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Dimension estimate of global attractors for a chemotaxis-growth system and its discretizations

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1. Introduction

This survey is concerned with numerical schemes for nonlinear evolution equations, such as reaction-diffusion equations or chemotaxis systems, which model pattern formation processes. Numerical schemes are necessary to observe dynamics of solutions of such equations numerically or visually. It is also well-known that the asymptotic behavior of solutions relating to patterns can be described by the dynamical systems of equations, and that the degrees of freedom of such processes, which characterize the richness of emerging patterns, correspond to the dimensions of their attractors.

Since numerical schemes work as transformers of equations and dynamical systems, they will transform also the solution trajectories and the structure of attractors to some others: they can violate some important properties of solutions, which may spoil the attractors. Hence, good numerical schemes is needed to produce good numerical analysis, and to reveal suitably the profiles of solutions and the structure of the attractors.

To this end a good question comes to our mind: which scheme can preserve the structure of attractors. We are now in position to study such numerical schemes from the viewpoint of attractor dimension.

In the present survey we consider the following chemotaxis-growth system [13]:

$$
\text{(CG)}\quad
\begin{cases}
\dfrac{\partial u}{\partial t} = a\triangle u - \nabla\cdot\{u\nabla\chi(\rho)\} + g(u) & \text{in } \Omega\times(0,\infty),\\[2mm]
\dfrac{\partial \rho}{\partial t} = b\triangle\rho - c\rho + du & \text{in } \Omega\times(0,\infty),\\[2mm]
\dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\[2mm]
u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in } \Omega
\end{cases}
$$

in a two-dimensional bounded convex domain $\Omega \in \mathbb{R}^2$. Here, a, b, c and d are positive constants. For simplicity, $\chi(\rho)$ is assumed to be linear,

$$\chi(\rho) = \nu\rho$$

with a constant $\nu > 0$, and $g(u)$ is assumed to be a cubic function

$$g(u) = fu^2(1-u)$$

with $f > 0$, respectively.

The system (CG) was presented by Mimura and Tsujikawa [13] as a model to study aggregating patterns of bacteria due to chemotaxis and growth. Such a pattern formation by chemotaxis is considered as to be a prototype of various phenomena of development or morphogenesis in biology [14]. Here, $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of a chemical substance, respectively, at the position $x \in \Omega \subset \mathbb{R}^2$ and time $t \in [0,\infty)$. The constants a and b are the diffusion rates of u and ρ, respectively; c and d are the degradation and production rates of ρ, respectively.

The function $\chi(\rho)$ is a sensitivity function due to chemotaxis. The function $g(u)$ denotes a growth rate of u.

Several authors have already studied the system (CG), and it is well known [1, 2, 18] that, the asymptotic behavior of solutions of (CG) is described by the dynamical system $(S_t, \mathfrak{X}, X)$ in the universal space $X = L^2(\Omega) \times H^1(\Omega)$, where the phase space $\mathfrak{X}$ is a bounded set of $H^2(\Omega) \times \mathcal{D}((-\Delta + 1)^{3/2})$ and, hence, a compact subset of X, and S_t is a nonlinear semigroup acting on $\mathfrak{X}$ which is continuous in the X-norm. Therefore, the dynamical system $(S_t, \mathfrak{X}, X)$ possesses a global attractor $\mathfrak{A} = \bigcap_{0 \le t < \infty} S_t\mathfrak{X}$. The existence of the exponential attractors has been also studied in [18]. Aida et al. [2] showed, with some numerical simulations, that the dimensions of attractors for this system increase as the chemotaxic coefficient ν increases.

On the other hand, in the papers [17, 20], some discretization schemes for chemotaxis systems have been presented, and it was shown that the solutions of the original systems can be well approximated by those of appropriate finite dimensional systems. We will show that, in the case of the system (CG), suitable approximations is essential to preserve the degrees of freedom in the pattern formation process. In this sense, our results might guarantee global reliability of corresponding numerical computations.

The aim of the paper is to estimate from above and below the fractal dimension of the global attractor for (CG) and its semi-discrete approximations in terms of the coefficients a, b, c, d, f and ν in (CG) and the approximation parameter h.

The paper is organized as follows: Section 2 is devoted to show the upper and lower estimate of dimension of the global attractor for (CG), and to explain briefly the estimation schemes. In Section 3 we present two approximations to (CG), and show the upper and lower estimate of dimensions of the global attractors for the approximate systems.

