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ISOLATION PHENOMENA FOR QUATERNIONIC YANG-MILLS CONNECTIONS

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1. Introduction and statement of results

In this paper, we shall study a certain class of Yang-Mills connections on a quaternionic Kähler manifold, called *quaternionic Yang-Mills connections*.

Our basic setting is the following. Let E be an associated Riemannian vector bundle of a principal bundle with a compact Lie group G as the structure group over a compact oriented Riemannian manifold (M, g) . Let \mathcal{A} be the space of connections on E . For a connection $\nabla \in \mathcal{A}$, we denote by d^∇ and δ^∇ the covariant exterior derivative and its formal adjoint respectively acting on $\text{End}(E)$ -valued p -forms.

The Yang-Mills energy functional $YM : \mathcal{A} \rightarrow \mathbb{R}$ is defined by

$$YM(\nabla) = \frac{1}{2} \int_M \|F^\nabla\|^2 dv_g,$$

where F^∇ is the curvature of a connection $\nabla \in \mathcal{A}$. A connection ∇ is called a *Yang-Mills connection*, if ∇ is a critical point of the Yang-Mills energy functional $YM(\nabla)$; namely, if it satisfies the Euler-Lagrange equation

$$\delta^\nabla F^\nabla = 0.$$

By the Bianchi identity $d^\nabla F^\nabla = 0$, the Euler-Lagrange equation is equivalent to

$$\Delta^\nabla F^\nabla = 0,$$

which says that F^∇ is harmonic, where $\Delta^\nabla = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$.

Nitta ([6]), Mamone Capria-Salamon ([2]) independently found higher dimensional analogues of the notion of self-dual and anti-self-dual connections on a quaternionic Kähler manifold. A quaternionic Kähler manifold is a Riemannian $4n$ -manifold whose holonomy group lies in $Sp(n) \cdot Sp(1)$, $n > 1$. In the case of $n = 1$, we add the condition that M is Einstein and half-conformally flat. The bundle of 2-forms on a quaternionic Kähler manifold (M, g) has the following irreducible decomposition as a representation of $Sp(n) \cdot Sp(1)$:

$$(1.1) \quad \wedge^2 T^*M = S^2\mathbb{H} \oplus S^2\mathbb{E} \oplus (S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp,$$

where \mathbb{H} and \mathbb{E} are the vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. Corresponding to the decomposition (1.1), we write the curvature F^∇ as

$$F^\nabla = F^1 + F^2 + F^3,$$

where $F^1 \in \Gamma(M; S^2\mathbb{H} \otimes \text{End}(E))$, $F^2 \in \Gamma(M; S^2\mathbb{E} \otimes \text{End}(E))$ and $F^3 \in \Gamma(M; (S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp \otimes \text{End}(E))$. A connection ∇ is said to be c_i -self-dual ($i=1, 2$ or 3) if $F^j = 0$ for all $j \neq i$. In the case of $n = 1$, we have $F^1 = F^+$, $F^2 = F^-$ and $F^3 = 0$ where F^+ (resp. F^-) is the (resp. anti-) self-dual part of the curvature F^∇ . We shall confine ourself to the case where (M, g) is a compact quaternionic Kähler $4n$ -manifold.

Recall that each c_i -self-dual connection is a Yang-Mills connection (cf. [6], [2], [3]). Moreover, if M is compact, a c_1 or c_2 -self-dual connection is minimizing the Yang-Mills energy functional $YM(\nabla)$ (cf. [3], [2]). As far as we know, there is no example of non-flat c_3 -self-dual connections. If they exist, they are believed to be unstable. Indeed, it is known ([7]) that any non-flat c_3 -self-dual connection over the quaternionic projective space $\mathbb{H}P^n$ is, if it exists, unstable. Nagatomo ([5]) proved that there is a unique non-flat c_1 -self-dual connection over any simply-connected quaternionic Kähler $4n$ -manifold with $n > 1$.

Let us recall some results on Yang-Mills connections. Bourguignon and Lawson ([1]) discussed gap-phenomena for Yang-Mills connections. They gave explicit C^0 -neighborhoods of the minimal Yang-Mills fields which contain no other Yang-Mills fields up to gauge equivalent. They obtained the following.

Theorem A. ([1]) *Let ∇ be a Yang-Mills connection on (S^4, g_0) . If the self-dual part F^+ of the curvature of ∇ satisfies the pointwise inequality $\|F^+\|^2 < 3$, then $F^+ = 0$. The same is true for the anti-self-dual part F^- of the curvature of ∇ .*

They next examined the case where the inequality $\|F^\nabla\|^2 < 3$ is relaxed on (S^4, g_0) .

Theorem B. ([1]) *Let ∇ be a Yang-Mills connection on a Riemannian vector bundle E over (S^4, g_0) . If F^∇ satisfies the pointwise inequality $\|F^\nabla\|^2 \leq 3$, then either E is flat or $E = E_0 \oplus S$ where E_0 is flat and where S is one of the 4-dimensional bundles of tangent spinors with the canonical Riemannian connections.*

The purpose of this paper is to generalize these results to quaternionic Kähler manifolds. We introduce the following notion for connections:

DEFINITION 1.1. A connection ∇ on a Riemannian vector bundle over a compact quaternionic Kähler manifold is called a *quaternionic Yang-Mills connection*

if $\Delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ where Ω is the fundamental 4-form on (M, g) (See §2).

Note that in the case of $n = 1$, the quaternionic Yang-Mills connections are the Yang-Mills connections, and vice versa. It is easy to see that the c_1 -, c_2 - and c_3 -self-dual connections introduced above are quaternionic Yang-Mills connections.

Proposition 1.1. *If a connection ∇ is a quaternionic Yang-Mills connection, then ∇ is a Yang-Mills connection.*

We shall give a proof of Proposition 1.1 in § 3.

Wolf ([9]) classified the compact simply-connected quaternionic Kähler symmetric spaces, called *Wolf spaces*. The only examples of the Wolf spaces are the following.

$$\begin{matrix} \mathbb{H}P^n, & Gr_2(\mathbb{C}^{n+2}), & Gr_4(\mathbb{R}^{n+4}), & \frac{G_2}{SO(4)}, \\ \frac{F_4}{Sp(3) \cdot Sp(1)}, & \frac{E_6}{SU(6) \cdot Sp(1)}, & \frac{E_7}{Spin(12) \cdot Sp(1)}, & \frac{E_8}{E_7 \cdot Sp(1)}. \end{matrix}$$

From now on, we suppose that (M, g) is a Wolf space. Note that the Riemannian curvature operator R acting on $\wedge^2 TM$ has also a splitting $R = R_1 + R_2 + R_3$ with respect to the decomposition (1.1). By ([4]) we can write the curvature operator R_i as $R_i = \mu_i I_{\wedge^2 TM}$ where μ_i ($i = 1$ or 2) is a positive constant. Since R_3 is negative semi-definite, we put $\mu_3 = 0$. We set $\lambda_i = s/(2n) - 2\mu_i$ ($i = 1, 2$ or 3) where s is the scalar curvature of (M, g) . Then we shall state the following.

