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ON THE EQUIVALENCE PROBLEM FOR A CERTAIN CLASS OF GENERALIZED SIEGEL DOMAINS, III

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Introduction. The notion of "generalized Siegel domains in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent c" was introduced by Kaup, Matsushima and Ochiai [3]. In the previous paper [6], we studied exclusively the structure of generalized Siegel domains in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent 1/2. Since then, as an application of the results obtained in [6], we considered the equivalence problem and showed that two generalized Siegel domains in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent 1/2 are holomorphically equivalent only if they are linearly equivalent [7], [8].

In this paper we study the equivalence problem for generalized Siegel domains in $\boldsymbol{C}\times\boldsymbol{C}^m$ with arbitrary exponent. To state our results, we need a few preparations. Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent c and $\mathfrak{g}(\mathcal{D})$ the real Lie algebra consisting of all complete holomorphic vector fields on $\mathcal{D}$. Then, by the definition of $\mathcal{D}$, the Lie algebra $\mathfrak{g}(\mathcal{D})$ contains the following vector field E on $\mathcal{D}$ (see section 1):

$$E=\sum_{k=1}^{n} z_k\frac{\partial}{\partial z_k}+c\sum_{\alpha=1}^{m} w_\alpha\frac{\partial}{\partial w_\alpha},$$

where $(z_1,\cdots,z_n,w_1,\cdots,w_m)$ is the natural coordinate system in $\boldsymbol{C}^n\times\boldsymbol{C}^m$. We put, for any $\lambda\in\boldsymbol{R}$

$$\mathfrak{g}_\lambda=\{X\in\mathfrak{g}(\mathcal{D})\,|\,[E,X]=\lambda X\}\,.$$

Now we can state our results. First of all, we shall prove the following proposition in section 2 (see Proposition 2.6):

Proposition. *Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent $c=1$. Then $\mathfrak{g}(\mathcal{D})$ has the following graded structure:*

$$\mathfrak{g}(\mathcal{D})=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1,\quad [\mathfrak{g}_\lambda,\mathfrak{g}_\mu]\subset\mathfrak{g}_{\lambda+\mu}\,.$$

Combining the results obtained in [3], [4] and [13] with this fact, we obtain the following

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Theorem 1. *Let $\mathcal{D}$ be a generalized Siegel domain in $C^n \times C^m$ with exponent c.*

(1) *If $c \neq 1/2$, then we have*

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}.$$

Moreover, in the case when $n=1$, we have the following direct sum decomposition of $\mathfrak{g}(\mathcal{D})$:

$$\mathfrak{g}(\mathcal{D}) = (\mathfrak{g}_{-1} + \mathfrak{g}_0' + \mathfrak{g}_1) + \mathfrak{g}_0'',$$

where $\mathfrak{g}_0'$ and $\mathfrak{g}_0''$ are vector subspaces of $\mathfrak{g}_0$ such that both $\mathfrak{g}_{-1} + \mathfrak{g}_0' + \mathfrak{g}_1$ and $\mathfrak{g}_0''$ are ideals of $\mathfrak{g}(\mathcal{D})$.

(2) *If $c = 1/2$, then we have*

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1, \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}.$$

Making use of this theorem, we can prove the following

Theorem 2. *Let $\mathcal{D}$ and $\mathcal{D}'$ be generalized Siegel domains in $C \times C^m$ with exponent c and c', respectively. Then $\mathcal{D}$ and $\mathcal{D}'$ are holomorphically equivalent if and only if there exists a non-singular linear mapping $\mathcal{L}$: $C \times C^m \rightarrow C \times C^m$ such that $\mathcal{L}(\mathcal{D}) = \mathcal{D}'$. Moreover, if $\mathcal{D}$ and $\mathcal{D}'$ are holomorphically equivalent, we have $c = c'$.*

After some preliminaries in sections 1 and 2, these two theorems will be proved in sections 3 and 4 respectively.

Now, the following generalization of the classical result due to H. Cartan [2], which states that two bounded circular domains D and D' in C^N containing the origin o are linearly equivalent if there exists a biholomorphic isomorphism f of D onto D' such that $f(o)=o$, will play an important role in the proof of Theorem 2.

Theorem 3. *Let D and D' be two circular domains in C^N containing the origin o of C^N. Suppose that D admits an $Aut(D)$-invariant Kähler metric ds_D^2, where $Aut(D)$ denotes the group of all biholomorphic transformations of D onto itself.*

Then D and D' are holomorphically equivalent if and only if they are linearly equivalent.

In his letter of May 16, 1978, Dr, K. Nakajima kindly announced, but without proof, that this fact is true for bounded circular domains in C^N containing the origin. Since we do not know his proof, we present our proof of this theorem in section 1.

1. Preliminaries

Let $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denotes the field of real (resp. complex) numbers as usual.

Let $(z_1, \cdots, z_n, w_1, \cdots, w_m)$ be the natural coordinate system in $\boldsymbol{C}^n \times \boldsymbol{C}^m$.

DEFINITION 1. A domain $\mathcal{D}$ in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ is called a *generalized Siegel domain with exponent c* if the following conditions are satisfied:

(1) $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{n+m}$ and $\mathcal{D} \cap (\boldsymbol{C}^n \times \{o\}) \neq \phi$, where $\{o\}$ denotes the origin of $\boldsymbol{C}^m$.

(2) $\mathcal{D}$ is invariant by the transformations of $\boldsymbol{C}^{n+m}$ of the following types:

(a) $(z, w) \mapsto (z+a, w)$ for all $a \in \boldsymbol{R}^n$;
(b) $(z, w) \mapsto (z, e^{\sqrt{-1}t} w)$ for all $t \in \boldsymbol{R}$;
(c) $(z, w) \mapsto (e^t z, e^{ct} w)$ for all $t \in \boldsymbol{R}$,

where c is a fixed real number depending only on $\mathcal{D}$. We call c the *exponent of* $\mathcal{D}$.

It is obvious from the definition that the following vector fields on $\mathcal{D}$ are contained in $\mathfrak{g}(\mathcal{D})$:

(a)′ $\partial/\partial z_k$ for $k = 1, 2, \cdots, n$;

(b)′ $I = \sqrt{-1} \sum_{\alpha=1}^{m} w_\alpha \dfrac{\partial}{\partial w_\alpha}$;

(c)′ $E = \sum_{k=1}^{n} z_k \dfrac{\partial}{\partial z_k} + c \sum_{\alpha=1}^{m} w_\alpha \dfrac{\partial}{\partial w_\alpha}$.

For later use, we here study the structure of circular domains in $\boldsymbol{C}^N$.

DEFINITION 2. A domain D in $\boldsymbol{C}^N$ is called a *circular domain* if D is invariant by the rotations

$$l_t\colon (z_1, \cdots, z_N) \mapsto (e^{\sqrt{-1}t} z_1, \cdots, e^{\sqrt{-1}t} z_N), \quad t \in \boldsymbol{R}, \tag{1.1}$$

where $(z_1, \cdots, z_N)$ is a fixed coordinate system in $\boldsymbol{C}^N$.

