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BEHAVIOR OF 1-DIMENSIONAL REINFORCED RANDOM WALK

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1. Introduction

In this paper, we discuss recurrence of a 1-dimensional reinforced random walk. This walk was first introduced by Diaconis and was generalized by Davis [3]. Here, we follow Davis' formulation. The transition mechanism of this walk is as follows. First, at each edge $[j, j+1]$, we assign an initial weight $w(0, j)$. If the walk starts at a site $k \in \mathbf{Z}$, then after a unit of time it will jump to its nearest neighbor sites $k-1$, $k+1$ with probabilities;

$$P[X_1 = k+1 \mid X_0 = k] = \frac{w(0, k)}{w(0, k-1) + w(0, k)},$$
$$P[X_1 = k-1 \mid X_0 = k] = \frac{w(0, k-1)}{w(0, k-1) + w(0, k)}.$$

After the first jump, the weight of the edge just crossed by the walk is increased and weights of other edges are left unchanged. Let $\{w(1, j)\}_{j \in \mathbf{Z}}$ be the new weights of edges at time 1. Then we have $w(1, j) = w(0, j)$ if the edge $[j, j+1]$ is not crossed by the walk at the first jump. At each time the walk crosses an edge, the weight of this edge is increased. Thus, if $w(n, j)$ stands for the weight of an edge $[j, j+1]$ at time n, then transition probabilities are defined by

$$\begin{aligned} &P[X_{n+1} = j_n + 1 | (X_0, \dots, X_n) = (j_0, \dots, j_n)] \\ &= 1 - P[X_{n+1} = j_n - 1 | (X_0, \dots, X_n) = (j_0, \dots, j_n)] \\ &= \frac{w(n, j_n)}{w(n, j_n - 1) + w(n, j_n)}. \end{aligned}$$

We call this process a reinforced random walk.

We remark that the walk is the same as the one Diaconis introduced when each increase of weights $w(n, j)$ is equal to one for every $j \in \mathbf{Z}$ and initial weights are all equal to one. The precise definition of a reinforced random walk is given in Section 2.

Davis discussed in [3] recurrence of a reinforced random walk mainly under the condition that (i) the weight process $\{w(n, j)\}_{j \in \mathbf{Z}}$ is of sequence type, and (ii) initial weights are all equal to one. One of our aim of this paper is to extend his results to matrix type weight processes. This was possible by the help of Rubin's construction used in [8]. Another aim of this paper is related to Theorem 3.1 of [3]. It is shown that almost surely the walk is not transient if the initial weights are all equal to one regardless of the increase mechanism of the weights. We want to know under what condition on the initial weights we can observe transience of the walk. This is not easy to answer, and it seems that conditions on the initial weights and increase mechanism both contribute to the asymptotic behavior of the walk. Only in the case of the walk introduced by Diaconis with general initial weights, we can give necessary and sufficient conditions for recurrence and transience. To our surprise, these conditions are just the same as in non-reinforced case.

In Section 2, we give definitions and notations. In Section 3, we show that a path of a reinforced random walk with arbitrary initial condition is recurrent, transient or of finite range almost surely. Specially, if a path is not recurrent or transient, then the path eventually stays only one edge and goes back and forth there. In Section 4, we discuss recurrence of a Diaconis walk. We will give a recurrence criterion of a Diaconis walk in terms of initial weights. This is possible by means of Pemantle's representation of a Diaconis walk using a random walk in a random environment (see [7]). In the final section, we try to prove the law of large numbers; $X_n/n \to 0$ as $n \to \infty$ under a slightly more general condition than the condition that all initial weights are equal to one, which is the case Davis treated.

2. Definitions

Let $\overrightarrow{X} = \{X_n\}_{n\geq 0}$ be a sequence of integer valued random variables and $[w] = \{w(n, j)\}_{n\geq 0, j\in \mathbf{Z}}$ be a matrix of positive random variables all defined on some probability space $(\Omega, \mathcal{G}, P)$. Let $\mathcal{G}_n$ be the sub-σ-field $\sigma\{X_m, w(m, j) \mid 0 \leq m \leq n, j \in \mathbf{Z}\}$ for every $n \in \mathbf{N}$.

We define a reinforced random walk as a pair $(\overrightarrow{X}, [w])$ on the probability space $(\Omega, \mathcal{G}, P)$ satisfying the following two conditions.

(i) For all $n \geq 0$, $j \in \mathbf{Z}$,

$$w(n+1, j) - w(n, j) \geq 0 \quad \text{a.s.} \tag{2.1}$$

with equality if (X_n, X_{n+1}) is not either $(j, j+1)$ or $(j+1, j)$.

(ii) For all $n \geq 0$, $j \in \mathbf{Z}$,

$$\begin{aligned} P[X_{n+1} = j+1 \mid X_n = j, \mathcal{G}_n] &= \frac{w(n, j)}{w(n, j-1) + w(n, j)} \quad \text{a.s.,} \\ P[X_{n+1} = j-1 \mid X_n = j, \mathcal{G}_n] &= \frac{w(n, j-1)}{w(n, j-1) + w(n, j)} \quad \text{a.s.} \end{aligned} \tag{2.2}$$

In the present paper, we abbreviate reinforced random walk to RRW.

We say that the walk $\overrightarrow{X}$ crosses $[j, j+1]$ in a time interval $[n, n+1]$, if $X_n = j, X_{n+1} = j+1$ or $X_n = j+1, X_{n+1} = j$. For $l \geq 0$ and $j \in \mathbf{Z}$, define $v(l, j)$ as $w(n, j)$ if one can find smallest integer n satisfying that $(X_0, \dots, X_n)$ crosses the edge $[j, j+1]$ l times. We say that the walk $\overrightarrow{X}$ is of matrix type if there is a matrix $A = \{a(m, j)\}_{m\geq 1, j\in\mathbf{Z}}$ of non-negative numbers such that

$$v(l, j) = w(0, j) + \sum_{m=1}^{l} a(m, j) \quad \text{a.s.} \tag{2.3}$$

for every $l \geq 0$, $j \in \mathbf{Z}$. We call A a reinforcing matrix. We say that the walk $\overrightarrow{X}$ is of sequence type if there is a sequance $\vec{a} = \{a(m)\}_{m\geq 1}$ of non-negative numbers such that

$$v(l, j) = w(0, j) + \sum_{m=1}^{l} a(m) \quad \text{a.s.}$$

for every $l \geq 0$, $j \in \mathbf{Z}$. We call $\vec{a}$ a reinforcing sequence. In particular, if $a(m) = 1$ for every $m \geq 1$, we call the walk $\overrightarrow{X}$ a Diaconis walk.