2. Dynamical system and global attractor

As is already mentioned, the system (CG) is globally well-posed in the space $X = L^2(\Omega) \times H^1(\Omega)$, and admits a unique global solution $U(\cdot, U_0)$ for each pair of initial functions $U_0 = (u_0, \rho_0)$ in the phase space $\mathfrak{X}$ which is compact in X. Hence we can define the semigroup of solution operator

$$S_t : U_0 \mapsto U(t, U_0) \quad \text{for } t \ge 0$$

acting on $\mathfrak{X}$ which is continuous in the X-norm. The triplet $(S_t, \mathfrak{X}, X)$ is called the dynamical system governed by the system (CG).

Then a nonempty subset $\mathfrak{A}$ of $\mathfrak{X}$ is called the global attractor for the dynamical system $(S_t, \mathfrak{X}, X)$ if

(i) it is compact in X;
(ii) it is invariant under S_t, that is, $S_t\mathfrak{A} = \mathfrak{A}$ for every $t \ge 0$;
(iii) and it attracts any bounded subset B of X in the sense that

$$\lim_{t \to \infty} h_X(S_t B, \mathfrak{A}) = 0,$$

where h_X denotes the Hausdorff semi-distance between subsets of X, defined by

$$h_X(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X.$$

It follows from its definition that the global attractor, if it exists, is unique, although it is not a smooth manifold, in general, and can have a very complicated geometric structure. According to [21], if $\mathfrak{X}$ is compact in X, we see $\mathfrak{A} = \bigcap_{0 \le t < \infty} S_t \mathfrak{X}$.

If one proves that the global attractor has finite fractal dimension, even though the initial phase space is infinite-dimensional, the dynamics, reduced to the global attractor, is in some specific sense finite-dimensional and can be described by a finite number of parameters. It thus follows that the global attractor appears as a suitable object in view of the study of the long-time behaviour of the system. We refer the reader to [5, 12, 21].

In the paper [15] the authors have obtained the following theorem.

Theorem 1 ([15]). *The dimension of the global attractor* $\mathfrak{A}$ *for the dynamical system governed by* (CG) *satisfy the estimate:*

$$C_1 \nu d \le \dim \mathfrak{A} + 1 \le C_2 \left((\nu d)^2 + 1 \right) \tag{1}$$

with some positive constants C_1 *and* C_2.

To prove the theorem we just apply volume contraction scheme (Babin and Vishik [5, Theorem 10.1.1]) for the upper bound and unstable manifold scheme (Aida et al. [2]) for the lower bound. We recall briefly their ideas for the reader's convenience, since we utilize them commonly in the theorems in the following sections. For the detail of the proof of Theorem 1, see [7, 8, 15].

2.1. **Volume contraction scheme.** Let X be a Hilbert space with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$, let $\mathfrak{X}$ be a compact subset of X, and consider a continuous dynamical system $(S_t, \mathfrak{X}, X)$ with a nonlinear semigroup S_t acting on $\mathfrak{X}$ which is continuous in X. As is seen above, the global attractor of $(S_t, \mathfrak{X}, X)$ is given by $\mathfrak{A} = \bigcap_{0 \le t < \infty} S_t \mathfrak{X}$. Assume that, for each $t \ge 0$, S_t is uniformly quasidifferentiable [5, Definition 10.1.3] on $\mathfrak{X}$ in the norm of X in the sense that, for each $U \in \mathfrak{X}$, there exists a linear operator $S_t'(U)$ in X, called the quasidifferential, such that

$$\|S_t(U_1) - S_t(U) - S_t'(U)(U_1 - U)\|_X \le \gamma_t(\|U_1 - U\|_X)\|U_1 - U\|_X$$

holds for any $t > 0$ and for any $U_1 \in \mathfrak{X}$, where the function $\gamma_t(\zeta)$ is independent of U and U_1 and satisfies $\gamma_t(\zeta) \to 0$ as $\zeta \to 0$. Also assume that, for each $U_0 \in \mathfrak{X}$, the quasidifferential $S_t'(U_0)$ is generated by the evolution equation