Let D be a circular domain in $\boldsymbol{C}^N$ which admits an Aut(D)-invariant Kähler metric ds_D^2. Then we have $\mathrm{Aut}(D) \subset \mathrm{Iso}(D)$, where Iso($D$) denotes the group of isometries of D with respect to ds_D^2. Therefore, being a closed subgroup of the Lie group Iso(D), Aut(D) is also a real Lie group. Moreover, the isotropy subgroup of Aut(D) at a point p of D is compact, since the isotropy subgroup of Iso(D) at p is so. We may identify the Lie algebra of Aut(D) with the real Lie algebra $\mathfrak{g}(D)$ consisting of all complete holomorphic vector fields on D. Using the coordinate system $(z_1, \cdots, z_N)$, any vector field X in $\mathfrak{g}(D)$ can be written in the form

$$X = \sum_{k=1}^{N} f_k \frac{\partial}{\partial z_k}, \tag{1.2}$$

where $f_k (k=1, 2, \cdots, N)$ are holomorphic functions on D. Now, suppose further that D contains the origin o of C^N. Then it is easy to see (cf. [6], section 1) that any vector field $X \in \mathfrak{g}(D)$ is a polynomial vector field, that is, in the expression (1.2) of X every component f_k is a polynomial. A vector field X is called a homogeneous polynomial vector field of degree ν, if any component f_k of X in (1.2) is a homogeneous polynomial of degree ν. In this terminology, we put

$$(1.3) \qquad \mathfrak{Z}_\nu = \begin{cases} \text{the set of all homogeneous polynomial vector} \\ \text{fields of degree } \nu \end{cases}$$

and

$$(1.4) \qquad \partial = \sqrt{-1} \sum_{k=1}^{N} z_k \frac{\partial}{\partial z_k},$$

which is the vector field in $\mathfrak{g}(D)$ induced by the global one-parameter subgroup $\{l_t\}_{t \in R}$ defined in (1.1). Then we can show the following

Lemma 1.1 (cf. [6], [11]). *With the same assumptions on D and notation as above, we define an endomorphism J of $\mathfrak{g}(D)$ by $J(X)=[\partial, X]$ for $X \in \mathfrak{g}(D)$. Then, denoting by $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}(D)$ corresponding to the isotropy subgroup K of $Aut(D)$ at the origin $o \in D$, we have*

$$(1.5) \qquad \mathfrak{k} = Ker\, J = \mathfrak{g}(D) \cap \mathfrak{Z}_1,$$

where $Ker\, J$ denotes the kernel of J; and

(1.6) *if we put* $\mathfrak{p} = \{X \in \mathfrak{g}(D) \mid J^2(X) = -X\}$, *then*

$$\begin{cases} \mathfrak{p} = \mathfrak{g}(D) \cap (\mathfrak{Z}_0 + \mathfrak{Z}_2); \\ \mathfrak{g}(D) = \mathfrak{k} + \mathfrak{p} \quad (\textit{direct sum}). \end{cases}$$

Lemma 1.2. *Let D be a circular domain in C^N containing the origin, which admits an $Aut(D)$-invariant Kähler metric ds_D^2. Let G be the identity component of $Aut(D)$ and D_0 the G-orbit passing through the origin o. Then D_0 is a complex submanifold of D. Moreover, it is a Hermitian symmetric space of non-compact type.*

Proof. First we notice that, being a G-orbit passing through the origin $o \in D$, D_0 is a Riemannian submanifold of D. Let g be the Riemannian metric on D_0 induced from ds_D^2. For each element σ of G, we denote by $r(\sigma)$ the restriction of σ to D_0, that is, $r(\sigma) \cdot x = \sigma \cdot x$ for all $x \in D_0$. It is then obvious that r is a Lie group homomorphism of G into the Lie group $\mathrm{Iso}(D_0)$ consisting of all isometries of D_0 with respect to the metric g and $r(G)$ acts transitively on D_0.

Now, assuming that $D_0 \supsetneqq \{o\}$, we shall show that the orbit D_0 is a non-compact complex submanifold of D. Let K be the isotropy subgroup of G at the origin o. We may identify D_0 with the quotient space G/K. Let $\mathfrak{g}(D)=\mathfrak{k}+\mathfrak{p}$ be the direct sum decomposition of $\mathfrak{g}(D)$ as in Lemma 1.1 and $T_0(D_0)$ the tangent space to D_0 at the origin o. Then we have

$$(1.7) \qquad T_0(D_0) = \{X(o) \mid X \in \mathfrak{p}\} ,$$

where $X(o)$ denotes the value at o of the vector field X. We now assert that

$$(1.8) \qquad T_0(D_0) \text{ is a complex subspace of } T_0(D) ,$$

where $T_0(D)$ is the tangent space to D at the origin o. In view of (1.7), it is sufficient to verify the following

$$(1.9) \qquad \sqrt{-1}X(o) \in T_0(D_0) \qquad \text{for every } X \in \mathfrak{p} .$$

For this, take an arbitrary vector field X on D belonging to $\mathfrak{p}$. Then, by Lemma 1.1, X can be wirtten in the form

$$(1.10) \qquad X = X_0 + X_2 \qquad \text{for some} \quad X_0 \in \mathfrak{z}_0 \text{ and } X_2 \in \mathfrak{z}_2 .$$

By a straightforward computation we have

$$(1.11) \qquad J(X) = -\sqrt{-1}X_0 + \sqrt{-1}X_2 \in \mathfrak{p} ,$$

where J is the endomorphism of $\mathfrak{g}(D)$ defined in Lemma 1.1. It follows then that

$$(1.12) \qquad \sqrt{-1}X(o) = \sqrt{-1}X_0(o) = (-J(X))(o) \in T_0(D_0) ,$$

as desired. Now, let I be the G-invariant complex structure on D. By virtue of (1.8) we can define an $r(G)$-invariant tensor field $\tilde{I}$ on $D_0 = G \cdot o$ by requiring that, for any point p of D_0 and any vector $X \in T_p(D_0)$,

$$(1.13) \qquad \tilde{I}_p(X) = I_p(X) .$$

Obviously $\tilde{I}$ then defines a complex structure on D_0, so that D_0 is a complex submanifold of D. Since D is an open subset of $\boldsymbol{C}^N$ and D_0 is a complex submanifold of D of positive dimension, it is evident that D_0 is non-compact.

It remains to prove that D_0 is a Hermitian symmetric space. Let g be the Kähler metric on D_0 induced from ds_D^2. Since ds_D^2 is G-invariant, the group $r(G)$ acts transitively on D_0 as a group of holomorphic isometries. Now, recalling that D is a circular domain in $\boldsymbol{C}^N$, we see that G contains the following element

$$(1.14) \qquad l_\pi \colon (z_1, \cdots, z_N) \mapsto (e^{\sqrt{-1}\pi} z_1, \cdots, e^{\sqrt{-1}\pi} z_N) .$$

It is an easy matter to see that $r(l_\pi)$ is an involutive holomorphic isometry of D_0

and the origin $o \in D_0$ is an isolated fixed point of $r(l_\pi)$. From this our last assertion is obvious, since the group of all holomorphic isometries acts transitively on D_0. q.e.d.

Lemma 1.3. *Let D be a circular domain in C^N as in Lemma 1.2. We denote by G the identity component of Aut(D) and K the isotropy subgroup of G at the orgin o as before. Let K_1 be any compact subgroup of G. Then there exists an element $g \in G$ such that $g^{-1} \cdot K_1 \cdot g \subset K$. In particular, K is a maximal compact subgroup of G and every maximal compact subgroup of G is conjugate to K under an inner automorphism of G.*

Proof. If $G=K$, our assertion is trivial. So we may assume that $G \supsetneqq K$. Then, by Lemma 1.2 the G-orbit $D_0=G/K$ passing through the origin o is a Hermitian symmetric space of non-compact type with $r(G)$-invariant Kähler metric g, where $r: G \to \mathrm{Iso}(D_0)$ is the Lie group homomorphism defined in the proof of Lemma 1.2. Consequently, $D_0=G/K$ is a complete simply connected Riemannian manifold of non-positive sectional curvature. On the other hand, being a subgroup of G, $r(K_1)$ acts on $D_0=G/K$ as a group of isometries. Hence, by a classical result due to E. Cartan [1] we conclude that $r(K_1)$ has a fixed point $p=g \cdot o \in D_0$ $(g \in G)$, that is, $k \cdot g \cdot o = r(k) \cdot (g \cdot o) = g \cdot o$ for every $k \in K_1$. Clearly this implies our assertion. q.e.d.

Proof of Theorem 3. It is trivial that D and D' are holomorphically equivalent, if they are linearly equivalent. Thus we have only to prove the converse.