Let $\Phi : (0, \infty)^{\mathbf{N}\cup\{0\}} \to (0, \infty]$ be given by

$$\Phi(\vec{\alpha}) = \sum_{k=0}^{\infty} \alpha(k)^{-1} \tag{2.4}$$

for every infinite dimensional positive vector $\vec{\alpha} = \{\alpha(k)\}_{k=0}^{\infty}$. This function plays an important role in this paper. We write column vectors $\vec{\alpha}_j = \{\alpha(n, j)\}_{n\geq 0}$, $\vec{\alpha}_{j,e} = \{\alpha(2n, j)\}_{n\geq 0}$, $\vec{\alpha}_{j,o} = \{\alpha(2n+1, j)\}_{n\geq 0}$ and initial weights vectors $\vec{w}_+ = \{w(0, j)\}_{j\geq 0}$, $\vec{w}_- = \{w(0, -j-1)\}_{j\geq 0}$. Let $(\overrightarrow{X}, [w])$ be a RRW. Define $\tau(j, l)$ as the l-th hitting time at j, that is,

$$\begin{cases} \tau(j, 1) = \inf\{n \geq 0 \mid X_n = j\}, \\ \tau(j, l+1) = \inf\{n > \tau(j, l) \mid X_n = j\} \quad \text{for } l \in \mathbf{N}. \end{cases}$$

For simplicity, we put $\tau_j = \tau(j, 1)$.

Throughout this paper, we understand that if $a > b$, $\sum_{n=a}^{b} \alpha(n) = 0$ and $\prod_{n=a}^{b} \alpha(n) = 1$ for every sequence $\{\alpha(n)\}_{n\geq 0}$.

3. Recurrence of a RRW in general

In this section, we discuss asymptotic behavior of X_n as $n \to \infty$, in particular, recurrence of a matrix type $\overrightarrow{X}$. We call the path $\overrightarrow{X}$ recurrent if for every $j \in \mathbf{Z}$, X_n visits j infinitely often. We call the path $\overrightarrow{X}$ transient if for every $j \in \mathbf{Z}$, X_n visits j only finitely many times. If there exist $\alpha < \beta$ such that $\alpha \leq X_n \leq \beta$ for all n, then we say that the path $\overrightarrow{X}$ has finite range. We want to know under what condition we can tell recurrence, transience or finiteness of the range of our RRW. Using the function Φ and column vectors $\vec{v}_j = \{v(l, j)\}_{l \geq 0}$, defined by (2.4) and (2.3), Sellke ([8]; Theorem 6) proved the next theorem.

Theorem 3.1 (Sellke). *Let $(\vec{X}, [w])$ be a matrix type RRW with a matrix A. If $\Phi(\vec{v}_j) = \infty$ for all $j \in \mathbf{Z}$, then*

$$P[\vec{X} \text{ is transient}] + P[\vec{X} \text{ is recurrent}] = 1.$$

In particular, this means that $P[\vec{X}$ has finite range$] = 0$.

Thus naturally we ask the question: what is the asymptotic behavior of RRW when $\Phi(\vec{v}_i) < \infty$ for some $i \in \mathbf{Z}$. The following theorem is our answer to this question.

Theorem 3.2. *Let $(\vec{X}, [w])$ be a matrix type RRW with a matrix A. If $\Phi(\vec{v}_i) < \infty$ for some $i \in \mathbf{Z}$, then*

$$P[\vec{X} \text{ is transient}] + P[\vec{X} \text{ has finite range}] = 1.$$

The above theorems tell us that the possible asymptotic behavior of the RRW is recurrent, transient or of finite range, that is, we do not have such situation that there exists a point such that our RRW visits each point which is to the right of this point infinitely often, and visits each point which is to the left of this point only finitely often.

Corollary 3.3. *Let $(\vec{X}, [w])$ be a matrix type RRW. Then*

$$P[\vec{X} \text{ is not recurrent, transient nor of finite range}] = 0.$$

For the proof of Theorem 3.2, we need Rubin's theorem. To state Rubin's theorem, we need some more notation. Let $\vec{r} = \{r(n)\}_{n \geq 0}$ and $\vec{l} = \{l(n)\}_{n \geq 0}$ be two increasing sequences such that $r(0) > 0$ and $l(0) > 0$. Let $\vec{S} = \{S(n)\}_{n \geq 0}$ be a sequence of random variables, each of which takes only two values u and d. The transition rule of $\vec{S}$ is given by

$$(3.1) \qquad \begin{aligned} P[S(n+1) = u \mid \Lambda_{m,n}] &= \frac{r(m)}{r(m) + l(n-m)}, \\ P[S(n+1) = d \mid \Lambda_{m,n}] &= \frac{l(n-m)}{r(m) + l(n-m)} \end{aligned}$$

for every $n \geq 0$ and $m \geq 0$, where $\Lambda_{m,n} = \{\omega \in \Omega \mid \#\{1 \leq l \leq n \mid S(l) = u\} = m\}$. We define the following three events:

$R = \{$there exists some number M such that $S(n) = u$ for all $n > M\}$,
$L = \{$there exists some number M such that $S(n) = d$ for all $n > M\}$,
$I = \Omega \setminus \{R \cup L\}$.

Proposition 3.4 (Rubin).

1. *If $\Phi(\vec{r}) < \infty$ and $\Phi(\vec{l}) < \infty$, then we have that $P[R] > 0$, $P[L] > 0$, $P[I] = 0$ and $P[R] + P[L] = 1$.*
2. *If $\Phi(\vec{r}) < \infty$ and $\Phi(\vec{l}) = \infty$, then we have that $P[R] = 1$, $P[L] = 0$ and $P[I] = 0$.*

2′. *If $\Phi(\vec{r}) = \infty$ and $\Phi(\vec{l}) < \infty$, then we have that $P[R] = 0$, $P[L] = 1$ and $P[I] = 0$.*

3. *If $\Phi(\vec{r}) = \infty$ and $\Phi(\vec{l}) = \infty$, then we have that $P[R] = 0$, $P[L] = 0$ and $P[I] = 1$.*

The proof of this proposition is given in [3]; 227–228.