Theorem 1.1. *Let ∇ be a quaternionic Yang-Mills connection on a Wolf space (M, g) , ($n \geq 1$), and assume $F^3 = 0$, i.e., the c_3 -self-dual part F^3 of the curvature of ∇ vanishes.*

(1) *If the c_1 -self-dual part F^1 of the curvature of ∇ satisfies the pointwise inequality*

$$\|F^1\|^2 < \frac{n(4n-1)\lambda_1^2}{16(2n-1)^2},$$

then $F^1 = 0$, that is, ∇ is a c_2 -self-dual connection.

(2) *If the c_2 -self-dual part F^2 of the curvature of ∇ satisfies the pointwise inequality*

$$\|F^2\|^2 < \frac{n(4n-1)\lambda_2^2}{16(2n-1)^2},$$

then $F^2 = 0$, that is, ∇ is a c_1 -self-dual connection.

Theorem 1.1 for $M = \mathbb{H}P^1$ coincides with Theorem A. It seems that the assumption $F^3 = 0$ is necessary to get the generalization of Theorem A. We next show that the c_3 -self-dual connections can be characterized as follows if they exist.

Theorem 1.2. *Let ∇ be a quaternionic Yang-Mills connection on a Wolf space (M, g) , ($n \geq 1$). If the c_1 -self-dual part F^1 and the c_2 -self-dual part F^2 of the curvature of ∇ respectively satisfy the pointwise inequalities*

$$\|F^1\|^2 < \frac{n(4n-1)\lambda_1^2}{16(2n-1)^2}, \quad \|F^2\|^2 < \frac{n(4n-1)\lambda_2^2}{16(2n-1)^2},$$

then $F^1 = F^2 = 0$, that is, ∇ is a c_3 -self-dual connection.

To generalize Theorem B, we suppose that the base manifold M is a quaternionic projective space $(\mathbb{H}P^n, g_0)$. Let g_0 be the Riemannian metric on $\mathbb{H}P^n$ with the scalar curvature $s = 4n(2n-1)(n+2)$. With respect to this metric g_0 , we calculate λ_1 and λ_2 of Theorem 1.1. Then we can read Theorem 1.1 as follows.

Corollary 1.1. *Let ∇ be a quaternionic Yang-Mills connection on $(\mathbb{H}P^n, g_0)$, ($n \geq 1$), and assume that $F^3 = 0$.*

(1) *If F^1 satisfies the pointwise inequality*

$$\|F^1\|^2 < n(4n-1),$$

then $F^1 = 0$, that is, ∇ is a c_2 -self-dual connection.

(2) *If F^2 satisfies the pointwise inequality*

$$\|F^2\|^2 < \frac{n(4n-1)(n+1)^2}{4},$$

then $F^2 = 0$, that is, ∇ is a c_1 -self-dual connection.

Using Corollary 1.1, we examine what happens when the inequality $\|F^\nabla\|^2 < n(4n-1)$ is relaxed on $(\mathbb{H}P^n, g_0)$.

Theorem 1.3. *Let ∇ be a quaternionic Yang-Mills connection on a Riemannian vector bundle E with any structure group G over $(\mathbb{H}P^n, g_0)$, ($n \geq 1$), and assume that $F^3 = 0$. If F^∇ satisfies the pointwise inequality $\|F^\nabla\|^2 \leq n(4n-1)$, then either E is a flat vector bundle or $E = E_0 \oplus \mathbb{H}$, where E_0 is a flat vector bundle and where \mathbb{H} is the tautological quaternion line bundle.*

In the case of $n = 1$, Theorem 1.3 coincides with Theorem B. We next obtain the following theorem in which the assumption of $F^3 = 0$ is not necessary.

Theorem 1.4. *Let ∇ be a quaternionic Yang-Mills connection on a Wolf space (M, g) , $(n \geq 2)$.*

(1) *If F^1 , F^2 and F^3 satisfy the pointwise inequalities*

$$\|F^1\| < \frac{\lambda_1}{\sqrt{2}},$$

$$\|F^3\| < \frac{\lambda_3}{\sqrt{2}} - (n+3)\|F^1\| - \|F^2\|,$$

then $F^1 = F^3 = 0$, that is, ∇ is a c_2 -self-dual connection. Moreover if ∇ is non-flat, then the c_2 -self-dual part F^2 satisfies

$$\frac{\lambda_2}{\sqrt{2}} \leq \|F^2\| < \frac{\lambda_3}{\sqrt{2}}.$$

(2) *If F^1 , F^2 and F^3 satisfy the pointwise inequalities*

$$\|F^2\| < \frac{\lambda_2}{\sqrt{2}},$$

$$\|F^3\| < \frac{\lambda_3}{\sqrt{2}} - \frac{3n+4}{n}\|F^2\| - \|F^1\|,$$

then $F^2 = F^3 = 0$, that is, ∇ is a c_1 -self-dual connection. Moreover if ∇ is non-flat, then the c_1 -self-dual part F^1 satisfies

$$\frac{\lambda_1}{\sqrt{2}} \leq \|F^1\| < \frac{\lambda_3}{\sqrt{2}}.$$

2. Preliminaries

In this section, we fix notation. Let (M, g) be a compact quaternionic Kähler $4n$ -manifold, and P a principal G -bundle over (M, g) with a compact Lie group G as structure group. We denote by \mathfrak{g} the Lie algebra of G . For a faithful orthogonal representation $\rho : G \rightarrow O(N)$, we consider a Riemannian vector bundle $E = P \times_{\rho} \mathbb{R}^N$ associated with P by ρ . Each connection on P corresponds to a connection ∇ on E . We denote by \mathcal{A} the set of the connections on E . To each connection ∇ on E , the curvature F^{∇} , given by the formula $F_{X,Y}^{\nabla} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for tangent vectors X, Y , is a 2-form on M with values in the bundle \mathfrak{so}_E whose fibre $\mathfrak{so}_{E,x}$, $x \in M$, consists of skew-symmetric endomorphisms of the fibre E_x of E . The pointwise norm of F^{∇} at each point x is given by

$$\|F^{\nabla}\|^2 = \sum_{i < j} \|F_{e_i, e_j}^{\nabla}\|^2,$$

where $\{e_1, \dots, e_{4n}\}$ is an orthonormal basis of the tangent space $T_x M$, $x \in M$, and the inner product of the fibre $\mathfrak{so}_{E,x}$ is given by

$$(2.1) \quad \langle A, B \rangle = -\frac{1}{2} \text{tr}(A \circ B)$$

for $A, B \in \mathfrak{so}_{E,x}$. There exists a subbundle \mathfrak{g}_E of \mathfrak{so}_E corresponding to a bundle $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$ through ρ . Let $A^p(\mathfrak{g}_E)$, $0 \leq p \leq 4n$, be the space of \mathfrak{g}_E -valued p -forms on M . We get the exterior differential $d^\nabla : A^p(\mathfrak{g}_E) \rightarrow A^{p+1}(\mathfrak{g}_E)$ and the adjoint operator $\delta^\nabla : A^p(\mathfrak{g}_E) \rightarrow A^{p-1}(\mathfrak{g}_E)$ corresponding to $\nabla \in \mathcal{A}$. $\Delta^\nabla = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$ is the *Laplacian* for \mathfrak{g}_E -valued p -forms. There is another second order operator $\nabla^* \nabla$, called the *rough Laplacian*, acting on \mathfrak{g}_E -valued differential forms. It is given by the formula $\nabla^* \nabla \varphi = -\sum_{j=1}^{4n} (\nabla_{e_j}^2 \varphi)$, $\varphi \in A^p(\mathfrak{g}_E)$, where $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{D_X Y}$.

