Nagase, M.
Osaka J. Math.
23 (1986), 425–440

ON SOME CLASSES OF L^p-BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

To the memory of Professor Hitoshi Kumano-go

Michihiro NAGASE

(Received March 22, 1985)

1. Introduction

In the present paper we shall give some sufficient conditions for the boundedness of pseudo-differential operators in $L^p=L^p(R^n)$ for $2\leqq p\leqq\infty$. We treat the classes of non-regular symbols, which generalize the Hörmander's class $S^m_{\rho,\delta}$. There have already been many L^p-boundedness theorems of pseudo-differential operators with symbols which belong to generalized classes of $S^m_{\rho,\delta}$ and are at least $n+\varepsilon$ differentiable in the covariables $\xi=(\xi_1, \cdots, \xi_n)$. In the present paper we study the boundedness for operators with symbols $p(x, \xi)$ which are only up to $\kappa=[n/2]+1$ differentiable in ξ.

Recently in [16], Wang-Li showed an L^p-boundedness theorem for pseudo-differential operators with symbols which belong to a generalized class of $S^{-m_p}_{\rho,\rho}$, where $0<\rho<1$ and $m_p=n(1-\rho)|1/2-1/p|$. Moreover in [12] and [13], the author has obtained L^p-boundedness theorems for the operators which have symbols of generalized class of $S^0_{1,\delta}(0\leqq\delta<1)$. In these paper the L^p-boundedness theorems for $p\geqq 2$ are proved under the assumptions that the symbols are only up to $\kappa=[n/2]+1$ differentiable and satisfy some additional conditions.

The main theorem of the present paper is Theorem 4.5 in Section 4, which is given for operators in the generalized class of Hörmander's $S^{-m_p}_{\rho,0}$. We note that Theorem 4.5 is obtained under $\kappa=[n/2]+1$ differentiability in ξ and Hölder continuity condition in the space varaibles $x=(x_1, \cdots, x_n)$ when p is sufficiently large or ρ is sufficiently near to 1.

As pointed out by Hörmander in [5], $m_p=n(1-\rho)|1/2-1/p|$ is the critical decreasing order for the L^p-boundedness of pseudo-differential operators with symbols in $S^m_{\rho,\delta}$. Furthermore we note that $\kappa=[n/2]+1$ differentiability of symbols in ξ does not always imply the L^p-boundedness of the operators when $1\leqq p<2$ (see [16] and [17]).

In Section 2 we give notation and preliminary lemmas. In Section 3, we show L^p-boundedness theorems for the operators with symbols which have higher decreasing order than the critical decreasing order m_p, as $|\xi|\to\infty$. In

Section 4, we investigate the L^p-boundedness of operators with symbols which have the critical decreasing order as $|\xi|\to\infty$. The main theorem is proved by using an approximation (regularization) of symbols (see [8]).

2. Preliminaries

We use a standard notation which is used in the theory of pseudo-differential operators (see [7] and [15]). Let $p(x, \xi)$ be a function defined on $R_x^n\times R_\xi^n$. Then the pseudo-differential operator $p(X, D_x)$ associated with symbol $p(x, \xi)$ is defined, formally, by

$$p(X, D_x)\, u(x) = \int e^{ix\cdot\xi}\, p(x, \xi)\, \hat{u}(\xi)\, d\xi\,,$$

where $\hat{u}(\xi)$ denotes the Fourier transform of the function $u(x)$, that is, $\hat{u}(\xi)=\int e^{-ix\cdot\xi}\, u(x)\, dx$, and $d\xi=(2\pi)^{-n}\, d\xi$. For $p(x, \xi)$ we denote $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$ $=(-i)^{|\beta|}\, \partial_\xi^\alpha \partial_x^\beta p(x, \xi)$ for any multi-indices α and β. Moreover we write $\langle\xi\rangle=$ $(1+|\xi|^2)^{1/2}$. Then the Hörmander's class $S_{\rho,\delta}^m$ of symbols is defined by $S_{\rho,\delta}^m=$ $\{p(x, \xi)\in C^\infty(R_x^n\times R_\xi^n);\ |p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$ for any α and $\beta\}$. Here and hereafter we denote by C, C_α, $C_{\alpha,\beta}$, c_n etc., the constants which are independent of the variables (x, ξ) and are not always the same at each occurence. We denote by N, N_0, N_1 etc., the semi-norms of symbols. Moreover we denote $\kappa=[n/2]+1$.