$$\frac{dV}{dt} = -\mathcal{A}(S_t U_0)V.$$

It is supposed that the operators $\mathcal{A}(U)$ are densely defined, closed linear operators acting on X and are defined for all $U \in \mathfrak{X}$, and that the domains $\mathcal{D}(\mathcal{A}(U)) \equiv \mathcal{D}$ are constant. Then, by Babin and Vishik [5, Theorem 10.1.1], the dimension $\dim \mathfrak{A}$ of the global attractor $\mathfrak{A}$ is estimated from above by

$$\dim \mathfrak{A} \le \min\{N \in \mathbb{N};\ q_N < 0\}.$$

Here, the number q_N is defined by

$$q_N = \liminf_{T \to \infty} \sup_{U_0 \in \mathfrak{A}} \frac{1}{T} \int_0^T \mathrm{Tr}_N(-\mathcal{A}(S_t U_0))dt,$$

and $\mathrm{Tr}_N(L)$ denotes the N-dimensional trace of the linear operator L defined by

$$\mathrm{Tr}_N(L) = \inf_{\{\phi_j\}} \sum_{j=0}^{N} (L\phi_j, \phi_j)_X \, dt,$$

where $\{\phi_j\} = \{\phi_j \in \mathcal{D}\}_{j=1,2,\ldots}$ are arbitrary orthonormal systems in X. For more detail we refer the reader [5, 21].

2.2. **Unstable manifold scheme.** We will just follow the machinery suggested by Babin and Vishik [5] and by Temam [21]. Their main idea is to construct a smooth unstable manifold $\mathfrak{W}_-^{loc}(U^{eq})$ localized in an open neighborhood $\mathfrak{O}$ of an equilibrium U^{eq} under the assumptions that the associated semigroup is Fréchet differentiable in $\mathfrak{O}$ with derivative of the Hölder class $\mathcal{C}^\alpha$ $(0 < \alpha < 1)$, and that U^{eq} is a hyperbolic equilibrium with finite instability dimension $\dim X_e(U^{eq}) < \infty$, Here, $X_e(U^{eq})$ is the unstable subspace of $-\mathcal{A}(U^{eq})$ which is tangent to $\mathfrak{W}_-^{loc}(U^{eq})$ at the point U^{eq}. We must notice that the global attractor always contains localized unstable manifolds. Since $\mathfrak{W}_-^{loc}(U^{eq})$ is a $\mathcal{C}^{1,\alpha}$ manifold of dimension $\dim X_e(U^{eq})$, we deduce that

$$\dim \mathfrak{A} \geq \dim X_e(U^{eq}).$$

For the system (CG) we can employ this machinery by taking the homogeneous equilibrium $U^{eq} = (1, d/c)$. For more detail, see [2].

3. Global attractors of approximate systems

Nakaguchi and Yagi [16, 17] has formulated a full-discrete approximation for the system (CG) by the consistent-mass finite element scheme and implicit Euler or Runge-Kutta scheme. The scheme is full-implicit, well-posed with no step-size control, and has the error estimate of order $O(h^{1-\varepsilon} + \tau)$ in $H^{1+\varepsilon}$-space for $0 < \varepsilon < 1/2$. The authors have already studied in [9] the dynamics of a semi-discrete approximation to (CG) by consistent-mass finite element scheme. However, that approximation does not preserve an important property, the nonnegativity of the solutions. Consequently, in general one can only obtain much coarser than (1) for (CG) the upper and lower bound for the dimension of global attractor of the approximate system, which will be stated below in Theorem 2.

Recently, Saito in [20] has formulated a full-discrete approximation for a simplified Keller-Segel system, (CG) with $g(u) = 0$ and $\partial\rho/\partial t = 0$, by the conservative upwind finite element scheme by Baba and Tabata [4] and semi-implicit Euler scheme. The scheme is well-posed under some time step-size control $\tau \sim O(h^2)$, and has the error estimate of order $O(h^{1-2/p} + \tau)$ in L^p-space. Moreover, Saito [20] proved conservation of mass and preservation of nonnegativity for the approximate solutions. Since the simplified Keller-Segel system is a variation of (CG), this scheme should be applied to the present system (CG). In the paper [10] the authors have employed this scheme for (CG), constructed the dynamical system for the approximate system, and shown that we can recover the dimension estimate of the global attractor just the same as (1) for (CG), which will be stated below in Theorem 3.

3.1. **Consistent-mass approximation.** We present here a consistent-mass finite element discretization for (CG). For convenience we refer the reader to [16, 17] for the scheme, and to [6, 11, 22] for the general theory of finite element method.