Suppose that there exists a biholomorphic isomorphism $\Phi: D \to D'$ of D onto D'. Let G (resp. G') be the identity component of $\mathrm{Aut}(D)$ (resp. $\mathrm{Aut}(D')$) and K (resp. K') the isotropy subgroup of G (resp. G') at the origin o. Now we have two cases to consider. Consider first the case where $G \cdot o = o$, that is, the origin o is invariant under G. In this case we have

$$\Phi(o) = \Phi(G \cdot o) = G' \cdot \Phi(o). \tag{1.15}$$

Since the group G' contains the global one-parameter subgroup $\{l_t\}_{t \in R}$ as defined in (1.1), this means that $\Phi(o)=o$. Taking a real number θ arbitrarily, we now consider the following biholomorphic transformation f of D onto itself defined by the composition

$$f = \Phi^{-1} \cdot l_{-\theta} \cdot \Phi \cdot l_\theta \tag{1.16}$$

where $\Phi^{-1}: D' \to D$ denotes the inverse mapping of Φ and l_θ the rotation defined in (1.1). Then we have

(1.17) $f(o) = o$ and

(1.18) the differential $(f_*)_0$: $T_0(D)\to T_0(D)$ of f at o is the identity mapping. Noting that the isotropy subgroup K is compact, we can see from the proof of Theorem 3.3, Chap. V of [5] that, under these two conditions, f must be the identity transformation of D. Hence, if we put $\Phi=(\Phi_1, \cdots, \Phi_N)$, we have from (1.16) that

$$\Phi_j(e^{\sqrt{-1}\theta}z) = e^{\sqrt{-1}\theta}\Phi_j(z) \qquad \text{for all } \theta\in \boldsymbol{R}\,, \tag{1.19}$$

from which we conclude that each component Φ_j is linear (cf. [10], p. 67).

We next consider the case where $G\cdot o \supsetneqq \{o\}$. By virtue of Lemma 1.3, we can choose an element g of G' in such a way that $g^{-1}\cdot(\Phi\cdot K\cdot\Phi^{-1})\cdot g=K'$, since $\Phi\cdot K\cdot\Phi^{-1}$ is a maximal compact subgroup of G'. Considering a biholomorphic isomorphism $\tilde{\Phi}$: $D\to D'$ defined by $\tilde{\Phi}=g^{-1}\cdot\Phi$, we have

$$\tilde{\Phi}(o) = \tilde{\Phi}(K\cdot o) = K'\cdot\tilde{\Phi}(o)\,. \tag{1.20}$$

Since the isotropy subgroup K' contains the global one-parameter subgroup as defined in (1.1), it follows from (1.20) that $\tilde{\Phi}(o)=o$. Repeating exactly the same arguments as in the first case, we conclude that $\tilde{\Phi}$ is linear, completing the proof.

We finish this section by a recent result on the cancellation problem due to Urata. This will be used in the proof of Theorem 2 in section 4.

Theorem U (Urata [12]). *Let X, Y and V be complex analytic spaces such that $V\times X$ is biholomorphic to $V\times Y$. If V is hyperbolic in the sense of Kobayashi* [6], *then X is biholomorphic to Y.*

2. The structure of generalized Siegel domains in $C^n\times C^m$ with exponent $c=1$

Throughout this section we denote by $\mathcal{D}$ a generalized Siegel domain in $C^n\times C^m$ with exponent $c=1$.

Let $Z_{\mu\nu}$(resp. $W_{\mu\nu}$) be a polynomial vector field on $\mathcal{D}$ having the following form

$$Z_{\mu\nu} = \sum_{k=1}^{n} P^k_{\mu\nu}\frac{\partial}{\partial z_k} \quad \left(\text{resp.}\quad W_{\mu\nu} = \sum_{\alpha=1}^{m} Q^\alpha_{\mu\nu}\frac{\partial}{\partial w_\alpha}\right), \tag{2.1}$$

where $P^k_{\mu\nu}$ (resp. $Q^\alpha_{\mu\nu}$) are homogeneous polynomials of degree μ in $z_l\,(1\leqq l\leqq n)$ and of degree ν in w_β $(1\leqq\beta\leqq m)$. We denote by $\mathfrak{Z}_{\mu\nu}$ (resp $\mathfrak{W}_{\mu\nu}$) the set of all vector fields of the form (2.1), that is,

$$\mathfrak{Z}_{\mu\nu} = \{Z_{\mu\nu}\} \qquad (\text{resp. } \mathfrak{W}_{\mu\nu}=\{W_{\mu\nu}\})\,. \tag{2.2}$$

Then, as we have observed in section 1 of [9], we have the following bracket relation in the case $c=1$:

$$(2.3)\qquad \begin{cases} [E,\ Z_{\mu\nu}] = (\mu+\nu-1)Z_{\mu\nu}; \\ [E,\ W_{\mu\nu}] = (\mu+\nu-1)W_{\mu\nu}; \\ [I,\ Z_{\mu\nu}] = \sqrt{-1}\nu Z_{\mu\nu}; \\ [I,\ W_{\mu\nu}] = \sqrt{-1}(\nu-1)W_{\mu\nu}, \end{cases}$$

where E and I are vector fields on $\mathcal{D}$ defined in section 1.

Now, as in the case where $\mathcal{D}$ is a generalized Siegel domain with exponent 1/2 (cf. [3], Lemma 3.1), we can see that every holomorphic vector field X in $\mathfrak{g}(\mathcal{D})$ can be written in the form

$$(2.4)\qquad X = \sum_{\mu\geqq 0} \{Z_{\mu 0}+Z_{\mu 1}+W_{\mu 0}+W_{\mu 1}+W_{\mu 2}\}.$$

Using (2.3), we have then

$$(2.5)\qquad adE\cdot X = \sum_{\mu\geqq 0} \{(\mu-1)Z_{\mu 0}+\mu Z_{\mu 1}+(\mu-1)W_{\mu 0}+\mu W_{\mu 1}+(\mu+1)W_{\mu 2}\}.$$

Hence, putting

$$(2.6)\qquad X_\mu = Z_{(\mu+1)0}+Z_{\mu 1}+W_{(\mu+1)0}+W_{\mu 1}+W_{(\mu-1)2}$$

for $\mu=-1, 0, 1, 2, \cdots$, we can verify easily that

$$(2.7)\qquad X = \sum_{\mu\geqq -1} X_\mu$$

and

$$(2.8)\qquad \Phi(adE)\cdot X = \sum_{\mu\geqq -1} \Phi(\mu)X_\mu$$

for every polynomial $\Phi(x)\in \boldsymbol{R}[x]$. Thus, by the same reasoning as in section 3 of [3], we obtain the following proposition (cf. [3], Theorem 2):

Proposition 2.1. *Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ with exponent $c=1$. For each $\mu\geqq -1$, let $\mathfrak{g}_\mu$ be the vector subspace of $\mathfrak{g}(\mathcal{D})$ consisting of all vector fields in $\mathfrak{g}(\mathcal{D})$ of the form* (2.6). *Then we have*

(2.9) *$\mathfrak{g}_\mu$ is the eigen space of adE for the eigen-value μ;*

(2.10) $\mathfrak{g}(\mathcal{D}) = \sum_{\mu\geqq -1} \mathfrak{g}_\mu$;

(2.11) $[\mathfrak{g}_\mu, \mathfrak{g}_\nu]\subset \mathfrak{g}_{\mu+\nu}$.

Lemma 2.2. *For $\mu=-1, 0, 1, 2, \cdots$, we have*

$$\mathfrak{g}_\mu = \mathfrak{g}'_\mu+\mathfrak{g}''_\mu \ \textit{(direct sum), where}$$
$$\begin{cases} \mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D})\cap(\mathfrak{Z}_{(\mu+1)0}+\mathfrak{W}_{\mu 1}); \\ \mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D})\cap(\mathfrak{Z}_{\mu 1}+\mathfrak{W}_{(\mu-1)2}). \end{cases}$$

Moreover, we have $\mathfrak{g}_{-1}=\{\sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \mid (a_1, \cdots, a_n) \in \boldsymbol{R}^n\}$.