Lemma 2.1. *Let $0\leqq\rho<1$ and let $\omega(t)$ be a non-negative and non-decreasing function defined on $[0, \infty)$ and satisfy*

$$\int_0^1 \frac{\omega(t)^2}{t}\, dt = M_2<\infty\,. \tag{2.1}$$

Suppose that a symbol $p(x, \xi)$ satisfies

$$\begin{cases} N_0 = \sup\limits_{|\alpha|\leqq\kappa,\ (x,\xi)} |p^{(\alpha)}(x, \xi)|\langle\xi\rangle^{|\alpha|}<\infty\,, \\ N_1 = \sup\limits_{|\alpha|\leqq\kappa,\ (x,y,\xi)} |p^{(\alpha)}(x, \xi)-p^{(\alpha)}(y, \xi)|\,\omega(|x-y|\langle\xi\rangle^\delta)^{-1}\langle\xi\rangle^{|\alpha|}<\infty\,. \end{cases} \tag{2.2}$$

Then $p(X, D_x)$ is L^2-bounded and we have

$$\|p(X, D_x)\, u\|_{L^2}\leqq C(N_0+N_1\, M_2)\|u\|_{L^2}\,. \tag{2.3}$$

Lemma 2.1 is shown in [9] and [10] for $\delta=0$ and in [13] for $0\leqq\delta<1$.

Lemma 2.2. *Let $0\leqq\rho<1$. Suppose that a symbol $p(x, \xi)$ satisfies*

$$N = \sup_{|\alpha|\leqq\kappa,\ |\beta|\leqq\kappa,\ (x,\xi)} |p_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle^{\rho(|\alpha|-|\beta|)}<\infty\,. \tag{2.4}$$

Then $p(X, D_x)$ is L^2-bounded and we have

$$\|p(X, D_x)\, u\|_{L^2} \leqq C\, N \|u\|_{L^2}\,. \tag{2.5}$$

When $\rho=0$, the lemma is obtained by Cordes in [2]. In [6] Kato proved the L^2-boundedness for $0<\rho<1$ when the semi-norm N in (2.4) is defined for $|\alpha| \leqq \kappa$ and $|\beta| \leqq \kappa+1$. In [1] Coifman-Meyer obtained the Lemma 2.2.

We use the following lemma in Section 4 in order to smooth the non-regular symbols. The lemma is shown in [8] and [11].

Lemma 2.3. *Let τ be a positive number. Then for any α there exists $\{\phi_{\alpha,\beta}(\xi)\}_{|\beta|\leqq|\alpha|}$ in $S_{1,0}^{-|\alpha|}$ such that for any C^∞ function function ψ we have*

$$\partial_\xi^\alpha \{\psi(\langle\xi\rangle^\tau z)\} = \sum_{|\beta|\leqq|\alpha|} \phi_{\alpha,\beta}(\xi) \{\langle\xi\rangle^\tau z\}^\beta\, \psi^{(\beta)}(\langle\xi\rangle^\tau z)\,, \tag{2.6}$$

where $\psi^{(\beta)}(y)=\partial_y^\beta \psi(y)$.

3. L^p-boundedness for operators with lower order symbols

In this section we treat pseudo-differential operators associated with symbols which decrease as $|\xi|\to\infty$ faster than the critical decreasing order for L^p-boundedness. We denote the norm of $L^p=L^p(R^n)$ by $\|\cdot\|_p$ and denote by $L(L^p)$ the space of bounded linear operators on L^p. Let $H^s=H^s(R^n)$ denote the Sobolev space of order s with norm $\|\cdot\|_{H^s}$ defined by

$$\|u\|_{H^s} = \|\langle D_x\rangle^s u\|_{L^2} = \{\int |\langle\xi\rangle^s \hat{u}(\xi)|^2 d\xi\}^{1/2}\,,$$

and let $\|\cdot\|_{H^s(a)}$ denote the equivalent norm with positive parameter a defined by

$$\|u\|_{H^s(a)} = \|\langle aD_x\rangle^s\, u\|_{L^2} = \{\int |\langle a\xi\rangle^s \hat{u}(\xi)|^2 d\xi\}^{1/2}\,.$$

Proposition 3.1. *Let $s>n/2$ and let $2\leqq p\leqq\infty$. We assume that a symbol $p(x,\xi)$ belongs to the Sobolev space H^s and satisfies*

$$\sup_x \|p(x,\cdot)\|_{H^s} = N_0 < \infty\,. \tag{3.1}$$

Then the operator $p(X, D_x)$ belongs to $L(L^p)$ and satisfies

$$\|p(X, D_x)\, u\|_p \leqq C\, a^{-n/2} \sup_x \|p(x,\cdot)\|_{H^s(a)} \|u\|_p \tag{3.2}$$

for any $a>0$, where the constant C is independent of $2\leqq p\leqq\infty$.

