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THE GIRTH OF CONVERGENCE GROUPS AND MAPPING CLASS GROUPS

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Abstract

We give a sufficient condition for the girth of finitely generated groups to be infinite by using a technique to prove a so-called ping-pong lemma or table tennis lemma. We show that some convergence groups and subgroups of mapping class groups satisfy the condition. Therefore the girth of each of them is infinite.

1. Introduction

DEFINITION 1.1 ([1], [2], [14]). Let G be a finitely generated group and $X(G)$ the family of finitely generating (not necessarily symmetric) sets of G. For any $S \in X(G)$, we define

$$U(S, G) = \min\{n \mid s_1 s_2 \cdots s_n = e,\ s_1 s_2 \cdots s_n \text{ is reduced},\ s_i \in S\}$$

and call $U(S, G)$ the girth of G with respect to S. In the above, the element e is the unit element of G. If there exists no reduced word $s_1 s_2 \cdots s_n$ with $s_i \in S$ which is trivial as an element of G, then we set $U(S, G) = \infty$. The *girth* of G is defined as

$$U(G) = \sup\{U(S, G) \mid S \in X(G)\}.$$

Let us list known facts on the girth of a finitely generated group G.

- If G is finite, then $U(G)$ is finite by definition.
- If G is an abelian group which is not isomorphic to $\mathbb{Z}$, then $U(G)$ is finite by definition.
- If G is isomorphic to $\mathbb{Z}$, then $U(G)$ is infinite by definition. More generally, if G is a free group of rank n, where n is any positive integer, then $U(G)$ is infinite.
- If G is a solvable group which is not isomorphic to $\mathbb{Z}$, then $U(G)$ is finite [14, Corollary 4.4].
- Let G be a subgroup of $GL(n, k)$, where k is any field. The girth of G is finite if and only if G contains a finite index solvable subgroup which is not isomorphic to $\mathbb{Z}$ [2, Theorem 4.4].

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- The girth of a one-relator group G is infinite if and only if G is not solvable [2, Theorem 3.1].

Akhmedov proved the following theorem on the girth of some subgroups of Gromov-hyperbolic groups by using a technique to prove a ping-pong lemma. See [5, Chapter III.Γ 2.1] for the definition of Gromov-hyperbolic groups. A group is said to be *virtually cyclic* if it contains a cyclic subgroup of finite index.

Theorem 1.2 ([2] Theorem 2.6). *Let G be a Gromov-hyperbolic group and H a finitely generated subgroup of G which is not virtually cyclic. Then the girth of H is infinite.*

Our result on the girth of convergence groups is stated as follows. We will define a convergence group in Definition 3.1.

Theorem 1.3. *Let G be a convergence group. Suppose that H is a finitely generated subgroup of G which is not virtually cyclic and that the limit set of H contains at least two points. Then the girth of H is infinite.*

Let G and H be groups as in Theorem 1.2. It is known that G acts on its ideal boundary as a convergence group [17, Theorem 3A], [3, Proposition 1.13]. The limit set of H contains at least two points since H contains a free group of rank two as a subgroup by [7, Chapitre 8, Théorème 37]. Hence G and H in Theorem 1.2 satisfy the assumption of Theorem 1.3.

There exists another class of examples satisfying the assumption of Theorem 1.3. Let G be a group and $\mathcal{G}$ a family of infinite subgroups of G. Suppose that G is hyperbolic relative to $\mathcal{G}$ in the sense of Bowditch [4, Definition 1]. By definition, since G acts properly discontinuously and isometrically on a Gromov-hyperbolic space satisfying some topological condition, G acts on its ideal boundary as a convergence group [17, Theorem 3A]. See [5, Chapter III.H] for Gromov-hyperbolic spaces. One of the examples of relatively hyperbolic groups is a free product of groups. Any free product of finitely generated infinite groups is hyperbolic relative to its factors in the sense of Bowditch. (Consider the action on its Bass–Serre tree [15]. This action satisfies [4, Definition 2] which is equivalent to [4, Definition 1].) Consequently, Theorem 1.3 is an extension of Theorem 1.2.

We also investigate the girth of mapping class groups. Let M be a compact, orientable and connected surface (we admit that it has non-empty boundary). Let Mod_M be the group of isotopy classes of orientation preserving diffeomorphisms on M. We call Mod_M the *mapping class group* of the surface M. See Section 4 for more details. We show the following result.

Theorem 1.4. *Suppose that G is a finitely generated irreducible subgroup of Mod_M which is not virtually cyclic. Then the girth of G is infinite.*

REMARK 1.5. In this paper, we consider the action of a mapping class group Mod_M on its Thurston boundary. Reducible elements of Mod_M can have more than two fixed points in it. Therefore, in general, subgroups of Mod_M do not act on its Thurston boundary as convergence groups since any infinite order element of convergence groups has at most two fixed points. Hence, Theorem 1.3 does not apply to this action.

Let X be a set and G a group acting on X. Given $g \in G$, we denote by $\mathrm{fix}(g)$ the fixed point set of g, that is, $\mathrm{fix}(g) = \{x \in X \mid g(x) = x\}$. The cardinality of $\mathrm{fix}(g)$ is denoted by $\#\,\mathrm{fix}(g)$. We prove Theorems 1.3 and 1.4 by checking the conditions of the following proposition.

Proposition 1.6. *Let X be a Hausdorff topological space containing infinite elements. Let G be a finitely generated group which is not virtually cyclic. Suppose that G acts on X by homeomorphisms and satisfies the following conditions*:

(1) *There exists $g \in G$ such that $\#\,\mathrm{fix}(g) = 2$;*

(2) *Let g be any element of G such that $\mathrm{fix}(g)$ consists of exactly two points, denoted by a and b. If we choose a neighborhood A of a and a neighborhood B of b with $A \cap B = \emptyset$, then we have the inclusions*

$$g^n(X \setminus B) \subset A$$

and

$$g^{-n}(X \setminus A) \subset B$$

for any sufficiently large n, after exchanging a and b if necessary;

(3) *For any elements $g, h \in G$ such that $\#\,\mathrm{fix}(g) = \#\,\mathrm{fix}(h) = 2$, the fixed point sets $\mathrm{fix}(g)$ and $\mathrm{fix}(h)$ satisfy either $\mathrm{fix}(g) = \mathrm{fix}(h)$ or $\mathrm{fix}(g) \cap \mathrm{fix}(h) = \emptyset$;*

(4) *For any element $g \in G$ such that $\#\,\mathrm{fix}(g) = 2$, the stabilizer of $\mathrm{fix}(g)$,*

$$\mathrm{Stab}(\mathrm{fix}(g)) = \{h \in G \mid h(\mathrm{fix}(g)) = \mathrm{fix}(g)\},$$

is virtually cyclic.