Now, we construct a sequence $\vec{S}_j = \{S_j(n)\}_{n\geq 1}$ from $\vec{X}$ for every $j \in \mathbf{Z}$ in the folloing way. If $X_{\tau(j,l)+1} = j + 1$, then we set $S_j(l) = u$ and otherwise $S_j(l) = d$. It is easy to see that the law of $\vec{S}_j$ is given by (3.1) with $S(n), r(m), l(n-m)$ replaced by $S_j(n), r_j(m), l_j(n-m)$ for some $\vec{r}_j = \{r_j(m)\}_{m\geq 0}$ and $\vec{l}_j = \{l_j(m)\}_{m\geq 0}$. In fact, $\vec{r}_j$ and $\vec{l}_j$ are explicitly given as in the following way.

1. If $j > X_0$, then we can adopt $\vec{r}_j$ and $\vec{l}_j$ as $\vec{v}_{j,e}$ and $\vec{v}_{j-1,o}$ respectively.
2. If $j = X_0$, then we can adopt $\vec{r}_j$ and $\vec{l}_j$ as $\vec{v}_{j,e}$ and $\vec{v}_{j-1,e}$ respectively.
3. If $j < X_0$, then we can adopt $\vec{r}_j$ and $\vec{l}_j$ as $\vec{v}_{j,o}$ and $\vec{v}_{j-1,e}$ respectively.

Let R_j, L_j, I_j be events defined by
$R_j = \{$there exists $N_j \in \mathbf{N}$ such that if $X_n = j$, then $X_{n+1} = j+1$ for all $n \geq N_j\}$,
$L_j = \{$there exists $N_j \in \mathbf{N}$ such that if $X_n = j$, then $X_{n+1} = j-1$ for all $n \geq N_j\}$,
$I_j = \Omega \setminus \{R_j \cup L_j\}$.
For each $j \in \mathbf{Z}$, the event R_j (resp. L_j, I_j) corresponds to the event that $S(n)$ is replaced with $S_j(n)$ in the definition of R (resp. L, I). Then, by Proposition 3.4, we have

Corollary 3.5.

1. *If $\Phi(\vec{v}_j) < \infty$ and $\Phi(\vec{v}_{j-1}) < \infty$, then we have that $P[R_j] > 0$, $P[L_j] > 0$, $P[I_j] = 0$ and $P[R_j] + P[L_j] = 1$.*
2. *If $\Phi(\vec{v}_j) < \infty$ and $\Phi(\vec{v}_{j-1}) = \infty$, then we have that $P[R_j] = 1$, $P[L_j] = 0$ and $P[I_j] = 0$.*

2′. *If $\Phi(\vec{v}_j) = \infty$ and $\Phi(\vec{v}_{j-1}) < \infty$, then we have that $P[R_j] = 0$, $P[L_j] = 1$ and $P[I_j] = 0$.*

3. *If $\Phi(\vec{v}_j) = \infty$ and $\Phi(\vec{v}_{j-1}) = \infty$, then we have that $P[R_j] = 0$, $P[L_j] = 0$ and $P[I_j] = 1$.*

In particular, we have

(3.2) *If $P[R_j] > 0$, then at least $\Phi(\vec{v}_j)$ is finite.*

(3.3) *If* $P[L_j] > 0$, *then at least* $\Phi(\vec{v}_{j-1})$ *is finite.*

(3.4) *If* $P[I_j] > 0$, *then both* $\Phi(\vec{v}_{j-1})$ *and* $\Phi(\vec{v}_j)$ *are infinite.*

As a consequence, we obtain

$$P[R_j \cap I_{j+1}] = 0, \tag{3.5}$$

$$P[I_{j-1} \cap L_j] = 0 \tag{3.6}$$

for every $j \in \mathbf{Z}$.

Proof of Corollary 3.5. The statements 1, 2, 2′, 3 are direct consequences of Proposition 3.4. We only have to note that if one of $\Phi(\vec{v}_{j,o})$, $\Phi(\vec{v}_{j,e})$, $\Phi(\vec{v}_j)$ is finite, then all of them are finite, because a sequence $\vec{v}_j$ is increasing. (3.2), (3.3) and (3.4) are then obvious. It remains to prove (3.5) and (3.6). We show $P[R_j]P[I_{j+1}] = 0$, which is sufficient to show (3.5). Assume that $P[R_j] > 0$. Then by (3.2), we have $\Phi(\vec{v}_j) < \infty$. This, together with (3.4), implies that $P[I_{j+1}] = 0$. In the same way, we can prove that $P[I_{j-1}]P[L_j] = 0$. □

Before going into the proof of Theorem 3.2, we introduce a new notation. Let F and G be subsets of Ω. We write by $F \sqsubset G$ if $P[F \setminus G] = 0$.

Proof of Theorem 3.2. We may show that

$$P[\{\overrightarrow{X} \text{ is not transient}\} \cap \{\overrightarrow{X} \text{ has finite range}\}^c] = 0. \tag{3.7}$$

If $P[\overrightarrow{X}$ is not transient$] = 0$, then (3.7) is trivially true, and thus we may concentrate on the case that $P[\overrightarrow{X}$ is not transient$] > 0$. Then by definition, there exists a $j \in \mathbf{Z}$ such that $P[\overrightarrow{X}$ visits j infinitely often$] > 0$. Take one of such j arbitrarily and fix it. Let $B_j = \{\overrightarrow{X}$ visits j infinitely often$\}$. It suffices to show that

$$B_j \sqsubset \{\overrightarrow{X} \text{ has finite range}\}.$$

We divide the set B_j into three parts, $B_j \cap R_j$, $B_j \cap L_j$ and $B_j \cap I_j$. Since $P[B_j] > 0$, at least one of them has positive probability.

