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ON SPECIAL TYPE OF HEREDITARY ABELIAN CATEGORIES

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In the book of Mitchell [5] he has defined a category of a commutative diagrams over an abelian category $\mathfrak{A}$. Especially he has developed this idea to a finite commutative diagrams and obtained many interesting results on global dimension of this diagram. Among them he has shown in [5], p. 237, Corollary 10. 10 that if I is a linearly ordered set, then gl dim $[I, \mathfrak{A}]=1+$gl dim $\mathfrak{A}$ for an abelian category $\mathfrak{A}$ with projectives. This is a generalization of Eilenberg, Rosenberg and Zelinsky [1], Theorem 8.

On the other hand, the author has studied a semi-primary hereditary ring and shown that it is a special type of generalized triangular matrix ring in [2].

In this note we shall generalize the notion of a generalized triangular matrix ring to an abelian category of generalized commutative diagram $[I, \mathfrak{A}_i]$ over abelian categories $\mathfrak{A}_i$ and obtain the similar results in it to [2], Theorem 1, where I is a finite linearly ordered set. The method in this note is quite similar to [5], IX, §10 and different from that of [2]. Finally we shall show that if the $\mathfrak{A}_i$ are the abelian category of right R_i-modules, then $[I, \mathfrak{A}_i]$ is equivalent to a generalized triangular matrix ring over R_i in [2], where R_i is a ring.

The author has shown many applications of generalized triangular matrix ring to semi-primary rings with suitable conditions in [2], [3] and [4]. However we do not study any applications of our results in this note and he hopes to continue this work on some other day.

1. Abelian categories of generalized commutative diagrams

Let $I=\{1, 2, \cdots, n\}$ be a linearly ordered set and $\mathfrak{A}_i$ be abelian categories. We consider additive covariant functors T_{ij} of $\mathfrak{A}_i$ to $\mathfrak{A}_j$ for i<j. For objects $A_i\in\mathfrak{A}_i$, $A_j\in\mathfrak{A}_j$ we define an arrow D_{ij}: $A_i\to A_j$ as follows:

$$(1) \qquad D_{ij}=d_{ij}T_{ij}, \qquad \text{where } d_{ij} \text{ is a morphism in } \mathfrak{A}_j.$$

Using those D_{ij} we can define a category $[I, \mathfrak{A}_i]$ of diagrams over $\{\mathfrak{A}_i\}_{i\in I}$. Namely, the objects of $[I, \mathfrak{A}_i]$ consist of sets $\{A_i\}_{i\in I}$ with $D_{ij}(A_i\in\mathfrak{A}_i)$ and the morphism of $[I, \mathfrak{A}_i]$ consist of sets $(f_i)_{i\in I}(f_i\in\mathfrak{A}_i)$ such that

$$d'_{ij}\mathrm{T}_{ij}(f_i) = f_j d_{ij}\,, \tag{2}$$

where f_i: $A_iA'_i$ and $\mathrm{D}_{ij}=d_{ij}\mathrm{T}_{ij}$, $\mathrm{D}'_{ij}=d'_{ij}\mathrm{T}_{ij}$ are arrows in $\boldsymbol{A}=(A_i)$ and $\boldsymbol{A}'=(A'_i)$, respectively.

Let $\boldsymbol{f}=(f_i)_{i\in I}$ be a morhphism of $\boldsymbol{A}$ to $\boldsymbol{A}'$. Then we define a set $(\mathrm{Im}\, f_i)$, $(\mathrm{coker}\, f_i)$ and so on. If $(\mathrm{Im}\, f_i)$, $(\mathrm{coker}\, f_i)\cdots$coincide with $\mathrm{Im}\,\boldsymbol{f}$, coker $\boldsymbol{f}\cdots$in $[I, \mathfrak{A}_i]$, respectively, we shall call $[I, \mathfrak{A}_i]$ a category induced naturally from $\mathfrak{A}_i$.

Proposition 1.1. *Let I and $\mathfrak{A}_i$ be as above. $[I, \mathfrak{A}_i]$ is an abelian category induced naturally from $\mathfrak{A}_i$ if and only if T_{ij} is cokernel preserving.*

Proof. We assume that T_{ij} is cokernel preserving. Let $f=(f_i)_{i\in I}$: $(A_i)\to(A'_i)$ be a morphism in $\mathfrak{A}=[I, \mathfrak{A}_i]$. Then we can easily see that $(\ker f_i)_{i\in I}$ is Kerf in $\mathfrak{A}$ and that $(\mathrm{coker}\, f_i)_{i\in I}$ is in $\mathfrak{A}$ since T_{ij} is cokernel preserving. Hence, we know from [1], p. 33, Theorem 20.1 that $\mathfrak{A}$ is an abelian category. Conversely, we assume $\mathfrak{A}$ is an abelian category as above. We may assume $I=(1, 2)$. Let f: $A_1\to C_1$ be an epimorphism in $\mathfrak{A}_1$ and $\mathrm{B}_2=\mathrm{im}\,\mathrm{T}(f)$, where $\mathrm{T}=\mathrm{T}_{1,2}$. Put $A=(A_1, \mathrm{T}(A_1))$ $C=(C_1, \mathrm{T}(C_1))$ and $\boldsymbol{f}=(f, \mathrm{T}(f))$. Then $\mathrm{Im}\,\boldsymbol{f}=(C_1, B_2)$, ($\boldsymbol{f}$: $A \xrightarrow{\boldsymbol{f}'} \mathrm{Im}\,\boldsymbol{f} \xrightarrow{i} C$). By the assumption $\boldsymbol{f}'$ and i are morphisms in $\mathfrak{A}$. Hence, there exists an morphism d: $\mathrm{T}(C_1)\to B_2$ in $\mathfrak{A}_2$ such that $d\mathrm{T}$ is an arrow in $\mathrm{im}\,\boldsymbol{f}$. Namely

$$\begin{array}{ccc} \mathrm{T}(A_1) & \xrightarrow{\mathrm{T}(f)} & \mathrm{T}(C_1) \\ \Big\Downarrow d_{12} & & \Big\downarrow d \\ \mathrm{T}(A_1) & \xrightarrow{\boldsymbol{f}'_2} & B_2 \end{array} \tag{3}$$

is commutative, where $i\boldsymbol{f}'_2=\mathrm{T}(f)$.

Therefore, $\boldsymbol{f}'_2=dT(f)=di\boldsymbol{f}'_2$. Since $\boldsymbol{f}'_2$ is epimorphic $di=\mathrm{I}_{B_2}$. On the other hand, we obtain similarly from an morphism i that $id=\mathrm{I}_{T(C_1)}$. Hence, d is isomorphic and T is an epimorphic functor. Let $A''_1 \xrightarrow{g} A_1 \xrightarrow{f} A_1/\mathrm{g}(A''_1)\to 0$ be exact and $B''_2=\mathrm{im}\,\mathrm{T}(g)$. Put $A=(A''_1, B''_2)$, $\mathrm{C}=(A_1, \mathrm{T}(A_1))$, and $f=(g, i)$, where $\mathrm{T}(g)$: $\mathrm{T}(A'')\to B''_2 \xrightarrow{\mathrm{i}} \mathrm{T}(A_1)$. From the assumption coker $f=(A_1/\mathrm{g}(A''_1), \mathrm{T}(A_1)/B''_2)$. Hence there exists d: $\mathrm{T}(A_1/\mathrm{g}(A''_1))\to\mathrm{T}(A_1)/B''_2$ such that $d\mathrm{T}(f)=h$, where $h=\mathrm{coker}\,(B''_2 \xrightarrow{i} \mathrm{T}(A_1))$, (cf. (3)). Hence, $\ker \mathrm{T}(f)\subseteqq B''_2$. $B''_2\subseteq\mathrm{Ker}\,\mathrm{T}(f)$ is clear, since $fg=0$. Therefore, T is cokernel preserving.

From this proposition we always assume that T_{ij} is cokernel preserving.