Let $\{\mathcal{T}_h;\ h > 0\}$ be a family of triangulations of Ω with the meshwidth parameter $h = \max\{d_\sigma;\ \sigma \in \mathcal{T}_h\} > 0$, where σ denotes the triangles defining $\mathcal{T}_h$ and d_σ their diameters. We use the following notations: let V_σ be the set of vertices of each triangle $\sigma \in \mathcal{T}_h$; let $\mathcal{P}_h = \bigcup \{V_\sigma;\ \sigma \in \mathcal{T}_h\}$ the set of all vertices in $\mathcal{T}_h$, and $\Lambda_P = \bigcup\{V_\sigma \setminus \{P\};\ \sigma \in \mathcal{T}_h$ such that $P \in V_\sigma\}$ the set of vertices neighboring the vertex $P \in \mathcal{P}_h$. In this paper we assume that

(G1): $\{\mathcal{T}_h\}_{h>0}$ is regular, that is, there exists a positive number $\mu_1 > 0$ independent of h such that $\mu_1 h_\sigma \le \rho_\sigma \le h_\sigma$ holds for every $\sigma \in \mathcal{T}_h$, where ρ_σ is the diameter of the inscribed circle of σ;

(G2): $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, that is, there exists a positive number $\mu_2 > 0$ independent of h such that $\mu_2 h \le h_\sigma \le h$ holds for every $\sigma \in \mathcal{T}_h$

The space of Courant elements is given by

$$Y_h = \left\{v \in C(\overline{\Omega});\ v|_\sigma \text{ is linear in each } \sigma \in \mathcal{T}_h\right\}$$

This space has the finite dimension $M_h = \dim Y_h = \#\mathcal{P}_h$. For each $P \in \mathcal{P}_h$ we associate a function $\phi_P \in Y_h$ by

$$\phi_P(Q) = \delta_{PQ} \quad \text{for } Q \in \mathcal{P}_h,$$

where δ_{xy} denotes Kronecker's delta. Then the set $\{\phi_P;\ P \in \mathcal{P}_h\}$ consists a basis of Y_h, that is, the vector space spanned by $\{\phi_P;\ P \in \mathcal{P}_h\}$ coinsides with Y_h. The interpolation operator $\pi_h : C(\overline{\Omega}) \to Y_h$ is introduced by

$$\pi_h v = \sum_{P \in \mathcal{P}_h} v(P)\phi_P \quad \text{for } v \in C(\overline{\Omega}).$$

We also equip Y_h with the usual L^2-inner product and consider it as a closed subspace of $L^2(\Omega)$. The L^2-orthogonal projection $p_h : L^2(\Omega) \to Y_h$ is introduced by

$$\langle p_h v, \hat{w}\rangle_{L^2} = \langle v, \hat{w}\rangle_{L^2} \quad \text{for } v \in C(\overline{\Omega}) \text{ and } \hat{w} \in Y_h.$$

Then the finite element approximation to (CG) on $Y_h \times Y_h$ is given by

$$(\mathrm{CG}_h)\quad \begin{cases} \dfrac{\partial \hat{u}}{\partial t} = a\triangle_h \hat{u} - \nu\beta_h(\hat{\rho})\hat{u} + f p_h\left[\hat{u}^2(1-\hat{u})\right] & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \hat{\rho}}{\partial t} = b\triangle_h \hat{\rho} - c\hat{\rho} + d\hat{u} & \text{in } \Omega \times (0,\infty), \\ \hat{u}(x,0) = \hat{u}_0(x), \quad \hat{\rho}(x,0) = \hat{\rho}_0(x) & \text{in } \Omega \end{cases}$$

with the initial functions $\hat{u}_0(x), \hat{\rho}_0(x) \in Y_h$, where $\triangle_h$ is the approximate Laplacian operator on Y_h defined by

$$\langle \triangle_h \hat{v}, \hat{w}\rangle_{L^2} = -\langle \nabla\hat{v}, \nabla\hat{w}\rangle_{L^2} \quad \text{for } \hat{v}, \hat{w} \in Y_h,$$

and, for each $\rho \in H^1(\Omega)$, the approximate chemotactic operator $\beta_h(\rho)$ on Y_h is defined by

$$\langle \beta_h(\rho)\hat{v}, \hat{w}\rangle_{L^2} = -\langle \hat{v}\nabla\rho, \nabla\hat{w}\rangle_{L^2}, \quad \text{for } \hat{v}, \hat{w} \in Y_h.$$

As already noticed in [3], the approximate system (CG_h) admits unique global solutions. But we must note that the nonnegativity of solutions to (CG_h) cannot be assured in general.