The bundle of 2-forms on a quaternionic Kähler manifold M has the following irreducible decomposition as a representation of $Sp(n) \cdot Sp(1)$:

$$(2.2) \quad \wedge^2 T^* M = S^2 \mathbb{H} \oplus S^2 \mathbb{E} \oplus (S^2 \mathbb{H} \oplus S^2 \mathbb{E})^\perp,$$

where \mathbb{H} and \mathbb{E} are the vector bundles associated to the standard representations of $Sp(1)$ and $Sp(n)$, respectively. A connection whose \mathfrak{g}_E -valued curvature 2-form lies in $S^2 \mathbb{H}$, $S^2 \mathbb{E}$ or $(S^2 \mathbb{H} \oplus S^2 \mathbb{E})^\perp$ is called a c_1 , c_2 or c_3 -self-dual connection respectively. Corresponding to the decomposition (2.2), we write the curvature F^∇ as

$$F^\nabla = F^1 + F^2 + F^3.$$

In the case of $n = 1$, corresponding to the fact that $SO(4) = Sp(1) \cdot Sp(1)$, $\wedge^2 T^* M$ is decomposed as

$$(2.3) \quad \wedge^2 T^* M = \wedge_+^2 \oplus \wedge_-^2.$$

A connection whose \mathfrak{g}_E -valued curvature 2-form lies in \wedge_+^2 or \wedge_-^2 is called a self-dual or anti-self-dual connection respectively. Corresponding to the decomposition (2.3), we write the curvature F^∇ as

$$F^\nabla = F^+ + F^-.$$

The associated bundles \mathbb{H} , \mathbb{E} for this case are precisely the half-spinor bundles of M . The vector bundle $S^2 \mathbb{H}$ is a subbundle of $\text{End}(TM)$ of real rank 3. Locally $S^2 \mathbb{H}$ has a basis $\{I, J, K\}$ satisfying

$$I^2 = J^2 = -1, \quad IJ = -JI = K.$$

The metric g on M satisfies $g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y)$ for all $X, Y \in T_x M$. Local 2-forms $\{\omega_I, \omega_J, \omega_K\}$ are defined by

$$\omega_I(X, Y) = g(IX, Y), \quad \omega_J(X, Y) = g(JX, Y), \quad \omega_K(X, Y) = g(KX, Y).$$

$\{\omega_I, \omega_J, \omega_K\}$ is a local orthogonal frame of $S^2\mathbb{H}$. We define a global 4-form Ω by

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

Ω is a nondegenerate and parallel form on M , called the *fundamental 4-form* on M . A connection ∇ on the quaternionic Kähler $4n$ -manifold (M, g) is a c_i -self-dual connection ($i = 1, 2$ or 3) if and only if its curvature F^∇ satisfies

$$(2.4) \quad *F^\nabla = c_i F^\nabla \wedge \Omega^{n-1},$$

where $*$ is the Hodge star operator and $c_1 = (6n)/((2n+1)!)$, $c_2 = -1/((2n-1)!)$ and $c_3 = 3/((2n-1)!)$ ([3]). Note that the equation (2.4) can be viewed as the self-dual or anti-self-dual equation on a oriented Riemannian 4-manifold.

Let (M, g) be a compact quaternionic Kähler $4n$ -manifold. At each point, we consider F^∇ as a linear map

$$F^\nabla : \wedge^2 TM \longrightarrow \mathfrak{g}_E.$$

In $\wedge^2 TM$ we have the identities

$$(2.5) \quad [e_i \wedge e_j, e_k \wedge e_l] = \delta_{il} e_k \wedge e_j + \delta_{jl} e_i \wedge e_k + \delta_{ik} e_j \wedge e_l + \delta_{jk} e_l \wedge e_i$$

for all i, j, k, l , where $\{e_1, \dots, e_{4n}\}$ is an orthonormal basis of the tangent space $T_x M$. For any φ in $A^2(\mathfrak{g}_E)$, the Bochner-Weitzenböck formula is

$$\langle \Delta^\nabla \varphi, \varphi \rangle - \langle \nabla^* \nabla \varphi, \varphi \rangle = \langle \varphi \circ \left(\frac{s}{2n} I - 2R \right), \varphi \rangle - \rho(\varphi),$$

where

$$\rho(\varphi) = \langle \kappa(\varphi), \varphi \rangle = \langle [F^\nabla, \varphi], \varphi \rangle, \quad \kappa(\varphi)_{X,Y} = \sum_{i=1}^{4n} \{ [F^\nabla_{e_i, X}, \varphi_{e_i, Y}] - [F^\nabla_{e_i, Y}, \varphi_{e_i, X}] \}$$

and R is the Riemannian curvature operator acting on $\wedge^2 TM$.

For $\varphi = F^\nabla$, this formula implies that

$$(2.6) \quad \langle \Delta F^\nabla, F^\nabla \rangle - \langle \nabla^* \nabla F^\nabla, F^\nabla \rangle = \langle F^\nabla \circ \left(\frac{s}{2n} I - 2R \right), F^\nabla \rangle - \rho(F^\nabla),$$

where

$$(2.7) \quad \rho(F^\nabla) = \sum_{i,j,k=1}^{4n} \langle [F^\nabla_{e_i, e_j}, F^\nabla_{e_j, e_k}], F^\nabla_{e_k, e_i} \rangle.$$

We now examine the term ρ given by (2.7). We now introduce an inner product on the bundle \mathfrak{g}_E as follows. Recall that we have $\mathfrak{g}_E \subseteq \mathfrak{so}_E$, the bundle of skew-symmetric endomorphisms of E . Given two endomorphisms A and B of E_x , we

define $\langle A, B \rangle := 1/2\text{tr}({}^tA \circ B)$. There is a natural bundle isomorphism $\wedge^2 E \simeq \mathfrak{so}_E$ determined by the requirement that

$$(u \wedge v)(w) = \langle u, w \rangle v - \langle v, w \rangle u$$

for $u, v, w \in E_x$. The elements $\{\xi_i \wedge \xi_j\}_{i < j}$ form an orthonormal basis of $(\mathfrak{so}_E)_x$ whenever (ξ_1, \dots, ξ_N) is an orthonormal basis of E_x . In particular, there is a canonical isometry $\wedge^2 TM \simeq \mathfrak{so}_M$. We have also $\mathfrak{g} \subseteq \wedge^2 T_x M \simeq \mathfrak{so}(N)$. For any Lie algebra \mathfrak{g} with a fixed Ad -invariant inner product $\langle \cdot, \cdot \rangle$, we have the associated fundamental 3-form $\Phi_{\mathfrak{g}}$ given by $\Phi_{\mathfrak{g}}(X, Y, Z) = \langle [X, Y], Z \rangle$ for $X, Y, Z \in \mathfrak{g}$ and $\Phi_{\wedge^2 TM}(\alpha, \beta, \gamma) = \langle [\alpha, \beta], \gamma \rangle$ for $\alpha, \beta, \gamma \in \wedge^2 TM$. We may rewrite (2.7) as