We shall define functors T_i: $\mathfrak{A}\to\mathfrak{A}_i$ and $\tilde{S}_i$: $\mathfrak{A}_i\to\mathfrak{A}$ as follows:

Let $A=(A_i)_{i\in I}$

$$\begin{aligned} &\mathrm{T}_i(A) = A_i \\ &\mathrm{T}_j\tilde{S}_i(A_i)) = 0 \qquad \text{for} \quad \mathrm{j} < \mathrm{i}\,, \end{aligned} \tag{4}$$

$$\mathrm{T}_j\tilde{S}_i(A_i)=\sum_{i<i_1<\cdots<i_k<j}\oplus \mathrm{T}_{i_kj}\mathrm{T}_{i_{k-1}i_k}\cdots\mathrm{T}_{ii_1}(A_i) \qquad \text{for} \quad \mathrm{i}<\mathrm{j}\,,$$

with arrow $\mathrm{D}_{ik}=\mathrm{T}_{jk}$ for j<k.

Then we have a natural equivalence η: $[\tilde{S}_i(\mathrm{A}_i),\ \mathrm{D}]\approx[\mathrm{A}_i,\ \mathrm{T}_i(\mathrm{D})]$ for any $\mathrm{A}_i\in\mathfrak{A}_i$ and $\mathrm{D}\in\mathfrak{A}$. Hence, we have from [5], p. 138, Coro. 7.4.

Proposition 1.2. *We assume that each $\mathfrak{A}_i$ has a projective class $\mathcal{E}_i$, and T_{ij} is cokernel preserving. Then $\cap T_i^{-1}(\mathcal{E}_i)$ is a projective class in $\mathfrak{A}=[I.\ \mathfrak{A}_i]$, whose projectives are the objects of the form $\bigoplus_{i\in I}\tilde{S}_i(P_i)$ and their retracts, where P_i is $\mathcal{E}_i$-projective for all $i\in I$.*

2. Commutative diagrams with special arrows

In the previous section we study a general case of abelian categories of commutative diagrams. However, it is too general to discuss them. Hence, we shall consider the following conditions:

[I] *T_{ij} is cokernel preserving.*

[II] *There exist natural transformations*

$$\psi_{ijk}\colon \mathrm{T}_{jk}\mathrm{T}_{ij}\to\mathrm{T}_{ik} \qquad \textit{for any} \quad i<j<k\,.$$

[III] *For any $i<j<k<l$ and N in A_i*

$$\begin{array}{ccc}
\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(N) & \xrightarrow{\mathrm{T}_{kl}(\psi_{ijk})} & \mathrm{T}_{kl}\mathrm{T}_{ik}(N)\\
\big\downarrow{\scriptstyle \psi_{jkl}} & & \big\downarrow{\scriptstyle \psi_{ikl}}\\
\mathrm{T}_{jl}\mathrm{T}_{ij}(N) & \xrightarrow{\psi_{ijl}} & \mathrm{T}_{il}(N)
\end{array}$$

is commutative

[IV] *For arrows $d_{ij}\colon \mathrm{T}_{ij}(A_i)\to A_j$* in $\mathfrak{A}=[I,\ \mathfrak{A}_i]$

$$\begin{array}{ccc}
\mathrm{T}_{jk}\mathrm{T}_{ij}(A_i) & \xrightarrow{\mathrm{T}_{jk}(d_{ij})} & \mathrm{T}_{jk}(A_j)\\
\big\downarrow{\scriptstyle \psi_{ijk}} & & \big\downarrow{\scriptstyle d_{ik}}\\
\mathrm{T}_{ik}(A_i) & \xrightarrow{d_{ik}} & A_k
\end{array}$$

is *commutative.*

From now on we always assume I, II and for any arrows in $\mathfrak{A}$, we require the condition IV.

We note that IV implies $\mathrm{D}_{jk}\mathrm{D}_{ij}(A_i)\subseteq\mathrm{D}_{ik}(A_i)$ for any $A=(A_i)_{i\in I}$ in $\mathfrak{A}$.

First we shall show that $\mathfrak{A}$ is still an abelian category under the assumption I even if we require IV in $\mathfrak{A}$.

Proposition 2.1. *Let $(\mathfrak{A}_i)_{i\in I}$ be abelian categories. We assume II. Then*

$\mathfrak{A}=[I, \mathfrak{A}_i]$ *requiring IV is abelian if and only if I is satisfied.*

Proof. Let $f=(f_i)\colon (A_i)\to(A'_i)$ in $\mathfrak{A}$. We consider a diagram

$$
(5)\qquad
\begin{tikzcd}
T_{jk}T_{ij}(A_i) \arrow[rr, "T(d_{ij})"] \arrow[dr, "TT'(f_i)"] \arrow[dd, "\psi"'] & & T'_{jk}(A_j) \arrow[dr, "T(f_j)"] \arrow[dd, "d_{jk}"] & \\
& T_{jk}T_{ij}(A'_i) \arrow[rr, "T(d'_{ij})"] \arrow[dd, "\psi"] & & T_{jk}(A'_j) \arrow[dd, "d'_{jk}"] \\
T_{ik}(A_i) \arrow[rr, "d_{ik}"] \arrow[dr, "T(f_i)"'] & & A_k \arrow[dr, "f_k"] & \\
& T_{ik}(A'_i) \arrow[rr, "d'_{ik}"] & & A'_k
\end{tikzcd}
$$

We only prove from Proposition 1.1 that for any morphism $g=(g_i)$, $(\ker g_i)_{i\in I}$ $(\operatorname{coker} g_i)_{i\in I}$ satisfy IV. Put $A_i=\ker g_i$ and $f_i=$inclusion morphism in the above. Then all squares except the rear in (5) are commutative from II, IV and (2). Since f_k is monomorphic, the rear one is commutative. Which shows $(\ker g_i)_{i\in I}$ satisfies IV. Similarly if $A_i=(\operatorname{coker} g_i)$ and f_i epimorphism of cokernel, then $(\operatorname{coker} g_i)$ satisfies IV, since $T_{jk}T_{ij}(f_i)$ is epimorphic from I.

Next, we shall define functors similarly to $\tilde{S}_i$. For $A_i\in\mathfrak{A}_i$ we put

$$
(6)\qquad
\begin{aligned}
&S_i(A_i) = (0, 0, \cdots, A_i, T_{ii+1}(A_i), \cdots, T_{in}(A_i)) \text{ with arrows}\\
&D_{tk} = 0 \qquad \text{for} \quad t<i\\
&D_{ik} = T_{ik} \qquad \text{for} \quad k>i\\
&D_{jk} = \psi_{ijk}T_{jk} \qquad \text{for} \quad k>j>i\,.
\end{aligned}
$$

If T_{ij}'s satisfy III, then $S_i(A_i)$ is an object in $[I, \mathfrak{A}_i]$ requiring IV. Furthermore, we can prove easily $[S_i(A_i), D]\approx[A_i, T_i(D)]$ for $D\in[I, \mathfrak{A}_i]$. Hence, we have similarly to Proposition 1.2

Proposition 1.2′. *We assume that each $\mathfrak{A}_i$ has a projective class $\mathcal{E}_i$ and I$\sim$III are satisfied. Then $\mathfrak{A}=[I, \mathfrak{A}_i]$ requiring IV has a projective class $\cap T_i^{-1}(\mathcal{E}_i)$ whose projectives are the objects of the form $\bigoplus_{i\in I} S_i(P_i)$ and their retracts, where P_i is $\mathcal{E}_i$-projective for all $i\in I$.*

In the rest of the paper we always assume that $[I, \mathfrak{A}_i]$ is an abelian category of the commutative diagrams whose arrows are required IV and that I$\sim$III are satisfied.

Proposition 2.2. $(D_{kl}D_{jk})D_{ij}=D_{kl}(D_{jk}D_{ij})$ *for* $i<j<k<l$.