Also noticed in [3], similarly to the original system (CG), the asymptotic behavior of solutions of (CG_h) is described by the dynamical system $(S_{h,t}, \mathfrak{X}_h, X_h)$ in the universal space $X_h = Y_h \times Y_h$ with the metric of the $L^2 \times H^1$-norm, where the phase space $\mathfrak{X}_h$ is a bounded and, hence, a compact subset of X_h; and $S_{h,t}$ is a nonlinear semigroup acting on $\mathfrak{X}_h$ which is continuous in the X_h-norm. Hence, again according to [21], the dynamical system $(S_{h,t}, \mathfrak{X}_h, X_h)$ possesses a global attractor $\mathfrak{A}_h = \bigcap_{0\le t<\infty} S_{h,t}\mathfrak{X}_h$.

The dimension of $\mathfrak{A}_h$ can be estimated as follows.

Theorem 2 (see [9]). *Let the assumptions* (G1)–(G2) *be fulfilled, and the discretization parameter $h > 0$ be sufficiently small. Then the dimensions of global attractors $\mathfrak{A}_h$ satisfy uniformly with respect to h the estimate:*

$$C_1\nu d \le \dim \mathfrak{A}_h + 1 \le C_2\left((\nu d)^6 + 1\right) \tag{2}$$

with some positive constants C_1 and C_2 which are independent of h.

For the proof, see [9].

3.2. **Nonnegativity-preserving approximation.** Let us now present the conservative upwind finite-element discretization for (CG). First we introduce the scheme of barycentric lumping of masses. For convenience we refer the reader to [11, Sec.5.1], [19, 20] and [22, Chap.15].

We assume in addition to (G1) and (G2) that

(G3): $\{\mathcal{T}_h\}_{h>0}$ is of acute type, that is, every angle of each triangle $\sigma \in \mathcal{T}_h$ is right-angle or acute.

Let $\bar{D}_P$ denote the barycentric domain corresponding to the vertex $P \in \mathcal{P}_h$, and $\bar{\phi}_P$ the characteristic function of $\bar{D}_P$. Let us define the barycentric lumped mass space $\bar{Y}_h$ by the vector space spanned by $\{\bar{\phi}_P; P \in \mathcal{P}_h\}$, that is,

$$\bar{Y}_h = \{\bar{v} \in L^\infty(\Omega);\ \bar{v}|_{\bar{D}_P} \text{ is constant in } \bar{D}_P \text{ for each } P \in \mathcal{P}_h\}.$$

The operator $\bar{L}_h : Y_h \to \bar{Y}_h$ defined by

$$\bar{L}_h\hat{v} = \sum_{P\in\mathcal{P}_h} \hat{v}(P)\bar{\phi}_P \quad \text{for } \hat{v} \in Y_h.$$

is called as the lumping operator. Now we can introduce Y_h a new inner product by

$$(\hat{v}, \hat{w})_b = \langle \bar{L}_h\hat{v}, \bar{L}_h\hat{w}\rangle_{L^2} \quad \text{for } \hat{v}, \hat{w} \in Y_h.$$

Then, by [11, Sec.5.1],

$$\|\hat{v}\|_b = (\hat{v}, \hat{v})_b^{1/2} = \|\bar{L}_h\hat{v}\|_{L^2} \quad \text{for } \hat{v}, \hat{w} \in Y_h$$

is a new norm equivalent to ordinary L^2-norm on Y_h. We denote by $\bar{W}_h$ the space Y_h equipped with the inner product $(\cdot,\cdot)_b$ and the norm $\|\cdot\|_b$ (the subscript "b" means "barycentric lumping").