Then the girth of G is infinite.

In the proof of Proposition 1.6, we use the technique to prove the ping-pong lemma (for example, see [8, II.B.24]). This lemma is applied to prove that certain class of groups satisfies the Tits alternative. The Tits alternative originates from Tits' result on linear groups [16, Theorem 1]. We say that a class of groups satisfies the Tits alternative if any group in the class is virtually solvable or it contains a free group of rank two as a subgroup. For example, all subgroups of mapping class groups satisfy the Tits alternative [11, Theorem A]. It is not known whether convergence groups satisfy the Tits alternative or not. However, if the limit set of a convergence group G contains more than two points, then G contains a free subgroup of rank two [17, Theorem 2U].

2. Groups of infinite girth

We will prove the following lemma which plays an important role in the proof of Proposition 2.6.

Lemma 2.1. *Suppose that X is a set and that G is not a virtually cyclic group acting on X. For any $i, j = 1, 2, \ldots, r$, let g_i and g_j be elements of G such that $\# \operatorname{fix}(g_i) = \# \operatorname{fix}(g_j) = 2$ and either $\operatorname{fix}(g_i) = \operatorname{fix}(g_j)$ or $\operatorname{fix}(g_i) \cap \operatorname{fix}(g_j) = \emptyset$. For each $i = 1, 2, \ldots, r$, we suppose that the stabilizer $\operatorname{Stab}(\operatorname{fix}(g_i)) = \{g \in G \mid g(\operatorname{fix}(g_i)) = \operatorname{fix}(g_i)\}$ of $\operatorname{fix}(g_i)$ is virtually cyclic. If we put $\mathcal{A} = \bigcup_{i=1}^{r} \operatorname{fix}(g_i)$, then there exists $h \in G$ such that $h\mathcal{A} \cap \mathcal{A} = \emptyset$.*

Before proving Lemma 2.1, we quote the following lemma.

Lemma 2.2 ([13] Lemma 4.1). *Suppose that a group G is a union of n cosets of subgroups $H_1, H_2, \ldots, H_n$ of G, i.e., $G = \bigcup_{i=1}^{n} g_i H_i$. Then there exists a subgroup H_i such that the index of H_i in G does not exceed n.*

Proof of Lemma 2.1. For any $i, j = 1, 2, \ldots, r$, the subset $A_{i,j} = \{g \in G \mid g(\operatorname{fix}(g_i)) = \operatorname{fix}(g_j)\}$ of G is a left coset of the stabilizer of $\operatorname{fix}(g_i)$, i.e., $A_{i,j} = g \cdot \operatorname{Stab}(\operatorname{fix}(g_i))$ for any $g \in A_{i,j}$. For any $i = 1, 2, \ldots, r$, the stabilizer $\operatorname{Stab}(\operatorname{fix}(g_i))$ is infinite index in G since G is not virtually cyclic. By Lemma 2.2, we have $G \supsetneq \bigcup_{i,j=1}^{r} A_{i,j}$. For any $h \in G \setminus \bigcup_{i,j=1}^{r} A_{i,j}$, we know $h(\mathcal{A}) \cap \mathcal{A} = \emptyset$ since the fixed point sets $\operatorname{fix}(g_i)$ and $\operatorname{fix}(g_j)$ satisfy $\operatorname{fix}(g_i) = \operatorname{fix}(g_j)$ or $\operatorname{fix}(g_i) \cap \operatorname{fix}(g_j) = \emptyset$ for any $i, j = 1, 2, \ldots, r$. $\square$

Before proving Proposition 2.6, we consider reduced words on

$$\Psi = \{g, f, g^M g_1 f^M, g^{2M} g_2 f^{2M}, \ldots, g^{kM} g_k f^{kM}\}$$

and

$$\Psi' = \{g, f, g_1, g_2, \ldots, g_k\},$$

where M is a positive integer. We will use these reduced words in the proof of Proposition 2.6 to apply the technique to prove the ping-pong lemma.

A non-empty reduced word w on Ψ is denoted by $w = u_1$ or

$$w = u_1 (g^{m_1 M} g_{m_1} f^{m_1 M})^{\varepsilon_1} u_2 (g^{m_2 M} g_{m_2} f^{m_2 M})^{\varepsilon_2} \cdots u_s (g^{m_s M} g_{m_s} f^{m_s M})^{\varepsilon_s} u_{s+1}$$

satisfying the following three conditions:

- $m_i \in \{1, 2, \ldots, k\}$ for any $i = 1, 2, \ldots, s$;
- $\varepsilon_i = 1$ or -1 for any $i = 1, 2, \ldots, s$;

- For any $i = 1, 2, \ldots, s+1$, the word u_i is the empty word or a reduced word only containing f and g.

REMARK 2.3. If $u_i = \emptyset$ and $m_{i-1} = m_i$, it is impossible to take $\varepsilon_{i-1} = 1$ and $\varepsilon_i = -1$. If $\varepsilon_{i-1} = 1$ and $\varepsilon_i = -1$, then the reduced word w contains a word

$$(g^{m_{i-1}M} g_{m_{i-1}} f^{m_{i-1}M})^{\varepsilon_{i-1}} u_i (g^{m_i M} g_{m_i} f^{m_i M})^{\varepsilon_i} = (g^{m_i M} g_{m_i} f^{m_i M})(g^{m_i M} g_{m_i} f^{m_i M})^{-1}.$$

This is a contradiction since w is a reduced word on Ψ. For the same reason, if $u_i = \emptyset$ and $m_{i-1} = m_i$, it is also impossible to take $\varepsilon_{i-1} = -1$ and $\varepsilon_i = 1$.

Note that if u_i is not empty, then u_i is also a reduced word on Ψ'.

For all $i = 1, 2, \ldots, k$, if we regard the letter $g^{iM} g_i f^{iM}$ in Ψ as a word on Ψ', it consists of the letters g, g_i and f in Ψ'. Regarding the letter $g^{iM} g_i f^{iM}$ in Ψ as a word on Ψ', we reduce w as a word on Ψ' and denote it by w'. The word w' is possibly empty although w is a non-empty reduced word on Ψ. We will show the following two lemmas on w and w'.

