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Application of Plastic Node Method to Thermal Elastic-Plastic and Dynamic Problems†

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Abstract

In 1968, the author and his colleagues developed the new mechanism of plastic hinge based on the incremental theory of plasticity and derived the elasitc-plastic stiffness matrices for one dimensional members. In 1979, extending the basic idea of the plastic hinge, the authors developed the new method of plastic analysis of plates and solid bodies using the ordinary finite element method and derived the elastic-plastic stiffness matrices. Later, the new method of plastic analysis was named "the plastic node method".

In this paper, the basic theory of the plastic node method is further developed for analyses of thermal elastic-plastic and dynamic behaviors of structures and the stiffness equations are derived. Using the new theory, several examples were analyzed and the good applicability of this method is demonstrated.

KEY WORDS: (Plastic Node Method) (Thermal Elastic-Plastic Problem) (Dynamic Elastic-Plastic Problem) (Welding Residual Stress) (Inertia Force)

1. Introduction

For the elastic-plastic analysis of space-framed structures, the authors developed a new mechanism of plastic hinge based on the plastic flow theory and derived the elastic-plastic and plastic stiffness matrices including the effect of the large deflection[1), 2)].

Recently, this plastic hinge method was generalized to be applicable to a continuum of any geometrical shape[3), 4)]. In this method, using the finite elements of the ordinary finite element method, plastification is examined only at the nodes. Regarding the yield conditions at the nodes as plastic potentials, and applying the plastic flow theory, the elastic-plastic stiffness matrices of the elements for plates and solid bodies as well as one-dimensional members were derived. In the above sense, the authors named this method "the plastic node method"[4)].

In this paper, in order to apply the plastic node method to dynamic elastic-plastic problems accompanied with temperature change, the basic theory is further developed. In derivation of the basic equations, treatment of inertia forces becomes important problems. Here, two types of basic equations are presented, and the analytical results are compared.

Several examples including dynamic and thermal effects are analyzed and the validity and usefulness of this method will be demonstrated.

2. Theory and Analysis

In this chapter, the basic theory of the plastic node method is extended to be applicable to dynamic elastic-plastic problems accompanied with temperature change.

When temperature changes, thermal elastic or elastic-plastic behavior occurs in an object. Then the following effects should be considered comparing with the ordinary elastic-plastic analysis.

(i) Production of thermal strain
(ii) Dependence of material properties (Young's modulus, Poisson's ratio, instantaneous linear expansion coefficient, yield stress, etc.) on temperature.
(iii) Changes of metallic structure (e.g. phase transformation, latent heat)

On the other hand, dynamic behavior should be evaluated in consideration of the following effects. That is,
(iv) Effect of inertia force

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(v) Effect of damping force
(vi) Dependence of material properties (especially yield stress) on strain rate

Within these effects, changes of metallic structure (iii) can be accounted as those of instantaneous linear expansion coefficient and yield stress. The dependence of material properties on strain rate (vi) is not to be considered here, since it can be treated in the same way as the effects of strain hardening which is presently under study. Then the basic theory is developed considering the effects of (i) through (v).

For the concrete formulation, the phenomena are classified into elastic and elastic-plastic ones, and the temperature and dynamic effects are introduced for each case.

The plastic node method is based on the ordinary finite element method (the stiffness method) as mentioned before. Therefore, in the elastic behavior, this method is the ordinary finite element method itself. In this paper, the behavior of an element is described within the framework of infinitesimal deformation.

2.1 Incremental relation between stress and nodal force

Based on the principle of virtual work, the relation between stress increments in an element and nodal force increments can be expressed as follows,

$$\{dx\} = \int_V [B]^t \{d\sigma\} dV \qquad (1)$$

where, $\{dx\}$: nodal force increments corresponding to stress increments
$\{d\sigma\}$: stress increments
$[B]$: nodal displacement-strain matrix

The above relation should be satisfied whether the element is elastic or plastic, temperature does change or not, and the behavior is static or dynamic.

2.2 Elastic stiffness equations

In this section, the elastic stiffness equations of a finite element will be derived.

2.2.1 Static elastic stiffness equation without temperature change

The basic expressions are shown in the following.

(1) Total strain and stress-strain relation

$$\{d\epsilon\} = \{d\epsilon^e\} \qquad (2)$$

where, $\{d\epsilon\}$, $\{d\epsilon^e\}$: total strain increments and elastic strain increments

$$\{d\sigma\} = [D^e]\ \{d\epsilon^e\} = [D^e]\ \{d\epsilon\} \qquad (3)$$

where, $[D^e]$: elastic stress-strain matrix

(2) Total nodal displacement and strain-nodal displacement relation

$$\{du\} = \{du^e\} \qquad (4)$$

$$\{d\epsilon\} = [B]\ \{du\},\ \{d\epsilon^e\} = [B]\ \{du^e\} \qquad (5)$$

where, $\{du\}$, $\{du^e\}$: total nodal displacement increments and elastic nodal displacement increments

(3) Nodal force

$$\{dx\} = \{dx_u\} \qquad (6)$$

where, $\{dx_u\}$: static nodal force increments

(4) Stiffness equation

Using Eqs. (1) through (6), the stiffness equation is derived as follows,

$$\{dx\} = [K^e]\ \{du\} \qquad (7)$$

where, $[K^e] = \int_V [B]^t [D^e] [B] dV$: elastic stiffness matrix

2.2.2 Dynamic elastic stiffness equation with temperature change

(1) Total strain and stress-strain relation[5)]

When temperature change, thermal strain is produced and material properties such as Young's modulus, Poisson's ratio and instantaneous linear expansion coefficient may vary. In this case, total strain increments can be obtained as the sum of the elastic and thermal components.

$$\{d\epsilon\} = \{d\epsilon^e\} + \{d\epsilon^T\} \qquad (8)$$

where, $\{d\epsilon^T\} = \{\alpha\}dT$: thermal strain increments
$\{\alpha\}$: instantaneous linear expansion coefficients
dT : temperature increment

The incremental relation between stresses and strains is expressed as,

$$\{d\sigma\} = [D_d{}^e]\ \{d\epsilon^e\} + \frac{d[D^e]}{dT}\ dT\{\epsilon^e\} \qquad (9)$$

where, $\{\epsilon^e\}$: total elastic strains before the latest load increment
$[D_d{}^e]$: elastic stress-strain matrix at the latest temperature
$\frac{d[D^e]}{dT}dT$: increment of stress-strain matrix due to temperature change