Proof.
$$\begin{aligned}
(\mathrm{D}_{kl}\mathrm{D}_{jk})\mathrm{D}_{ij}(A) &= d_{jl}\psi_{jkl}(\mathrm{T}_{kl}\mathrm{T}_{jk})(d_{ij})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \\
&= d_{jl}\mathrm{T}_{jl}(d_{ij})\psi_{jkl}\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \quad \text{(naturality of } \psi) \\
&= d_{il}\psi_{ijl}\psi_{jkl}\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \qquad \text{(IV)} \\
&= d_{il}\psi_{ikl}\mathrm{T}_{kl}(\psi_{ijk})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \qquad \text{(III)} \\
&= d_{kl}\mathrm{T}_{kl}(d_{ik})\mathrm{T}_{kl}(\psi_{ijk})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \quad \text{(IV)} \\
&= d_{kl}\mathrm{T}_{kl}(\mathrm{d}_{ik}\psi_{ijk})\mathrm{T}_{kl}\mathrm{T}_{jk}\mathrm{T}_{ij}(A) \\
&= \mathrm{D}_{kl}(\mathrm{D}_{jk}\mathrm{D}_{ij})(A) \qquad \text{for any} \quad A\in\mathfrak{A}_i.
\end{aligned}$$

Theorem 2.3. (cf. [1], p. 234, Lemma 9.3) *Let* $I=I_1\cup I_2$ *and* $I_1=\{1, 2\cdots, i-1\}$, $I_2=\{i, \cdots, n\}$. *Then* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{A}'=[(1, 2), [I_1\mathfrak{A}_k], [I_2, \mathfrak{A}_{k'}]]$ *with a suitable functor* $\boldsymbol{T}_{12}$: $[I_1, \mathfrak{A}_k]\to[I_2, \mathfrak{A}_{k'}]$.

Proof. First we define a functor $\boldsymbol{T}_{12}$. Let $\boldsymbol{A}_1=(A_i)_{i\in I_1}$. For any $k\geqslant i$ we consider a diagram $\mathrm{D}_k=\{\mathrm{T}_{lk}(A_l),\ \mathrm{T}_{l'k}\mathrm{T}_{ll'}A_l)$ for $l<l'<i<k$ with arrows $\mathrm{T}_{l'k}\mathrm{T}_{ll'}(A_l) \xrightarrow{\psi} \mathrm{T}_{lk}(A_l)$ and $\mathrm{T}_{l'k}\mathrm{T}_{ll'}(A_l) \xrightarrow{\mathrm{T}_{l'k}(d_{ll'})} \mathrm{T}_{l'k}A_{l'})\}$. D_k has a colimit A_k in $\mathfrak{A}_k$ by [1], p. 46, Coro. 2.5, $(\{D_k\} \xrightarrow{\alpha_k} A_k)$. Put $\boldsymbol{A}_2=(A_i, \cdots, A_n)$. We shall show that $\boldsymbol{A}_2$ is in $[I_2, \mathfrak{A}_{k'}]$. We have to define $\mathrm{D}_{kk'}$ for $i\leq k<k'$. Consider a diagram

$$\left.\begin{array}{ccc}
\mathrm{T}_{kk'}\mathrm{T}_{lk}(A_l) & \xrightarrow{\psi_{lkk'}} & \mathrm{T}_{l'k'}(A_l) \\
\Big\uparrow{\scriptstyle \mathrm{T}(\psi)} & & \Big\uparrow{\scriptstyle \psi} \\
\mathrm{T}_{kk'}\mathrm{T}_{l'k}\mathrm{T}_{ll'}(A_l) & \xrightarrow{\psi_{lkk'}} & \mathrm{T}_{l'k'}\mathrm{T}_{ll'}(A_l) \\
\Big\downarrow{\scriptstyle \mathrm{T}_{k'k}\mathrm{T}_{ll'}(d_{ll'})} & & \Big\downarrow{\scriptstyle \mathrm{T}_{l'k'}(d_{ll'})} \\
\mathrm{T}_{kk'}\mathrm{T}_{l'k}(A_{l'}) & \xrightarrow{\psi_{l'kk'}} & \mathrm{T}_{l'k}(A_{l'})
\end{array}\right\} \to A_{k'} \tag{7}$$

The upper and lower squares are commutative by III and naturality of ψ, respectively. Then (7) implies that these exist compatible morphism: $\{\mathrm{T}_{kk'}(D_k)\}\to A_{k'}$. Since $\mathrm{T}_{kk'}$ is colimit preserving by [5], p. 55. Proposition 6.4, we have a unique morphism $d_{kk'}$: $\mathrm{T}_{kk'}(A_k)\to A_{k'}$. Hence we can define $\mathrm{D}_{kk'}=d_{kk'}\mathrm{T}_{kk'}$. Next we show that those $\mathrm{D}_{kk'}$ satisfy IV. For $i\leqslant k<k'<k''$ we have a diagram

(8)

$$\begin{tikzcd}
\mathrm{T}_{k'k''}\mathrm{T}_{kk'}(D_k) \arrow[rr, "\mathrm{T}_{k'k''}(\psi)"] \arrow[dd, "\mathrm{TT}(\alpha_k)"'] \arrow[dr, "\psi"] & & \mathrm{T}_{k'k''}(D_{k'}) \arrow[dd, "\mathrm{T}(\alpha_{k'})" near start] \arrow[dr, "\psi"] & \\
& \mathrm{T}_{kk''}(D_k) \arrow[rr, crossing over, "\psi" near start] & & D_{k''} \arrow[dd, "\alpha_{k''}"] \\
\mathrm{T}_{k'k''}\mathrm{T}_{kk'}(A_k) \arrow[rr, "\mathrm{T}_{kk'}(d_{kk'})" near end] \arrow[dr, "\psi"'] & & \mathrm{T}_{k'k''}(A_{k'}) \arrow[dr, "d_{k'k''}"] & \\
& \mathrm{T}_{kk''}(A_k) \arrow[rr, "d_{kk''}"] \arrow[from=uu, crossing over, "\mathrm{T}_{kk''}(\alpha_k)" near end] & & A_{k''}
\end{tikzcd}$$

All squares except bottom are commutative by III and the definitions $d_{kk'}$, $d_{kk''}$ and $d_{k'k''}$. On the other hand, it is clear that φ_k: $\mathrm{T}_{k'k''}\mathrm{T}_{kk'}(D_k) \xrightarrow{\mathrm{TT}(\alpha_k)} \mathrm{T}_{k'k''}\mathrm{T}_{kk'}(A_k) \xrightarrow{\psi} \mathrm{T}_{kk''}(A_k) \xrightarrow{d_{kk''}} \mathfrak{A}_{k''}$ is compatible. Since $\mathrm{T}_{k'k''}\mathrm{T}_{kk'}$ is colimit preserving, we have a unique morphism Φ: $\mathrm{T}_{k'k''}\mathrm{T}_{kk'}(A_k) \to A_{k''}$ such that $\psi_k = \Phi\mathrm{TT}(\alpha_k)$. Therefore, the bottom square is also commutative, which means II. Thus we have shown that $\boldsymbol{T}_{12}$ is a functor. Let $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ be in $\mathfrak{A}'$, where $\boldsymbol{A}_1 = (A_i)_{i \in I_1}$ and $\boldsymbol{A}_2 = (B_j)_{j \in I_2}$. From the definition of $\boldsymbol{T}_{12}$ we have a morphism: $\mathrm{T}_{jk}(A_j) \xrightarrow{\alpha_k} A_k \xrightarrow{d_k} B_k$ for $j \in I_1$, $k \in I_2$, where $(d_i)_{i \in I}$: $\boldsymbol{T}_{12}(\boldsymbol{A}_1) \to \boldsymbol{A}_2$. We put

$$\begin{aligned} \mathrm{D}'_{jk} &= d_k \alpha_k \mathrm{T}_{jk} \qquad \text{for } \; j<i<k \text{ and} \\ \mathrm{D}'_{st} &= \mathrm{D}_{st} \qquad \text{for } \; s, t \in I_1 \; \text{ or } \; T_2 . \end{aligned}$$