Proof. We have only to prove L^2- and L^∞-boundedness of the operator because of the Riesz-Thorin interpolation theorem (see [18]). First we show L^∞-boundedness. We can write

$$p(X, D_x)\, u(x) = \int K(x, x-y)\, u(y)\, dy\,, \tag{3.3}$$

where the integral kernel $K(x, z)$ is defined by

$$K(x, z) = \int e^{iz\cdot\xi} p(x, \xi)\, d\xi \,. \tag{3.4}$$

It follows from the Schwarz inequality that

$$\int |K(x, z)|\, dz \leqq \{\int \langle az\rangle^{-2s} dz\}^{1/2} \{\int \langle az\rangle^{2s} |K(x, s)|^2\, dz\}^{1/2}$$
$$= c_n a^{-n/2} ||p(x, \cdot)||_{H^s(a)} \leqq c_n a^{-n/2} \sup_x ||p(x, \cdot)||_{H^s(a)} \,,$$

and this implies that the operator $p(X, D_x)$ is L^∞-bounded.

Next we show L^2-boundedness. By (3.3) we have

$$\int |p(X, D_x)\, u(x)|^2\, dx \leqq \int (\int |K(x, x-y)\, u(y)|\, dy)^2\, dx$$
$$\leqq \int \{\int \langle a(x-y)\rangle^{2s} |K(x, x-y)|^2\, dy\} \{\int \langle a(x-y)\rangle^{-2s} |u(y)|^2\, dy\}\, dx$$
$$\leqq c_n^2\, a^{-n} (\sup_x ||p(x, \cdot)||_{H^s(a)})^2\, ||u||_2^2 \,.$$

This means that the operator $p(X, D_x)$ belongs to $L(L^2)$. Q.E.D.

We note that the symbol in Proposition 3.1 is uniformly bounded by the Sobolev inequality, however, the derivatives of the symbols are not always bounded. As a special case we have

Corollary 3.2. *Let $2 \leqq p \leqq \infty$. If the support of a symbol $p(x, \xi)$ is contained in $\{\xi;\ |\xi| \leqq r\}$ for some positive constant r and if $p(x, \xi)$ satisfies*

$$N_0 = \sup_{|\alpha| \leqq \kappa,\ (x, \xi)} |p^{(\alpha)}(x, \xi)| < \infty \,, \tag{3.5}$$

then the operator $p(X, D_x)$ is L^p-bounded and we have

$$||p(X, D_x)\, u||_p \leqq C\, N_0 ||u||_p \,, \tag{3.6}$$

where the constant C is independent of $2 \leqq p \leqq \infty$.

By this corollary, hereafter we may assume that the support of the symbols are contained in $\{\xi;\ |\xi| \geqq R\}$ for some positive R.

Theorem 3.3. *Let $0 \leqq \rho \leqq 1$ and let $\omega(t)$ be a non-negative and non-decreasing function which satisfies*

$$\int_0^1 \frac{\omega(t)}{t}\, dt = M_1 < \infty \,. \tag{3.7}$$

If a symbol $p(x, \xi)$ satisfies

$$N = \sup_{|\alpha| \leqq \kappa,\, (x,\xi)} |p^{(\alpha)}(x, \xi)|\, \omega(\langle\xi\rangle^{-1})^{-1} \langle\xi\rangle^{n(1-\rho)/2+\rho|\alpha|} < \infty\,, \tag{3.8}$$

then $p(X, D_x)$ belongs to $L(L^p)$ for $2 \leqq p \leqq \infty$ and we have

$$||p(X, D_x)\, u||_p \leqq C(N\, M_1 + N_0)\, ||u||_p\,, \tag{3.9}$$

where the constant C is independent of $2 \leqq p \leqq \infty$, and N_0 is defined by

$$N_0 = \sup_{|\alpha| \leqq \kappa, |\xi| \leqq 4} |p^{(\alpha)}(x, \xi)|\,. \tag{3.10}$$

Proof. By Corollary 3.2 we may assume that the support of $p(x, \xi)$ is contained in $\{\xi;\, |\xi| \geqq 2\}$, because of (3.10). Then since $\omega(t)$ is non-decreasing, (3.8) can be replaced by

$$|p^{(\alpha)}(x, \xi)| \leqq N\, \omega(|\xi|^{-1})\, |\xi|^{-n(1-\rho)/2-\rho|\alpha|} \quad (|\xi| \geqq 2) \tag{3.8$'$}$$

for $|\alpha| \leqq \kappa$. We take a smooth function $f(t)$ on R^1 so that the support is contained in the interval $[1/2,1]$, $f(t) \geqq 0$ and

$$\int_0^\infty \frac{f(t)}{t}\, dt = 1\,. \tag{3.11}$$

Then since

$$\int_0^\infty \frac{f(t|\xi|)}{t}\, dt = 1 \quad \text{for} \quad |\xi| \neq 0\,,$$

we can write

$$p(X, D_x)\, u(x) = \int_0^{1/2} p(t, X, D_x)\, u(x)\, \frac{dt}{t}\,,$$

where $p(t, x, \xi) = p(x, \xi) f(t|\xi|)$, since $p(t, x, \xi) = 0$ for $t > 1/2$.