(2) Total nodal displacement and strain-nodal displacement relation

Due to production of thermal strains, thermal nodal displacements are also produced as follows,

$$\{du\} = \{du^e\} + \{du^T\} \tag{10}$$

where, $\{du^T\}$: thermal nodal displacement increments
The incremental relation between nodal displacements and the corresponding strains in an element are expressed for respective component as follows,

$$\{d\epsilon\} = [B]\ \{du\},\ \{d\epsilon^e\} = [B]\ \{du^e\},\ \{d\epsilon^T\} = [B]\ \{du^T\} \tag{11}$$

(3) Nodal force

In the dynamic problems, both inertia and damping forces should be considered in addition to the static ones.
(i) Inertia force

Inertia force acting at any point in an element can be expressed as the product of the mass by acceleration. In the finite element method, it is necessary to concentrate the inertia forces acting in an element at its nodal points in the form of equivalent nodal inertia forces. Here, two types of methods are employed, which are the lumped mass method and the consistent mass method. Generally, easiness to use is with the former, while higher accuracy of the result is with the latter.

(i-i) Inertia force based on the lumped mass method: In this method, the mass is concentrated at the nodes. Usually, the total mass of an element is equally distributed at the nodes and the rotating inertia forces are neglected. Then, the quivalent nodal inertia force increments can be expressed as follows, considering the existence of thermal strains.

$$\{dx_m\} = -[M_l](\{d\ddot{u}^e\} + \{d\ddot{u}^T\}) = -[M_l]\ \{d\ddot{u}\} \tag{12}$$

where, $\{dx_m\}$: equivalent nodal inertia force increments
$\{d\ddot{u}^e\}$, $\{d\ddot{u}^T\}$, $\{d\ddot{u}\}$: increments of elastic, thermal and total nodal displacement acceleration
$[M_l]$: the lumped mass matrix (diagonal matrix)

(i-ii) Inertia force based on the consistent mass method: In this method, the nodal inertia forces are evaluated equivalent to the inertia forces distributed over the entire element. According to the principle of virtual work, the equivalent nodal inertia force increments can be obtained as follows,

$$\{dx_m\} = -[M_c](\{d\ddot{u}^e\} + \{d\ddot{u}^T\}) = -[M_c]\ \{d\ddot{u}\} \tag{13}$$

where, $[M_c] = \int_V \rho [A]^t [A] dV$: the consistent mass matrix
$[A]$: displacement function, $\{du_{el}\} = [A]\ \{du\}$
$\{du_{el}\}$: displacement increments in an element
ρ : density
(ii) Damping force

Regarding damping forces distributed in an element as proportion to velocity, the equivalent nodal damping force increments are expressed as follows,

$$\{dx_h\} = -[H](\{d\dot{u}^e\} + \{d\dot{u}^T\}) = -[H]\{d\dot{u}\} \tag{14}$$

where, $\{dx_h\}$: equivalent nodal damping force increments
$\{d\dot{u}^e\}$, $\{d\dot{u}^T\}$, $\{d\dot{u}\}$: increments of elastic, thermal and total nodal displacement velocity
$[H]$: damping matrix

The above damping matrix, $[H]$, cannot be obtained theoretically at present but is usually evaluated experimentally[6]. In this paper, the details of the damping matrix will not be discussed, and Eq. (14) is used for the expression of the nodal damping force increments in the following derivation.
(iii) Total nodal force increment

The total nodal force increments in the dynamic problems can be expressed as the sum of the static nodal force increments, $\{dx_u\}$, the nodal inertia force increments, $\{dx_m\}$, and the nodal damping force increments, $\{dx_h\}$. That is,

$$\{dx\} = \{dx_u\} + \{dx_m\} + \{dx_h\} \tag{15}$$

(4) Stiffness equation

As mentioned above, the effect of temperature change appears on $\{d\sigma\}$ represented by Eq. (9). On the other hand, the dynamic effect appears on $\{dx\}$ of Eqs. (1) and (15) including the effect of temperature change in the form of Eqs. (13) and (14). Using Eqs. (1) and (8) through (15), the elastic stiffness equations are obtained as follows.
(i) When both temperature change and dynamic effects are considered together:

$$\{dx_u\} = [K_d^e]\ \{du\} + [H]\ \{d\dot{u}\} + [M]\ \{d\ddot{u}\} - [K_d^e]\ \{du^T\} + [dK^e]\ \{u^e\} \tag{16-a}$$

where, $[K_d^e] = \int_V [B]^t [D_d^e]\ [B] dV$: elastic stiffness matrix at the latest temperature
$[dK^e] = \int_V [B]^t [dD^e]\ [B] dV$: increment of elastic stiffness matrix due to temperature charge
$\{u^e\}$: total elastic nodal displacements before the latest load increment

(ii) When only the effect of temperature change is considered:

$$\{dx_u\} = [K_d{}^e]\ \{du\} - [K_d{}^e]\ \{du^T\} + [dK^e]\ \{u^e\} \tag{16-b}$$

(iii) When only the dynamic effect is considered:

$$\{dx_u\} = [K^e]\ \{du\} + [H]\ \{d\dot{u}\} + [M]\ \{d\ddot{u}\} \tag{16-c}$$

2.3 Elastic-plastic stiffness equations

Applying the plastic node method, the elastic-plastic stiffness equations are derived for a finite element with plastic region.

2.3.1 Static elastic-plastic stiffness equation without temperature change[3), 4)]

(1) Total strain and stress-strain relation in an element

In the plastic node method, no plastic strain is produced in an element. Then, strain increments and their relations to stress increments are also expressed by Eqs. (2) and (3).

(2) Total nodal displacement and strain-nodal displacement relation

Plastic displacements being concentrated only at the nodes, the total nodal displacement increments are expressed by the sum of elastic and plastic components.

$$\{du\} = \{du^e\} + \{du^p\} \tag{17}$$

where, $\{du^p\}$: plastic nodal displacement increments

The relation between elastic nodal displacements and elastic strains is the same as Eq. (5) in the elastic condition, while plastic displacements do not have a similar relation.