Proof. Let X be an arbitrary vector field on $\mathcal{D}$ belonging to $\mathfrak{g}_\mu$. Then, assuming that X has the form as in (2.6), we have by a routine calculation that

$$(2.12) \qquad \begin{cases} adI \cdot X = \sqrt{-1} Z_{\mu 1} - \sqrt{-1} W_{(\mu+1)0} + \sqrt{-1} W_{(\mu-1)2} ; \\ (adI)^2 \cdot X = -\{Z_{\mu 1} + W_{(\mu+1)0} + W_{(\mu-1)2}\} , \end{cases}$$

from which we obtain $\mathfrak{g}_\mu = \mathfrak{g}'_\mu + \mathfrak{g}''_\mu$ (direct sum), where

$$(2.13) \qquad \begin{cases} \mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{(\mu+1)0} + \mathfrak{W}_{\mu 1}) ; \\ \mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu 1} + \mathfrak{W}_{(\mu+1)0} + \mathfrak{W}_{(\mu-1)2}) . \end{cases}$$

Now, we shall prove that $\mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu 1} + \mathfrak{W}_{(\mu-1)2})$ in (2.13). The proof is by induction on μ. Let W_{00} be an arbitrary element of $\mathfrak{g}''_{-1}$. Then we see from (2.12) that W_{00} and $\sqrt{-1} W_{00}$ are contained in $\mathfrak{g}(\mathcal{D})$, so that $W_{00}=0$ by Cartan's principle: $\mathfrak{g}(\mathcal{D}) \cap \sqrt{-1}\mathfrak{g}(\mathcal{D}) = \{0\}$. Thus, our assertion is really true for $\mu=-1$. Supposing that our assertion holds for $\mu \geqq -1$, we take an arbitrary vector field X on $\mathcal{D}$ belonging to $\mathfrak{g}''_{\mu+1}$. By (2.13) X may be written in the form

$$(2.14) \qquad X = Z_{(\mu+1)1} + W_{(\mu+2)0} + W_{\mu 2} .$$

Then, since $[\partial/\partial z_k, X] \in \mathfrak{g}''_\mu$, $[\partial/\partial z_k, Z_{(\mu+1)1}] \in \mathfrak{Z}_{\mu 1}$, $[\partial/\partial z_k, W_{(\mu+2)0}] \in \mathfrak{W}_{(\mu+1)0}$, $[\partial/\partial z_k, W_{\mu 2}] \in \mathfrak{W}_{(\mu-1)2}$ for every $k=1, 2, \cdots, n$ and the $\mathfrak{W}_{(\mu+1)0}$-component of any vector field belonging to $\mathfrak{g}''_\mu$ does not appear by the induction assumption, we conclude that

$$(2.15) \qquad [\partial/\partial z_k, W_{(\mu+2)0}] = 0 \qquad \text{for} \quad k=1, 2, \cdots, n ,$$

which implies that $W_{(\mu+2)0}=0$. We have thus proved that $X=Z_{(\mu+1)1}+W_{\mu 2}$, and so $\mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu 1} + \mathfrak{W}_{(\mu-1)2})$ for every μ. As a consequence of this fact, we also see that $\mathfrak{g}_{-1} = \mathfrak{g}(\mathcal{D}) \cap \mathfrak{Z}_{00}$. Once it is shown that the coefficient of every vector field on $\mathcal{D}$ belonging to $\mathfrak{g}_{-1}$ are real, our proof is completed. But this follows from the proof of Theorem 3 in [3]. q.e.d.

Lemma 2.3. *Let* $\mathfrak{r}$ *be the radical of* $\mathfrak{g}(\mathcal{D})$. *Then we have*

$$\mathfrak{r} = \sum_{\mu \geqq -1} \mathfrak{r}_\mu , \qquad \textit{where} \quad \mathfrak{r}_\mu = \mathfrak{r} \cap \mathfrak{g}_\mu .$$

Moreover, $\mathfrak{r}_\mu = \mathfrak{g}_\mu$ *for* $\mu \geqq 2$.

Proof. This can be proved in exactly the same way as Lemma 4.1 in [3]. q.e.d.

Now, let $A=\sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k}$ $(a_k \in \boldsymbol{R})$ be an element of $\mathfrak{g}_{-1}$. According to Kaup, Matsushima and Ochiai [3], we shall define the linear mapping $\Phi_A: \mathfrak{g}_1 \rightarrow \mathfrak{g}_{-1}$ by

$$\Phi_A(X) = (1/2)(adA)^2 \cdot X \qquad \text{for} \quad X \in \mathfrak{g}_1 . \tag{2.16}$$

Then, using the concrete expression of X as in (2.6), we can show by a straightforward computation that

$$X(\sqrt{-1}a, 0) = -\Phi_A(X)(\sqrt{-1}a, 0) \qquad \text{for all} \quad X \in \mathfrak{g}_1 , \tag{2.17}$$

where $a=(a_1, \cdots, a_n)$. From this we can verify easily the following lemmas with the same arguments as in the proofs of Lemmas 4.2 and 4.3 in [3]. So we will omit the proofs.

Lemma 2.4. $\mathfrak{r} \cap \mathfrak{g}_1 = \{0\}$.

Lemma 2.5. $\mathfrak{g}_\mu = \{0\}$ *for* $\mu = 2, 3, \cdots$

Thus, summing up we have the following

Proposition 2.6. *Let $\mathcal{D}$ be a generalized Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ with exponent $c=1$. For each $\mu \geqq -1$, let $\mathfrak{g}_\mu$ be the subspace of $\mathfrak{g}(\mathcal{D})$ as defined in Proposition* 2.1. *Then we have*

$$\mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1 , \qquad [\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subset \mathfrak{g}_{\mu+\nu} , \tag{2.18}$$

where

$$\mathfrak{g}_{-1} = \{\sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \mid (a_1, \cdots, a_n) \in \boldsymbol{R}^n\} , \tag{2.19}$$

(2.20) *for* $\mu=0, 1$, $\mathfrak{g}_\mu=\mathfrak{g}'_\mu+\mathfrak{g}''_\mu$ (*direct sum*), *where*

$$\begin{cases} \mathfrak{g}'_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{(\mu+1)0}+\mathfrak{W}_{\mu 1}) ; \\ \mathfrak{g}''_\mu = \mathfrak{g}(\mathcal{D}) \cap (\mathfrak{Z}_{\mu 1}+\mathfrak{W}_{(\mu-1)2}) . \end{cases}$$

3. Proof of Theorem 1

Throughout this section we denote by $\mathcal{D}$ a generalized Siegel domain in $\boldsymbol{C} \times \boldsymbol{C}^m$ with exponent c, unless otherwise stated. By change of linear coordinates if necessary, we may assume without loss of generality that $(\sqrt{-1}, 0) \in \mathcal{D}$.

Lemma 3.1. *If $(z, w) \in \mathcal{D}$, then $Im.z>0$.*

Proof. In the case where the exponent c of $\mathcal{D}$ is non-zero, this can be verified in the same way as in the proof of Lemma 1 in [6].

We next consider the case $c=0$. Suppose that there exists a point

$(z_0, w_0)\in\mathcal{D}$ such that $\mathrm{Im}.z_0\leqq 0$. Then, by the same reasoning as in the proof of Lemma 1 in [6], $\mathcal{D}$ contains a point of the form $(0, \tilde{w}_0)$. Then, by the definition 1, $\mathcal{D}$ also contains the set $\{(a, \tilde{w}_0)\in \boldsymbol{C}\times\boldsymbol{C}^m \mid a\in\boldsymbol{R}\}$. Moreover, since $\mathcal{D}$ is open in $\boldsymbol{C}\times\boldsymbol{C}^m$, we can choose a positive number r_0 in such a way that the points $-(\sqrt{-1}r_0, \tilde{w}_0)$ and $(\sqrt{-1}r_0, \tilde{w}_0)$ are contained in $\mathcal{D}$. Then $\mathcal{D}$ also contains the set $\{(z, \tilde{w}_0)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid \mathrm{Im}.z\neq 0\}$. As a result, we conclude that $\mathcal{D}$ contains the set $\{(z, \tilde{w}_0)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid z\in\boldsymbol{C}\}$, which is naturally identified with $\boldsymbol{C}$. But, since $\mathcal{D}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{m+1}$, this is a contradiction. q.e.d.

Lemma 3.2. *We put* $\mathcal{D}_{\sqrt{-1}}=\{w\in\boldsymbol{C}^m \mid (\sqrt{-1}, w)\in\mathcal{D}\}$. *Then*

(3.1) $\mathcal{D}_{\sqrt{-1}}$ *is a circular domain in* $\boldsymbol{C}^m$ *containing the origin* o;

(3.2) $\mathcal{D}=\{(z, w)\in\boldsymbol{C}\times\boldsymbol{C}^m \mid Im.z>0,\ w/(Im.z)^c\in\mathcal{D}_{\sqrt{-1}}\}$.