We shall show that D'_{ij} satisfy IV. Take $j<h<k$. If $j \in I_2$ or $k \in I_1$, then it is obvious. We assume $j \in I_1$ and $h, k \in I_2$. Then we have

$$(9) \qquad \begin{array}{ccccc} \mathrm{T}_{hk}\mathrm{T}_{jh}(A_j) & \xrightarrow{\mathrm{T}(\alpha_h)} & \mathrm{T}_{hk}(A_h) & \xrightarrow{\mathrm{T}(d_h)} & \mathrm{T}_{hk}(B_h) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle d_{hk}} & & \downarrow{\scriptstyle d'_{hk}} \\ \mathrm{T}_{jk}(\mathrm{A}_j) & \xrightarrow{\alpha_k} & A_k & \xrightarrow{d_k} & B_k \end{array}$$

where d'_{hk} is a given morphism in $\boldsymbol{A}_2$. The left side is commutative by the definition of $\boldsymbol{T}_{12}$ and so is the right side, since h, $k \in \mathrm{I}_2$. Hence, the out side square means IV. We can easily see by the defininition of $\{D_k\}$ that IV is satisfied for j, $h \in \mathrm{I}_1$ and $k \in \mathrm{I}_2$. Hence, $\boldsymbol{T}(\boldsymbol{A}_1, \boldsymbol{A}_2) = (A_1, \cdots, A_{i-1}, B_i, \cdots, B_n)$ is an object in $\mathfrak{A}$. Conversely, for $\boldsymbol{A} = (A_1, \cdots, A_n)$ we put $\boldsymbol{S}(\boldsymbol{A}) = ((A_1, \cdots, A_{i-1}), (A_i, \cdots, A_n))$. Then it is clear that $\boldsymbol{S}(\boldsymbol{A}) \in \mathfrak{A}'$ and $\boldsymbol{TS} = \mathrm{I}_{\mathfrak{A}}$, $\boldsymbol{ST} = \mathrm{I}_{\mathfrak{A}'}$. This shows that $\boldsymbol{T}_{12}$ is cokernel preserving by Proposition 1.1.

3. Hereditary categories

In this section, we always assume that I$\sim$IV are satisfied and every $\mathfrak{A}_i$ has projectives and hence $\mathfrak{A}=[I, \mathfrak{A}_i]$ has projectives by Proposition 1.2$'$.

If every object in an abelian category $\mathfrak{B}$ is projective, we call $\mathfrak{B}$ a semi-simple category, which is equivalent to a fact gl dim $\mathfrak{B}=0$. If gl dim $\mathfrak{B}\leqq 1$ we call $\mathfrak{B}$ hereditary.

Proposition 3.1. ([5], p. 235, Coro. 10.3). *We assume that $\mathfrak{A}_i$ has projectives and that T_{ij} is projective preserving. Let $D=(D_i)_{i\in I}$ be an object in $[I, \mathfrak{A}_i]$* and $m=\max$ *$(hd\ D_i)$, n=the number of elements of I. Then $hd\ D\leqslant n+m-1$.*

Since T_{ij} is projective preserving, we can prove it similarly to [1], p. 235.

Corollary. *Let $I=(1,2)$ and T_{12} be projective preserving. Then*

$$\max(\text{gl dim}\,\mathfrak{A}_1, \text{gl dim}\,\mathfrak{A}_2)\leqq \text{gl dim}\,[(1, 2), \mathfrak{A}_1, \mathfrak{A}_2]\leqslant \max(\text{gl dim}\,\mathfrak{A}_i)+1\ .$$

Proof. The right side inequality is clear from Proposition 3.1. Let A be an object in $\mathfrak{A}_1$. It is clear that $\mathrm{hd}(A, 0)\geqq \mathrm{hd}\,A$. Since T_{12} is projective preserving, we have similarly $\mathrm{hd}(0, A')\geqq \mathrm{hd}A'$ for $A'\in\mathfrak{A}_2$.

Lemma 3.2. *Let $\mathfrak{A}=[(1, 2), \mathfrak{A}_1, \mathfrak{A}_2]$. If* gl dim $\mathfrak{A}\leqq 1$, *then T_{12} is projective preserving.*

Proof. Let P_1 be projective in $\mathfrak{A}_1$. Then $(P_1, \mathrm{T}_{12}(P_1))$ is projective in $\mathfrak{A}$ by Proposition 1.2. Let $0\leftarrow \mathrm{T}_{12}(P_1)\leftarrow Q$ be an exact sequence in $\mathfrak{A}_2$ with Q projective. Then $(0. 0)\leftarrow(P_1, 0)\leftarrow(P_1, \mathrm{T}_{12}(P_1))\leftarrow(0, Q)$ is exact in $\mathfrak{A}$. Since gl dim $\mathfrak{A}\leqq 1$, $(0, \mathrm{T}_{12}(P_1))$ is projective in $\mathfrak{A}((0, \mathrm{T}_{12}(P_1))\subset(P_1, \mathrm{T}_{12}(P_1))$. Hence, $\mathrm{T}(P_1)\leftarrow Q$ is retract and $\mathrm{T}_{12}(P_1)$ is projective in $\mathfrak{A}_2$.

Similarly to the category of modules we have

Lemma 3.3. *Let A be an abelian category. If $A\oplus B=A'\oplus C$ and $A\supset A'$, then $A'=A\oplus A''$, $A''=A\cap C$ and $C=A''\oplus C'$.*

Lemma 3.4. *Let $I=(1, 2)$ and $\mathfrak{A}=[I, \mathfrak{A}_i]$. If T_{12} is projective preserving, then every projective object A in $\mathfrak{A}$ is of a form $(P_1, T_{12}(P_1)\oplus P_2)$ and the arrow d_{12} in A is monomorphic, where P_i is projective in $\mathfrak{A}_i$.*

Proof. Since $\boldsymbol{A}=(A_1, A_2)$ is a retraction of an object of a form $\boldsymbol{P}=(P_1, \mathrm{T}_{12}(P_1)\oplus P_2)$ with P_i projective in $\mathfrak{A}_i$. Hence, $0\to\boldsymbol{A}\to\boldsymbol{P}$ splits. Let $P_1=A_1\oplus Q_1$. Then $\mathrm{T}_{12}(P_1)=\mathrm{T}_{12}(A_1)\oplus\mathrm{T}_{12}(Q_1)$ and A_2 is a coretract of $\mathrm{T}_{12}(A_1)\oplus \mathrm{T}_{12}(Q_1)\oplus P_2$. Furthermore, $\mathrm{T}_{12}(A_1)\xrightarrow{d_{12}} A_2\to \mathrm{T}_{12}(P_1)\oplus P_2=\mathrm{T}_{12}(A_1)\to \mathrm{T}_{12}(P_1)\oplus P_2$, and the right side is monomorphic. Hence, d_{12} is monomorphic. Thus we may assume $T_{12}(A_1)\subset A_2\subset T_{12}(P_1)\oplus P_2$. Therefore, $A_2=T_{12}(A_1)\oplus A_2'$ by Lemma 3.3. Since P_1 is projective and T_{12} is projective preserving, $T_{12}(P_1)\oplus P_2$ is projective in $\mathfrak{A}_2$. Hence, A_2' is projective by Lemma 3.3.

Lemma 3.5. *Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be hereditary and T_{12} projective preserving. If $T_{12}(P_2)$ is a coretract of $T_{12}(P_1)$ for any projective objects $P_1\supset P_2$ in $\mathfrak{A}_1$, then $\mathfrak{A}=[(1, 2), \mathfrak{A}_1, \mathfrak{A}_2]$ is hereditary.*

Proof. Let (A_1, A_2) be any object in $\mathfrak{A}$ and $0\leftarrow(A_1, A_2)\xleftarrow{f} P$ be exact, where P $\mathfrak{A}$-projective. Then $P=(P_1, T_{12}(P_1)\oplus P_2)$ with P_i projective by Lemma 3.4. Put ker $f=(K_1, K_2)$. Since $\mathfrak{A}_1$ is hereditary, K_1 is projective. Hence, $T_{12}(K_1)$ is a coretract of $T_{12}(P_1)$ by the assumption. Hence, $K_2=T_{12}(K_1)\oplus K_2'$ by Lemma 3.3. Since K_2 is projective, (K_1, K_2) is $\mathfrak{A}$-projective.