$$\begin{aligned} \rho(F^\nabla) &= \sum_{i,j,k=1}^{4n} \Phi_{\mathfrak{g}_E}(F_{e_i, e_j}^\nabla, F_{e_j, e_k}^\nabla, F_{e_k, e_i}^\nabla) \\ &= \sum_{i,j,k=1}^{4n} (F^{\nabla*} \Phi_{\mathfrak{g}_E})(e_i \wedge e_j, e_j \wedge e_k, e_k \wedge e_i) \\ &= (F^{\nabla*} \Phi_{\mathfrak{g}_E}, \Phi_{\wedge^2 TM}), \end{aligned}$$

where, for notational convenience, we define the inner product in $\wedge^3(\wedge^2 T^*M)$ by $(\Phi, \Psi) = \sum_{U, V, W} \Phi(U, V, W)\Psi(U, V, W)$, where U, V and W are an orthonormal basis of $\wedge^2 TM$. Therefore, we have the following basic result. Let F^∇ be a curvature 2-form on E and let λ be the minimal eigenvalue of the operator $(s/2n)I - 2R$ on 2-forms over a compact quaternionic Kähler manifold M . Then

$$(2.8) \quad \langle \nabla^* \nabla F^\nabla, F^\nabla \rangle - \langle \Delta^\nabla F^\nabla, F^\nabla \rangle \leq -\{\lambda \|F^\nabla\|^2 - (F^{\nabla*} \Phi_{\mathfrak{g}_E}, \Phi_{\wedge^2 TM})\}.$$

At each point $x \in M$, we want to estimate $(F_x^{\nabla*} \Phi_{\mathfrak{g}_E}, \Phi_{\wedge^2 T_x M})$ in terms of $\|F^\nabla\|^2$ where $F_x^\nabla : \mathfrak{so}(4n) \rightarrow \mathfrak{g}$ is a linear map and where \mathfrak{g} is any Lie subalgebra of $\mathfrak{so}(N)$. Recall that an inner product on \mathfrak{g} is induced from the canonical one on $\mathfrak{so}(N)$ defined by $\langle A, B \rangle = -(1/2)\text{tr}(\rho(A) \cdot \rho(B))$. Consequently $F_x^{\nabla*} \Phi_{\mathfrak{g}} = F_x^{\nabla*} \Phi_{\mathfrak{so}(N)}$. Therefore, in the argument of this paper, we can ignore \mathfrak{g} .

The norm $\|\cdot\|$ induced by the inner product (2.1) has the property that

$$(2.9) \quad \|[A, B]\| \leq \sqrt{2}\|A\| \cdot \|B\|$$

for all A, B in which the equality holds if and only if the pair A, B is orthogonally equivalent to the following matrices:

$$(2.10) \quad \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$i = \left(\begin{array}{cc|cc} 0 & -t & 0 & 0 \\ t & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -t \\ 0 & 0 & t & 0 \end{array} \right), \quad j = \left(\begin{array}{cc|cc} 0 & 0 & -t & 0 \\ 0 & 0 & 0 & t \\ \hline t & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{array} \right).$$

We shall also state the following result, which is used in proving our theorems.

Lemma 2.1. *Let $S = ((s_{ij}))$ be a symmetric $4n \times 4n$ matrix with $s_{ij} \geq 0$ and $s_{ii} = 0$. If $\text{tr}S^2 = (4n(4n - 1)\lambda^2)/((4n - 2)^2 2^2)$ for any positive real number λ , then*

$$\text{tr}S^3 \leq \frac{4n(4n - 1)\lambda^3}{(4n - 2)^2 2^3}$$

with equality holding if and only if $s_{ij} = (\lambda)(4(2n - 1))$, $i \neq j$.

The proof of Lemma 2.1 is entirely similar to the argument for Lemma (5.14) in [1].

Denoting F_{e_i, e_j}^α by F_{ij}^α , we have the following.

Proposition 2.1 ([3]). *Let F^1 , F^2 and F^3 be respectively the c_1 -self-dual, c_2 -self-dual and c_3 -self-dual parts.*

(1) *The c_1 -self-dual part F^1 satisfies*

$$\begin{aligned} F_{4k+1, 4k+2}^1 &= F_{4k+3, 4k+4}^1 = F_{4l+1, 4l+2}^1 = F_{4l+3, 4l+4}^1, \\ F_{4k+1, 4k+3}^1 &= F_{4k+4, 4k+2}^1 = F_{4l+1, 4l+3}^1 = F_{4l+4, 4l+2}^1, \\ F_{4k+1, 4k+4}^1 &= F_{4k+2, 4k+3}^1 = F_{4l+1, 4l+4}^1 = F_{4l+2, 4l+3}^1, \\ F_{4p+1, 4q+1}^1 &= F_{4p+2, 4q+2}^1 = F_{4p+3, 4q+3}^1 = F_{4p+4, 4q+4}^1 = 0, \\ &(\forall k, l, p, q), \\ F_{4p+\alpha, 4q+\beta}^1 &= 0, \quad (\forall p \neq q, \forall \alpha, \beta). \end{aligned}$$

(2) *The c_2 -self-dual part F^2 satisfies*

$$\begin{aligned} F_{4k+1, 4k+2}^2 &= -F_{4k+3, 4k+4}^2, \\ F_{4k+1, 4k+3}^2 &= F_{4k+2, 4k+4}^2, \\ F_{4k+1, 4k+4}^2 &= -F_{4k+2, 4k+3}^2, \\ F_{4p+1, 4q+1}^2 &= F_{4p+2, 4q+2}^2 = F_{4p+3, 4q+3}^2 = F_{4p+4, 4q+4}^2, \\ F_{4p+1, 4q+2}^2 &= -F_{4p+2, 4q+1}^2 = -F_{4p+3, 4q+4}^2 = F_{4p+4, 4q+3}^2, \\ F_{4p+1, 4q+3}^2 &= F_{4p+2, 4q+4}^2 = -F_{4p+3, 4q+1}^2 = -F_{4p+4, 4q+2}^2, \\ F_{4p+1, 4q+4}^2 &= -F_{4p+2, 4q+3}^2 = F_{4p+3, 4q+2}^2 = -F_{4p+4, 4q+1}^2, \\ &(\forall k), \quad (0 \leq p < q \leq n - 1). \end{aligned}$$

(3) The c_3 -self-dual part F^3 satisfies

$$\begin{aligned} \sum_{k=0}^{n-1} F_{4k+1,4k+2}^3 &= \sum_{k=0}^{n-1} F_{4k+1,4k+3}^3 = \sum_{k=0}^{n-1} F_{4k+1,4k+4}^3 = 0, \\ F_{4p+1,4q+2}^3 + F_{4q+1,4p+2}^3 &= F_{4p+3,4q+4}^3 + F_{4q+3,4p+4}^3, \\ F_{4p+1,4q+3}^3 + F_{4q+1,4p+3}^3 &= -(F_{4p+2,4q+4}^3 + F_{4q+2,4p+4}^3), \\ F_{4p+1,4q+4}^3 + F_{4q+1,4p+4}^3 &= F_{4p+2,4q+3}^3 + F_{4q+2,4p+3}^3, \\ \sum_{\alpha=1}^4 F_{4p+\alpha,4q+\alpha}^3 &= 0, \end{aligned}$$

($\forall p, q$).