(3) Plasticity condition and plastic nodal displacement increment

The plasticity condition at the ith node of an element can be expressed by the stress components at the ith node, $\sigma_{xi}, \sigma_{yi}, \ldots\ldots, \tau_{zxi}$ and the yield stress of the material, σ_Y, as follows,

$$f_i(\sigma_{xi}, \sigma_{yi}, \ldots, \tau_{zxi}, \sigma_Y) = 0 \tag{18}$$

Generally, stresses at the ith node of a finite element are represented as a function of the nodal forces at the ith and the other j−1 nodes depending upon the assumed displacement function ($j \leq n$, n: the number of nodes in an element). Therefore, the plasticity condition at the ith node can be expressed by the nodal forces as follows,

$$F_i(\{x\}_1, \{x\}_2 \ldots, \{x\}_j, \sigma_Y) = 0 \tag{19}$$

When plastifications occur at the first through the kth nodes in an element, the plasticity conditions are expressed as follows,

$$F_i(\{x\}, \sigma_Y) = 0 \qquad (i = 1, 2, \ldots, k) \tag{20}$$

As long as the plastic nodes are under loading, the following conditions must be satisfied.

$$0 = dF_i = \{\phi_i\}^t\{dx\} \qquad (i = 1, 2, \ldots, k) \tag{21}$$

where,
$$\{\phi_i\} = \left\{\frac{\partial F_i}{\partial\{x\}_1}\ \frac{\partial F_i}{\partial\{x\}_2} \cdots \frac{\partial F_i}{\partial\{x\}_j}\right\}$$
$$= \left\{\frac{\partial F_i}{\partial\{x\}_1}\ \frac{\partial F_i}{\partial\{x\}_2} \cdots \frac{\partial F_i}{\partial\{x\}_n}\right\}$$
$$= \left\{\frac{\partial F_i}{\partial\{x\}}\right\}, \ \frac{\partial F_i}{\partial\{x\}_l} = \{0\} \ (j < l \leq n)$$

Regarding the plasticity function, F_i, as a plastic potential and applying the plastic flow theory, the plastic nodal displacement increments can be obtained as follows,

$$\{du^p\} = \sum_{i=1}^{k} d\lambda_i\{\phi_i\} = [\phi]\ \{d\lambda\} \tag{22}$$

where, $\{d\lambda\} = \{d\lambda_1 d\lambda_2 \ldots d\lambda_k\}$

$[\Phi] = [\phi_1\phi_2 \ldots \phi_k]$

(4) Nodal force

Nodal force increments, $\{dx\}$, are the same as in the case of static and elastic behavior.

$$\{dx\} = \{dx_u\} \tag{23}$$

(5) Stiffness equation

Using the basic equations presented above, the elastic-plastic stiffness equation is derived. Substituting Eqs. (3), (5), (17) and (22) into Eq. (1), the following equation is obtained.

$$\{dx\} = [K^e]\ (\{du\} - \{du^p\}) = [K^e]\ (\{du\} - [\Phi]\{d\lambda\}) \tag{24}$$

Further substitution of Eq. (24) into Eq. (21) presents as,

$$\{\phi_i\}^t[K^e](\{du\} - [\Phi]\ \{d\lambda\}) = 0 \quad (i = 1, 2, \ldots, k) \tag{25}$$

Equation (25) can be rewritten as,

$$[\Phi]^t[K^e](\{du\} - [\Phi]\{d\lambda\}) = \{0\} \tag{26}$$

Equation (26) composes a set of simultaneous linear equations for $\{d\lambda\}$. Solving this equation, $\{d\lambda\}$ can be obtained as the function of $\{du\}$ as follows,

$$\{d\lambda\} = ([\Phi]^t[K^e]\ [\phi])^{-1}[\Phi]^t[K^e]\ \{du\} \tag{27}$$

Substituting Eq. (27) into Eq. (24) again, the elastic-plastic stiffness equation is obtained as follows,

$$\{dx\} = [K^p]\ \{du\} \tag{28}$$

where, $[K^p] = [K^e] - [K^e]\ [\Phi]\,([\Phi]^t [K^e]\ [\Phi])^{-1}\ [\Phi]^t [K^e]$
: elastic-plastic stiffness matrix

When all nodes become plastic, Eq. (28) gives the plastic stiffness matrix.

2.3.2 Dynamic elastic-plastic stiffness equation with temperature change

(1) Total strain and stress-strain relation in an element

No plastic strain being produced in an element, both Eqs. (8) and (9) are also applicable.

(2) Total nodal displacement and strain-nodal displacement relation

Nodal displacement increments consist of elastic, plastic and thermal displacement components, as

$$\{du\} = \{du^e\} + \{du^p\} + \{du^T\} \tag{29}$$

The incremental relations between elastic displacements and elastic strains and that between thermal displacements and thermal strains are expressed by Eq. (11).

(3) Plasticity condition and plastic nodal displacement increment

Generally, since the yield stress of the material is influenced by temperature, the plasticity condition at the ith node expressed Eq. (20) should be replaced by

$$F_i(\{x\}, \sigma_{Yi}(T_i)) = 0 \quad (i = 1, 2, \ldots, k) \tag{30}$$

where, T_i : temperature at the ith node

In oder that the plastic nodes are under loading, Eq. (21) should be also replaced by

$$0 = dF_i = \{\phi_i\}^t \{dx\} + \frac{\partial F_i}{\partial \sigma_{Yi}} d\sigma_{Yi} \quad (i = 1, 2, \ldots, k) \tag{31}$$

On the other hand, the plastic nodal displacement increments can be expressed by the same form as Eq. (22).

(4) Nodal force

(i) Inertia force

Here the nodal inertia forces of an element which is in the elastic-plastic condition are discussed. Like the case of an elastic element, the lumped mass method and the consistent mass method are applied. When the consistent mass method is used, the nodal inertia force increments are estimated by two approaches.

(i-i) Equivalent nodal inertia forces by the lumped mass method: In this method, the equivalent nodal inertia forces are evaluated as the product of the concentrated nodal mass and total nodal displacement acceleration. Then, their increments, are,

$$\begin{aligned}\{dx_m\} &= -\ [M_l]\,(\{d\ddot{u}^e\} + \{d\ddot{u}^T\} + \{d\ddot{u}^p\}) \\ &= -\ [M_l]\ \{d\ddot{u}\}\end{aligned} \tag{32}$$

where, $\{d\ddot{u}^p\}$: increments of plastic nodal displacement acceleration

(i-ii) Equivalent nodal inertia forces by the consistent mass method-I (plastic displacements are considered to be distributed in an element): In the plastic node method, the discontinuous fields are introduced to the plastic regions for the evaluation of the rigidity of an element. On the other hand, considering the actual elastic-plastic behaviors of the structures, there exists a mass even in the plastic region. From this point of view, in the consistent mass method-I, the continuous fields are introduced only for the evaluation of the equivalent nodal inertia forces.