Next we introduce an upwind approximation (cf. [4, 20]) for the chemotaxis term $\nabla \cdot \{v\nabla\rho\}$. For each $\rho \in H^1(\Omega)$, let us define a linear operator $\bar{\beta}_h(\rho)$ on $\bar{W}_h$ by

$$\begin{aligned}(\bar{\beta}_h(\rho)\hat{v}, \hat{w})_b &= \sum_{P\in\mathcal{P}_h} \hat{w}(P) \sum_{Q\in\Lambda_P} \{\bar{\beta}^+_{PQ}(\rho)\hat{v}(P) - \bar{\beta}^-_{PQ}(\rho)\hat{v}(Q)\} \\ &= \sum_{P\in\mathcal{P}_h} \hat{v}(P) \sum_{Q\in\Lambda_P} \bar{\beta}^+_{PQ}(\rho)\left(\hat{w}(P) - \hat{w}(Q)\right) \quad \text{for } \hat{v}, \hat{w} \in \bar{W}_h\end{aligned}$$

with the upwind coefficient

$$\bar{\beta}^{\pm}_{PQ}(\rho) = \int_{\bar{\Gamma}_{PQ}} [\bar{n}_{PQ} \cdot \nabla \rho]_{\pm}\, dx,$$

where $[x]_{\pm} = \max\{\pm x, 0\}$ denotes the positive/negative part of the number x, $\bar{\Gamma}_{PQ} = \partial \bar{D}_P \cap \partial \bar{D}_Q$ the boundary of adjacent barycentric domains, and $\bar{n}_{PQ}$ the normal vector on $\bar{\Gamma}_{PQ}$ outward from $\bar{D}_P$.

Then the approximation to (CG) on $\bar{W}_h \times \bar{W}_h$ is given by

$$(\mathrm{CG}_h^b)\quad \begin{cases} \dfrac{\partial \hat{u}}{\partial t} = a\bar{\triangle}_h \hat{u} - \nu \bar{\beta}_h(\hat{\rho})\hat{u} + f\pi_h[\hat{u}^2(1-\hat{u})] & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \hat{\rho}}{\partial t} = b\bar{\triangle}_h \hat{\rho} - c\hat{\rho} + d\hat{u} & \text{in } \Omega \times (0,\infty), \\ \hat{u}(x,0) = \hat{u}_0(x), \quad \hat{\rho}(x,0) = \hat{\rho}_0(x) & \text{in } \Omega \end{cases}$$

with the initial functions $\hat{u}_0(x), \hat{\rho}_0(x) \in \bar{W}_h$. where $\bar{\triangle}_h$ is the approximate Laplacian operator on $\bar{W}_h$ defined by

$$(\bar{\triangle}_h \hat{v}, \hat{w})_b = -\langle \nabla \hat{v}, \nabla \hat{w} \rangle_{L^2} \quad \text{for } \hat{v}, \hat{w} \in \bar{W}_h.$$

The unique global existence of nonnegative solutions to (CG_h^b) has been already mentioned in [10]. See also [20].

Then, similarly to the case of consistent-mass case (CG_h), the asymptotic behavior of solutions of (CG_h^b) is described by the dynamical system $(\bar{S}_{h,t}, \bar{\mathfrak{X}}_h, \bar{X}_h)$ in the universal space $\bar{X}_h = \bar{W}_h \times \bar{W}_h$ with the metric of the $L^2 \times H^1$-norm, where the phase space $\bar{\mathfrak{X}}_h$ is a bounded and, hence, a compact subset of $\bar{X}_h$; and $\bar{S}_{h,t}$ is a nonlinear semigroup acting on $\bar{\mathfrak{X}}_h$ which is continuous in the $\bar{X}_h$-norm. Hence, according to [21] again, the dynamical system $(\bar{S}_{h,t}, \bar{\mathfrak{X}}_h, \bar{X}_h)$ possesses a global attractor $\bar{\mathfrak{A}}_h = \bigcap_{0 \le t < \infty} \bar{S}_{h,t}\bar{\mathfrak{X}}_h$.

The dimension of $\bar{\mathfrak{A}}_h$ can be estimated as follows.

Theorem 3 ([10]). *Let the assumptions* (G1)–(G3) *be fulfilled, and the discretization parameter* $h > 0$ *be sufficiently small. Then the dimensions of global attractors* $\bar{\mathfrak{A}}_h$ *satisfy uniformly with respect to* h *the estimate:*

$$(3)\qquad C_1 \nu d \le \dim \bar{\mathfrak{A}}_h + 1 \le C_2\left((\nu d)^2 + 1\right)$$

with some positive constants C_1 *and* C_2 *which are independent of* h.

For the proof, see [10].

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