Theorem 3.6. *Let $I=(1, 2, \cdots, n)$ be a linearly ordered set, $\mathfrak{A}_i$ abelian categories with projectives. Let $\mathfrak{A}=[I, \mathfrak{A}_i]$ be the abelian category of commutative diagrams over $\mathfrak{A}_i$ with functors T_{ij} satisfying $I\sim IV$. If $\mathfrak{A}$ is hereditary, then we have:*

i) *Every projective object of $\mathfrak{A}$ is of a form $\bigoplus_{i\in I} S_i(P_i)$, where P_i is projective in $\mathfrak{A}_i$.*

ii) *T_{ij} is projective preserving for any $i<j$.*

iii) *$T_{ij}(P_2)$ is a coretract of $T_{ij}(P_1)$ for any projective objects $P_1\supset P_2$ in $\mathfrak{A}_i$.*

iv) *$[(i_1, i_2, \cdots, i_t), A_{i_1}, A_{i_2}, \cdots, A_{i_t}]\equiv\mathfrak{A}(i_1, i_2, \cdots, i_t)$ is hereditary for any $i_1<i_2<\cdots<i_t$.*

v) *If $P=(P_i)_{i\in I}$ is projective in $\mathfrak{A}$, then every d_{ij} in P is a coretract. $(P_{i_1}, P_{i_2}, \cdots, P_{i_t})$ is $\mathfrak{A}(i_1, i_2, \cdots, i_t)$-projective.*

Proof. We shall prove the theorem by the induction on the number n of element of I. We obtain $\mathfrak{A}\approx[(1,2), \mathfrak{A}_1, \mathfrak{A}(I-1)]\equiv\mathrm{A}'$ from Theorem 3.2. Then $\mathfrak{A}(I-1)$ is hereditary by Lemma 3.2 and Corollary to Proposition 3.1. Furthermore, $\boldsymbol{T}_{12}$ in $\mathfrak{A}'$ is projective preserving. i) Let $P=(P_i)_{i\in I}$ be projective in $\mathfrak{A}$. Then $P=(P_1, \boldsymbol{T}_{12}(P_1)\oplus \boldsymbol{P}_2)$ by Lemma 3.4, where $\boldsymbol{P}_2$ is projective in $\mathfrak{A}(I-1)$. We obtain, by the definition of $\boldsymbol{T}_{12}$, that $\boldsymbol{T}_{12}(P_1)=(T_{1i}(P_1))_{i\in I-1}$. Hence, $P=\bigoplus_{i\in I} S_i(P_i)$ by the induction hypothesis. ii) Every component of projective object in $\mathfrak{A}(I-1)$ is projective by the induction. Hence, $T_{1i}(P_1)$ is projective in $\mathfrak{A}_i$. iii) Let $P_1\supset P_2$ be projective in $\mathfrak{A}_1$. Put $A=(P_1/P_2, 0, \cdots, 0)$. Then we have an exact sequence $0\leftarrow A\leftarrow(P_1, \boldsymbol{T}_{12}(P_1))$. Since $\mathfrak{A}$ is hereditary, its kernel $(P_2, \boldsymbol{T}_{12}(P_1))$ is projective. Therefore, $T_{1i}(P_2)$ is a coretract from i). iv) We may show that $\mathfrak{A}(I-i)$ is hereditary for any i. $\mathfrak{A}\approx[I_1, i, I_2, \mathfrak{A}_1', \mathfrak{A}_i, \mathfrak{A}_2']$, where $I_1=(1, \cdots, i-1)$, $I_2=(i+1,\cdots, n)$, $\mathfrak{A}_1=\mathfrak{A}(I_1)$ and $\mathfrak{A}_2=\mathfrak{A}(I_2)$. From Lemma 3.2 $\boldsymbol{T}_{13}$ is projective preserving and hence $\mathfrak{A}(I-i)$ is hereditary from iii) and Lemma 3.5 and the definition of $\boldsymbol{T}_{13}$. v) Since $P=(P_1, \boldsymbol{T}_{12}(P_1)\oplus P_2)$, d_{1i}: $T_{1i}(P_1)\rightarrow P_i$ is a coretract.

$P\approx(\boldsymbol{P}_1', P_2, \boldsymbol{P}_3')$, where $\boldsymbol{P}_1'=(P_j)_{j\in I_1}$ and $P_3'=(P_j)_{j\in I_2}$. Then it is clear from i) and induction that $(\boldsymbol{P}_1', \boldsymbol{P}_3')$ is $\mathfrak{A}(I-i)$-projective.

Next we shall study a condition of every projective objects in $\mathfrak{A}$ being of a form $\oplus \mathrm{S}_i(P_i)$, when T_{ij} is projective preserving.

Lemma 3.7. *Let $\mathfrak{A}$ and $\mathfrak{A}_i$ be as above and T_{ij} projective preserving. If we have*

$$(*) \qquad \mathrm{T}_{ij}(P_i)=\mathrm{T}_{i+1j}\mathrm{T}_{i,i+1}(P_i)\oplus\mathrm{T}_{i+2j}(\mathrm{K}^{i+2}(P_i))\oplus\cdots\oplus\mathrm{T}_{j-1j}(\mathrm{K}^{j-1}(P_i))\oplus\mathrm{K}^j(P_i)$$

for any projective object P_i in $\mathfrak{A}_i$ for all i, then every object $A=(A_i)_{i\in I}$ in $\mathfrak{A}$ is of a form $\oplus S_i(Q_i)$ whenever A is subobject of $P=(Q_i')_{i\in I}$ and A_i is a coretract of Q_i' for all i, where $K^j(P_i)$ is an object in $\mathfrak{A}_j$, Q_i and Q_i' are $\mathfrak{A}_i$-projective, and the equality in (*) is given by taking suitable transformation from the right side to the left in (*).

Proof. We may assume $P=\underset{i\in I}{\oplus}\mathrm{S}_i(P_i)$ and P_i is $\mathfrak{A}_i$-projective. Put $P=(\boldsymbol{P}_i)_{i\in I}$. From the assumption $\boldsymbol{P}_1=A_1\oplus Q_1$. We shall show the following fact by the induction on i.

$$\text{i)} \qquad A_i=\mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{2i}(K^2)\oplus\cdots\oplus\mathrm{T}_{i-1i}(K^{i-1})\oplus K^i$$

$$\text{ii)} \qquad K^i\oplus Q_i=P_i\oplus\mathfrak{K}^i(Q_1)\oplus\mathfrak{K}^i(Q_2)\cdots\oplus\mathrm{K}^1(Q_{i-2})\oplus\mathrm{T}_{i-1i}(Q_{i-1}),$$

and this is a coretract of P_i, where $\mathrm{K}^i(Q_j)$ is the object in (*) for projective Q_i and the equalities are considered in P_i by suitable imbedding mappings. If $i=1, 2$, i) and ii) are clear (see the proof of Lemma 3.4). We assume i) and ii) are true for $k<i$. Using this assumption we first show for $2<j<i-1$ that

$$\begin{aligned}\text{iii)} \qquad \boldsymbol{P}_i=\;&\mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{2i}(K^2)\oplus\cdots\oplus\mathrm{T}_{ji}(K^j)\\&\oplus\mathrm{T}_{j+1i}(P_{j+1}\oplus(\mathrm{K}^{j+1}(Q_1)\oplus\cdots\oplus\mathrm{K}^{j+1}(Q_{j-1})\oplus\mathrm{T}_{jj+1}(Q_j))\\&\oplus\mathrm{T}_{j+2i}(P_{j+2}\oplus\mathrm{K}^{j+2}(Q_1)\oplus\cdots\oplus\mathrm{K}^{j+2}(Q_{j-1})\oplus\mathrm{K}^{j+2}(Q_j))\\&\oplus\cdots\cdots\cdots\cdots\\&\oplus\mathrm{T}_{i-1i}(P_{i-1}\oplus(\mathrm{K}^{i-1}(Q_1)\oplus\cdots\oplus\mathrm{K}^{i-1}(Q_{j-1})\mathrm{K}^{i-1}(Q_j))\\&\oplus P_i\oplus\mathrm{K}^i(Q_1)\oplus\cdots\oplus\mathrm{K}^i(Q_{j-1})+\mathrm{K}^i(Q_j)\,.\end{aligned}$$