Here, it is assumed that the plastic displacements are distributed over the entire element in the same pattern of the elastic ones. That is,

$$\{du^p_{el}\} = [A]\ \{du^p\} \tag{33}$$

where, $\{du^p_{el}\}$: plastic displacement increments in an element

Then,

$$\{d\epsilon^p\} = [B]\ \{du^p\} \tag{34}$$

Upon this assumption, the following equation corresponding to Eq. (24) is developed.

$$\begin{aligned}\{dx\} &= \int_V [B]^t \{d\sigma\}\,dV = \int_V [B]^t [D^e]\ \{d\epsilon^e\}\,dV \\ &= \int_V [B]^t [D^e](\{d\epsilon\} - \{d\epsilon^p\})\,dV \\ &= \int_V [B]^t [D^e]\ [B]\ \{du\}\,dV \\ &\quad - \int_V [B]^t [D^e]\ [B]\ \{du^p\}\,dV \\ &= [K^e]\ \{du\} - [K^e]\ \{du^p\} = [K^e]\ \{du^e\}\end{aligned} \tag{35}$$

Equation (35) presents just the same stiffness equation as Eq. (24) which is derived on the assumption that the plastic displacements are concentrated only at the nodes. Accordingly, such assumption as Eq. (33) never be inconsistent with the evaluation of the rigidity in the plastic node method.

Applying the principle of virtual work, the equivalent nodal inertia force increments can be obtained as follows,

$$\begin{aligned}\{dx_m\} &= -\ [M_c]\ (\{d\ddot{u}^e\} + \{d\ddot{u}^p\} + \{d\ddot{u}^T\}) \\ &= -\ [M_c]\ \{d\ddot{u}\}\end{aligned} \tag{36}$$

$\{dx_m\}$ is represented as the function of the total nodal displacement accelerations similarly to Eq. (32).

In Eq. (36), the thermal displacements are also assumed to be distributed with the same displacement function as of the elastic displacements like in Eq. (11).

(i-iii) Equivalent nodal inertia forces by the consistent mass method-II (Plastic displacements are considered to be concentrated only at nodes): In this approach, such characteristic of the plastic node method is strictly applied that although the plastic displacements are developed at the nodes, no region with the mass exists there. Then,

(a) The inertia forces occur only in the elastic region excluding plastic nodes.

(b) The forces acting from the outside of plastic nodes (nodal force) equal to those acting in the elastic region (inside of plastic nodes).

Based on the assumption of (a), the inertia force increments produced in the elastic region are replaced by the equivalent nodal inertia force increments at the tentative nodes (i' and j' in **Fig. 1**) set at the points where the elastic region connects to the plastic nodes (Fig. 1). Based on the principle of virtual work, the following equation is derived.

$$\{du'\}^{*t}\{dx_m'\} = -\int_V \{du_{el}\}^{*t}\rho\{d\ddot{u}_{el}\}dV_{elastic}$$
$$= -\int_V \{du'\}^{*t}[A]^t\rho[A]\{d\ddot{u}'\}dV_{elastic}$$
$$= -\{du'\}^{*t}\int_V [A]^t\rho[A]\,dV_{elastic}\cdot$$
$$\times\{d\ddot{u}'\} = -\{du'\}^{*t}[M_c]\{d\ddot{u}'\} \quad (37)$$

where,

$\{dx_m'\}$: equivalent nodal intertia force increments at tentative nodes

$\{du'\}^*$: virtual displacements at tentative nodes

$\{du_{el}\}^*$: virtual displacements in the elastic region,

$$\{du_{el}\}^* = [A]\ \{du'\}^*$$

$\{d\ddot{u}_{el}\}$: increments of displacement acceleration in the elastic region, $\{d\ddot{u}_{el}\} = [A]\ \{d\ddot{u}'\}$

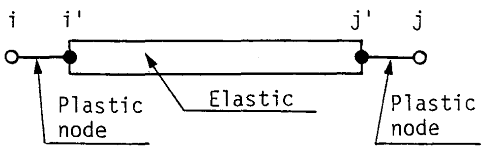


Fig. 1 Element with plastic nodes and inertia forces

Using Eq. (37), the equivalent nodal inertia force increments at the tentative nodes, $\{dx_m'\}$, are obtained. In this case, the tentative nodal displacements are the total nodal displacements minus plastic nodal displacements, that is the sum of elastic nodal displacements and thermal nodal displacements. Therefore, $\{dx_m'\}$ is expressed as follows,

$$\{dx_m'\} = -[M_c]\ \{d\ddot{u}'\} = -[M_c]\ (\{d\ddot{u}^e\}+\{d\ddot{u}^T\})$$
$$= -[M_c]\ (\{d\ddot{u}\}-\{d\ddot{u}^P\}) \quad (38)$$

Here, the tentative nodes are attached to the inside of the plastic nodes. Then, based on the assumption of (b), $\{dx_m'\}$ are regarded as the forces acting at the actual nodes. That is,

$$\{dx_m\} = \{dx_m'\} = -[M_c]\ (\{d\ddot{u}\}-\{d\ddot{u}^P\}) \quad (39)$$

(ii) Damping force

As mentioned in section 2.2.2-(3)-(ii), Eq. (14) presents the equivalent nodal damping force increments. However, in the elastic-plastic condition, it should be noted that the plastic nodal displacement velocity increments are also included in the total ones.

(iii) Total nodal force increment

The total nodal force increments are the sum of the static nodal force increments, $\{dx_u\}$, the nodal inertia force increments, $\{dx_m\}$ and the nodal damping force increments $\{dx_h\}$.

$$\{dx\} = \{dx_u\}+\{dx_m\}+\{dx_h\} \quad (40)$$

(5) Stiffness equation

In the plastic node method, since no plastic strain is developed in an element, the effects of temperature chang except for those on the yield stress can be accounted as the change of the elastic stiffness matrix. Then, substituting Eqs. (9), (11), (29) and (22) into Eq. (1), the elastic characteristics of an element are presented as follows,

$$\{dx\} = [K_d^e](\{du\}-[\Phi]\{d\lambda\}-\{du^T\})$$
$$+[dK^e]\{u^e\} \quad (41)$$