Proposition 2.1 follows from the argument for Theorem 2.2 in [3].

3. Some properties of quaternionic Kähler manifolds

In this section, we prepare a few propositions. First, we shall give a proof of Proposition 1.1.

Proof of Proposition 1.1. We see that $d^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ by $d^\nabla F^\nabla = 0$ and $d\Omega = 0$. Hence if M is compact, then the connection ∇ satisfies $\Delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ if and only if $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$. We shall prove that ∇ satisfies $\delta^\nabla F^\nabla = 0$ if $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$. We take an orthonormal frame field $\{e_i; i = 1, 2, \dots, 4n\}$ such that $Ie_{4k+1} = e_{4k+2}$, $Je_{4k+1} = e_{4k+3}$, $Ke_{4k+1} = e_{4k+4}$, ($k = 0, 1, \dots, n-1$), and denote the dual frame by $\{\theta^i; i = 1, 2, \dots, 4n\}$. The vector bundle $S^2\mathbb{H}$ has the following frame field, $\{\omega_I, \omega_J, \omega_K\}$:

$$\begin{aligned} \omega_I &= \sum_{k=0}^{n-1} (\theta^{4k+1} \wedge \theta^{4k+2} + \theta^{4k+3} \wedge \theta^{4k+4}), \\ \omega_J &= \sum_{k=0}^{n-1} (\theta^{4k+1} \wedge \theta^{4k+3} + \theta^{4k+4} \wedge \theta^{4k+2}), \\ \omega_K &= \sum_{k=0}^{n-1} (\theta^{4k+1} \wedge \theta^{4k+4} + \theta^{4k+2} \wedge \theta^{4k+3}). \end{aligned}$$

The fundamental 4-form is $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$. Using the orthonormal frame $\{\theta^i; i = 1, 2, \dots, 4n\}$, we can write the curvature 2-form F^∇ as $F^\nabla = \sum_{i < j} F_{ij} \theta^i \wedge \theta^j$. From $\Omega^{n-1} = ((2n-1)!/6) * \Omega$ ([3]), $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ is equivalent to $\delta^\nabla(F^\nabla \wedge * \Omega) = 0$. It is easy to see that the quaternionic Yang-Mills equation $\delta^\nabla(F^\nabla \wedge * \Omega) = 0$ is equivalent to

$$\nabla_i F_{ij} = 0, \quad (i, j = 1, \dots, 4n).$$

On the other hand, the Yang-Mills equation $\delta^\nabla F^\nabla = 0$ is equivalent to

$$\sum_i \nabla_i F_{ij} = 0, \quad (j = 1, \dots, 4n).$$

Therefore, if ∇ satisfies $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$, then $\delta^\nabla F^\nabla = 0$. \square

Proposition 3.1. *Let F^1 , F^2 and F^3 be respectively the c_1 -self-dual, c_2 -self-dual and c_3 -self-dual parts of the curvature F^∇ on a compact quaternionic Kähler manifold. Then the following are equivalent:*

$$(3.1) \quad \Delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0;$$

$$(3.2) \quad \Delta^\nabla \left(\frac{c_\alpha - c_\gamma}{c_\alpha} F^\alpha + \frac{c_\beta - c_\gamma}{c_\beta} F^\beta \right) = 0$$

for any permutation (α, β, γ) of $\{1, 2, 3\}$.

Proof. Let

$$(3.3) \quad F^\nabla = F^\alpha + F^\beta + F^\gamma$$

denote the curvature, for any (α, β, γ) as above. From (3.3), we have

$$(3.4) \quad c_\gamma F^\nabla \wedge \Omega^{n-1} = c_\gamma F^\alpha \wedge \Omega^{n-1} + c_\gamma F^\beta \wedge \Omega^{n-1} + c_\gamma F^\gamma \wedge \Omega^{n-1}.$$

Hence, we get

$$(3.5) \quad c_\gamma * (F^\nabla \wedge \Omega^{n-1}) = \frac{c_\gamma}{c_\alpha} F^\alpha + \frac{c_\gamma}{c_\beta} F^\beta + F^\gamma.$$

It follows from (3.3) and (3.5) that

$$(3.6) \quad \left(1 - \frac{c_\gamma}{c_\alpha}\right) F^\alpha + \left(1 - \frac{c_\gamma}{c_\beta}\right) F^\beta = F^\nabla - c_\gamma * (F^\nabla \wedge \Omega^{n-1}).$$

Applying d^∇ and δ^∇ to (3.6), respectively, and using Bianchi identity $d^\nabla F^\nabla = 0$ and $d\Omega^{n-1} = 0$, we obtain

$$\begin{aligned} d^\nabla \left[\left(1 - \frac{c_\gamma}{c_\alpha}\right) F^\alpha + \left(1 - \frac{c_\gamma}{c_\beta}\right) F^\beta \right] &= -c_\gamma * \delta^\nabla(F^\nabla \wedge \Omega^{n-1}), \\ \delta^\nabla \left[\left(1 - \frac{c_\gamma}{c_\alpha}\right) F^\alpha + \left(1 - \frac{c_\gamma}{c_\beta}\right) F^\beta \right] &= \delta^\nabla F^\nabla. \end{aligned}$$

From Proposition 1.1, ∇ fullfills $\delta^\nabla F^\nabla = 0$ if it satisfies $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$. Hence, (3.1) and (3.2) are equivalent. This completes the proof of Proposition 2.1. \square

In the case $n = 1$, we conclude that the following three conditions are equivalent ([1]):

$$(1) \quad \delta^\nabla F^\nabla = 0, \quad (2) \quad \Delta^\nabla F^+ = 0, \quad (3) \quad \Delta^\nabla F^- = 0.$$

Proposition 3.2. *Let F^1 and F^2 be respectively the c_1 -self-dual and c_2 -self-dual parts. Then for vectors $X, Y \in T_x M$, the quantity*

$$\sum_{j=1}^{4n} F_{e_j, X}^1 \cdot F_{e_j, Y}^2$$

is symmetric in X and Y .