Now
$$\begin{aligned}\boldsymbol{P}_i&=\mathrm{T}_{1i}(P_1)\oplus\mathrm{T}_{2i}(P_2)\oplus\cdots\oplus\mathrm{T}_{i-1i}(P_{i-1})\oplus P_i\\&=\mathrm{T}_{1i}(P_1)\oplus\mathrm{T}_{2i}(P_2)\oplus\boldsymbol{P}_i' \qquad (\boldsymbol{P}_i'=\mathrm{T}_{3i}(P_3)\oplus\cdots\oplus P_i)\\&=\mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{1i}(Q_1)\oplus\mathrm{T}_{2i}(Q_1)\oplus\mathrm{T}_{2i}(P_2)\oplus\boldsymbol{P}_i'\\&=\mathrm{T}_{1i}(A_1)\oplus(\mathrm{T}_{2i}\mathrm{T}_{12}(Q_1)\oplus\mathrm{T}_{3i}(\mathrm{K}^3(Q_1))\oplus\cdots\oplus\mathrm{T}_{i-1i}(\mathrm{K}^{i-1}(Q_1))\\&\quad\oplus\mathrm{K}^i(Q_1))\oplus\mathrm{T}_{2i}(P_2)\oplus\boldsymbol{P}_i' \qquad ((*))\\&=\mathrm{T}_{1i}(A_1)\oplus(\mathrm{T}_{2i}(P_2\oplus\mathrm{T}_{12}(Q_1))\oplus(\mathrm{T}_{3i}(\mathrm{K}^3(Q_1))\oplus\cdots\oplus\mathrm{K}^i(Q_1)+\boldsymbol{P}_i'\\&=\mathrm{T}_{1i}(A_1)\oplus\mathrm{T}_{2i}(K^2)\\&\quad\oplus\mathrm{T}_{3i}(P_3\oplus\mathrm{K}^3(Q_1)\oplus\mathrm{T}_{23}(Q_2))\end{aligned}$$

$$\oplus \mathrm{T}_{4i}(P_4 \oplus \mathrm{K}^3(Q_1) \oplus \mathrm{K}^4(Q_2)) \oplus \cdots$$
$$\oplus \mathrm{T}_{i-1i}(P_{i-1} \oplus \mathrm{K}^{i-1}(Q_1) \oplus \mathrm{K}^{i-1}(Q_2))$$
$$\oplus P_i \oplus \mathfrak{K}^i(Q_1) \oplus \mathrm{K}^i(Q_2) .$$

This is a case of $j=2$ in iii). We assume iii) is true for $k \leqq j$. Since $j+1<i$, we obtain from ii) and taking T_{j+1i}

$$\mathrm{T}_{j+1i}(K^{j+1}) \oplus \mathrm{T}_{j+1i}(Q_{j+1}) = \mathrm{T}_{j+1i}(P_{j+1} \oplus \mathrm{K}^{j+1}(Q_1) \oplus \mathrm{K}^{j+1}(Q_2) \oplus \cdots \oplus \mathrm{K}^{j+1}(Q_{j-1}) \oplus \mathrm{T}_{jj+1}(Q_j)) .$$

On the other hand,

$$\mathrm{T}_{j+1i}(Q_{j+1}) = \mathrm{T}_{j+2i}\mathrm{T}_{j+1j+2}(Q_{j+1}) \oplus \mathrm{T}_{j+3i}(\mathrm{K}^{j+3}(Q_{j+1})) \oplus \cdots \oplus \mathrm{T}_{i-1i}(\mathrm{K}^{i-1}(Q_{i+1})) \oplus \mathrm{K}^i(Q_{j+1})$$

Since Q_{j+1} is a coretract of $\boldsymbol{P}_{j+1}$ and $\mathrm{T}_{j+1i}(\boldsymbol{P}_{j+1})$ is a coretract of $\boldsymbol{P}_i$ by the following Lemma 3.8, we may regard the above objects on the both sides as sub objects in $\boldsymbol{P}_i$. Hence, we obtain

$$\begin{aligned}\boldsymbol{P}_i = {} & \mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{ji}(K^i) \oplus \mathrm{T}_{j+1i}(K^{j+1}) \\ & \oplus \mathrm{T}_{j+2i}(P_{j+2} \oplus \mathrm{K}^{j+2}(Q_1) \oplus \cdots \oplus \mathrm{K}^{j+2}(Q_j)) \oplus \mathrm{T}_{j+1j+2}(Q_{j+1})) \oplus \cdots \\ & \oplus \mathrm{T}_{i-1i}(P_i \oplus \mathrm{K}^{i-1}(Q_1) \oplus \cdots \oplus \mathrm{K}^{i-1}(Q_j)) \oplus \mathrm{K}^{i-1}(Q_{j+1})) \\ & \oplus P_i \oplus \mathrm{K}^i(Q_1) \oplus \cdots \oplus \mathrm{K}^i(Q_j) \oplus \mathrm{K}^i(Q_{j+1}) .\end{aligned}$$

Thus we obtain from i) and ii)

$$\begin{aligned}\boldsymbol{P}_i = {} & \mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{i-2i}(K^{i-2}) \oplus \mathrm{T}_{i-1i}(P_{i-1} \oplus \mathrm{K}^{i-1}(Q_1) \oplus \cdots \\ & \oplus \mathrm{K}^{i-1}(Q_{i-3}) \oplus \mathrm{T}_{i-2i-1}(Q_{i-2})) \oplus (P_i \oplus \mathrm{K}^i(Q_1)) \oplus \cdots \oplus \mathrm{K}^i(Q_{i-2})) \\ = {} & \{\mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{i-2i}(K^{i-2}) \oplus \mathrm{T}_{i-1i}(K^{i-1})\} \oplus \{P_i \oplus \mathrm{K}^i(Q_1) \oplus \cdots \\ & \oplus \mathrm{K}^i(Q_{i-2}) \oplus \mathrm{T}_{i-1i}(Q_{i-1})\} .\end{aligned}$$

Since $A_l \supset K^l$ and $A_i \supset \mathrm{T}_{1i}(A_1) \oplus \mathrm{T}_{2i}(K^2) \oplus \cdots \oplus \mathrm{T}_{i-1i}(K^{i-1}) = A'_i$, we obtain $A_i = A'_i \oplus K^i$ and Q_i in $\mathfrak{A}_i$ such that

$$K^i \oplus Q_i = P_i \oplus \mathrm{K}^i(Q_1) \oplus \cdots \oplus \mathrm{K}^i(Q_{i-2}) \oplus \mathrm{T}_{i-1i}(Q_{i-1}) ,$$

and hence, $K^i \oplus Q_i$ is a coretract of $\boldsymbol{P}_i$. Therefore, $A = \bigoplus_{i \geqslant 2} \mathrm{S}_i(K^i) \oplus \mathrm{S}_1(A_1)$. Since T_{ij} is projective preserving, each K^i is $\mathfrak{A}_i$-projective.

Lemma 3.8. *Let $\mathfrak{A}$ and $\mathfrak{A}_i$ and T_{ij} be as above. We assume that T_{ij} satisfies the condition (*). Then $T_{ij}(P_i)$ is a coretract of P_j for any projective object $P=(P_i)_{i \in I}$.*

Proof. We may assume $P = \bigoplus_{i \in I} \mathrm{S}_i(Q_i)$ by Lemma 3.3, where Q_i is $\mathfrak{A}_i$-projective. Then $P_i=\sum_{i=1}^{k-1}\oplus \mathrm{T}_{ki}(Q_k)\oplus Q_i$. We shall show under the assumption of Lemma 3.8 that $\mathrm{T}_{jl}\mathrm{T}_{ij}(P_i)\xrightarrow{\psi_{ijl}}\mathrm{T}_{il}(P_i)$ is a coretract. Let $t=l-i$. If $t=2$, then the fact is clear from (*). We assume it for $t<k$ and $k=l-i$. $\mathrm{T}_{jl}\mathrm{T}_{ij}(P_i)=\mathrm{T}_{jl}\mathrm{T}_{i+1j}\mathrm{T}_{ii+1}(R_i)\oplus\mathrm{T}_{jl}(\mathrm{T}_{i+2j}(\mathrm{K}^{i+2}(P_i))\oplus\cdots\oplus\mathrm{T}_{j-1j}(\mathrm{K}^{j-1}(P_i))\oplus\mathrm{K}^{j}(P_j))$ and

$$\begin{aligned}\mathrm{T}_{il}(P_i)=&\ \mathrm{T}_{i+1l}\mathrm{T}_{ii+l}(P_i)\oplus\mathrm{T}_{i+2l}(\mathrm{K}^{i+2}(P_i))\oplus\cdots\oplus\mathrm{T}_{jl}(\mathrm{K}^{j}(P_i))\\&+\mathrm{T}_{j+1l}(\mathrm{K}^{j+1}(P_i)\oplus\cdots\oplus\mathrm{K}^{l}(P_i)\,.\end{aligned}$$

Hence, we obtain ψ_{ijl} is a coretract from the assumption III, naturality of ψ and induction hypothesis. From those facts we can easily prove Lemma 3.8.