When this equation is substituted into Eq. (31), the resulting equation yields,

$$[\Phi]^T\{[K_d^e](\{du\}-[\Phi]\{d\lambda\}-\{du^T\})$$
$$+[dK^e]\{u^e\}\}+[\Psi]\ \{d\sigma_Y\} = \{0\} \quad (42)$$

where, $[\Psi]$: diagonal matrix of which diagonal elements are

$$\frac{\partial F_1}{\partial\sigma_{Y1}},\ \frac{\partial F_2}{\partial\sigma_{Y2}},\ \ldots,\ \frac{\partial F_k}{\partial\sigma_{Yk}}$$

$$\{d\sigma_Y\} = \{d\sigma_{Y1}\,d\sigma_{Y2}\ \ldots\ d\sigma_{Yk}\}^T,$$

$$d\sigma_{Yi} = \frac{d\sigma_{Yi}}{dT_i}\,dT_i$$

Solving this set of simultaneous equations for $\{d\lambda\}$, $\{d\lambda\}$ can be obtained as follows,

$$\{d\lambda\} = ([\Phi]^t[K_d^e]\,[\Phi])^{-1}[\Phi]^t[K_d^e]\{du\}$$
$$+([\Phi]^t[K_d^e]\,[\Phi])^{-1}(-[\Phi]^t[K_d^e]\{du^T\}$$
$$+[\Phi]^t[dK^e]\{u^e\}+[\Psi]\{d\sigma_Y\}) \quad (43)$$

In section 2.3.2-(4)-(i), three types of equations were derived for evaluation of the equivalent nodal inertia force increments. Formally, they can be classified into two types of equations, that is, [M] $\{d\ddot{u}\}$ for Eqs. (32) and (36), [M] $(\{d\ddot{u}\}-\{d\ddot{u}^P\})$ for Eq. (39). Then, the stiffness equations are derived for the respective type as follows.

(i) When [M] $\{d\ddot{u}\}$ type is used as inertia force increments:

Substituting Eqs. (14), (43) and (32) or (36) into Eq. (41), the elastic-plastic stiffness equations are obtained as follows.

(a) When both temperature change and dynamic effects are considered together:

$$\{dx_u\} = [K_d^p]\{du\} + [H]\{d\dot{u}\} + [M]\{d\ddot{u}\} - [K_d^p]\{du^T\} + [G_{I1}][dK^e]\{u^e\} - [K_d^p][Q]\{d\sigma_Y\} \quad (44\text{-a})$$

where, $[G_{I1}] = [I] - [K_d^e][G]$,

$[Q] = [\Phi]([\Phi]^t[K_d^e][\Phi])^{-1}[\Psi]$,

$[G] = [\Phi]([\Phi]^t[K_d^e][\Phi])^{-1}[\Phi]^t$,

$[I]$: identity matrix

(b) When only the effect of temperature change is considered:

$$\{dx_u\} = [K_d^p]\{du\} - [K_d^p]\{du^T\} + [G_{I1}][dK^e]\{u^e\} - [K_d^e][Q]\{d\sigma_Y\} \quad (44\text{-b})$$

(c) When only the dynamic effect is considered:

$$\{dx_u\} = [K^p]\{du\} + [H]\{d\dot{u}\} + [M]\{d\ddot{u}\} \quad (44\text{-c})$$

(ii) When [M] $(\{d\ddot{u}\} - \{d\ddot{u}^p\})$ type is used as inertia force increments:

Substitution of Eqs. (39) and (14) into Eq. (41) presents as,

$$\{dx_u\} - [M](\{d\ddot{u}\} - \{d\ddot{u}^p\}) - [H]\{d\dot{u}\} = [K_d^e](\{du\} - [\Phi]\{d\lambda\} - \{du^T\}) + [dK^e]\{u^e\} \quad (45)$$

$\{d\ddot{u}^p\}$ included in Eq. (45) can be expressed using Eqs. (22) and (43) as,

$$\{d\ddot{u}^p\} = [\Phi]\{d\ddot{\lambda}\} = [\Phi]\{([\Phi]^t[K_d^e][\Phi])^{-1} \times [\Phi]^t[K_d^e]\{d\ddot{u}\} + ([\Phi]^t[K_d^e][\Phi])^{-1} \times (-[\Phi]^t[K_d^e]\{d\ddot{u}^T\} + [\Phi]^t[d\ddot{K}^e]\{u^e\} + [\Psi]\{d\ddot{\sigma}_Y\})\} \quad (46)$$

Substituting Eqs. (43) and (46) into Eq. (45), the elastic-plastic stiffness equations are obtained as follows.

(a) When both temperature change and dynamic effects are considered together:

$$\{dx_u\} = [K_d^p]\{du\} + [H]\{d\dot{u}\} + [M][G_{I2}]\{d\ddot{u}\} - [K_d^p]\{du^T\} + [G_{I1}][dK^e]\{u^e\} - [K_d^e][Q]\{d\sigma_Y\} + [M]([G][K_d^e]\{d\ddot{u}^T\} - [G][d\ddot{K}^e]\{u^e\} - [Q]\{d\ddot{\sigma}_Y\}) \quad (47\text{-a})$$

where, $[G_{I2}] = [I] - [G][K_d^e]$

(b) When only the effect of temperature change is considered:

$$\{dx_u\} = [K_d^p]\{du\} - [K_d^p]\{du^T\} + [G_{I1}][dK^e]\{u^e\} - [K_d^e][Q]\{d\sigma_Y\} \quad (47\text{-b})$$

(c) When only the dynamic effects is considered:

$$\{dx_u\} = [K^p]\{du\} + [H]\{d\dot{u}\} + [M][G_{I2}]\{d\ddot{u}\} \quad (47\text{-c})$$

Equations (44) and (47) are a set of simultaneous second differential equations with respect to the nodal displacement increments $\{du\}$. In this paper, for integration of these equations, the Newmark's β-method will be used.

3. Examples of Analysis

In this chapter, the validity and usefulness of the plastic node method for the thermal elastic-plastic and dynamic problems will be demonstrated with several examples.

3.1 Thermal elastic-plastic analysis (Analysis of welding residual stress)

The thermal elastic-plastic behavior in a long butt joint of plate by welding is analyzed. An instantaneous heat source is applied along the weld line. In **Fig. 2**, the analytical model and conditions are shown.

For actual analysis of the middle part of the joint, a strip which is prependicular to the weld line and is expressed by the broken lines in Fig. 2 is treated. The assumed geometrical boundary conditions are as follows. The sections of the strip expressed by the broken lines in Fig. 2 are always kept plane, and when there is no rigidity in the weld zone, both parallel and rotating deformations of the sections are allowed. While, when the rigidity is recovered, the rotating deformation is restricted.

