A_i$, $A$ and $T_{ij}$ be as above. Then

$$gl dim \mathcal{A}_i \geq \max (gl dim \mathcal{A}_i) + n - 1$$

Proof. Let $A$ be in $\mathcal{A}(I')$ and $0 \to M(A) \to P_1 \to P_2 \to \cdots$ be a projective resolution of $M(A)$ in $\mathcal{A}$. Then $0 \to A \to F(P_1) \to F(P_2) \to \cdots$ is a projective resolution of $A$ in $\mathcal{A}(I')$ from Proposition 3.10.

We recall that $\mathcal{A}$ is semi-simple if and only if every object of $\mathcal{A}$ is projective.

Theorem 3.12. Let $\mathcal{A}_i$ be semi-simple abelian categories and $I$ a linearly ordered finite set. Then $\mathcal{A} = [I, \mathcal{A}_i]$ with $T_{ij}$ satisfying I~IV is hereditary if and only if

$$T_{ij}(M) = T_{ij}T_{ij+1}(M) \oplus T_{ij+2}(K^{i+2}(M)) \oplus \cdots \oplus T_{j-i}(K^{j-i}(M)) \oplus K^j(M)$$

for every object $M$ in $\mathcal{A}$ for all $i$, where $K^i(M) \in \mathcal{A}_i$. Furthermore, $gl dim \mathcal{A} = 1$ if and only if there exists not a zero functor $T_{ij}$, (cf. [2], Theorem 1).

Proof. The first half is clear from Lemmas 3.7 and 3.8 and Proposition 3.11. If $T_{ij}$ is not a zero functor, then $A = (A, 0)$ is not projective in $\mathcal{A}(i, j)$ for any $\mathcal{A}$ such that $T_{ij}(\mathcal{A}) \neq 0$ by Proposition 3.10. Hence, $gl dim \mathcal{A} \geq gl dim \mathcal{A}(i, j) \geq 1$. If $T_{ij}$ is a zero functor for all $i<j$, then $\mathcal{A} = \bigoplus \mathcal{A}_i$. Hence, $gl dim \mathcal{A} = 0$. 


Let \( \{R_i\}_{i \in I} \) be rings. Finally we assume that \( \mathcal{A} \) is the abelian category of right \( R_i \)-modules. By [5], p. 121., Prop. 1.5 we know \( U = \oplus S_i(R_i) \) is a small, projective generator in \( \mathcal{A} \). Put \( R = [U, U] \). Let \( r, r' \) be elements in \( R_i \), respectively. By \( T_{ij}(R_i) \) we denote morphisms in \( [R_i, R_i] \) and \( [R_j, T_{ij}(R_i)] \) such that \( r_i(x_i) = r x_i \) and \( r'_i(x_i) = r' x_i \), respectively where \( x_i \in R_i \). We can naturally regard \( T_{ij}(R_i) \) a left \( R_i \)-module by setting \( \varphi_y = T_{ij}(r_i)y \) for any \( r \in R_i \) and \( y \in T_{ij}(R_i) \). Furthermore, we define \( \varphi'z = \psi_{ijk}T_{ik}(r_i) \) for any \( k > j \) and \( z \in T_{jk}(R_j) \), where we assume \( T_{ii} = I_{R_i} \). Then we identify \( R \) with the set

\[
R = \left\{ \begin{array}{c}
\begin{bmatrix}
r_1 & \cdots & r_m \\
r_2 & \cdots & r_m \\
\vdots & \ddots & \vdots \\
0 & \cdots & r_m \\
\end{bmatrix}, \quad r_{ij} \in T_{ij}(R_i), \ r_i \in R_i
\end{array} \right\}.
\]

**Lemma 3.13.** \( \varphi_{ij} \varphi_{jk} = \varphi_{ijk} \) and \( \varphi_{ij} \varphi' = \varphi'_{ij} \), \( \varphi_{ij} = \varphi'_{ij} \).

**Proof.** For any \( k \geq j \) we have \( \varphi_{ij} \varphi_{jk} = \psi_{ijk} T_{jk}(r_{ij})T_{jk}(r_{ij}) = \psi_{ijk} T_{jk}(r_{ij})T_{jk}(r_{ij})0_i \) (naturality of \( \varphi \))

\[
= \psi_{ijk} T_{jk}(T_{ij}(r_{ij}))(r_{ij}) = (r_{ij}) T_{jk}(r_{ij}R_i) = r_{ij} \varphi_{ij}.
\]

\[
\psi_{ijk} T_{jk}(r_{ij}) T_{jk}(r_{ij})(r_{ij}) = \psi_{ijk} T_{jk}(r_{ij}) T_{jk}(r_{ij})(r_{ij}) = (r_{ij}) T_{jk}(r_{ij}R_i) = r_{ij} \varphi_{ij}.
\]

On the other hand we put

\[
r_{jk} = \varphi_{ij} (r_{jk}) = (\psi_{ijk} T_{jk}(r_{ij}))(r_{jk}) = (r_{jk}) \gamma_{ij} \xrightarrow{T_{jk}(r_{ij})} T_{jk}(R_j) \xrightarrow{T_{ij}(r_{ij})} T_{ij}(R_i) \xrightarrow{\psi} T_{ij}(R_i). \]

Hence,

\[
r_{jk} = (\psi_{ijk} T_{jk}(r_{ij}))(r_{jk}) \quad (\psi_{ijk} T_{jk}(r_{ij}))(r_{jk}) \]

Therefore, \( \varphi_{ij} \varphi_{jk} = \varphi_{ij}(r_{jk}) \) by the assumption III.

If we define a multiplication on \( R \) by setting

\[
(*) \quad r_{ij} r_{jk} = \varphi_{ij}(r_{jk})
\]

we have from [5], p. 104, Theorem 4.1 and p. 106, Theorem 5.1.

**Theorem 3.14.** Let \( \mathcal{C} \) be the abelian category of right \( R_i \)-module. Then \( \mathcal{C} \) is equivalent to the abelian category of a left \( R \)-module, where
And $T_{ij}(M_i) \approx M \otimes T_{ij}(R_i)$ for any $M_i \in A_i$ (** is given by an $R_i$-$R_j$ homomorphism $\psi_{ik}T_{ij}(R_i) \otimes T_{jk}(R_j) \rightarrow T_{ik}(P_i)$ (cf. [2], Theorem 1).

References