Lemma 3.9. *Let $\mathfrak{A}_i$ and $\mathfrak{A}$ be as above, and I'a subset of I. Then there exist functors M: $[I', \mathfrak{A}]\to[I, \mathfrak{A}]$, F: $[I, \mathfrak{A}]\to[I', \mathfrak{A}_i]$ such that $FM=I_{[I', \mathfrak{A}_i]}$, where F is the restriction functor.*

Proof. We may assume $I=I'\cup\{i\}$ by the induction. Let $I_1=\{j\mid\in I, j<i\}$ $I_2=\{j\mid\in I, j>i\}$ and $A=(A_j)_{j\in I'}$. If $I_1=\phi$, we put $A_1=0$. We assume $I_1=\phi$. We consider a family $D_i=\{\mathrm{T}_{ki}(A_k)$, $\mathrm{T}_{k'i}\mathrm{T}_{kk'}(A_k)\xrightarrow{\psi_{kk'i}}\mathrm{T}_{ki}(A_k)$ and $\mathrm{T}_{ki}\mathrm{T}_{kk'}(A_k)$ $\xrightarrow{\mathrm{T}_{ki}(d_{kk'})}\mathrm{T}_{ki}(A_k)$ for $k<k'<i\}$. Put A_i is a colimit of D_i. Then we have defined arrows D_{ki} and D_{il} for $k\in I_1$, $l\in I_2$ from (7). It is easily seen from the definition of colimit that those D_{ij} satisfy IV. Then $\mathrm{M}(A)=(A_k)_{k\in I}$ is a desired functor.

REMARK. We note that if $A=(A_k)$ is a coretract of $B=(B_k)_{k\in I'}$, then $\mathrm{M}(A)$ is a coretract of $\mathrm{M}(B)$, (cf. [5], p. 47, Coro. 2.10).

Proposition 3.10. *Let $\{\mathfrak{A}_i\}_{i\in I}$ be abelian categories with projective class $\mathcal{E}_i$ and $\mathfrak{A}(I)=[I, \mathfrak{A}_i]$. We assume T_{ij} is projective preserving. Then every projective object $P=(P_i)_{i\in I'}$ in $\mathfrak{A}(I')$ is of a form $\bigoplus_{i\in I'}S_i(Q_i)$ with Q_i projective in $\mathfrak{A}_i$ for any subset I' of I and $(P_j)_{j\in I''}$ is $\mathfrak{A}(I'')$-projective for any subset I'' of I' if and only if (*) is satisfied.*

Proof. "only if". Let P_i be projective in $\mathfrak{A}$. Then $\mathrm{S}_i(P_i)$ is $\mathfrak{A}$-projective, and hence, $\boldsymbol{P}'=(\mathrm{T}_{ii+1}(P_i), \cdots, \mathrm{T}_{in}(P_i))$ is $\mathfrak{A}(I\text{-}\{1, \cdots, i\}]$-projective. Therefore, the fact $\boldsymbol{P}'=\bigoplus_{k\geqslant i+1}\mathrm{S}_k(Q_k)$ from the assumption is equivalent to (*). "if". Let $\boldsymbol{P}'=(P'_k)_{k\in I'}$ be projective in $\mathfrak{A}(I')$. Then $\boldsymbol{P}'$ is a retract of $\bigoplus_{t\in I'}\bar{S}_t(P_t)$, where P_t is $\mathfrak{A}_t$-pojectrive and $\bar{\mathrm{S}}_t$ is functor: $\mathfrak{A}_t\to\mathfrak{A}(I')$ in (6). Let M be a functor in Lemma 3.9. Then $\mathrm{M}(\bigoplus_{t\in I'}\bar{S}_t(P_t))=\bigoplus_{i\in I'}\mathrm{S}_t(P_t)$ from the construction of M_t and $\mathrm{M}(P')$ is its retract from the above remark. Hence, $\mathrm{M}(P')$ is $\mathfrak{A}$-projective.

Therefore, $\mathrm{M}(P')=\bigoplus_{i\in I}\mathrm{S}_i(Q_i)$ with Q_i projective in $\mathfrak{A}_i$ by Lemma 3.7. Let $I'=\{i_1,\cdots,i_t\}$. We shall show $A_{i_k}=(\mathrm{T}_{i_k'i_k}(Q_{i_k'}))^t{}_{k=k'}=\sum_{k=k'}^{t}\oplus\bar{S}_{i_k}(P''_{i_{k'}})$, where $\mathrm{T}_{i_k'i_k'}=I_{\mathfrak{A}_{i_k'}}$ and P''_{i_k} is $\mathfrak{A}_{i_k}$-projective. We obtain from Lemma 3.7 that $\mathrm{T}_{i_k'i_{k-1}}(Q_{i_k'})=\mathrm{T}_{i_{t-1}i_k}\mathrm{T}_{i_k'i_{t-1}}(Q_{i_k'})\oplus P'_{i_k}$ and $\mathrm{T}_{i_k'i_{t-1}}(Q_{i_k'})=\mathrm{T}_{i_{t-2}i_{t-1}}\mathrm{T}_{i_k'i_{t-2}}(Q_{i_k'})\oplus P'_{i_{t-1}}$. Hence,

$$\begin{aligned}\mathrm{T}_{i_k'i_t}(Q_{i_k'})&=\mathrm{T}_{i_{t-1}i_t}\mathrm{T}_{i_{t-2}i_{t-1}}\mathrm{T}_{i_k'i_{t-2}}(Q_{i_k'})\oplus\mathrm{T}_{i_{t-1}i_t}(P'_{i_{t-1}})\oplus P'_{i_t}\\&=\mathrm{T}_{i_{t-2}i_t}\mathrm{T}_{i_k'i_{t-2}}(Q_{i_k'})\oplus\mathrm{T}_{i_{t-1}i_t}(P'_{i_{t-1}})\oplus P'_{i_t}\end{aligned}$$

from III. Repeating this argument we have $A_{i_k'}=\sum_{k=k'}^{t}\oplus\bar{S}_{i_k'}(P'_{i_k})$. Therefore, $\boldsymbol{P}=\sum_{k=1}^{t}\oplus A_{i_k'}=\bigoplus_{i_k'\in I'}\mathrm{S}_{i_k'}(P'''_{i_k})$. This completes the proof.

Proposition 3.11. *Let $\mathfrak{A}$ and $\mathfrak{A}_i$ be as above. We assume T_{ij} is projective preserving and satisfies (*), then for $D=(D_i)_{i\in I}$ in $\mathfrak{A}$*

$$hd\,D\leqq\max\,(hd\,D_i)+1\,.$$

Proof. Put $n=\max\,(hd\,D_i)$. Let $0\leftarrow D\leftarrow P_0\leftarrow\cdots\leftarrow P_{n-1}\xleftarrow{d_n}P_n$ be a projective resolution of D and $K_n=\ker d_n$. Since $n\geqq\mathrm{hd}\,D_i$, every component of im d_n is projective. Hence, K_n is $\mathfrak{A}$-projective by Lemma 3.7.

Corollary. *Let A_i, A and T_{ij} be as above. Then*

$$\mathrm{gl\,dim}\,\mathfrak{A}\geqq\mathrm{gl\,dim}\,\mathfrak{A}(I')$$

for any subset of I' and $\mathrm{gl\,dim}\,\mathfrak{A}\leqq\max\,(\mathrm{gl\,dim}\,\mathfrak{A}_i)+n-1$.

Proof. Let A be in $\mathfrak{A}(I')$ and $0\leftarrow\mathrm{M}(A)\leftarrow P_1\leftarrow P_2\leftarrow\cdots$ be a projective resolution of $\mathrm{M}(A)$ in $\mathfrak{A}$. Then $0\leftarrow A\leftarrow\mathrm{F}(P_1)\leftarrow\mathrm{F}(P_2)\leftarrow$ is a projective resolution of $\mathfrak{A}$ in $\mathfrak{A}(I')$ from Proposition 3.10.