Proof. This is immediate from the definition of $\mathcal{D}$ and Lemma 3.1. q.e.d.

Proof of Theorem 1. The second statement (2) of the theorem is nothing but a result due to Kaup, Matsushima and Ochiai [3]. Moreover, combining Theorem 3.2 in Kaup and Upmeier [4] and Proposition 7.1 in Vey [13] with Proposition 2.6 in section 2, we obtain the first assertion of (1).

In the following part of the proof, we denote by $\mathcal{D}$ a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent c. We have now two cases to consider. Consider first the case $c=0$. Then, by Lemma 3.2, $\mathcal{D}$ is the direct product $\mathfrak{H}\times\mathcal{D}_{\sqrt{-1}}$, where $\mathfrak{H}$ is the upper half plane $\{z\in\boldsymbol{C} \mid \mathrm{Im}.z>0\}$ and $\mathcal{D}_{\sqrt{-1}}$ is the circular domain defined in the same Lemma 3.2. Combining this fact with Propositions 7.1 and 8.1 in Vey [13], we can see that $\mathfrak{g}(\mathcal{D})$ has the following structure:

(3.3) $\mathfrak{g}(\mathcal{D})=\mathfrak{g}(\mathfrak{H})+\mathfrak{g}(\mathcal{D}_{\sqrt{-1}})$ (direct sum of ideals) ;

(3.4) $\mathfrak{g}(\mathfrak{H})=\mathfrak{g}_{-1}+\mathfrak{g}_0'+\mathfrak{g}_1$;

(3.5) $\mathfrak{g}(\mathcal{D}_{\sqrt{-1}})=\mathfrak{g}(\mathcal{D})\cap(\mathfrak{W}_{00}+\mathfrak{W}_{01}+\mathfrak{W}_{02})\subset\mathfrak{g}_0$,

where

(3.6) $\mathfrak{g}_{-1}=\{a\frac{\partial}{\partial z} \mid a\in\boldsymbol{R}\}$;

(3.7) $\mathfrak{g}_0'=\mathfrak{g}(\mathcal{D})\cap\mathfrak{Z}_{10}\subset\mathfrak{g}_0$;

(3.8) $\mathfrak{g}_1=\mathfrak{g}(\mathcal{D})\cap\mathfrak{Z}_{20}$.

Therefore, putting $\mathfrak{g}_0''=\mathfrak{g}(\mathcal{D}_{\sqrt{-1}})$, we have our assertion.

Consider next the case $c\neq 0$. By Theorem 3.2 in Kaup and Upmeier [4], the linear mapping $(ad(\partial/\partial z))^2\colon \mathfrak{g}_1\to\mathfrak{g}_{-1}$ is injective in the case $c\neq 1$. We also claim that this is true for the case $c=1$. Indeed, using the equality (2.17), this can be verified with the same arguments as in the proof of Lemme 6.4 in

Vey [13]. Consequently, we have $\dim \mathfrak{g}_1 \leqq 1$, because $\dim \mathfrak{g}_{-1}=1$ by Proposition 2.6. We want to show that $\dim \mathfrak{g}_1=1$. For this it is sufficient to prove that $\mathfrak{g}_1 \neq \{0\}$. We put $\mathcal{D}_0=\{(z, w)\in\mathcal{D} \mid w=0\}$. Then, $\mathcal{D}_0$ is identified with the upper half plane $\mathfrak{H}$ by Lemma 3.1. Now, it is well-known that each element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, \boldsymbol{R})$ acts on $\mathfrak{H}$ by a holomorphic transformation

$$(3.9) \qquad l_\gamma(z) = (az+b)\cdot(cz+d)^{-1},$$

and conversely each biholomorphic transformation of $\mathfrak{H}$ onto itself is obtained in the manner described in (3.9). For each element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, \boldsymbol{R})$, we here define a mapping $\tilde{l}_\gamma: \mathfrak{H}\times \boldsymbol{C}^m \to \mathfrak{H}\times \boldsymbol{C}^m$ by

$$(3.10) \qquad \tilde{l}_\gamma(z, w) = (l_\gamma(z), (cz+d)^{-2c}w)\,.$$

It is then checked easily that $\tilde{l}_\gamma$ is a holomorphic mapping and $\tilde{l}_\gamma(\mathcal{D})\subset\mathcal{D}$, so that $\tilde{l}_\gamma$ induces a biholomorphic transformation of $\mathcal{D}$ onto itself (cf. [6], Corollary 3). By the construction of $\tilde{l}_\gamma$, it is obvious that $\tilde{l}_\gamma=l_\gamma$ on $\mathcal{D}_0$. Therefore, the group $\mathrm{Aut}_0(\mathcal{D}_0)$ can be identified with a subgroup of $\mathrm{Aut}_0(\mathcal{D})$ via the correspondence $l_\gamma \mapsto \tilde{l}_\gamma$. Finally, consider the global one-parameter subgroup

$$(3.11) \qquad \tilde{l}_{\gamma_t}\colon (z, w) \mapsto (l_{\gamma_t}(z), (tz+1)^{-2c}w)\,, \quad t\in\boldsymbol{R}\,,$$

of $\mathrm{Aut}_0(\mathcal{D})$ defined by the one-parameter subgroup

$$(3.12) \qquad \gamma_t = \exp t\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t\in\boldsymbol{R}\,,$$

of $SL(2, \boldsymbol{R})$. Then, we can see by a direct computation that $\{\tilde{l}_{\gamma_t}\}_{t\in\boldsymbol{R}}$ defines a non-zero vector field on $\mathcal{D}$ belonging to $\mathfrak{g}_1$. Consequently, we have $\mathfrak{g}_1 \neq \{0\}$, as desired.

Now, noting that $\mathfrak{g}_0=\{X\in\mathfrak{g}(\mathcal{D}) \mid [E, X]=0\}$ and the group $\mathrm{Aut}_0(\mathcal{D}_0)$ can be considered as a subgroup of $\mathrm{Aut}_0(\mathcal{D})$ as above, we can show that $\mathrm{Aut}_0(\mathcal{D})$ leaves invariant the complex submanifold $\mathcal{D}_0$ of $\mathcal{D}$, and in fact $\mathcal{D}_0$ coincides with the $\mathrm{Aut}_0(\mathcal{D})$-orbit passing through the point $(\sqrt{-1}, 0)$: $\mathcal{D}_0=\mathrm{Aut}_0(\mathcal{D})\cdot(\sqrt{-1}, 0)$. Hence, there is a natural homomorphism $\pi\colon \mathfrak{g}(\mathcal{D})\to\mathfrak{g}(\mathcal{D}_0)$ induced by the Lie group homomorphism of $\mathrm{Aut}_0(\mathcal{D})$ to $\mathrm{Aut}_0(\mathcal{D}_0)$ defined by $g\mapsto g_{|\mathcal{D}_0}$, where $g_{|\mathcal{D}_0}$ denotes the restriction of $g\in\mathrm{Aut}_0(\mathcal{D})$ to $\mathcal{D}_0$. Let $\mathfrak{g}(\mathcal{D}_0)=\mathfrak{g}^0_{-1}+\mathfrak{g}^0_0+\mathfrak{g}^0_1$ be the decomposition of $\mathfrak{g}(\mathcal{D}_0)$ as in Kaup, Matsushima and Ochiai [3]. Then, since $\pi(E)=z\dfrac{\partial}{\partial z}$, π preserves the gradition, i.e., $\pi(\mathfrak{g}_\lambda)\subset\mathfrak{g}^0_\lambda$. Moreover, it is clear that $\pi(\mathfrak{g}_\lambda)=\mathfrak{g}^0_\lambda$ for $\lambda=-1$ and 1. On the other hand, since $\mathfrak{g}(\mathcal{D}_0)$ is a simple Lie algebra isomorphic to $sl(2, \boldsymbol{R})$, we have $\mathfrak{g}^0_0=[\mathfrak{g}^0_{-1}, \mathfrak{g}^0_1]$, so that $\mathfrak{g}^0_0=\pi([\mathfrak{g}_{-1}, \mathfrak{g}_1])\subset\pi(\mathfrak{g}_0)$. Therefore, π is surjective. Put $\mathfrak{g}_0''=\mathrm{Ker}\,\pi$ and $\mathfrak{g}_0'=$

$[\mathfrak{g}_{-1}, \mathfrak{g}_1]\subset\mathfrak{g}_0$. Since π is injective on $\mathfrak{g}_\lambda$, for $\lambda=-1$ and 1, we see $\mathfrak{g}_0''\subset\mathfrak{g}_0$. From this we conclude that $[\mathfrak{g}_0'', \mathfrak{g}_\lambda]=\{0\}$ for $\lambda=-1$ and 1, and hence $[\mathfrak{g}_0'', \mathfrak{g}_0']=\{0\}$ by the Jacobi identity. Finally it is an easy matter to see that $\mathfrak{g}_{-1}+\mathfrak{g}_0'+\mathfrak{g}_1$ and $\mathfrak{g}_0''$ are ideals of $\mathfrak{g}(\mathscr{D})$ satisfying the condition: $\mathfrak{g}(\mathscr{D})=(\mathfrak{g}_{-1}+\mathfrak{g}_0'+\mathfrak{g}_1)+\mathfrak{g}_0''$ (direct sum), completing the proof.