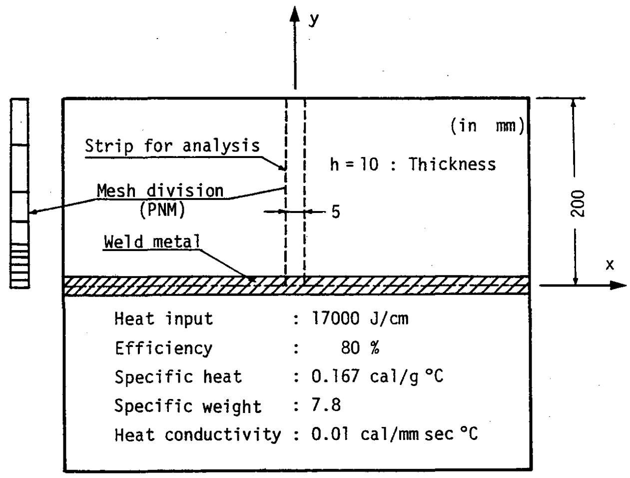


Fig. 2 Model for thermal elastic-plastic analysis (Butt weld)

Both the ordinary finite element method and the plastic node method are adopted for comparison. In the finite element method, 80 constant strain triangular elements are used to regard the solutions by this method as very accurate ones. On the other hand, in the plastic node method, since the plastification is examined at the nodes and the inside of an element is completely elastic, it is simple to obtain accurate solutions by large elements in which changes of strains, stresses and temperature can be considered. Here, 10 rectangular elements in which temperature and strains linearly vary are adopted as shown in Fig. 2.

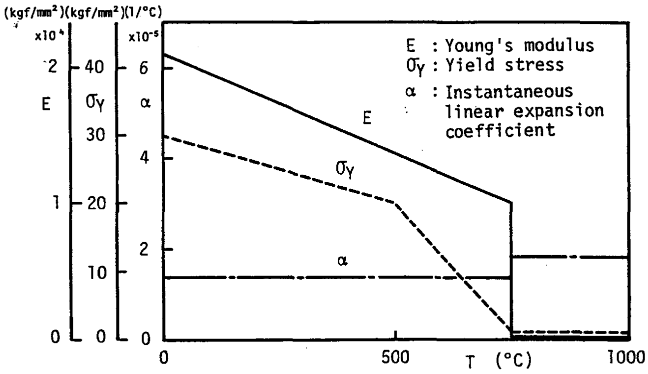
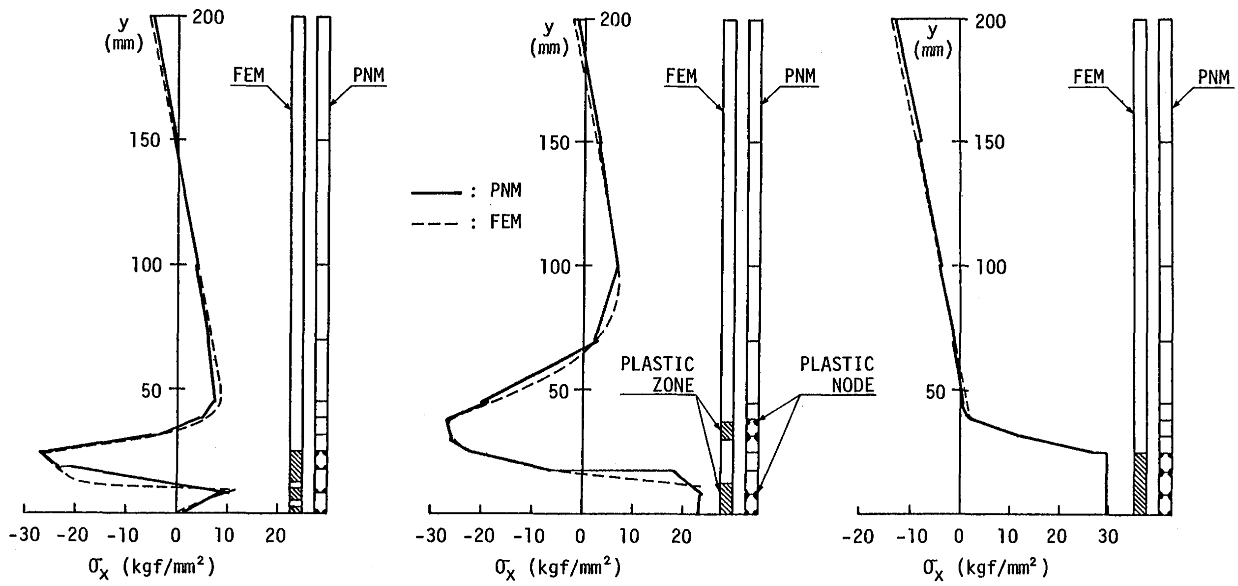


Fig. 3 Mechanical properties of the model



(a) Transient stress 1 (t = 12 sec)
(b) Transient stress 2 (t = 64 sec)
(c) Residual stress (t = ∞)

Fig. 4 Transient and residual stresses, and plastic zone and plastic nodes

The mechanical properties of the material used in the analysis are shown in **Fig. 3**. The mechanically melting temperature (stiffness recovery temperature at cooling stage) is set at 750°C. Changes of metallic structure are neglected in this example. The dependence of an instantaneous linear expansion coefficient on temperature being very small, it is assumed to be constant as shown in Fig. 3. This assumption and the temperature field which varies linearly in an element presents a compatibility of thermal strains in this rectangular element. Under 750°C, both Young's modulus and yield stress are varied linearly in an element depending on temperature. While, above 750°C, for the stability of analysis, Young's modulus is assumed to be constant in an element and evaluated according to the average temperature.

In **Fig. 4**, the resulting transient and residual stress distributions in the transverse direction are shown accompanied with the plastic zones or nodes at each state. Figure 4 (a) shows the state just after the rigidity of the weld zone is recovered, Fig. 4 (b) the following state and Fig. 4 (c) the distribution of residual stresses. The solid and broken lines indicate the solutions obtained by the plastic node method and the finite element method respectively. When the plastic node method is applied, it should be noted that there exist both plastic and elastic nodes in an element owing to the variation of stresses and the difference of the yield stresses at each node. Comparison between the solutions by two methods about the distributions of stresses and plastic zones indicates the good applicability of the plastic node method in spite of rough meshes.

3.2 Analysis of dynamic elastic-plastic behavior

The infinitesimal elastic-plastic dynamic behaviors of beam and plate under impact lateral loads are analyzed using three types of the nodal inertia forces in the plastic node method.

For comparison, the ordinary finite element method is also adopted. The effects of damping force are neglected here.

(1) Time increment

As mentioned before, for time integration of Eqs. (44) and (47), the Newmark's β-method is applied. In this method, when β is set at 1/4, the solutions are unconditionally stable, while at 1/6, conditionally. Therefore, the time increment, dt, can be selected to a comparatively large value in the former case, but is restricted in the latter to be less than the shortest natural period of the analytical model, t_{min}.