We first show that

$$B_j \cap R_j \sqsubset \{\overrightarrow{X} \text{ has finite range}\}. \tag{3.8}$$

It is not difficult to see that on the set $B_j \cap R_j$, $\overrightarrow{X}$ visits $j+1$ infinitely often, i.e.,

$$B_j \cap R_j \subset B_{j+1}.$$

Furthermore, on the set $B_{j+1} \cap R_{j+1}$, $\overrightarrow{X}$ can not visit j infinitely often, i.e.,

$$B_{j+1} \cap R_{j+1} \subset B_j^c.$$

By Corollary 3.5, this means that

$$B_j \cap R_j \sqsubset B_{j+1} \cap (L_{j+1} \cup I_{j+1}).$$

But we have $P[B_j \cap R_j \cap B_{j+1} \cap I_{j+1}] = 0$ by (3.5). Therefore we have

$$B_j \cap R_j \sqsubset B_{j+1} \cap L_{j+1}.$$

Thus on the set $B_j \cap R_j \cap B_{j+1} \cap L_{j+1}$, $\overrightarrow{X}$ eventually sticks to the edge $\{j, j+1\}$, and goes back and forth, so $\overrightarrow{X}$ has finite range. In the same way, we obtain that

$$B_j \cap L_j \sqsubset \{\overrightarrow{X} \text{ has finite range}\} \tag{3.9}$$

by using (3.6).

We will next show that

$$P[B_j \cap I_j] = 0 \tag{3.10}$$

under the condition that there is some i with $\Phi(\vec{v}_i) < \infty$. Let us assume that there is such an i to the right of j. Let j_1 be the smallest $i > j$ with $\Phi(\vec{v}_i) < \infty$. In this case, we have $\Phi(\vec{v}_{j_1-1}) = \infty$, $\Phi(\vec{v}_{j_1}) < \infty$. By Corollary 3.5, we remark that

$$P[R_{j_1}] = 1. \tag{3.11}$$

We only have to consider the case that $P[I_j] > 0$ since (3.10) is trivially true if not satisfies. In this case, by Corollary 3.5, we note that $\Phi(\vec{v}_{j-1}) = \Phi(\vec{v}_j) = \infty$. It is easy to see that

$$B_j \cap I_j \subset B_{j+1}.$$

If $j+1 < j_1$, then $\Phi(\vec{v}_j) = \Phi(\vec{v}_{j+1}) = \infty$ and therefore $P[I_{j+1}] = 1$. Thus we have that

$$B_j \cap I_j \sqsubset B_{j+1} \cap I_{j+1}.$$

By induction and (3.11), we obtain that

$$B_j \cap I_j \sqsubset \cdots \sqsubset B_{j_1-1} \cap I_{j_1-1} \sqsubset B_{j_1} \cap R_{j_1}. \tag{3.12}$$

But on $B_{j_1} \cap R_{j_1}$, (3.11) tells us that almost surely $\overrightarrow{X}$ can not visit $j_1 - 1$ infinitely often. Thus we have

$$B_{j_1-1} \cap I_{j_1-1} \sqsubset B_{j_1} \cap R_{j_1} \sqsubset B_{j_1-1}^c.$$

Therefore $P[B_{j_1-1} \cap I_{j_1-1}] = 0$. This combined with (3.12) means that $P[B_j \cap I_j] = 0$.

If there is no i with $\Phi(\vec{v}_i) < \infty$ to the right of j, then there is one to the left of j by the assumption of Theorem 3.2, and we can argue in a similar way to obtain that $P[B_j \cap I_j] = 0$. □

By this proof, we have proved a stronger statement than that in Theorem 3.2. Namely we have the following corollary.

Corollary 3.6. *Let* $(\overrightarrow{X}, [w])$ *be a matrix type RRW. If* $P[\overrightarrow{X}$ *has finite range*$] = 1$, *then there are random integers* N *and* j *such that* $X_n \in \{j, j+1\}$ *for every* $n > N$.

4. Recurrence of a Diaconis Walk

In the present section, we consider recurrence of a Diaconis walk. We obtain the following result.

Theorem 4.1. *Let* $(\overrightarrow{X}, [w])$ *be a Diaconis walk.*

1. *If* $\Phi(\vec{w}_+) < \infty$ *or* $\Phi(\vec{w}_-) < \infty$, *then*

$$P[\overrightarrow{X} \text{ is transient}] = 1.$$

2. *If* $\Phi(\vec{w}_+) = \Phi(\vec{w}_-) = \infty$, *then*

$$P[\overrightarrow{X} \text{ is recurrent}] = 1.$$

The remainder of this section is devoted to the proof of Theorem 4.1.

We can assume without loss of generality that $X_0 = 0$. It suffices to prove the following statements.

[A] If $\Phi(\vec{w}_+) < \infty$, then $P[\tau_j = \infty] > 0$ for every $j < 0$.

[A′] If $\Phi(\vec{w}_-) < \infty$, then $P[\tau_j = \infty] > 0$ for every $j > 0$.

[B] If $\Phi(\vec{w}_+) = \infty$, then $P[\tau_j < \infty] = 1$ for every $j < 0$.

[B′] If $\Phi(\vec{w}_-) = \infty$, then $P[\tau_j < \infty] = 1$ for every $j > 0$.

We prepare a sequence of independent random variables $\vec{\theta} = \{\theta_j\}_{j \in \mathbf{Z}}$ in the following way.

If $j > 0$, then θ_j has the beta distribution with parameters $(w(0, j)/2, (w(0, j-1)+1)/2)$.

If $j = 0$, then θ_j has the beta distribution with parameters $(w(0, j)/2, w(0, j-1)/2)$.

If $j < 0$, then θ_j has the beta distribution with parameters $((w(0, j)+1)/2, w(0, j-1)/2)$.

Let Q_j be the distribution of θ_j given as above and let $Q = \otimes_{j=-\infty}^{\infty} Q_j$. We construct the Markov chain $\overrightarrow{Z} = \{Z_n(\vec{\theta})\}_{n \geq 0}$ started at $Z_0 = 0$, with transition probability $P[Z_{n+1}(\vec{\theta}) = j+1 \mid Z_n(\vec{\theta}) = j] = \theta_j$ for every $j \in \mathbf{Z}$. In this case, we can check easily that for every $n \geq 0$ and $(y_1, \dots, y_n) \in \mathbf{Z}^n$,

$$P[(X_1, \dots, X_n) = (y_1, \dots, y_n) \mid X_0 = 0] = \int P[(Z_1(\vec{\theta}), \dots, Z_n(\vec{\theta})) = (y_1, \dots, y_n) \mid Z_0(\vec{\theta}) = 0] dQ(\vec{\theta}).$$

(See [5], [7]; 1231–1233 and [3]; 226–228.)