Lemma 2.4. *If $w = u_1$, then the reduced word w' on Ψ' is also denoted by $w' = u_1$.*

If $w = u_1(g^{m_1 M} g_{m_1} f^{m_1 M})^{\varepsilon_1} u_2 (g^{m_2 M} g_{m_2} f^{m_2 M})^{\varepsilon_2} \cdots u_s (g^{m_s M} g_{m_s} f^{m_s M})^{\varepsilon_s} u_{s+1}$, then the reduced word w' on Ψ' is denoted by $w' = u_1' g_{m_1}^{\varepsilon_1} u_2' g_{m_2}^{\varepsilon_2} \cdots u_s' g_{m_s}^{\varepsilon_s} u_{s+1}'$, where the word u_i' is the empty word or a reduced word only containing f and g for all $i = 1, 2, \ldots, s+1$.

Before proving this lemma, we explain the assertion of this lemma by taking an example. Take $w = g^M g_1 f^M (g^{2M} g_2 f^{2M})^{-1}$. The word w consists of the two letters $g^M g_1 f^M$ and $g^{2M} g_2 f^{2M}$ in Ψ. Then $w' = g^M g_1 f^{-M} g_2^{-1} g^{-2M}$ and w' consists of the letters g, f, g_1 and g_2 in Ψ'. In this case, $u_1' = g^M$, $u_2' = f^{-M}$, $u_3' = g^{-2M}$, $m_1 = 1$, $m_2 = 2$, $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$.

Proof. Recall that for all $i = 1, 2, \ldots, s+1$, the word u_i is a reduced word not only on Ψ but also on Ψ'. Hence if $w = u_1$, then $w' = u_1$.

In the following context, we treat the case of

$$w = u_1(g^{m_1 M} g_{m_1} f^{m_1 M})^{\varepsilon_1} u_2 (g^{m_2 M} g_{m_2} f^{m_2 M})^{\varepsilon_2} \cdots u_s (g^{m_s M} g_{m_s} f^{m_s M})^{\varepsilon_s} u_{s+1}.$$

The word $u_1' g_{m_1}^{\varepsilon_1} u_2' g_{m_2}^{\varepsilon_2} \cdots u_s' g_{m_s}^{\varepsilon_s} u_{s+1}'$ is obviously a reduced word on Ψ'. We will show that w' is denoted by this word.

For all $i = 1, 2, \ldots, k$, if the letter $g^{iM} g_i f^{iM}$ in Ψ is regarded as a reduced word on Ψ', it contains the letters g, g_i and f in Ψ'. For all $i = 1, 2, \ldots, k$, if a reduced word $(g^{iM} g_i f^{iM})^{-1}$ on Ψ is regarded as a reduced word on Ψ', it is denoted by $f^{-iM} g_i^{-1} g^{-iM}$. It also contains the letters f, g_i and g in Ψ'. For all $i = 1, 2, \ldots, s$, after we carry out this process for $(g^{m_iM}g_{m_i}f^{m_iM})^{\varepsilon_i}$, we regard w as a (possibly not reduced) word w'' on Ψ'. After w'' is reduced with respect to Ψ', the reduced word $w' = u'_1 g^{\varepsilon_1}_{m_1} u'_2 g^{\varepsilon_2}_{m_2} \cdots u'_s g^{\varepsilon_s}_{m_s} u'_{s+1}$ on Ψ' is obtained. We will explain the process of reducing w'' to w' in the following.

If $\varepsilon_1 = 1$, then

$$w = u_1(g^{m_1M}g_{m_1}f^{m_1M})u_2(g^{m_2M}g_{m_2}f^{m_2M})^{\varepsilon_2}\cdots u_s(g^{m_sM}g_{m_s}f^{m_sM})^{\varepsilon_s}u_{s+1}.$$

Reduce $u_1 g^{m_1M}$ with respect to Ψ' and denote it by u'_1. Moreover if $u_1 = g^{-m_1M}$, then

$$u'_1 = \emptyset.$$

If $\varepsilon_1 = -1$, then

$$w = u_1(g^{m_1M}g_{m_1}f^{m_1M})^{-1}u_2(g^{m_2M}g_{m_2}f^{m_2M})^{\varepsilon_2}\cdots u_s(g^{m_sM}g_{m_s}f^{m_sM})^{\varepsilon_s}u_{s+1}.$$

Reduce $u_1 f^{-m_1M}$ with respect to Ψ' and denote it by u'_1. Moreover if $u_1 = f^{m_1M}$, then

$$u'_1 = \emptyset.$$

Hence the word u'_1 is a reduced word on Ψ' containing only f and g, or the empty word in both the case of $\varepsilon_1 = 1$ and $\varepsilon_1 = -1$.

For all $i = 2, 3, \dots, s$, we denote by u'_i the word obtained by reducing the following words.

- $f^{m_{i-1}M}u_i g^{m_iM}$ for $\varepsilon_{i-1} = 1$ and $\varepsilon_i = 1$.
- $f^{m_{i-1}M}u_i f^{-m_iM}$ for $\varepsilon_{i-1} = 1$ and $\varepsilon_i = -1$.
- $g^{-m_{i-1}M}u_i g^{m_iM}$ for $\varepsilon_{i-1} = -1$ and $\varepsilon_i = 1$.
- $g^{-m_{i-1}M}u_i f^{-m_iM}$ for $\varepsilon_{i-1} = -1$ and $\varepsilon_i = -1$.

Note that if $u_i = \emptyset$ and $m_{i-1} = m_i$, it is impossible to take ε_{i-1} and ε_i with $\varepsilon_{i-1}\cdot\varepsilon_i = -1$ by Remark 2.3.

For all $i = 2, 3, \dots, s$, the word u'_i is also possibly empty. For example, if $\varepsilon_{i-1} = 1$, $\varepsilon_i = 1$ and $u_i = f^{-m_{i-1}M}g^{-m_iM}$, then u'_i is the empty word. By the above argument, for all $i = 2, 3, \dots, s$, we know that u'_i is the empty word or a reduced word on Ψ' containing only f and g.

If $\varepsilon_s = 1$, the word u'_{s+1} is obtained by reducing $f^{m_sM}u_{s+1}$. In addition, if $u_{s+1} = f^{-m_sM}$, then $u'_{s+1} = \emptyset$. If $\varepsilon_s = -1$, the word u'_{s+1} is obtained by reducing $g^{-m_sM}u_{s+1}$. Moreover if $u_{s+1} = g^{m_sM}$, then $u'_{s+1} = \emptyset$. Hence u'_{s+1} is the empty word or a reduced word on Ψ' containing only f and g. □

If the word length of w with respect to Ψ is not greater than M, the following lemma holds. Using the technique to prove the ping-pong lemma in the proof of Proposition 2.6, we make use of this lemma.