In the analyses of elastic response using 1/4 for β, such dt as 4 times t_{min} yielded the exact solutions comparing with the analytical ones. This indicates that 1/4 for β is more useful than 1/6 in the framework of elastic analyses.

On the other hand, in the elastic-plastic analyses, dt can not be always selected to be unconditionally large even when 1/4 is used for β, since both the accuracy of evaluation of yielding and a deviation of stresses from the yields surface are effected by dt. Then, a check for convergence becomes indispensable.

From the above view points, in this paper, 1/6 is adopted for β and approximately one half of t_{min} for dt. t_{min} in the lumped mass method is usually larger than that in the consistent mass method. However, the same value of dt is adopted in the both methods for comparison of their difference. In the elastic-plastic conditions, since the rigidity is reduced, t_{min} obtained by the analysis of eigen value in the elastic condition is also applicable.

(2) Beam under impact lateral load

The dynamic elastic-plastic behavior of rectangular beam fixed at both ends under a centrally concentrated step load is analyzed.

Using one dimensional beam elements, the beam is divided into 20 elements in the longitudinal direction. In the finite element method, two types of further divisions are adopted. In the first type, each element is equally divided into 20 layers in the direction of the thickness, (type EL-I). In the second type, the element is divided not only in the above direction but also the longitudinal into 10 parts in order to strictly distinguish the elastic and plastic regions, (type EL-II). The applied load is 80% of the static collapse load obtained by the plastic analysis of beam and applied in the first time increment.

In the analysis by the plastic node method, the plastic node is developed when the nodal mement reaches the fully plastic moment of the member. In the finite element method, each region in an element yields when the stress at its center reaches the yield stress. The analyses are carried out strictly evaluating both unloading and reloading and the equilibrium conditions are satisfied at each time step.

The adopted combinations of the analysis methods are indicated in **Table 1** and the resulting relations between time and deflection at the center are shown in **Fig. 5**.

Table 1 Methods of analysis (beam)

Method	El-pl analysis (Mesh type)	Inertia force
1	P.N.M.	Lumped
2	P.N.M.	Consistent-I
3	P.N.M.	Consistent-II
4	F.E.M. (El-I)	Lumped
5	F.E.M. (El-II)	Lumped
6	F.E.M. (El-I)	Consistent
7	F.E.M. (El-II)	Consistent

Concerning the solutions by the finite element method, when type E1-II in which the elastic and plastic regions are further strictly distinguished is used, both results by the lumped mass and the consistent mass method approximately coincide with each other. When type E1-I is used, the lumped mass method presents the behavior including oscillation of higher mode. By the consistent mass method, the stable behavior is obtained, but the deflection is comparatively small.

Then, attention is paid to the solutions by the plastic node method. When the lumped mass method and the consistent mass method-I are applied, the solutions by the respective methods coincide almost completely and the stable responses are obtained. Besides, they are close to the solutions by the finite element method using type EL-II. In this example, although the mechanism of static collapse is established by the plastifications at the center and both ends, unlimited deformation does not occur, since the work done by the specific external force is absorbed by the works of the plastic nodes. On the other hand, when the consistent mass method-II is applied, the following behavior is obtained. Right after the initial plastification at the center, the inertia forces are increased rapidly and the neighboring two nodes are plastified. Then, the local mechanism of collapse is established and the solution diverges.

(3) Square plate under impact lateral load

The behavior of a simply supported square plate under uniformly distributed lateral step loads is analyzed. The applied loads are 80% of the static collapse loads. Assuming the deformation of the model can be expressed only

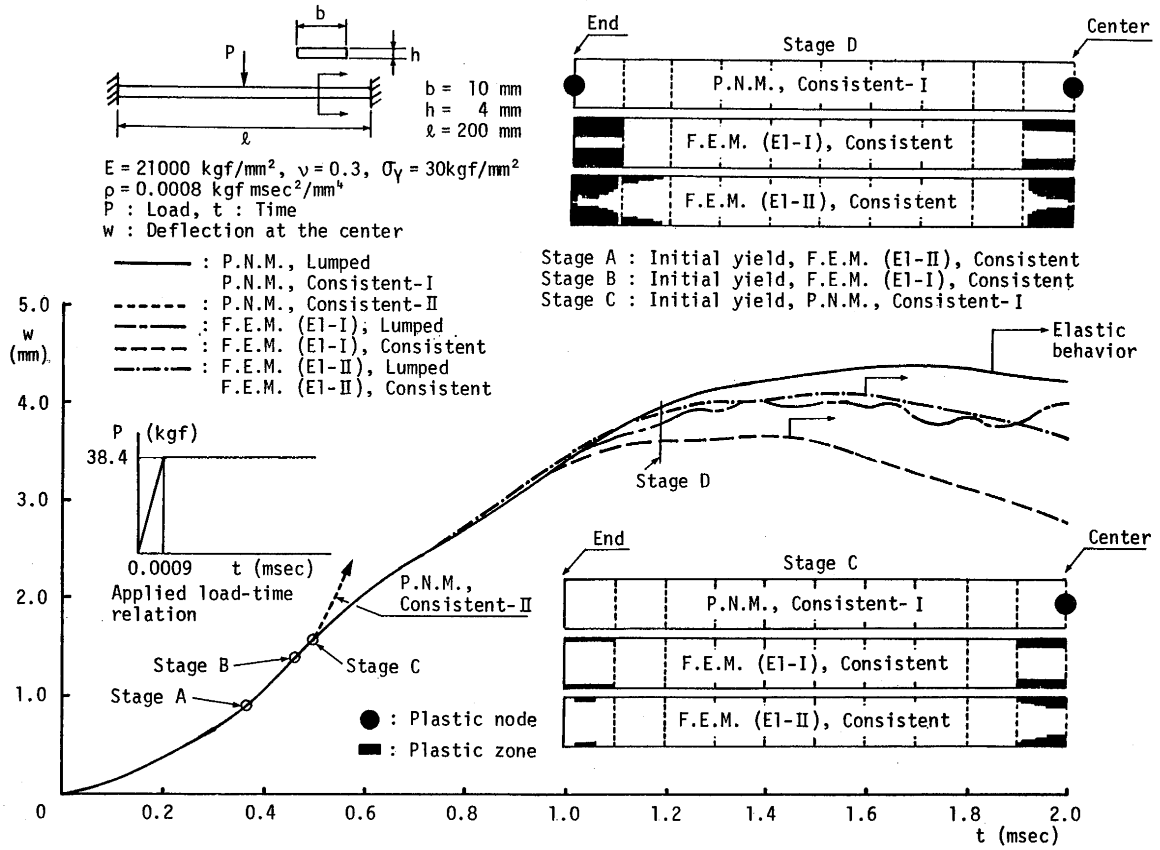


Fig. 5 Time-deflection relations of beams under impact lateral load, and plastic zone and plastic nodes

by the symmetric mode, a 1/8 part of the square plate is analyzed.