If the environment is non-random, then by the classical difference equation method, we obtain the following result. (See [1]; 65–71.)

Lemma 4.2. *Let $\vec{\theta} = \{\theta_j\}_{j \in Z}$ be a fixed environment and $\vec{Z}$ be the random walk whose transition probability is defined by $\theta_j = P[Z_{n+1} = j+1 \mid Z_n = j]$. We define $f_j = P[Z_n = j$ for some $n \geq 1 \mid Z_0 = 0]$ for all $j \neq 0$. Let $Y_j = (1-\theta_j)/\theta_j$ and we put*

$$\Theta_+ = \sum_{j=1}^{\infty} \prod_{i=1}^{j} Y_i, \quad \Theta_- = \sum_{j=1}^{\infty} \prod_{i=1}^{j} (Y_{-i})^{-1}.$$

1. *If $j > 0$, then we obtain that $f_j < 1$ for $\Theta_- < \infty$ and that $f_j = 1$ for $\Theta_- = \infty$.*
2. *If $j < 0$, then we obtain that $f_j < 1$ for $\Theta_+ < \infty$ and that $f_j = 1$ for $\Theta_+ = \infty$.*

Specially we remark that for all $j \neq 0$,

$$P[\tau_j < \infty \mid X_0 = 0] = \int P[Z_n = j \text{ for some } n \geq 1 \mid Z_0 = 0] dQ.$$

We rewrite Θ_+ and Θ_- as follows:

$$\Theta_+ = \sum_{j=1}^{\infty} \exp\left(\sum_{i=1}^{j} \ln[Y_i]\right), \quad \Theta_- = \sum_{j=1}^{\infty} \exp\left(-\sum_{i=1}^{j} \ln[Y_{-i}]\right),$$

and let $S_j = \sum_{i=1}^{j} \ln[Y_i]$. We first caluculate the expectation and the variance of $\ln[(1-\theta)/\theta]$ when θ has the beta distribution with parameters (a, b).

$$\begin{aligned}
&\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} E\left[\ln\left[\frac{1-\theta}{\theta}\right]\right] \\
&= \int_0^1 \ln[1-x] x^{a-1}(1-x)^{b-1} dx - \int_0^1 \ln[x] x^{a-1}(1-x)^{b-1} dx \\
&= \int_0^1 \frac{\partial}{\partial b}\{x^{a-1}(1-x)^{b-1}\} dx - \int_0^1 \frac{\partial}{\partial a}\{x^{a-1}(1-x)^{b-1}\} dx \\
&= \frac{\partial}{\partial b} \int_0^1 x^{a-1}(1-x)^{b-1} dx - \frac{\partial}{\partial a} \int_0^1 x^{a-1}(1-x)^{b-1} dx \\
&= \frac{\partial}{\partial b}\left\{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right\} - \frac{\partial}{\partial a}\left\{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right\}
\end{aligned}$$

$$= \frac{\Gamma(a)\Gamma'(b)}{\Gamma(a+b)} - \frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}.$$

Thus

$$E\left[\ln\left[\frac{1-\theta}{\theta}\right]\right] = \frac{\Gamma'(b)}{\Gamma(b)} - \frac{\Gamma'(a)}{\Gamma(a)} = \psi(b) - \psi(a), \tag{4.1}$$

where ψ is the polygamma function defined by $\psi(z) = \Gamma'(z)/\Gamma(z)$. In the similar way,

$$\begin{aligned} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} E\left[\left\{\ln\left[\frac{1-\theta}{\theta}\right]\right\}^2\right] &= \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial a}\right)^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\ &= \frac{\Gamma(a)\Gamma''(b) - 2\Gamma'(a)\Gamma'(b) + \Gamma''(a)\Gamma(b)}{\Gamma(a+b)}. \end{aligned} \tag{4.2}$$

We remark

$$\frac{d\psi(z)}{dz} = \frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{\Gamma''(z)}{\Gamma(z)} - \left(\frac{\Gamma'(z)}{\Gamma(z)}\right)^2, \text{ i.e., } \frac{\Gamma''(z)}{\Gamma(z)} = \psi'(z) + \psi^2(z). \tag{4.3}$$

By (4.2) and (4.3),

$$\begin{aligned} E\left[\left\{\ln\left[\frac{1-\theta}{\theta}\right]\right\}^2\right] &= \frac{\Gamma''(b)}{\Gamma(b)} - 2\frac{\Gamma'(a)}{\Gamma(a)}\frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma''(a)}{\Gamma(a)} \\ &= \psi'(b) + \psi'(a) + (\psi(b) - \psi(a))^2. \end{aligned} \tag{4.4}$$

From (4.1) and (4.4), we consequently obtain that

$$V\left[\ln\left[\frac{1-\theta}{\theta}\right]\right] = E\left[\left\{\ln\left[\frac{1-\theta}{\theta}\right]\right\}^2\right] - \left\{E\left[\ln\left[\frac{1-\theta}{\theta}\right]\right]\right\}^2 = \psi'(b) + \psi'(a). \tag{4.5}$$

By the Weierstrass formula, we have that for every $z > 0$,

$$\psi(z) = -\gamma - \sum_{l=0}^{\infty}\left\{\frac{1}{z+l} - \frac{1}{l+1}\right\}, \tag{4.6}$$

$$\psi'(z) = \sum_{l=0}^{\infty}(z+l)^{-2}, \tag{4.7}$$

$$\psi''(z) = -2\sum_{l=0}^{\infty}(z+l)^{-3},$$

where γ is the Euler constant. Specially we remark that $\psi(z)$ is an increasing and concave function of $z > 0$. Furthermore we need two facts for every $z > 0$:

$$2\left(\psi\left(z+\frac{1}{2}\right) - \psi(z)\right) \geq \psi(z+1) - \psi(z), \tag{4.8}$$

$$\psi(z+1) - \psi(z) = z^{-1}. \tag{4.9}$$

From (4.6), (4.7), (4.8) and (4.9), we have the following estimates for all $z > 0$ and $y > 0$:

$$(2z)^{-1} \le \psi\left(z + \frac{1}{2}\right) - \psi(z) \le z^{-1}, \tag{4.10}$$

$$\ln[y] - \ln[z] - y^{-1} \le \quad \psi(y) - \psi(z) \quad \le \ln[y] - \ln[z] + z^{-1}, \tag{4.11}$$