Proof. Let $\{e_1, \dots, e_{4n}\}$ be an orthonormal frame field of $T_x M$. Substituting $X = e_{4k+1}$, $Y = e_{4k+2}$ into $\sum_{j=1}^{4n} F_{e_j, X}^1 \cdot F_{e_j, Y}^2$ and using Proposition 2.1, we see that

$$\begin{aligned} \sum_{j=1}^{4n} F_{e_j, e_{4k+1}}^1 \cdot F_{e_j, e_{4k+2}}^2 &= F_{e_{4k+3}, e_{4k+1}}^1 \cdot F_{e_{4k+3}, e_{4k+2}}^2 + F_{e_{4k+4}, e_{4k+1}}^1 \cdot F_{e_{4k+4}, e_{4k+2}}^2 \\ &= F_{e_{4k+2}, e_{4k+4}}^1 \cdot F_{e_{4k+1}, e_{4k+4}}^2 + F_{e_{4k+3}, e_{4k+2}}^1 \cdot F_{e_{4k+3}, e_{4k+1}}^2 \\ &= \sum_{j=1}^{4n} F_{e_j, e_{4k+2}}^1 \cdot F_{e_j, e_{4k+1}}^2 \end{aligned}$$

for each $0 \leq k \leq n - 1$. This completes the proof of Proposition 3.2. \square

The following is the key of the proofs of the theorems.

Proposition 3.3. *Let F^1 , F^2 and F^3 be respectively the c_1 -self-dual, c_2 -self-dual and c_3 -self-dual parts. Then*

$$(1) \quad [F^1, F^2]_{X, Y} = 0,$$

$$(2) \quad [F^2, F^3]_{X, Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g},$$

$$(3) \quad [F^1, F^3]_{X, Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g},$$

where $[F^\alpha, F^\beta]_{X, Y} = \sum_{j=1}^{4n} \{[F_{e_j, X}^\alpha, F_{e_j, Y}^\beta] - [F_{e_j, Y}^\alpha, F_{e_j, X}^\beta]\}$ for all $X, Y \in T_x M$, $\alpha, \beta = 1, 2, 3$.

Proof. (1) From Proposition 3.2, X and Y are symmetric. Hence, $[F^1, F^2]_{X, Y} = 0$.

(2) From the properties of the Killing form, we have

$$(3.7) \quad \langle [A, B], C \rangle = \langle A, [B, C] \rangle$$

for any $A, B, C \in \wedge^2 T_x^* M \otimes \mathfrak{g}$. Using Proposition 2.1, we see that $[F^1, F^1]_{X, Y} \in S^2\mathbb{H}_x \otimes \mathfrak{g}$, $[F^2, F^2]_{X, Y} \in S^2\mathbb{E}_x \otimes \mathfrak{g}$ and $[F^3, F^3]_{X, Y} \in \wedge^2 T_x^* M \otimes \mathfrak{g}$ and note that

$[F^\alpha, F^\beta] = [F^\beta, F^\alpha]$. Putting $A = F^1$, $B = F^2$ and $C = F^3$ in (3.7) and using $[F^1, F^2]_{X,Y} = 0$, we get

$$(3.8) \quad \langle F^1, [F^2, F^3] \rangle = 0.$$

Putting $A = F^1$, $B = F^3$ and $C = F^2$ in (3.7), we have

$$(3.9) \quad \langle [F^1, F^3], F^2 \rangle = \langle F^1, [F^3, F^2] \rangle.$$

Putting $A = F^1$, $B = F^1$ and $C = F^3$ in (3.7), we get $\langle [F^1, F^1], F^3 \rangle = \langle F^1, [F^1, F^3] \rangle$. From $[F^1, F^1]_{X,Y} \in S^2\mathbb{H}_x \otimes \mathfrak{g}$, we have

$$(3.10) \quad \langle F^1, [F^1, F^3] \rangle = 0.$$

Putting $A = F^2$, $B = F^2$ and $C = F^3$ in (3.7), we get $\langle [F^2, F^2], F^3 \rangle = \langle F^2, [F^2, F^3] \rangle$. From $[F^2, F^2]_{X,Y} \in S^2\mathbb{E}_x \otimes \mathfrak{g}$, we have

$$(3.11) \quad \langle F^2, [F^2, F^3] \rangle = 0$$

From (3.8) and (3.11), we conclude that

$$[F^2, F^3]_{X,Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g}.$$

(3) From (3.8) and (3.9), we get

$$(3.12) \quad \langle [F^1, F^3], F^2 \rangle = 0.$$

From (3.10) and (3.12), we conclude that

$$[F^1, F^3]_{X,Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g}$$

These complete the proof of Proposition 3.3. □

The proof of the following Proposition 3.4 is analogous to that of Proposition (5.6) in [1].

Proposition 3.4. *Let $F_x^\nabla : \mathfrak{so}(4n) \longrightarrow \mathfrak{so}(N)$ be a linear map and λ be a positive real number.*

(I) *If $\|F^\nabla\|^2 \leq (n(4n-1)\lambda^2)/(16(2n-1)^2)$, then*

$$(3.13) \quad (F_x^{\nabla*} \Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) \leq \lambda \|F^\nabla\|^2.$$

(II) *Putting $\lambda = 4(2n-1)$, we have the following:*

If $\|F^\nabla\|^2 \leq n(4n-1)$, then

$$(3.14) \quad (F_x^{\nabla*} \Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) \leq 4(2n-1) \|F^{\nabla}\|^2.$$

The equality holds if and only if there is an orthogonal splitting $\mathbb{R}^N = S_0 \oplus S_1$ ($\dim S_1 = 4$) with respect to which $F_x^{\nabla} = 0 \oplus \sigma$ where σ is a representation $\sigma : \mathfrak{sp}(1) \rightarrow \mathfrak{so}(4)$.

Proof. We shall prove the inequality for $\|F^{\nabla}\|^2 = (n(4n-1)\lambda^2)/(16(2n-1)^2)$. Let $\{e_i \wedge e_j\}_{i < j}$ be the orthonormal basis of $\mathfrak{so}(4n) \cong \wedge^2 T_x M$. Then $\|F^{\nabla}\|^2 = \sum_{i < j} \|F_x^{\nabla}(e_i \wedge e_j)\|^2$ and $(F_x^{\nabla*} \Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) = \sum_{i,j,k=1}^{4n} \langle [F_x^{\nabla}(e_i \wedge e_j), F_x^{\nabla}(e_j \wedge e_k)], F_x^{\nabla}(e_k \wedge e_i) \rangle$. We now denote $F_x^{\nabla}(e_i \wedge e_j)$ by F_{ij} . We introduce the $4n \times 4n$ -symmetric matrix $S = ((s_{ij}))$ with non-negative entries $s_{ij} = \sqrt{2} \|F_{ij}\|$. By the assumption, $\text{tr} S^2 = \sum_{i,j=1}^{4n} s_{ij}^2 = 4 \sum_{i < j} \|F_{ij}\|^2 = (4n(4n-1)\lambda^2)/((4n-2)^2 2^2)$. By Lemma 2.1 we have

$$\text{tr} S^3 = \sum_{i,j,k=1}^{4n} s_{ij} s_{jk} s_{ki} \leq \frac{4n(4n-1)\lambda^3}{(4n-2)^2 2^3}.$$

Therefore, using (2.9), we see that

$$\begin{aligned} (F_x^{\nabla*} \Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) &\leq \sum_{i,j,k=1}^{4n} |\langle [F_{ij}, F_{jk}], F_{ki} \rangle| \\ &\leq \sum_{i,j,k=1}^{4n} \|[F_{ij}, F_{jk}]\| \cdot \|F_{ki}\| \\ &\leq \sum_{i,j,k=1}^{4n} \sqrt{2} \|F_{ij}\| \cdot \|F_{jk}\| \cdot \|F_{ki}\| \\ &= \frac{1}{2} \sum_{i,j,k=1}^{4n} s_{ij} s_{jk} s_{ki} \leq \lambda \|F^{\nabla}\|^2. \end{aligned}$$

Hence, we complete the proof of (I).