Lemma 2.5. *Let w be the reduced word*

$$u_1(g^{m_1M}g_{m_1}f^{m_1M})^{\varepsilon_1}u_2(g^{m_2M}g_{m_2}f^{m_2M})^{\varepsilon_2}\cdots u_s(g^{m_sM}g_{m_s}f^{m_sM})^{\varepsilon_s}u_{s+1}$$

on Ψ. Suppose that the word length of w with respect to Ψ is at most M. Then for the reduced word $w' = u'_1 g^{\varepsilon_1}_{m_1} u'_2 g^{\varepsilon_2}_{m_2} \cdots u'_s g^{\varepsilon_s}_{m_s} u'_{s+1}$ on Ψ', the following assertions hold for all $i = 1, 2, \ldots, s$:

(1) *u'_i is not empty;*
(2) *If $\varepsilon_i = 1$, then the last letter of u'_i is g and the first letter of u'_{i+1} is f;*
(3) *If $\varepsilon_i = -1$, then the last letter of u'_i is f and the first letter of u'_{i+1} is g.*

Before proving this lemma, we show some examples to explain the assertion of this lemma. If $w = g^{-M+1}(g^M g_1 f^M)$ (the word length of w with respect to Ψ is M), then $w' = gg_1 f^M$, $u'_1 = g$ and $u'_2 = f^M$. The last letter of u'_1 is g and the first letter of u'_2 is f. This lemma is false if the word length of w with respect to Ψ is more than M. For example, if $w = g^{-M}(g^M g_1 f^M)$ (the word length of w with respect to Ψ is $M+1$), then $w' = g_1 f^M$, $u'_1 = \emptyset$ and $u'_2 = f^M$.

Proof. If $\varepsilon_1 = 1$, the word u'_1 is obtained by reducing a word $u_1 g^{m_1M}$. Especially, if $u_1 = \emptyset$, then $u'_1 = g^{m_1M}$ and the last letter of u'_1 is g. On the other hand, if $\varepsilon_1 = 1$ and $u_1 \neq \emptyset$, then we set $u_1 = f^{a_1}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{b_l}$, where if $a_1 = 0$, then $b_1 \neq 0$, and if $b_l = 0$, then $a_l \neq 0$. Since the word length of w with respect to Ψ is at most M, we know $|a_i|$ and $|b_i|$ are less than M for all $i = 1, 2, \ldots, l$. If $b_l = 0$, then $u_1 g^{m_1M} = u'_1$ because $u_1 g^{m_1M} = f^{a_1}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{m_1M}$ is already a reduced word on Ψ'. Therefore the last letter of u'_1 is g. If $b_l \neq 0$, then $u_1 g^{m_1M} = f^{a_1}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{b_l+m_1M}$. If the exponent $b_l + m_1M$ of g is equal to 0, then $|b_l| = |-m_1M| = m_1M \geq M$. This is a contradiction since $|b_l|$ is less than M. Therefore we know $b_l + m_1M \neq 0$ and the last letter of u'_1 is g.

If $\varepsilon_1 = -1$, we can prove that the last letter of u'_1 is f in the same way as above.

For any $i = 2, 3, \ldots, s$, if $\varepsilon_{i-1} = 1$ and $\varepsilon_i = 1$, the word u'_i is obtained by reducing a word $f^{m_{i-1}M}u_i g^{m_iM}$. If $u_i = \emptyset$, then $u'_i = f^{m_{i-1}M}g^{m_iM}$. If $u_i \neq \emptyset$, then u_i is denoted by $u_i = f^{a_1}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{b_l}$. In the above description, if $a_1 = 0$, then $b_1 \neq 0$, and if $b_l = 0$, then $a_l \neq 0$. Note that $|a_i|$ and $|b_i|$ is less than M for any $i = 1, 2, \ldots, l$ because the word length of w with respect to Ψ is at most M. Then we obtain

$$u'_i = \begin{cases} f^{m_{i-1}M}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{m_iM} & (a_1 = 0,\ b_l = 0), \\ f^{m_{i-1}M+a_1}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{m_iM} & (a_1 \neq 0,\ b_l = 0), \\ f^{m_{i-1}M}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{b_l+m_iM} & (a_1 = 0,\ b_l \neq 0), \\ f^{m_{i-1}M+a_1}g^{b_1}f^{a_2}g^{b_2}\cdots f^{a_l}g^{b_l+m_iM} & (a_1 \neq 0,\ b_l \neq 0). \end{cases}$$

If $m_{i-1}M + a_1 = 0$, then $|a_1| = |-m_{i-1}M| = m_{i-1}M \geq M$. This is a contradiction since $|a_1|$ is less than M, thus $a_1 + m_{i-1}M \neq 0$. We can show that $b_l + m_iM$ is not equal to 0 in the same way as above. Therefore the first letter of u'_i is f and the last letter of u'_i is g in both the case of $u'_i = \emptyset$ and $u'_i \neq \emptyset$. For any $i = 2, 3, \ldots, s$, we can prove that if $\varepsilon_{i-1} = -1$ and $\varepsilon_i = -1$, the first letter of u_i' is g and the last letter of u_i' is f in the same way.

For any $i = 2, 3, \ldots, s$, if $\varepsilon_{i-1} = -1$ and $\varepsilon_i = 1$, the word u_i' is obtained by reducing a word $g^{-m_{i-1}M} u_i g^{m_i M}$. If $u_i = \emptyset$, then $u_i' = g^{(-m_{i-1}+m_i)M}$. Note $m_{i-1} \neq m_i$ by Remark 2.3. Therefore $u_i' = g^{(-m_{i-1}+m_i)M}$ is not the empty word, and u_i' starts with g and ends with g. If $u_i \neq \emptyset$, then we can prove that the first letter of u_i' is g and the last letter of u_i' is also g in the same way as in the case of $\varepsilon_{i-1} = \varepsilon_i = 1$.

If $\varepsilon_{i-1} = 1$ and $\varepsilon_i = -1$, the first letter of u_i' is f and the last letter of u_i' is also f by the same argument as above.