$$z^{-1} \le \quad \psi'(z) \quad \le z^{-1} + z^{-2}. \tag{4.12}$$

Now, we show that if $\Phi(\vec{w}_+) < \infty$, then $\Theta_+ < \infty$ almost surely. Note that $V[S_j]$ converges as j tends to infinitely when $\Phi(\vec{w}_+) < \infty$. In fact, by (4.5) and (4.12), we have that for every $j \in \mathbf{N}$,

$$\begin{aligned} V[S_j] &= \sum_{i=1}^{j} \left\{ \psi'\left(\frac{w(0,i-1)+1}{2}\right) + \psi'\left(\frac{w(0,i)}{2}\right) \right\} \\ &\le \sum_{i=1}^{j} \left\{ \frac{2}{w(0,i-1)+1} + \frac{2}{w(0,i)} + \frac{4}{(w(0,i-1)+1)^2} + \frac{4}{w(0,i)^2} \right\} \\ &\le 4\Phi(\vec{w}_+) + 8(\Phi(\vec{w}_+))^2 + \frac{2}{w(0,0)+1} + \frac{4}{(w(0,0)+1)^2} < \infty. \end{aligned} \tag{4.13}$$

Therefore since $\{\theta_j\}_{j \in \mathbf{Z}}$ are independent, $S_n - E[S_n]$ converges almost surely by the three series theorem. That is, almost surely we can find a constant C_1 depending on the environment such that

$$S_j - E[S_j] \le C_1 \tag{4.14}$$

for all j. In order to derive an upper estimate of Θ_+, we need to get an upper bound for $E[S_j]$.

By (4.1), (4.10) and (4.11), we have

$$\begin{aligned} E[S_j] &= \sum_{i=1}^{j} \left\{ \psi\left(\frac{w(0,i-1)+1}{2}\right) - \psi\left(\frac{w(0,i)}{2}\right) \right\} \\ &= \psi\left(\frac{w(0,0)}{2}\right) - \psi\left(\frac{w(0,j)}{2}\right) \\ &\quad + \sum_{i=0}^{j-1} \left\{ \psi\left(\frac{w(0,i)+1}{2}\right) - \psi\left(\frac{w(0,i)}{2}\right) \right\} \\ &\le \ln\left[\frac{w(0,0)}{2}\right] - \ln\left[\frac{w(0,j)}{2}\right] + \frac{2}{w(0,j)} + \sum_{i=0}^{j-1} \frac{2}{w(0,i)} \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
&\le \ln[w(0,0)] - \ln[w(0,j)] + \frac{2}{w(0,0)} + 2\Phi(\vec{w}_+) \\
&= -\ln[w(0,j)] + C_2.
\end{aligned}
$$

From (4.14) and (4.15), we obtain that

$$
S_j \le -\ln[w(0,j)] + C_2 + C_1 \quad \text{a.s.}
$$

for all $j \ge 1$. Then we have that

$$
\begin{aligned}
\Theta_+ &= \sum_{j=1}^{\infty} \exp(S_j) \\
&\le \sum_{j=1}^{\infty} \exp\Big(-\ln[w(0,j)] + C_2 + C_1\Big) \\
&= \exp(C_1 + C_2) \sum_{j=1}^{\infty} w(0,j)^{-1} \quad \text{a.s.}
\end{aligned}
$$

Thus we have shown that if $\Phi(\vec{w}_+) < \infty$, then $\Theta_+ < \infty$ almost surely. Hence by Lemma 4.2, $P[Z_n = j \text{ for some } n] < 1$ for every $j < 0$ for almost all environments. This proves the statement [A].

Next, we show that if $\Phi(\vec{w}_+) = \infty$, then $\Theta_+ = \infty$ almost surely under a condition that we can take a constant $C_3 > 0$ such that $w(0,j) \ge C_3$ for all $j \ge 0$. Note that $V[S_j]$ diverges as j tends to infinitely when $\Phi(\vec{w}_+) = \infty$. In fact, by (4.5) and (4.12), we have that for every $j \in \mathbf{N}$,

$$
\begin{aligned}
V[S_j] &= \sum_{i=1}^{j} \left\{ \psi'\left(\frac{w(0,i-1)+1}{2}\right) + \psi'\left(\frac{w(0,i)}{2}\right) \right\} \\
&\ge \sum_{i=1}^{j} \left\{ \frac{2}{w(0,i-1)+1} + \frac{2}{w(0,i)} \right\} \\
&\ge \sum_{i=1}^{j} \frac{2}{w(0,i)}.
\end{aligned}
$$

Letting $j \to \infty$, we obtain

$$
\sum_{i=1}^{\infty} V\left[\ln\left[\frac{1-\theta_i}{\theta_i}\right]\right] \ge \sum_{i=1}^{\infty} \frac{2}{w(0,i)} \ge 2\Phi(\vec{w}_+) = \infty.
$$

We quote next form of the law of large numbers. (See [6]; 186–188.)

Lemma 4.3 (Kolmogorov). *Let* $\overrightarrow{X} = \{X_n\}_{n \ge 1}$ *be independent random variables* *and we put* $T_n = \sum_{m=1}^{n} X_m$. *If* $V[X_n] < \infty$ *for every* $n \in N$ *and* $V[T_n]$ *diverges as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \frac{T_n - E[T_n]}{V[T_n]} = 0 \quad a.s.$$

Appling this lemma, we immediately obtain that for all $\epsilon > 0$, there exists an integer N such that for all $j \geq N$,

$$S_j \geq E[S_j] - \epsilon V[S_j] \tag{4.16}$$

almost surely. In order to derive a lower estimate of Θ_+, we need to get a lower bound for $E[S_j]$ and an upper bound for $V[S_j]$.