We next prove (II). Putting $\lambda = 4(2n-1)$ in (I), we see that

$$\begin{aligned} (F_x^{\nabla*} \Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) &\leq \sum_{i,j,k=1}^{4n} |\langle [F_{ij}, F_{jk}], F_{ki} \rangle| \\ &\leq \sum_{i,j,k=1}^{4n} \|[F_{ij}, F_{jk}]\| \cdot \|F_{ki}\| \\ &\leq \sum_{i,j,k=1}^{4n} \sqrt{2} \|F_{ij}\| \cdot \|F_{jk}\| \cdot \|F_{ki}\| \end{aligned}$$

$$= \frac{1}{2} \sum_{i,j,k=1}^{4n} s_{ij}s_{jk}s_{ki} \leq 4(2n-1) \|F^\nabla\|^2.$$

Suppose now that we have the equality in each line. From the last line we see that $s_{ij} = 1$, and so $\|F_{ij}\| = (1/\sqrt{2})$ for $i \neq j$. From the first and second lines we conclude that, when i, j, k are mutually distinct, $[F_{ij}, F_{jk}] = tF_{ki}$ where $t > 0$. Taking the inner product with F_{ki} and using the equality in each line we see that $t = 1$. Hence, we have

$$(3.15) \quad [F_{ij}, F_{jk}] = F_{ki}$$

for all i, j, k distinct. This equation has a number of consequences. Setting $\alpha_{ijkl} = [F_{ij}, F_{kl}]$, we have $\alpha_{ijkl} = -\alpha_{jikl}$, $\alpha_{ijkl} = -\alpha_{ijlk}$, $\alpha_{ijkl} + \alpha_{iklj} + \alpha_{iljk} = 0$, $\alpha_{ijkl} = \alpha_{klij}$. However, from the definition we see $\alpha_{ijkl} = -\alpha_{klij}$, and so we conclude that

$$(3.16) \quad [F_{ij}, F_{kl}] = 0$$

for i, j, k, l distinct. Comparing (3.15) and (3.16) with (2.5) we conclude that $F_x^\nabla : \mathfrak{so}(4n) \rightarrow \mathfrak{so}(N)$ is a Lie algebra homomorphism. Finally, we observe that by (2.9) each pair (F_{ij}, F_{jk}) for i, j, k distinct is conjugate to a pair of matrices of (2.10). In particular, each of the endomorphisms F_{ij} is supported in the same 4-dimensional subspace. Therefore, we conclude that $F_{ij} : \mathfrak{so}(4n) \rightarrow \mathfrak{so}(4)$ is also a Lie algebra homomorphism. This homomorphism is injective. To see this directly we note that if i, j, k, l are mutually distinct, then it is easy to see that $\langle F_{ij}, F_{kl} \rangle = 0$. The matrices $\{F_{ij}\}_{i < j}$ are orthogonal. Hence F_{ij} is injective. Therefore, $F_{ij} : \mathfrak{so}(4n) \rightarrow \mathfrak{so}(4)$ reduce the Lie algebra homomorphism $F_{ij} : \mathfrak{sp}(1) \rightarrow \mathfrak{so}(4)$. Note that $\mathfrak{so}(4n) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n))^\perp$. This completes the proof of Proposition 3.4. \square

4. Proof of theorems

In this section, we shall give the proofs of theorems stated in Introduction.

Proof of Theorem 1.1. We shall rewrite the Bochner-Weitzenböck formula (2.4).

$$(4.1) \quad \langle \Delta^\nabla \varphi, \varphi \rangle - \langle \nabla^* \nabla \varphi, \varphi \rangle = \langle \varphi \circ \left(\frac{s}{2n} I - 2R \right), \varphi \rangle - \rho(\varphi),$$

where $\rho(\varphi) = \langle [F^\nabla, \varphi], \varphi \rangle$ for any $\varphi \in A^2(\mathfrak{g}_E)$. We put $A = (c_1 - c_2)/(c_1)$ and $B = (c_3 - c_2)/(c_3)$. Substituting $\varphi = AF^1 + BF^3$ into (4.1) and using Proposition 3.1, we have

$$(4.2) \quad -\|\nabla(AF^1 + BF^3)\|^2 = A^2\lambda_1\|F^1\|^2 + B^2\lambda_3\|F^3\|^2 - \rho(AF^1 + BF^3),$$

where $\lambda_i = ((s/(2n))I - 2R_i)_{X,Y} = s/(2n) - 2\mu_i$, $X, Y \in T_x M$. We see that

$$\begin{aligned} \rho(AF^1 + BF^3) &= \langle [F^\nabla, AF^1 + BF^3], AF^1 + BF^3 \rangle \\ &= A^2 \{ \langle [F^1, F^1], F^1 \rangle + \langle [F^2, F^1], F^1 \rangle + \langle [F^3, F^1], F^1 \rangle \} \\ &\quad + AB \{ \langle [F^1, F^1], F^3 \rangle + \langle [F^2, F^1], F^3 \rangle + \langle [F^3, F^1], F^3 \rangle \} \\ &\quad + AB \{ \langle [F^1, F^3], F^1 \rangle + \langle [F^2, F^3], F^1 \rangle + \langle [F^3, F^3], F^1 \rangle \} \\ &\quad + B^2 \{ \langle [F^1, F^3], F^3 \rangle + \langle [F^2, F^3], F^3 \rangle + \langle [F^3, F^3], F^3 \rangle \}. \end{aligned}$$

Using Proposition 3.3, we get

$$(4.3) \quad \begin{aligned} \rho(AF^1 + BF^3) &= A^2 \langle [F^1, F^1], F^1 \rangle + (2AB + B^2) \langle [F^3, F^1], F^3 \rangle \\ &\quad + B^2 \langle [F^2, F^3], F^3 \rangle + B^2 \langle [F^3, F^3], F^3 \rangle. \end{aligned}$$

Since we assume $F^3 = 0$, (4.2) implies that

$$(4.4) \quad -\|\nabla F^1\|^2 = \lambda_1 \|F^1\|^2 - \langle [F^1, F^1], F^1 \rangle.$$

By Proposition 3.4 (I), if $\|F^1\|^2 < (n(4n-1)\lambda_1^2)/(16(2n-1)^2)$, then $\langle [F^1, F^1], F^1 \rangle < \lambda_1 \|F^1\|^2$. Hence, the right hand side of (4.4) is non-negative. On the other hand, the left hand side of (4.4) is non-positive. This is a contradiction. This implies $F^1 = 0$. The same statement is true for F^2 . \square