As an element, a non-conforming triangular plate bending element is adopted. In this element, the stresses at each node are represented as the function of the all nodal forces in the element. Therefore, in the plastic node method, the virtual plastic nodal displacements are developed even at the elastic nodes as indicated in Eq. (22). In the finite element method, elements are divide into 20 layers.

The cases of analyses are indicated in **Table 2**, and the solutions are shown in **Fig. 6**.

Like the previous example (2), when the consistent mass method-II is applied in the plastic node method, collapse occurs right after the initial plastification, and the solutions diverges, (Fig. 6). From these phenomena, it can be concluded that the discontinuous fields can be introduced usefully for the evaluation of rigidity like in the plastic node method, but the nodal inertia forces in the consistent mass method should be always evaluated regarding the deformations as continuous.

On the other hand, the solutions by the other four methods well coincide with each other, and the validity of the plastic node method using the lumped mass method and the consistent mass method-I is demonstrated.

The characteristics of the above solutions are as follows. Since the plastification occurs at the earlier stage in the finite element method, the velocity of the deflection is larger than that obtained by the plastic node method until the first peak. However, the residual deflection by each method well coincides, since the expansion of the yield zones in this example reaches almost through the thickness of plate even in the case of the finite element method. The computation time by the plastic node method is approximately a half of that by the finite element method.

From the above results, the good applicabilities of the plastic node method to the dynamic elastic-plastic analyses are confirmed.

Table 2 Methods of analysis (plate)

Method	El-pl analysis (Mesh type)	Inertia force
1	P.N.M	Lumped
2	P.N.M.	Consistent-I
3	P.N.M.	Consistent-II
4	F.E.M.	Lumped
5	F.E.M.	Consistent

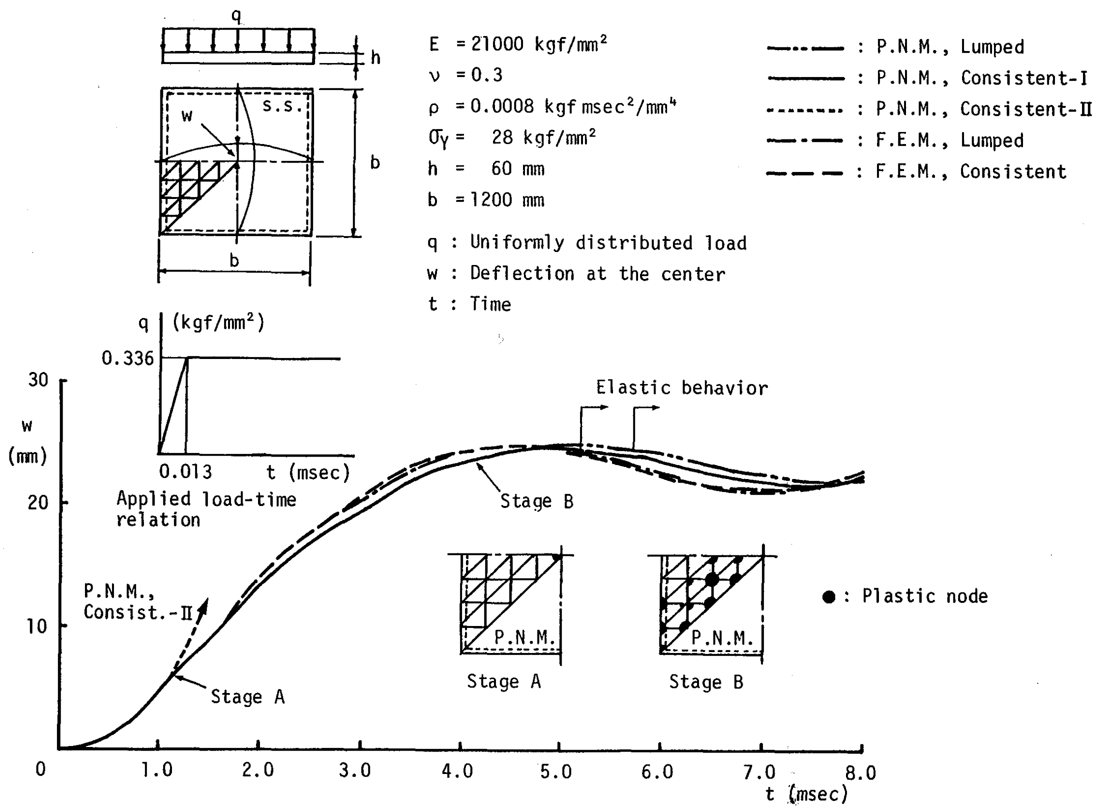


Fig. 6 Time-deflection relations of plates under impact lateral distributed load and plastic nodes

4. Conclusion

In this paper, the theory of the plastic node method is extended to be applicable to the dynamic thermal elastic-plastic problem and some examples of the analysis are demonstrated. The main results are summarized as follows.

(1) Theoretical equations were developed regarding thermal strains, dependence of material properties on temperature and changes of metallic structure as the effect of temperature change, and inertia and damping forces as the dynamic effect. As a result, interaction terms of the respective effects were formed. However, the problems were analyzed being divided into the thermal elastic-plastic and the dynamic elastic-plastic, then the applicability of the theory was investigated.

(2) As a thermal elastic-plastic problem, the transient and residual stresses in a butt joint of a plate by welding were analyzed. Using the rectangular elements in which both the strains and the temperature linearly vary, the highly accurate solutions were obtained in spite of rough meshes.

(3) As for the dynamic elastic-plastic problem, the beam fixed at both ends under centrally concentrated lateral step load and the simply supported square plate under uniformly distributed lateral step loads were analyzed. The solutions by the plastic node method in which the lumped mass method and the consistent mass method-I are used for the inertia forces were in good agreement with those by the finite element method. Then, the good applicability of the plastic node method was confirmed.

According to the result of beam, it also became clear that in spite of the field of an infinitesimal displacement, the member did not collapse even after the mechanism of static collapse was established.

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