4. Proof of Theorem 2

Throughout this section we denote by $\mathscr{D}$ (resp. $\mathscr{D}'$) a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent c(resp. c'). In general, for given two domains S and S' we employ the notation A' for denoting the onject for S' corresponding to an object A for S.

Now, we begin with the following

Lemma 4.1. *Let B and B' be two hyperbolic circular domains in $\boldsymbol{C}^N$ containing the origin o. Suppose that the following two conditions are satisfied*:

(4.1) *The Lie algebra $\mathfrak{g}(B)$ (resp. $\mathfrak{g}(B')$) contains the element $\bar{I}$ (resp. $\bar{I}'$) of the form*

$$\bar{I}=\sqrt{-1}\sum_{k=2}^{N} z_k\frac{\partial}{\partial z_k}\quad\left(\textit{resp. } \bar{I}'=\sqrt{-1}\sum_{k=2}^{N} z_k'\frac{\partial}{\partial z_k'}\right);$$

(4.2) *There exists a non-singular linear mapping Φ: $\boldsymbol{C}^N\to\boldsymbol{C}^N$ of the form*

$$\Phi:\begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_N' \end{pmatrix}=\begin{pmatrix} 1 & \Phi_2^1\cdots\cdots\Phi_N^1 \\ 0 & \Phi_2^2\cdots\cdots\Phi_N^2 \\ \vdots & \vdots \qquad\quad \vdots \\ 0 & \Phi_2^N\cdots\cdots\Phi_N^N \end{pmatrix}\cdot\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$$

such that $\Phi(B)=B'$.

Then $\Phi_k^1=0$ for $k=2, 3, \cdots, N$.

Proof. Let Λ: $B'\to B$ be the inverse mapping of Φ and put

$$\Lambda:\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}=\begin{pmatrix} 1 & \Lambda_2^1\cdots\cdots\Lambda_N^1 \\ 0 & \Lambda_2^2\cdots\cdots\Lambda_N^2 \\ \vdots & \vdots \qquad\quad \vdots \\ 0 & \Lambda_2^N\cdots\cdots\Lambda_N^N \end{pmatrix}\cdot\begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_N' \end{pmatrix} \tag{4.3}$$

Denoting by Λ_*: $\mathfrak{g}(B')\to\mathfrak{g}(B)$ the differential of Λ, we have by routine calculation that

$$\begin{aligned}\Lambda_*\bar{I}' &=\sqrt{-1}\sum_{s=1}^{N}\Big(\sum_{k,l=2}^{N}\Lambda_j^s\Phi_l^k z_l\Big)\frac{\partial}{\partial z_s} \\ &=\Big(-\sqrt{-1}\sum_{l=2}^{N}\Phi_l^1 z_l\Big)\frac{\partial}{\partial z_1}+\sqrt{-1}\sum_{k=2}^{N} z_k\frac{\partial}{\partial z_k}\end{aligned} \tag{4.4}$$

$$= (-\sqrt{-1}\sum_{l=2}^{N}\Phi_l^1 z_l)\frac{\partial}{\partial z_1}+\bar{I}.$$

Consequently, the vector field

$$X = (-\sqrt{-1}\sum_{l=2}^{N}\Phi_l^1 z_l)\frac{\partial}{\partial z_1} \tag{4.5}$$

also belongs to $\mathfrak{g}(B)$. Then, as we can see easily, the global one-parameter group $\{\phi_t\}_{t\in \boldsymbol{R}}$ generated by X is given by

$$\phi_t = \begin{pmatrix} 1 & -\sqrt{-1}\Phi_2^1 t \cdots\cdots & -\sqrt{-1}\Phi_N^1 t \\ 0 & 1 & 0\cdots\cdots\cdots 0 \\ \vdots & \vdots & \ddots \\ 0 & 0 & 0\cdots\cdots 0\ 1 \end{pmatrix}, \quad t\in \boldsymbol{R}, \tag{4.6}$$

so that ϕ_t acts on B by the following transformation

$$\begin{cases} z_1 \mapsto z_1 - \sqrt{-1}(\sum_{l=2}^{N}\Phi_l^1 z_l)t \\ z_k \mapsto z_k \quad \text{for} \quad k=2,3,\cdots,N. \end{cases} \tag{4.7}$$

Here we notice that the group $\mathrm{Aut}_0(B)$ contains the global one-parameter subgroup

$$\Psi_\theta\colon (z_1, z_2, \cdots, z_N) \mapsto (e^{\sqrt{-1}\theta}z_1, z_2, \cdots, z_N), \quad \theta\in \boldsymbol{R}. \tag{4.8}$$

In fact, $\{\Psi_\theta\}_{\theta\in\boldsymbol{R}}$ is the one-parameter subgroup of $\mathrm{Aut}_0(B)$ generated by the holomorphic vector field $\sqrt{-1}z_1\frac{\partial}{\partial z_1} = \sqrt{-1}\sum_{k=1}^{N} z_k\frac{\partial}{\partial z_k} - \bar{I}$ belonging to $\mathfrak{g}(B)$.

Now, suppose that $(\Phi_2^1, \Phi_3^1, \cdots, \Phi_N^1) \neq (0, 0, \cdots, 0)$. Then, choosing a point $p_0=(0, z_2^0, \cdots, z_N^0)$ of B such that $\sum_{l=2}^{N}\Phi_l^1 z_l^0 \neq 0$, we see that B contains the set

$$\{\Psi_\theta\cdot\phi_t\cdot p_0 \mid \theta, t\in \boldsymbol{R}\} = \{(z_1, z_2^0, \cdots, z_N^0 \in \boldsymbol{C}^N \mid z_1 \in \boldsymbol{C}\},$$

which is canonically identified with the complex plane $\boldsymbol{C}$. But this is impossible, because B is hyperbolic in the sense of Kobayashi [5]. Thus we have proved that $(\Phi_2^1, \Phi_3^1, \cdots, \Phi_N^1) = (0, 0, \cdots, 0)$. q.e.d.

We now consider a mapping φ; $\{z\in\boldsymbol{C} \mid \mathrm{Im}.z>0\}\times \boldsymbol{C}^m \to \boldsymbol{C}^{m+1}$ defined by

$$\varphi : \begin{cases} z_1 = (z-\sqrt{-1})\cdot(z+\sqrt{-1})^{-1} \\ z_k = \dfrac{4^c w_{k-1}}{(z-\sqrt{-1})^{2c}} \quad \text{for} \quad k=2,3,\cdots,m+1, \end{cases} \tag{4.9}$$

where c is the exponent of $\mathcal{D}$. As we can see easily, φ defines a biholomorphic isomorphism of $\mathscr{D}$ onto the image domain $\mathscr{B}=\varphi(\mathscr{D})$ in C^{m+1}.