We recall that $\mathfrak{A}$ is *semi-simple* if and only if every object of $\mathfrak{A}$ is projective.

Theorem 3.12. *Let $\mathfrak{A}_i$ be semi-simple abelian categories and I a linearly ordered finite set. Then $\mathfrak{A}=[I,\mathfrak{A}_i]$ with T_{ij} satisfying I$\sim$IV is hereditary if and only if*

$$\mathrm{T}_{ij}(M)=\mathrm{T}_{i+1j}\mathrm{T}_{ii+1}(M)\oplus\mathrm{T}_{i+2j}(\mathrm{K}^{i+2}(M))\oplus\cdots\oplus\mathrm{T}_{j-1j}(\mathrm{K}^{j-1}(M))\oplus\mathrm{K}^j(M)$$

for every object M in $\mathfrak{A}$ for all i, where $K^t(M)\in\mathfrak{A}_t$. Furthermore, $\mathrm{gl\,dim}\,\mathfrak{A}=1$ *if and only if there exists not a zero functor T_{ij}*, (cf. [2], Theorem 1).

Proof. The first half is clear from Lemmas 3.7 and 3.8 and Proposition 3.11. If T_{ij} is not a zero functor, then $\boldsymbol{A}=(A,0)$ is not projective in $\mathrm{A}(i,j)$ for any $\mathfrak{A}$ such that $\mathrm{T}_{ij}(\mathfrak{A})\neq0$ by Proposition 3.10. Hence, $\mathrm{gl\,dim}\,\mathfrak{A}\geqq\mathrm{gl\,dim}\,\mathfrak{A}(i,j)\geqq1$. If T_{ij} is a zero functor for all $i<j$, then $\mathfrak{A}=\sum\oplus\mathfrak{A}_i$. Hence, $\mathrm{gl\,dim}\,\mathfrak{A}=0$.

Let $\{R_i\}_{i\in I}$ be rings. Finally we assume that $\mathfrak{A}_i$ is the abelian category of right R_i-modules. By [5], p. 121., Propo. 1.5 we know $U=\bigoplus_i \mathrm{S}_i(R_i)$ is a small, projective generator in $\mathfrak{A}$. Put $R=[U, U]$. Let r, r' be elements in R_i and $\mathrm{T}_{ij}(R_i)$, respectively. By r_l, r'_l we denote morphisms in $[R_i, R_i]$ and $[R_j, \mathrm{T}_{ij}(R_i)]$ such that $r_l(x_i)=rx_i$ and $r'_l(x_j)=r'x_j$, respectively where $x_t\in R_t$. We can naturally regard $\mathrm{T}_{ij}(R_i)$ a left R_i-module by setting $\bar{r}y=\mathrm{T}_{ij}(r_l)y$ for any $r\in R_i$ and $y\in \mathrm{T}_{ij}(R_i)$. Furthermore, we define $\bar{r}'_l z=\psi_{ijk}\mathrm{T}_{ik}(r_l)$ for any $k>j$ and $z\in \mathrm{T}_{jk}(R_j)$, where we assume $\mathrm{T}_{ii}=\mathrm{I}_{\mathfrak{A}_i}$. Then we identify R with the set

$$\boldsymbol{R}=\left\{\begin{pmatrix} \mathrm{r}_1\mathrm{r}_{12}\cdots\cdots\cdots\mathrm{r}_{1n} \\ \quad\mathrm{r}_2\mathrm{r}_{22}\cdots\cdots\mathrm{r}_{2n} \\ \qquad\cdots\cdots\vdots \\ 0 \qquad\qquad \vdots \\ \qquad\qquad\quad \mathrm{r}_n \end{pmatrix},\quad \mathrm{r}_{ij}\in \mathrm{T}_{ij}(R_i),\ \mathrm{r}_i\in R_i\right\}.$$

Lemma 3.13. *$\bar{r}_{ij}\bar{r}_{jk}=\bar{r}_{ij}(r_{jk})$ and $\bar{r}_{ij}\bar{r}_j=\overline{r_{ij}r_j}$, $\bar{r}_i\bar{r}_{ij}=\overline{r_ir_{ij}}$.*

Proof. For any $k\geq j$ we have $\bar{r}_{ij}\bar{r}_j=\psi_{ijk}\mathrm{T}_{jk}((r_{ij})_l)\mathrm{T}_{jk}((r_j)_l)=\psi_{ijk}\mathrm{T}_{jk}(r'_{ij}r)_l)$ $=\overline{r_{ij}r}$, and

$$\begin{aligned}\bar{r}_i\bar{r}_{ij} &= \mathrm{T}_{ik}((r_i)_l)\psi_{ijk}\mathrm{T}_{jk}((r_{ij})_l) = \psi_{ijk}\mathrm{T}_{jk}\mathrm{T}_{ij}((r_i)_l)\mathrm{T}_{jk}((r_{ij})0_l) \quad \text{(naturality of } \psi)\\ &= \psi_{ijk}\mathrm{T}_{jk}(\mathrm{T}_{ij}((r_i)_l)(r_{ij})_l)\\ &= \psi_{ijk}\mathrm{T}_{jk}((r_ir_{ij})_l) \quad \text{(definition of } R_i \text{ module } \mathrm{T}_{ij}(R_i)\,.\\ &= \overline{r_ir_{ij}}\,.\end{aligned}$$

$$\begin{aligned}\bar{r}_{ij}\bar{r}_{jk} &= \psi_{ijt}\mathrm{T}_{jt}((r_{ij})_l)\psi_{ikl}\mathrm{T}_{kt}(r_{jk})_l)\\ &= \psi_{ijt}\psi_{jkl}\mathrm{T}_{kt}(\mathrm{T}_{jk}(r_{ij})_l)\mathrm{T}_{kt}((r_{jk})_l) \quad \text{(naturality of } \psi)\,.\end{aligned}$$

On the other hand we put

$$\begin{aligned} r_{ik} &= \bar{r}_{ij}(r_{jk}) = (\psi_{ijk}\mathrm{T}_{jk}(r_{ij})_l))(r_{jk})(r_{ik})_l \colon R_k \xrightarrow{(r_{jk})_l} \mathrm{T}_{jk}(R_j) \\ &\xrightarrow{\mathrm{T}_{jk}(r_{ij})} \mathrm{T}_{jk}\mathrm{T}_{ij}(R_i) \xrightarrow{\psi} \mathrm{T}_{ij}(R_i). \quad \text{Hence,}\\ \bar{r}_{ik} &= (\psi_{ijt}\mathrm{T}_{kt})(\psi_{ijk}\mathrm{T}_{jk}((r_{ij})_l)(r_{jk})_l)\,.\end{aligned}$$

Therefore, $\bar{r}_{ij}\bar{r}_{jk}=\overline{\bar{r}_{ij}(r_{jk})}$ by the assumption III.

If we define a multiplication on $\boldsymbol{R}$ by setting

$$(*)\qquad r_{ij}r_{jk} = \bar{r}_{ij}(r_{jk})$$

we have from [5], p. 104, Theorem 4.1 and p. 106, Theorem 5.1

Theorem 3.14. *Let $\mathfrak{G}^{R_i}$ be the abelian category of right R_i-module. Then $[I, \mathfrak{G}^{R_i}]$ is equivalent to the abelian category of a left R-module, where*

$$R = \begin{pmatrix} R_1 T_{12}(R_1) \cdots\cdots T_{1n}(R_1) \\ R_2 \cdots\cdots T_{2n}(R_2) \\ \ddots \quad \vdots \\ 0 \quad \ddots \quad \vdots \\ R_n \end{pmatrix} \quad \textit{with product } (^{**}) \,.$$

And $\mathrm{T}_{ij}(M_i) \approx M \otimes \mathrm{T}_{ij}(R_i)$ *for any* $M_i \in A_i$ (**) *is given by an* R_i-R_j *homomorphism* $\psi_{ik} \mathrm{T}_{ij}(R_i) \underset{R_j}{\otimes} \mathrm{T}_{jk}(R_j) \rightarrow \mathrm{T}_{ik}(P_i)$ (cf. [2], Theorem 1).

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