If $\varepsilon_s = 1$ (resp. $\varepsilon_s = -1$), the word u_{s+1}' is obtained by reducing the word $f^{m_s M} u_{s+1}$ (resp. $g^{-m_s M} u_{s+1}$). By the same argument as above, we know that the first letter of u_{s+1}' is f (resp. g). □

Proposition 2.6. *Let X be a Hausdorff topological space containing infinite elements. Let G be a finitely generated group which is not virtually cyclic. Suppose that G acts on X by homeomorphisms and satisfies the following conditions*:

(1) *There exists $g \in G$ such that $\#\operatorname{fix}(g) = 2$;*

(2) *Let g be any element of G such that $\operatorname{fix}(g)$ consists of exactly two points, denoted by a and b. If we choose a neighborhood A of a and a neighborhood B of b with $A \cap B = \emptyset$, then we have the inclusions*

$$g^n(X \setminus B) \subset A$$

and

$$g^{-n}(X \setminus A) \subset B$$

for any sufficiently large n, after exchanging a and b if necessary;

(3) *For any elements $g, h \in G$ such that $\#\operatorname{fix}(g) = \#\operatorname{fix}(h) = 2$, the fixed point sets $\operatorname{fix}(g)$ and $\operatorname{fix}(h)$ satisfy either $\operatorname{fix}(g) = \operatorname{fix}(h)$ or $\operatorname{fix}(g) \cap \operatorname{fix}(h) = \emptyset$;*

(4) *For any element $g \in G$ such that $\#\operatorname{fix}(g) = 2$, the stabilizer of $\operatorname{fix}(g)$,*

$$\operatorname{Stab}(\operatorname{fix}(g)) = \{h \in G \mid h(\operatorname{fix}(g)) = \operatorname{fix}(g)\},$$

is virtually cyclic.

Then the girth of G is infinite.

Proof. For any positive integer M, we will prove that there exists a finite generating set Ψ of G such that $U(\Psi, G) > M$.

Let $S = \{g_1, g_2, \ldots, g_k\}$ be a generating set of G. By condition (1), there exists $g \in G$ and $a, b \in X$ such that $a \neq b$ and $\operatorname{fix}(g) = \{a, b\}$. Note that for any $h \in G$, the conjugate element hgh^{-1} of g has two distinct fixed points $h(a)$ and $h(b)$. Thus there exists $h \in G$ such that

$$(*) \qquad \{h(a), h(b)\} \cap \{a, b, g_1^{\pm 1}(a), g_1^{\pm 1}(b), \ldots, g_k^{\pm 1}(a), g_k^{\pm 1}(b)\} = \emptyset$$

by conditions (3), (4) and Lemma 2.1.

Set $f = hgh^{-1} \in G$. The fixed point set of f is $\{h(a), h(b)\}$. There exists $x \in X$ such that

$$x \in X \setminus (\{a, b, h(a), h(b)\} \cup \{g_1(a), g_1(b), \ldots, g_k(a), g_k(b)\} \\ \cup \{g_1(h(a)), g_1(h(b)), \ldots, g_k(h(a)), g_k(h(b))\})$$

since X is an infinite set. Let U_1, U_2, U_3, U_4 and W be neighborhoods of $a, b, h(a), h(b)$ and x, respectively and disjoint each other. For all $l = 1, 2, \ldots, k$, we know $a \neq g_l^{\pm 1}(h(a))$, $a \neq g_l^{\pm 1}(h(b))$, $b \neq g_l^{\pm 1}(h(a))$ and $b \neq g_l^{\pm 1}(h(b))$ by $(*)$. Thus, for all $l = 1, 2, \ldots, k$, we can suppose that

$$(**) \qquad U_i \cap g_l^{\pm 1}(U_j) = \emptyset \quad \left(\begin{cases} i = 1, 2 \\ j = 3, 4 \end{cases} \text{or} \quad \begin{cases} i = 3, 4 \\ j = 1, 2 \end{cases} \right)$$

by continuity of the action of G on X. By condition (2), there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, we obtain $g^{\pm n}(X \setminus (U_1 \cup U_2)) \subset U_1 \cup U_2$ and $f^{\pm n}(X \setminus (U_3 \cup U_4)) \subset U_3 \cup U_4$. By $W \subset X \setminus (U_1 \cup U_2)$ and $W \subset X \setminus (U_3 \cup U_4)$, we obtain $g^{\pm n}(W) \subset U_1 \cup U_2$ and $f^{\pm n}(W) \subset U_3 \cup U_4$ for any $n \geq n_0$. Rewrite g^{n_0} and f^{n_0} by g and f, respectively, then $g^{\pm 1}(X \setminus (U_1 \cup U_2)) \subset U_1 \cup U_2$, $f^{\pm 1}(X \setminus (U_3 \cup U_4)) \subset U_3 \cup U_4$, $g^{\pm 1}(W) \subset U_1 \cup U_2$ and $f^{\pm 1}(W) \subset U_3 \cup U_4$.

A finite set

$$\Psi = \{g, f, g^M g_1 f^M, g^{2M} g_2 f^{2M}, \ldots, g^{kM} g_k f^{kM}\}$$

also generates G because G is generated by S.

We will show $U(\Psi, G) > M$.

Let w be a non-empty reduced word on Ψ whose word length with respect to Ψ is at most M. As we have seen, w is denoted by $w = u_1$ or

$$w = u_1 (g^{m_1 M} g_{m_1} f^{m_1 M})^{\varepsilon_1} u_2 (g^{m_2 M} g_{m_2} f^{m_2 M})^{\varepsilon_2} \cdots u_s (g^{m_s M} g_{m_s} f^{m_s M})^{\varepsilon_s} u_{s+1}$$

satisfying the following conditions:

- $m_i \in \{1, 2, \ldots, k\}$ for any $i = 1, 2, \ldots, s$;
- $\varepsilon_i = 1$ or -1 for any $i = 1, 2, \ldots, s$;
- For any $i = 1, 2, \ldots, s+1$, the word u_i is the empty word or a reduced word only containing f and g.

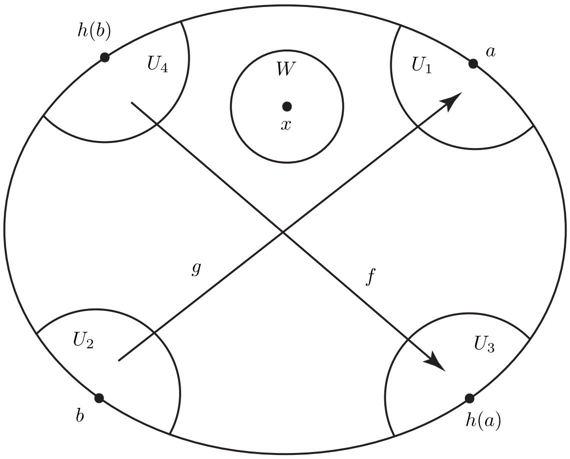

Fig. 1. The technique to prove the ping-pong lemma.