In the case $c\neq 0, 1/2$, we know already from the proof of Theorem 1 that

$$\mathrm{Aut}_0(\mathscr{D})\cdot(\sqrt{-1}, 0)=\{(z, 0)\in C\times C^m \mid \mathrm{Im}. z>0\}$$

and hence

$$\mathrm{Aut}_0(\mathscr{B})\cdot o=\{(z_1, 0, \cdots, 0)\in C^{m+1} \mid |z_1|<1\}$$

by (4.9). Moreover, by direct computations as in the proof of Theorem 2 in [6], the structure of $\mathrm{Aut}_0(\mathscr{B})$ is explicitly determined as follows. Let $SU(1, 1)$ be the matrix group defined by

$$(4.10)\qquad SU(1, 1)=\left\{\begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in SL(2, C)\ \middle|\ \begin{array}{l} |a|^2-|\mathfrak{c}|^2=1 \\ |\mathfrak{b}|^2-|d|^2=-1 \\ \bar{\mathfrak{b}}a-\bar{d}\mathfrak{c}=0\end{array}\right\}$$

and $K^0_{\sqrt{-1}}\subset GL(m, C)$ the identity component of the isotropy subgroup of $\mathrm{Aut}(\mathscr{D}_{\sqrt{-1}})$ at the origin o of C^m, where $\mathscr{D}_{\sqrt{-1}}$ is the circular domain defined in Lemma 3.2. Then we can verify that the group $\mathrm{Aut}_0(\mathscr{B})$ consists of all transformations of the following type (cf. [6], REMARK 3):

$$(4.11)\qquad \begin{cases} \mathfrak{z}\mapsto(a\mathfrak{z}+\mathfrak{b})\cdot(\mathfrak{c}\mathfrak{z}+d)^{-1} \\ \mathfrak{z}'\mapsto K\cdot(\mathfrak{c}\mathfrak{z}+d)^{-2c}\cdot\mathfrak{z}', \end{cases}$$

where $\mathfrak{z}=z_1$, $\mathfrak{z}'={}^t(z_2, \cdots, z_{m+1})$, $\begin{pmatrix} a & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}\in SU(1, 1)$ and $K\in K^0_{\sqrt{-1}}\subset GL(m, C)$.

The following lemma is essential to the proof of Theorem 2.

Lemma 4.2. *With the notation as above, the domain $\mathscr{B}$ is a hyperbolic circular domain in C^{m+1} containing the origin o. Moreover $\mathfrak{g}(\mathscr{B})$ contains the element $\bar{I}$ of the form $\bar{I}=\sqrt{-1}\sum_{k=2}^{m+1} z_k\frac{\partial}{\partial z_k}$.*

Proof. By using (4.11), the first statement can by verified in exactly the same way as in the proof of Lemma 1 in [8].

For the second assertion, we recall that $\mathrm{Aut}_0(\mathscr{D})$ contains the following global one-parameter subgroup

$$(4.12)\qquad l_\theta : (z, w)\mapsto(z, e^{\sqrt{-1}\theta}w), \quad \theta\in R.$$

By way of (4.9), $\{l_\theta\}_{\theta\in R}$ induces the global one-parameter subgroup

$$(4.13)\qquad \bar{l}_\theta : (z_1, z_2, \cdots, z_{m+1})\mapsto(z_1, e^{\sqrt{-1}\theta}z_2, \cdots, e^{\sqrt{-1}\theta}z_{m+1}), \quad \theta\in R,$$

of $\mathrm{Aut}_0(\mathscr{B})$, which defines the desired element $\bar{I}$. q.e.d.

We are now prepared to prove Theorem 2.

Proof of Theorem 2. Since it is trivial that $\mathcal{D}$ and $\mathcal{D}'$ are holomorphically equivalent if they are linearly equivalent, we have only to prove the converse.

Let $\varphi: \mathcal{D} \to \mathcal{B}$ be the biholomorphic isomorphism of $\mathcal{D}$ onto $\mathcal{B}$ defined in (4.9) and $\varphi': \mathcal{D}' \to \mathcal{B}'$ the corresponding isomorphism of $\mathcal{D}'$ onto the image domain $\mathcal{B}'$. Suppose that there exists a biholomorphic isomorphism $\Phi: \mathcal{D} \to \mathcal{D}'$ of $\mathcal{D}$ onto $\mathcal{D}'$. We put $\tilde{\Phi} = \varphi' \cdot \Phi \cdot \varphi^{-1}$. Then $\tilde{\Phi}$ gives rise to a biholomorphic isomorphism of $\mathcal{B}$ onto $\mathcal{B}'$. Now, we know already by Lemma 4.2 that $\mathcal{B}$ and $\mathcal{B}'$ are hyperbolic circular domains in $\boldsymbol{C}^{m+1}$ containing the origin o. Moreover, since $\mathcal{B}$ (resp. $\mathcal{B}'$) is holomorphically equivalent to a bounded domain, $\mathcal{B}$ (resp. $\mathcal{B}'$) has the Bergman metric $ds^2_{\mathcal{B}}$ (resp. $ds^2_{\mathcal{B}'}$), which is $\mathrm{Aut}(\mathcal{B})$ (resp. $\mathrm{Aut}(\mathcal{B}')$)-invariant Kähler metric. Hence, it follows immediately from Theorem 3 that there exists a non-singular linear mapping $\tilde{\mathcal{L}}: \boldsymbol{C}^{m+1} \to \boldsymbol{C}^{m+1}$ such that $\tilde{\mathcal{L}}(\mathcal{B}) = \mathcal{B}'$. We shall prove that this isomorphism $\tilde{\mathcal{L}}$ induces a linear isomorphism $\mathcal{L}: \boldsymbol{C} \times \boldsymbol{C}^m \to \boldsymbol{C} \times \boldsymbol{C}^m$ such that $\mathcal{L}(\mathcal{D}) = \mathcal{D}'$. The proof is divided into three cases as follows.

Case I: $c \neq 0,\ 1/2$.

In this case we know from the proof of Theorem 1 and (4.9) that

$$\mathrm{Aut}_0(\mathcal{B}) \cdot o = \{(z_1, 0, \cdots, 0) \in \boldsymbol{C}^{m+1} \mid |z_1| < 1\} . \tag{4.14}$$

Since $\dim_{\boldsymbol{C}}(\mathrm{Aut}_0(\mathcal{B}') \cdot o) = \dim_{\boldsymbol{C}}(\mathrm{Aut}_0(\mathcal{B}) \cdot o) = 1$, we obtain that

$$\tilde{\mathcal{L}}(\mathrm{Aut}_0(\mathcal{B}) \cdot o) = \mathrm{Aut}_0(\mathcal{B}') \cdot o = \{(z'_1, 0, \cdots, 0) \in \boldsymbol{C}^{m+1} \mid |z'_1| < 1\} , \tag{4.15}$$

from which we conclude that $\tilde{\mathcal{L}}: \boldsymbol{C}^{m+1} \to \boldsymbol{C}^{m+1}$ is of the form

$$\tilde{\mathcal{L}}: \begin{pmatrix} z'_1 \\ z'_2 \\ \vdots \\ z'_{m+1} \end{pmatrix} = \left(\begin{array}{c|c} a & * \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right) \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+1} \end{pmatrix}, \quad |a| = 1 , \tag{4.16}$$

where A is an $m \times m$ non-singular matrix. Since the group $\mathrm{Aut}_0(\mathcal{B})$ contains the linear transformations

$$\tilde{l}_t : (z_1, \cdots, z_{m+1}) \mapsto (e^{\sqrt{-1}t} z_1, \cdots, e^{\sqrt{-1}t} z_{m+1}) , \quad t \in \boldsymbol{R} ,$$

changing $\tilde{\mathcal{L}}$ by a suitable linear transformation $\tilde{\mathcal{L}} \cdot \tilde{l}_t$ if necessary, we may assume that $a = 1$. Then, as a consequence of Lemmas 4.2 and 4.1, $\tilde{\mathcal{L}}$ is reduced to the following form

$$\tilde{\mathcal{L}}: \begin{pmatrix} z'_1 \\ z'_2 \\ \vdots \\ z'_{m+1} \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+1} \end{pmatrix} . \tag{4.17}$$