From the assumption, we remark that $w(0,j)+1 \leq w(0,j)(C_3+1)/C_3$. By (4.1), (4.10) and (4.11), we have

$$\begin{aligned} E[S_j] &= \sum_{i=1}^{j} \left\{ \psi\left(\frac{w(0,i-1)+1}{2}\right) - \psi\left(\frac{w(0,i)}{2}\right) \right\} \\ &= \psi\left(\frac{w(0,0)}{2}\right) - \psi\left(\frac{w(0,j)+1}{2}\right) \\ &\quad + \sum_{i=0}^{j} \left\{ \psi\left(\frac{w(0,i)+1}{2}\right) - \psi\left(\frac{w(0,i)}{2}\right) \right\} \\ &\geq \ln\left[\frac{w(0,0)}{2}\right] - \ln\left[\frac{w(0,j)+1}{2}\right] - \frac{2}{w(0,0)} + \sum_{i=0}^{j} \frac{1}{w(0,i)} \\ &\geq \ln[w(0,0)] - \ln[w(0,j)] - \ln\left[\frac{C_3+1}{C_3}\right] - \frac{2}{w(0,0)} + \sum_{i=0}^{j} w(0,i)^{-1} \\ &= -\ln[w(0,j)] + C_4 + \sum_{i=0}^{j} w(0,i)^{-1}. \end{aligned} \tag{4.17}$$

Moreover by (4.13) we have

$$\begin{aligned} V[S_j] &\leq \sum_{i=0}^{j} \left\{ \frac{4}{w(0,i)} + \frac{8}{w(0,i)^2} \right\} \\ &\leq (4 + 8C_3^{-1}) \sum_{i=0}^{j} w(0,i)^{-1} \\ &= C_5 \sum_{i=0}^{j} w(0,i)^{-1}. \end{aligned} \tag{4.18}$$

We put $\Phi_j(\vec{w}_+) = \sum_{i=0}^{j} w(0,i)^{-1}$ for every $j \geq 0$. From definition, we remark that $\Phi_j(\vec{w}_+) \geq \Phi_N(\vec{w}_+)$ for all $j > N \geq 0$. From (4.16), (4.17) and (4.18), we obtain that

$$S_j \geq -\ln[w(0,j)] + C_4 + (1 - \epsilon C_5)\Phi_j(\vec{w}_+) \quad \text{a.s.}$$

for all $j \geq N$ and $\epsilon > 0$. Then we have that for any $0 < \epsilon < C_5{}^{-1}$,

$$\begin{aligned}\Theta_+ &= \sum_{j=1}^{\infty} \exp(S_j) \\ &\geq \sum_{j=N}^{\infty} \exp\Big(-\ln[w(0,j)] + C_4 + (1-\epsilon C_5)\Phi_j(\vec{w}_+)\Big) \\ &\geq \exp(C_4)\exp\left((1-\epsilon C_5)\Phi_N(\vec{w}_+)\right)\sum_{j=N}^{\infty} w(0,j)^{-1} = \infty \quad \text{a.s.}\end{aligned}$$

Hence we finish showing that if $\Phi(\vec{w}_+) = \infty$, then $\Theta_+ = \infty$ almost surely under the condition that $w(0,j) \geq C_3 > 0$ for all $j \geq 0$.

If we can not find such a constant C_3, then $\inf_{j\geq 0} w(0,j) = 0$. Therefore we can not use the previous argument. But in this case, we have the following lemma, whose proof is just the same as Lemma 3.0 of [3]. We remark that much stronger results are known in [4].

Lemma 4.4. *Let $(\overrightarrow{X}, [w])$ be a RRW. If we can take $C_6 > 0$ satisfying $w(0,i) \leq C_6$ for infinitely many $i > 0$, then for every $j < 0$,*

$$P[\tau_j < \infty] + P[\tau_j = \infty \text{ and } \overrightarrow{X} \text{ has finite range}] = 1.$$

However by Theorem 3.1, we obtain that $P[\overrightarrow{X}$ has finite range$] = 0$. Thus we show $P[\tau_j < \infty] = 1$ for every $j < 0$. Therefore we proved the statement [B].

In the same way, we can show that if $\Phi(\vec{w}_-) < \infty$, then $P[\tau_j < \infty] < 1$ for every $j > 0$, and that if $\Phi(\vec{w}_-) = \infty$, then $P[\tau_j < \infty] = 1$ for every $j > 0$. The proof is left to the reader. This completes the proof of Theorem 4.1.

5. Law of Large Numbers

In this section we prove the following theorem.

Theorem 5.1. *Let $(\overrightarrow{X}, [w])$ be a sequence-type recurrent RRW with a sequence $\vec{a} = \{a(n)\}_{n\geq 1}$, such that initial weights take values in a finite set $\{f_1, f_2, \dots, f_b\}$ of positive real numbers. Then we have that*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \quad a.s.$$

We need to use the next lemma to prove Theorem 5.1. For the proof, see [3]; 214–225.

Lemma 5.2. *Let $(\overrightarrow{X}, [w])$ be a matrix-type recurrent RRW with a matrix A, starting at $X_0 = k > 0$. Let τ_0 be the first hitting time at 0 and set*

$$T = \#\{n \geq 0 \mid X_n \in (0, k), X_{n+1} \in [0, k], n < \tau_0\}.$$

If there exists a positive constant C_7 such that $C_7^{-1} \leq w(0, j) \leq C_7$ for all $j \in \mathbf{Z}$, then there is a positive constant C_8, which depends on the set of initial weights $\{w(0, j)\}_{0 \leq j < k}$ and the matrix A, such that

$$E[\tau_0] \geq E[T] \geq C_8 k^{3/2}. \tag{5.1}$$

This lemma corresponds to Lemma 4.9. in [3]. Although $E[T]$ does not appear in the statement of Lemma 4.9. in [3], it is shown in the proof that the expectation of a smaller quantity than T is bounded from below by a constant multiple of $k^{3/2}$. Actually, Davis obtained that

$$E[U] \geq C_8 k^{3/2}$$

where

$$U = \#\{n \geq 0 \mid X_n \in (0, k), X_{n+1} \in [0, k], X_n < X_{n+1}, n < \tau_0\}.$$

Proof of Theorem 5.1. Let k be a positive integer and we put

$$\zeta_m = \#\{n \geq 0 \mid X_n \in (-mk, -(m-1)k), X_{n+1} \in [-mk, -(m-1)k], n < \tau_{-mk}\}$$

for all $m \geq 1$. The distribution of ζ_m depends on the set of initial weights $\{w(0, j)\}_{-mk \leq j < -(m-1)k}$ and the sequence $\vec{a} = \{a(l)\}_{l \geq 1}$. If there are numbers $m > n > 0$ such that $w(0, -mk + j) = w(0, -nk + j)$ for all $0 \leq j \leq k - 1$, then ζ_m and ζ_n are i.i.d. random variables. But each $w(0, j)$ takes one of the values $f_1, f_2, \dots, f_b$, and each interval $[-mk, -(m-1)k]$ has only k–edges. Hence there are at most b^k types of distributions for ζ_l's. Let $\{\check{\zeta}_l\}_{l=1}^{b^k}$ be the random variables corresponding to b^k possible types of arrangement of initial weights. From (5.1) and T is replaced with $\check{\zeta}_l$ we can see $E[\check{\zeta}_l] \geq C_{8,l} k^{3/2}$ for each $1 \leq l \leq b^k$. We put $C_9 = \min_{1 \leq l \leq b^k} C_{8,l}$. Thus we obtain the following inequality for all $1 \leq l \leq b^k$.