Put $\Psi' = \{g, f, g_1, g_2, \ldots, g_k\}$. Reduce the word w with respect to Ψ'. By Lemma 2.4 and Lemma 2.5, the word w' is denoted by $w' = u_1$ or $w' = u'_1 g^{\varepsilon_1}_{m_1} u'_2 g^{\varepsilon_2}_{m_2} \cdots u'_s g^{\varepsilon_s}_{m_s} u'_{s+1}$, where for all $i = 1, 2, \ldots, s+1$, the word u'_i is a non-empty reduced word only containing f and g. By Lemma 2.5, for all $i = 1, 2, \ldots, s$, the following assertions hold:

- If $\varepsilon_i = 1$, then the letter right before g_{m_i} is g and the letter right after g_{m_i} is f;
- If $\varepsilon_i = -1$, then the letter right before $g^{-1}_{m_i}$ is f and the letter right after $g^{-1}_{m_i}$ is g.

We move the point $x \in W \subset X$ by any element w of G whose word length with respect to Ψ is at most M and show $wx \neq x$ by using the same technique to prove the ping-pong lemma. If $wx \neq x$, the element w of G is not trivial in G and thus $U(\Psi, G) > M$.

If $w = u_1$, using the same technique in the proof of the ping-pong lemma, we see that $wx = u_1 x$ is a point of $U_1 \cup U_2 \cup U_3 \cup U_4$ because u_1 consists of only g and f. We see $wx \neq x$ since $wx \in U_1 \cup U_2 \cup U_3 \cup U_4$ and $x \in W \subset X \setminus (U_1 \cup U_2 \cup U_3 \cup U_4)$.

We consider the case of

$$w = u_1 (g^{m_1 M} g_{m_1} f^{m_1 M})^{\varepsilon_1} u_2 (g^{m_2 M} g_{m_2} f^{m_2 M})^{\varepsilon_2} \cdots u_s (g^{m_s M} g_{m_s} f^{m_s M})^{\varepsilon_s} u_{s+1}.$$

By Lemma 2.4, w is denoted by $w' = u'_1 g^{\varepsilon_1}_{m_1} u'_2 g^{\varepsilon_2}_{m_2} \cdots u'_s g^{\varepsilon_s}_{m_s} u'_{s+1}$ as a reduced word on Ψ'. As an element of G, w is equal to w' and thus moving x by $w' \in G$ is equal to moving x by $w \in G$.

In $u'_1 g^{\varepsilon_1}_{m_1} u'_2 g^{\varepsilon_2}_{m_2} \cdots u'_s g^{\varepsilon_s}_{m_s} u'_{s+1} x$ $(= w'x)$, note $u'_{s+1} x$ first. Recall that the element u'_{s+1} of G contains only f and g. If $\varepsilon_s = 1$, then the first letter of u'_{s+1} is f by Lemma 2.5. Using the same technique in the proof of the ping-pong lemma, we know $u'_{s+1} x \in U_3 \cup U_4$.

If $\varepsilon_s = -1$, then the first letter of u'_{s+1} is g by Lemma 2.5 and thus $u'_{s+1} x \in U_1 \cup U_2$.

In $u'_1 g_{m_1}^{\varepsilon_1} u'_2 g_{m_2}^{\varepsilon_2} \cdots u'_s g_{m_s}^{\varepsilon_s} u'_{s+1} x$ $(= w'x)$, we note $u'_s g_{m_s}^{\varepsilon_s} u'_{s+1} x$ next. If $\varepsilon_s = 1$ and $\varepsilon_{s-1} = 1$, then

$$u'_s g_{m_s} u'_{s+1} x \in u'_s g_{m_s}(U_3 \cup U_4).$$

By $(**)$, we see

$$U_1 \cap g_{m_s}(U_3 \cup U_4) = \emptyset, \quad U_2 \cap g_{m_s}(U_3 \cup U_4) = \emptyset$$

and thus

$$u'_s g_{m_s} u'_{s+1} x \in u'_s g_{m_s}(U_3 \cup U_4) \subset u'_s(X \setminus (U_1 \cup U_2)).$$

Since the last letter of u'_s is g and the first letter of u'_s is f by Lemma 2.5, using the same technique in the proof of the ping-pong lemma again, we see $u'_s(X \setminus (U_1 \cup U_2)) \subset U_3 \cup U_4$. Therefore $u'_s g_{m_s} u'_{s+1} x \in U_3 \cup U_4$.

By the same argument as above, we can see

$$u'_s g_{m_s}^{\varepsilon_s} u'_{s+1} x \in \begin{cases} U_1 \cup U_2 & (\varepsilon_s = 1,\, \varepsilon_{s-1} = -1), \\ U_3 \cup U_4 & (\varepsilon_s = -1,\, \varepsilon_{s-1} = 1), \\ U_1 \cup U_2 & (\varepsilon_s = -1,\, \varepsilon_{s-1} = -1). \end{cases}$$

Repeating this process, we obtain

$$\begin{aligned} wx = w'x &= u'_1 g_{m_1}^{\varepsilon_1} u'_2 g_{m_2}^{\varepsilon_2} \cdots u'_s g_{m_s}^{\varepsilon_s} u'_{s+1} x \\ &\in U_1 \cup U_2 \cup U_3 \cup U_4 \subset X \setminus W. \end{aligned}$$

This means that $wx \neq x$ since $wx \in X \setminus W$ and $x \in W$.

In conclusion, any element w of G whose word length with respect to Ψ is at most M is non-trivial in G and thus the inequality $U(\Psi, G) > M$ holds. □

3. Convergence groups

We will show that some convergence groups satisfy the conditions of Proposition 2.6 and thus they have infinite girth. We first recall some definitions and known results of convergence groups. Although there exist two equivalent definitions of convergence groups by Tukia [17] and Bowditch [3], we adopt Tukia's definition in this paper. We refer to [17] for more details.

DEFINITION 3.1. Let X be an infinite compact Hausdorff space and G an infinite group acting on X by homeomorphisms. We say that G acts on X as a *convergence group* (or simply say that G is a *convergence group*) if, whenever $\{g_n\}_{n\in\mathbb{N}}$ is a sequence consisting of mutually distinct elements of G, we can find a subsequence $\{g_{n_i}\}_i$

and points $a, b \in X$ (which may be equal) satisfying the following: For any compact set K in $X \setminus \{b\}$, we have

$$g_{n_i}(x) \to a \quad \text{uniformly } (i \to \infty,\ \forall x \in K),$$

and for any compact set K' in $X \setminus \{a\}$, we have

$$g_{n_i}^{-1}(x) \to b \quad \text{uniformly } (i \to \infty,\ \forall x \in K').$$

We say that a is an *attractive point* of $\{g_n\}_n$ and b is a *repelling point* of $\{g_n\}_n$.

Any subgroup of convergence groups is also a convergence group by definition.