Recalling the definitions of the isomorphisms $\varphi: \mathcal{D}\to\mathcal{B}$ and $\varphi': \mathcal{D}'\to\mathcal{B}'$, we put $\mathcal{L}=\varphi'^{-1}\cdot\tilde{\mathcal{L}}\cdot\varphi$. Then it is easily checked that $\mathcal{L}$ is a biholomorphic isomorphism of $\mathcal{D}$ onto $\mathcal{D}'$ of the following form

$$\mathcal{L}: \begin{cases} z' = z \\ w'_\alpha = \dfrac{4^{c-c'}}{(z+\sqrt{-1})^{2(c-c')}} \displaystyle\sum_{\beta=1}^{m} A_{\alpha\beta} w_\beta \qquad (1\leqq\alpha\leqq m), \end{cases} \tag{4.18}$$

where we put $A=(A_{\alpha\beta})_{1\leqq\alpha,\beta\leqq m}$. Once it is shown that the exponents c and c' are identical, we may conclude from (4.18) that the mapping $\mathcal{L}: \mathcal{D}\to\mathcal{D}'$ gives a desired linear equivalence between $\mathcal{D}$ and $\mathcal{D}'$. Now, we start out to prove $c=c'$. Let $\{\varphi'_t\}_{t\in\boldsymbol{R}}$ be the global one-parameter subgroup

$$\varphi'_t: (z', w') \mapsto (e^t z, e^{c't} w'), \quad t\in\boldsymbol{R} \tag{4.19}$$

of $\mathrm{Aut}(\mathcal{D}')$. By direct computations, we can show that the global one-parameter subgroup $\{\tilde{\varphi}_t\}_{t\in\boldsymbol{R}}$ of $\mathrm{Aut}(\mathcal{D})$ defined by $\tilde{\varphi}_t=\mathcal{L}^{-1}\cdot\varphi'_t\cdot\mathcal{L}$ is given by

$$\tilde{\varphi}_t: (z, w) \mapsto \left(e^t z, \frac{e^{c't}}{(e^t z+\sqrt{-1})^{2(c'-c)}(z+\sqrt{-1})^{2(c-c')}}\cdot w\right), \tag{4.20}$$

so that the complete holomorphic vector field X on $\mathcal{D}$ induced by $\{\tilde{\varphi}_t\}_{t\in\boldsymbol{R}}$ is of the following form

$$X = z\frac{\partial}{\partial z} + \sum_{\alpha=1}^{m}\left(c' - \frac{2(c'-c)z}{z+\sqrt{-1}}\right) w_\alpha \frac{\partial}{\partial w_\alpha}. \tag{4.21}$$

On the other hand, we know from [3] that every complete holomorphic vector field on $\mathcal{D}$ is a polynomial vector field. By (4.21), it is clear that X is a polynomial vector field only if $c=c'$, as desired.

Case II: $c = 0$.

By Lemma 3.2 $\mathcal{D}$ is the direct product $\mathcal{D}=\mathfrak{H}\times\mathcal{D}_{\sqrt{-1}}$ so that $\mathcal{B}=U\times\mathcal{D}_{\sqrt{-1}}$, where $\mathfrak{H}$ is the upper half plane and U is the unit disk $\{z_1\in\boldsymbol{C}\mid |z_1|<1\}$. We have two cases to consider. Consider first the case where $\dim_{\boldsymbol{C}}(\mathrm{Aut}_0(\mathcal{B})\cdot o)=1$. In this case we have

$$\mathrm{Aut}_0(\mathcal{B})\cdot o = \{(z_1, 0, \cdots, 0)\in\boldsymbol{C}^{m+1} \mid |z_1|<1\} \tag{4.22}$$

and

$$\tilde{\mathcal{L}}(\mathrm{Aut}_0(\mathcal{B})\cdot o) = \mathrm{Aut}_0(\mathcal{B}')\cdot o = \{(z'_1, 0, \cdots, 0)\in\boldsymbol{C}^{m+1} \mid |z'_1|<1\}. \tag{2.23}$$

From this, repeating the same arguments as in the Case I, we can see that $\mathcal{D}$ and $\mathcal{D}'$ are linearly equivalent and $c=c'$. Consider next the case where $\dim_{\boldsymbol{C}}(\mathrm{Aut}_0(\mathcal{B})\cdot o)>1$. We first claim that the exponent c' is also zero. It is evident that $c'=0$ or $1/2$, since $\dim_C(\mathrm{Aut}_0(\mathscr{B}')\cdot o)=1$ in the case $c'\neq 0$, $1/2$. Suppose that $c'=1/2$. Then, as we have observed in the previous paper [6], the orbit $\mathrm{Aut}_0(\mathscr{B}')\cdot o$ is a unit ball. In particular, $\mathrm{Aut}_0(\mathscr{B}')\cdot o$ is irreducible in the sense of Kähler geometry. On the other hand, by Theorem 3 and the fact that $\mathscr{B}$ is the direct product $\mathscr{B}=U\times \mathscr{D}_{\sqrt{-1}}$, we see that the orbit $\mathrm{Aut}_0(\mathscr{B})\cdot o$ is also the direct product $\mathrm{Aut}_0(\mathscr{B})\cdot o=U\times S$, where S is a positive dimensional Hermitian symmetric space of non-compact type. Since $\mathrm{Aut}_0(\mathscr{B})\cdot o$ and $\mathrm{Aut}_0(\mathscr{B}')\cdot o$ are holomorphically equivalent, this is a contradiction. Thus we have proved that $c'=0$, and hence $\mathscr{D}'$ is also the direct product $\mathscr{D}'=\mathfrak{H}\times \mathscr{D}'_{\sqrt{-1}}$ by Lemma 3.2. Since $\Phi\colon \mathscr{D}=\mathfrak{H}\times \mathscr{D}_{\sqrt{-1}}\to \mathscr{D}'=\mathfrak{H}\times \mathscr{D}'_{\sqrt{-1}}$ is a biholomorphic isomorphism and the upper half plane $\mathfrak{H}$ is of course a hyperbolic complex manifold in the sense of Kobayashi [5], it follows immediately from Theorem U in section 1 that $\mathscr{D}_{\sqrt{-1}}$ and $\mathscr{D}'_{\sqrt{-1}}$ are also holomorphically equivalent. Now, being isomorphic to a complex submanifold of $\mathscr{D}$ (resp. $\mathscr{D}'$), the domain $\mathscr{D}'_{\sqrt{-1}}$ (resp. $\mathscr{D}'_{\sqrt{-1}}$) a hyperbolic circular domain in C^m containing the origin. Moreover, noting the fact $\mathscr{D}_{\sqrt{-1}}=\mathfrak{H}\times \mathscr{D}_{\sqrt{-1}}$ (resp. $\mathscr{D}'=\mathfrak{H}\times \mathscr{D}'_{\sqrt{-1}}$) in our case, the domain $\mathscr{D}_{\sqrt{-1}}$ (resp. $\mathscr{D}'_{\sqrt{-1}}$) has the $\mathrm{Aut}(\mathscr{D}_{\sqrt{-1}})$ (resp. $\mathrm{Aut}(\mathscr{D}'_{\sqrt{-1}})$)-invariant Kähler metric induced from the Bergman metric of $\mathscr{D}$ (resp. $\mathscr{D}'$). Hence, it follows from Theorem 3 that $\mathscr{D}_{\sqrt{-1}}$ and $\mathscr{D}'_{\sqrt{-1}}$ are linearly equivalent. It is now trivial that $\mathscr{D}=\mathfrak{H}\times \mathscr{D}_{\sqrt{-1}}$ and $\mathscr{D}'=\mathfrak{H}\times \mathscr{D}'_{\sqrt{-1}}$ are linearly equivalent.

Case III: $c=1/2$.

In the case where $\dim_C(\mathrm{Aut}_0(\mathscr{B})\cdot o)=1$, our assertion can be proved in the same way as Case I. Next, consider the case where $\dim_C(\mathrm{Aut}_0(\mathscr{B})\cdot o)>1$. We assert that the exponent c' of $\mathscr{D}'$ is also $1/2$. In fact, replacing $\mathscr{D}$ by $\mathscr{D}'$ in the second case of the Case II, this can be verified easily. As a result, two domains $\mathscr{D}$ and $\mathscr{D}'$ are generalized Siegel domains in $C\times C^m$ with exponent $1/2$. Therefore, our assertion follows from the previous paper [8]. q.e.d.

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