Proof of Theorem 1.4. From (4.2), (4.3) and using (2.7), we obtain

$$\begin{aligned} -\|\nabla(AF^1 + BF^3)\|^2 &= A^2 \{ \lambda_1 \|F^1\|^2 - \langle [F^1, F^1], F^1 \rangle \} \\ &\quad + B^2 \{ \lambda_3 \|F^3\|^2 - (n+3) \langle [F^3, F^1], F^3 \rangle \\ &\quad - \langle [F^2, F^3], F^3 \rangle - \langle [F^3, F^3], F^3 \rangle \} \\ &\geq A^2 \{ \lambda_1 \|F^1\|^2 - \sqrt{2} \|F^1\|^3 \} \\ &\quad + B^2 \{ \lambda_3 \|F^3\|^2 - \sqrt{2}(n+3) \|F^1\| \|F^3\|^2 \\ &\quad - \sqrt{2} \|F^2\| \|F^3\|^2 - \sqrt{2} \|F^3\|^3 \} \\ &= A^2 \{ (\lambda_1 - \sqrt{2} \|F^1\|) \|F^1\|^2 \} \\ &\quad + B^2 \{ (\lambda_3 - \sqrt{2}(n+3) \|F^1\| - \sqrt{2} \|F^2\| - \sqrt{2} \|F^3\|) \|F^3\|^2 \}. \end{aligned}$$

Hence, if

$$\lambda_1 - \sqrt{2} \|F^1\| > 0 \quad \text{and} \quad \lambda_3 - \sqrt{2}(n+3) \|F^1\| - \sqrt{2} \|F^2\| - \sqrt{2} \|F^3\| > 0,$$

we see that $F^1 = F^3 = 0$. When $F^1 = F^3 = 0$, moreover, from the second inequality stated above, we have $\|F^2\| < \lambda_3/\sqrt{2}$. On the other hand, from the Bochner-

Weitzenböck formula for $F^\nabla = F^2$ and using Proposition 3.3 and (2.7), we get

$$\begin{aligned} \langle \Delta^\nabla F^2, F^2 \rangle - \|\nabla F^2\|^2 &= \lambda_2 \|F^2\|^2 - \langle [F^2, F^2], F^2 \rangle \\ &\geq (\lambda_2 - \sqrt{2} \|F^2\|) \|F^2\|^2. \end{aligned}$$

Since $\Delta^\nabla F^2 = 0$, we have

$$-\|\nabla F^2\|^2 \geq (\lambda_2 - \sqrt{2} \|F^2\|) \|F^2\|^2.$$

If $\|F^2\| < \lambda_2/\sqrt{2}$, then $F^2 = 0$. Thus if $F^2 \neq 0$, then $\lambda_2/\sqrt{2} \leq \|F^2\|$. Consequently, if ∇ is a non-flat, then the c_2 -self-dual part F^2 satisfies $\lambda_2/\sqrt{2} \leq \|F^2\| < \lambda_3/\sqrt{2}$, where λ_1, λ_2 and λ_3 always satisfy $\lambda_1 < \lambda_2 < \lambda_3$ on (M, g) . The same argument can be applied to (2) of Theorem 1.4. \square

Proof of Theorem 1.2. We put $A = (c_1 - c_3)/c_1$ and $B = (c_2 - c_3)/c_2$. Substituting $\varphi = AF^1 + BF^2$ into the Bochner-Weitzenböck formula (4.1) and using Proposition 3.1, we have

$$(4.5) \quad \begin{aligned} -\|\nabla(AF^1 + BF^2)\|^2 &= A^2\{\lambda_1\|F^1\|^2 - \langle [F^1, F^1], F^1 \rangle\} \\ &\quad + B^2\{\lambda_2\|F^2\|^2 - \langle [F^2, F^2], F^2 \rangle\}. \end{aligned}$$

By Proposition 3.4 (I), if

$$\|F^1\|^2 < \frac{n(4n-1)\lambda_1^2}{16(2n-1)^2} \quad \text{and} \quad \|F^2\|^2 < \frac{n(4n-1)\lambda_2^2}{16(2n-1)^2}$$

then

$$\langle [F^1, F^1], F^1 \rangle < \lambda_1 \|F^1\|^2 \quad \text{and} \quad \langle [F^2, F^2], F^2 \rangle < \lambda_2 \|F^2\|^2.$$

Hence, the right hand side of (4.5) is non-negative. Meanwhile, the left hand side of (4.5) is non-positive. This is a contradiction. This implies $F^1 = F^2 = 0$. \square

Proof of Corollary 1.1. Let $\mathbb{H}P^n = Sp(n+1)/Sp(n) \times Sp(1)$ be the quaternionic projective space. Let $sp(n+1) = sp(n) + sp(1) + \mathfrak{m}$ be the orthogonal decomposition of $sp(n+1)$ with respect to Killing form B . We identify \mathfrak{m} with the tangent space of $\mathbb{H}P^n$ at the origin in a natural manner. Let g_0 denote the invariant Riemannian metric on $\mathbb{H}P^n$ defined by $-2(2n-1)(n+2)B|_{\mathfrak{m}}$. The Ricci tensor of $(\mathbb{H}P^n, g_0)$ is given by $Ric(X, Y) = (2n-1)(n+2)g_0(X, Y)$ for $X, Y \in \mathfrak{m}$. Accordingly, the scalar curvature is given by $s = 4n(2n-1)(n+2)$. Corresponding to the decomposition (2.2), we can write the Riemannian curvature operator R as $R = R_1 + R_2 + R_3$. From [4] and for this metric g_0 we know

$$R_1 = n(2n-1)I, \quad R_2 = (2n-1)I, \quad R_3 = 0.$$

We have $\lambda_1 = (s/(2n)I - 2R_1)_{X,Y} = (s/(2n)I - 2n(2n - 1)I)_{X,Y} = 4(2n - 1)$. In the same way, we have $\lambda_2 = 2(2n - 1)(n + 1)$. Substituting λ_1 and λ_2 into Theorem 1.1, we get Corollary 1.1. \square

Proof of Theorem 1.3. When $\|F^\nabla\|^2 \leq n(4n - 1)$, by the Bochner-Weitzenböck formula (4.1) and Proposition 3.4 (II), we conclude that $F^\nabla = 0$. Hence E is flat bundle. When $\|F^\nabla\|^2 \equiv n(4n - 1)$, we get $\nabla F^\nabla = 0$. Proposition 3.4 (II) implies that there is an orthogonal splitting $E = E_0 \oplus S$ where E_0 is flat, where S is a 4-dimensional bundle. By Corollary 1.1 (2) and $\nabla F^\nabla = 0$, $F^\nabla : \wedge^2 TM \rightarrow P \times_{Ad} \mathfrak{so}(4)$ reduces to $F^\nabla : S^2\mathbb{H} \rightarrow P \times_{Ad} \mathfrak{so}(4)$. This implies that the connection ∇ is a c_1 -self-dual connection. The vector bundle \mathbb{H} on any simply-connected quaternionic Kähler manifold with non-zero scalar curvature admits a unique c_1 -self-dual connection ([5]). The vector bundle \mathbb{H} , only when $M = \mathbb{H}P^n$, is globally defined ([8]). Consequently $S \cong \mathbb{H}$, hence $E = E_0 \oplus \mathbb{H}$. \square

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