In this section, we assume that X is an infinite compact Hausdorff space and that G is an infinite group which acts on X as a convergence group.

DEFINITION 3.2. If $g \in G$ is a finite order element, then we say that g is *elliptic*. If $g \in G$ is an infinite order element and has two distinct fixed points in X, we say that g is *loxodromic*. If $g \in G$ is an infinite order element and has exactly one fixed point in X, g is said to be *parabolic*.

Lemma 3.3 ([17] Lemma 2A). *If H is an abelian convergence group and $\{h_n\}_n$ is an infinite sequence consisting of mutually distinct elements of H. Then the attractive and repelling point of $\{h_n\}_n$ are fixed by every $h \in H$.*

We know that a non-elliptic element $g \in G$ has at most two fixed points in X by considering the infinite sequence $\{g^n\}_n$ and using Lemma 3.3.

The following lemma holds by [17, Lemma 2D].

Lemma 3.4. *Let g be a loxodromic element of G. Let $a \in X$ be an attractive point of $\{g^n\}_n$ and let $b \in X$ be a repelling point of $\{g^n\}_n$. Then we have $\mathrm{fix}(g) = \{a, b\}$. Moreover, if we choose a neighborhood of A of a and a neighborhood B of b with $A \cap B = \emptyset$, then we have the inclusions $g^n(X \setminus B) \subset A$ and $g^{-n}(X \setminus A) \subset B$ for any sufficiently large n.*

Theorem 3.5 ([17] Theorem 2I). *Let $g \in G$ be loxodromic. Then the infinite cyclic group $\langle g \rangle$ generated by g is a finite index subgroup in the stabilizer $\mathrm{Stab}(\mathrm{fix}(g)) = \{h \in G \mid h(\mathrm{fix}(g)) = \mathrm{fix}(g)\}$ of the fixed point set of g.*

DEFINITION 3.6. We say that $x \in X$ is a *limit point* of G if x is an attractive point or a repelling point of an infinite sequence consisting of distinct elements of G. We call the set of limit points of G the *limit set* of G and denote it by $L(G)$.

The following two results are important for us.

Theorem 3.7 ([17] Theorem 2G). *Let $g \in G$ be a loxodromic or parabolic element. Let $h \in G$ be also a loxodromic or parabolic element. Then they satisfy either* $\mathrm{fix}(g) = \mathrm{fix}(h)$ *or* $\mathrm{fix}(g) \cap \mathrm{fix}(h) = \emptyset$.

Lemma 3.8 ([17] Lemma 2Q). *If $L(G)$ contains at least two points, then for given $x \in L(G)$ and a neighborhood U of x, there exists a loxodromic element $g \in G$ with one fixed point in U.*

Using the above results, we will prove one of our main theorems by showing to satisfy all the conditions of Proposition 2.6.

Theorem 3.9. *Let G be a convergence group. Suppose that H is a finitely generated subgroup of G which is not virtually cyclic and that the limit set $L(H)$ of H contains at least two points. Then the girth of H is infinite.*

Proof. Since a subgroup of a convergence group is also a convergence group, there exists a loxodromic element $g \in H$ by Lemma 3.8. Thus condition (1) in Proposition 2.6 is satisfied.

Condition (2) in Proposition 2.6 is satisfied by Lemma 3.4. Conditions (3) and (4) in Proposition 2.6 are satisfied by Theorem 3.7 and Theorem 3.5, respectively. By the above argument, all the conditions in Proposition 2.6 are satisfied, thus the girth of H is infinite. □

4. Mapping class groups

In this section, we will show that some mapping class groups have infinite girth. We first recall some properties of mapping class groups. We refer to [6], [9], [10] for more details.

DEFINITION 4.1. Let M be a compact orientable surface. We admit that M has non-empty boundary. Let Mod_M be the group of isotopy classes of orientation preserving diffeomorphisms on M. We call Mod_M the *mapping class group* of the surface M.

In the following context, the surface M is assumed compact, orientable and connected. We denote by g the genus of M and by p the number of boundary components of M. Put $\kappa(M) = 3g + p - 4$ and assume $\kappa(M) \geq 0$ unless otherwise stated.

Let $V(M)$ be the set of all non-trivial isotopy classes of non-peripheral (i.e., not isotopic to any boundary component of M) simple closed curves on M. We denote by $\mathcal{R}(M)$ the set of all non-negative real valued functions on $V(M)$ with product topology. Let $\mathcal{PR}(M)$ be the quotient space of $\mathcal{R}(M) \setminus \{0\}$ by the natural diagonal action of the multiplicative group $\mathbb{R}_{>0}$ of all positive real numbers.

Let $i\colon V(M)\times V(M)\to\mathbb{N}$ be the minimal geometric intersection number among two elements of $V(M)$. For all $\alpha\in V(M)$, the minimal geometric intersection number $i(\alpha,\alpha)$ is equal to zero.

We identify $V(M)$ to $\mathcal{R}(M)\setminus\{0\}$, i.e., identify α in $V(M)$ to $i(\alpha,\cdot)$ in $\mathcal{R}(M)\setminus\{0\}$. The map $V(M)\ni\alpha\mapsto i(\alpha,\cdot)\in\mathcal{R}(M)\setminus\{0\}$ is injective. The closure of the set $\{r\alpha\mid r\in\mathbb{R}_{>0},\alpha\in V(M)\}$ in $\mathcal{R}(M)$ is denoted by $\mathcal{MF}_M$. The space $\mathcal{MF}_M$ is homeomorphic to the Euclid space of dimension $6g-6+2p$. Each element of $\mathcal{MF}_M$ is identified to a foliation with some singularities on M with transverse measure [6, Exposé 5].

The map $V(M)\to\mathcal{R}(M)\setminus\{0\}\to\mathcal{PR}(M)$ is also injective. The closure of the image of $V(M)$ is denoted by $\mathcal{PMF}_M\subset\mathcal{PR}(M)$. The space $\mathcal{PMF}_M$ is called the *Thurston boundary* of M and homeomorphic to the sphere of dimension $6g-7+2p$. The function $i\colon V(M)\times V(M)\to\mathbb{N}$ can be extended continuously to a function $i\colon\mathcal{MF}_M\times\mathcal{MF}_M\to\mathbb{R}_{\geq 0}$ such that

$$i(r_1F_1,r_2F_2)=r_1r_2i(F_1,F_2)$$

for any $r_1,r_2\in\mathbb{R}_{>0}$ and $F_1,F_2\in\mathcal{MF}_M$. Hence for all elements $F_1,F_2\in\mathcal{PMF}_M$, it makes sense whether $i(F_1,F_2)$ is zero or not. We put

$$\mathcal{MIN}=\{F\in\mathcal{PMF}_M\mid i(F,\alpha)\neq 0\text{ for any }\alpha\in V(M)\}.$$

The mapping class group Mod_M of M continuously acts on $\mathcal{MF}_M$ and $\mathcal{PMF}_M$ because $\mathcal{R}(M)$ is equipped with the product topology. Hence all elements of Mod_M are homeomorphisms of $\mathcal{MF}_M$ and $\mathcal{PMF}_M$. Since the function $i\colon\mathcal{MF}_M\times\mathcal{MF}_M\to\mathbb{R}_{\geq 0}$ is Mod_M-invariant, i.e.,

$$i(gF_1,gF_2)=i(F_1,F_2)$$

for all $g\in Mod_M$ and $F_1,F_2\in\mathcal{MF}_M$, the space $\mathcal{MIN}$ is Mod_M-invariant.