$$E[\check{\zeta}_l] \geq C_9 k^{3/2} \tag{5.2}$$

For every $1 \leq l \leq b^k$ and $n \geq 1$, let n_l be the number of ζ_m's with $1 \leq m \leq n$ whose distribution is the same as that of $\check{\zeta}_l$. That is

$$n_l = \#\{1 \leq m \leq n \mid \zeta_m = \check{\zeta}_l \text{ (in law)}\}.$$

We note that $\sum_{l=1}^{b^k} n_l = n$. We set $J_1 = \{1 \le l \le b^k \mid n_l \to \infty \text{ as } n \to \infty\}$.

We define a non-random sequence $\{\sigma(l,t)\}_{t\ge 0}$ for each l as

$$\begin{cases} \sigma(l,0) = 0, \\ \sigma(l,t) = \inf\{m > \sigma(l,t-1) \mid \zeta_m = \check{\zeta}_l \text{ (in law)} \quad \text{for } t \in \mathbf{N}. \end{cases}$$

For simplicity, we put $\zeta_{l,t} = \zeta_{\sigma(l,t)}$. For each l, $\{\zeta_{l,t}\}_{t\ge 1}$ is a sequence of i.i.d. non-negative random variables, and we can apply the law of large numbers.

Lemma 5.3 (Kolmogorov). *Let $\overrightarrow{X} = \{X_n\}_{n\ge 1}$ be i.i.d. non-negative random variables and we put $T_n = \sum_{m=1}^{n} X_m$. Then*

$$\lim_{n\to\infty} \frac{T_n}{n} = E[T_1] \quad a.s.$$

For the idea of proof, for example, see [2]; 126–127.

Appling this lemma, we immediately obtain the following fact for every $l \in J_1$: with probability 1 it holds that for every positive number δ_1, there exists some large number $M_l \in \mathbf{N}$ such that for all $m > M_l$,

$$\sum_{t=1}^{m} \zeta_{l,t} \ge m(C_9 k^{3/2} - \delta_1). \tag{5.3}$$

Because if $E[\zeta_{l,1}] < \infty$ for some $l \in J_1$, then we know by Lemma 5.3. that with probability 1, for any δ_1, there exists a constant $M_l > 0$ such that $\sum_{t=1}^{m} \zeta_{l,t} \ge m(E[\zeta_{l,1}] - \delta_1)$ for $m > M_l$, we obtain (5.3) using (5.2). If $E[\zeta_{l,1}] = \infty$ for some $l \in J_1$, then we also know that with probability 1, for every $L > 0$ there exists a constant $M_l > 0$ such that $\sum_{t=1}^{m} \zeta_{l,t} \ge mL$ for $m > M_l$. We choose $L = C_9 k^{3/2} - \delta_1$, and obtain (5.3).

If $l \notin J_1$, then n_l stays bounded as $n \to \infty$. We set $M_l = \sup_{n\ge 1} n_l$ for $l \notin J_1$.

We put $M_0 = \max_{1\le l\le b^k} M_l$. Note that with probability 1, M_0 is finite. Let n be given and we set $J_2 = J_2(n) = \{1 \le l \le b^k \mid n_l > M_0\}$. Note that if $l \notin J_1$, then $l \notin J_2$. Then we have that

$$\begin{aligned} \sum_{l\in J_2} n_l &= n - \sum_{l\notin J_2} n_l \\ &\ge n - (b^k - |J_2|)M_0 \\ &\ge n - b^k M_0 \end{aligned}$$

for all n, where $|J_2|$ denotes the number of elements in J_2. Given a positive number δ_2, we can take $N > b^k M_0$ such that $b^k M_0 < \delta_2 n$ for $n \ge N$, i.e., for all $n \ge N$,

$$\sum_{l\in J_2} n_l \ge (1 - \delta_2)n. \tag{5.4}$$

From (5.3) and (5.4), we obtain that for all $n \geq N$,

$$\begin{aligned}\frac{\tau_{-nk}}{n} &\geq \frac{1}{n}\sum_{m=1}^{n}\zeta_m \\ &= \frac{1}{n}\sum_{l=1}^{b^k}\sum_{t=1}^{n_l}\zeta_{l,t} \\ &\geq \frac{1}{n}\sum_{l\in J_2} n_l(C_9k^{3/2}-\delta_1) \\ &\geq (1-\delta_2)(C_9k^{3/2}-\delta_1).\end{aligned}$$

For any $\epsilon > 0$, we can take k sufficiently large, such that

$$(1-\delta_2)(C_9k^{3/2}-\delta_1) > \frac{k}{\epsilon}.$$

Thus we have that for each $n \geq N$, $X_m > -nk$ for all m satisfying $0 < m < (k/\epsilon)n$. Let us take $m \geq kN/\epsilon$ arbitrarily and let n be the smallest integer greater than $m\epsilon/k$. Then we have that

$$\frac{k}{\epsilon}N \leq \frac{k}{\epsilon}(n-1) \leq m < \frac{k}{\epsilon}n$$

and that $n > N \geq 1$. By the above argument, this means that

$$\frac{X_m}{m} > -\frac{nk}{m} \geq -\frac{nk}{(n-1)k}\epsilon \geq -2\epsilon \quad \text{a.s.}$$

This implies that

$$\liminf_{m\to\infty}\frac{X_m}{m} \geq -2\epsilon \quad \text{a.s.}$$

In a similar way, considering τ_{nk} instead of τ_{-nk} we can show that

$$\limsup_{m\to\infty}\frac{X_m}{m} \leq 2\epsilon \quad \text{a.s.}$$

Hence we proved that

$$\lim_{m\to\infty}\frac{X_m}{m} = 0 \quad \text{a.s.}$$

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