In the following context, consider the action of the group Mod_M on $\mathcal{PMF}_M$. We call $g\in Mod_M$ a *pseudo-Anosov* element if g is infinite order and has distinct two fixed points in $\mathcal{PMF}_M$. These fixed points are contained in $\mathcal{MIN}$. A pseudo-Anosov element of Mod_M has the following properties.

Lemma 4.2 ([9] Lemma 8.3). *Let $f\in Mod_M$ be a pseudo-Anosov element and μ^u (resp. μ^s) the unstable (resp. stable) foliation of f in $\mathcal{MF}_M$. If U and V are disjoint neighborhoods in $\mathcal{PMF}_M$ of the projective points of μ^u and μ^s to $\mathcal{PMF}_M$, respectively, then we have the inclusions $f^n(\mathcal{PMF}_M\setminus V)\subset U$ and $f^{-n}(\mathcal{PMF}_M\setminus U)\subset V$ for any sufficiently large n.*

Lemma 4.3 ([9] Lemma 5.11). *If f and g are pseudo-Anosov elements of Mod_M, then either* $\mathrm{fix}(f)=\mathrm{fix}(g)$ *or* $\mathrm{fix}(f)\cap\mathrm{fix}(g)=\emptyset$.

We define a simplicial complex $C(M)$ called the *curve complex*. The set of vertices of $C(M)$ is $V(M)$. A simplex of $C(M)$ is a non-empty finite subset of $V(M)$ which can be realized disjointly on M. We denote by $S(M)$ the set of all simplices of $C(M)$. Note that Mod_M naturally acts on $C(M)$ as simplicial automorphisms. The set $S(M)$ is naturally embedded into $\mathcal{PMF}_M$ so that $\mathcal{MIN} \cap S(M) = \emptyset$.

A subgroup G of Mod_M is said to be *reducible* if there is an element of $S(M)$ fixed by all elements of G. Otherwise G is said to be *irreducible*.

Lemma 4.4 ([9] Corollary 7.14). *If G is an infinite irreducible subgroup of Mod_M, then G contains a pseudo-Anosov element.*

Theorem 4.5 ([12] Theorem 4.6). *If $\kappa(M) \geq 0$, then any subgroup G of Mod_M is one of the following types*:
(1) *G is finite*;
(2) *G is infinite and reducible*;
(3) *There exists a pseudo-Anosov element $g \in G$ such that $h(\mathrm{fix}(g)) = \mathrm{fix}(g)$ for any $h \in G$*;
(4) *There exist two pseudo-Anosov elements $g_1, g_2 \in G$ such that $\mathrm{fix}(g_1) \cap \mathrm{fix}(g_2) = \emptyset$.*

REMARK 4.6. A group G satisfying condition (3) is virtually infinite cyclic and G satisfying condition (4) contains a non-abelian free subgroup.

Using Theorem 4.5, we will prove the following lemma.

Lemma 4.7. *Let M be a surface with $\kappa(M) \geq 0$. If an element f of Mod_M is pseudo-Anosov, then the stabilizer of $\mathrm{fix}(f)$, that is, $\mathrm{Stab}(\mathrm{fix}(f)) = \{g \in Mod_M \mid g(\mathrm{fix}(f)) = \mathrm{fix}(f)\}$ is a virtually infinite cyclic group.*

Proof. The stabilizer $\mathrm{Stab}(\mathrm{fix}(f))$ is an infinite subgroup of Mod_M since f is an infinite order element. Hence $\mathrm{Stab}(\mathrm{fix}(f))$ is the group in case of (2) or (3) or (4) in Theorem 4.5. It is impossible that $\mathrm{Stab}(\mathrm{fix}(f))$ is the group in case (2). If there exists $\sigma \in S(M)$ such that $\mathrm{Stab}(\mathrm{fix}(f))\sigma = \sigma$, then $f\sigma = \sigma$ and thus $\sigma \subset \mathrm{fix}(f)$. Since $\mathcal{MIN}$ contains $\mathrm{fix}(f)$ and does not contain σ, this is a contradiction. Therefore $\mathrm{Stab}(\mathrm{fix}(f))$ is the group in case (3) or (4). In case (4), there exists a pseudo-Anosov element $g \in \mathrm{Stab}(\mathrm{fix}(f))$ such that $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$. This is impossible for f, $g \in \mathrm{Stab}(\mathrm{fix}(f))$. Therefore $\mathrm{Stab}(\mathrm{fix}(f))$ is the group in case (3), and it is a virtually infinite cyclic subgroup. □

We prove another main theorem such that the girth of some mapping class groups is infinite. Note that we do not assume $\kappa(M) \geq 0$ in this theorem.

Theorem 4.8. *Let G be a finitely generated irreducible subgroup of Mod_M and suppose that G is not virtually cyclic. Then the girth of G is infinite.*

Proof. For a surface M with $\kappa(M) < 0$, the mapping class group Mod_M of M is a finite group or isomorphic to $SL(2, \mathbb{Z})$. Since it is known that $SL(2, \mathbb{Z})$ is a Gromov-hyperbolic group, this theorem is true by Theorem 1.2.

We will show that all the conditions in Proposition 2.6 are satisfied if $\kappa(M) \geq 0$. Note that $\mathcal{PMF}_M$ is homeomorphic to the sphere of dimension $6g - 7 + 2p$ and Mod_M acts on $\mathcal{PMF}_M$ by homeomorphisms. Condition (1) in Proposition 2.6 is satisfied by Lemma 4.4. Condition (2) in Proposition 2.6 is satisfied by Lemma 4.2. We see that condition (3) in Proposition 2.6 is satisfied by Lemma 4.3 and condition (4) is satisfied by Lemma 4.7. □

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