To estimate the norm of $p(t, X, D_x)$ we make use of Proposition 3.1 with $s = \kappa$ and $a = t^{-\rho}$. Since $1/(2t) \leqq |\xi| \leqq 1/t$ on the support of $f(t|\xi|)$, we have

$$\sum_{|\alpha| \leqq \kappa} |t^{-\rho|\alpha|}\, \partial_\xi^\alpha \{p(x, \xi) f(t|\xi|)\}|^2 \leqq C^2\, N^2\, t^{n(1-\rho)}\, \omega(2t)^2\,.$$

Therefore we have

$$\begin{aligned} ||p(t, x, \cdot)||^2_{H^\kappa(t^{-\rho})} &\leqq C^2\, N^2\, t^{n(1-\rho)}\, \omega(2t)^2 \int_{1/(2t) \leqq |\xi| \leqq 1/t} d\xi \\ &= C^2\, N^2\, t^{-n\rho}\, \omega(2t)^2\,. \end{aligned}$$

Hence, by Proposition 3.1, we see that the norm of the operator $p(t, X, D_x)$ is not greater than $CN\omega(2t)$, which gives

$$||p(X, D_x)\, u||_p \leqq CN \int_0^{1/2} \omega(2t)\, \frac{dt}{t}\, ||u||_p = CNM_1 ||u||_p\,. \qquad \text{Q.E.D.}$$

REMARK 3.4. (i) In this theorem we did not assume the continuity of symbols in the space variables x. In fact we needed only the uniform boundedness and measurability of symbols in the space variables x in the proof of this theorem.
(ii) In the case $\rho=1$, Theorem 3.3 has already been proved in [12] and [13].

Now we give L^p-boundedness results in the case $0\leqq\rho<1$ as corollaries of Theorem 3.3.

Corollary 3.5. *Let* $0\leqq\rho<1$ *and* $2\leqq p\leqq\infty$. *We assume that a function* $\omega(t)$ *on* $[0, \infty)$ *is the same as in Theorem* 3.3 *and assume that a symbol* $p(x, \xi)$ *satisfies*

$$N=\sup_{|\alpha|\leqq\kappa,\, |\beta|\leqq\kappa,\, (x,\xi)} |p_{(\beta)}^{(\alpha)}(x, \xi)|\, \omega(\langle\xi\rangle^{-1})^{-1}\langle\xi\rangle^{m_p+\rho(|\alpha|-|\beta|)}<\infty\,, \tag{3.12}$$

where m_p *is the critical decreasing order for* L^p*-boundedness, that is,*

$$m_p=n(1-\rho)\,(1/2-1/p)\,. \tag{3.13}$$

Then $p(X, D_x)$ *is* L^p*-bounded and we have*

$$||p(X, D_x)\, u||_p\leqq C(N\, M_1+N_0)\, ||u||_p\,, \tag{3.14}$$

where the constant C *is independent of* $2\leqq p\leqq\infty$ *and* N_0 *is defined in* (3.10).

Proof. When $p=\infty$ and $p(x, \xi)$ satisfies (3.12) for $p=\infty$, by Theorem 3.3, $p(X, D_x)$ is L^∞-bounded. Since $\omega(\langle\xi\rangle^{-1})$ is a bounded function in ξ, if $p(x, \xi)$ satisfies (3.12) for $p=2$, then it follows from Lemma 2.2 that $p(X, D_x)$ is L^2-bounded. Then by the interpolation theorem of analytic families of operators (see, for example, [14]), we can get the corollary by defining the families of operators in a similar way to Wang-Li in [16] (see also [3]). Q.E.D.

Corollary 3.6. *Let* $0\leqq\rho\leqq1$ *and* $m>n(1-\rho)/2$. *If a symbol* $p(x, \xi)$ *satisfies*

$$N=\sup_{|\alpha|\leqq\kappa,\, (x,\xi)} |p^{(\alpha)}(x, \xi)|\langle\xi\rangle^{m+\rho|\alpha|}<\infty\,, \tag{3.15}$$

then $p(X, D_x)$ *belongs to* $L(L^p)$ *for* $2\leqq p\leqq\infty$, *and we have*

$$||p(X, D_x)\, u||_p\leqq C\, N||u||_p\,, \tag{3.16}$$

where we can take the constant C *independently of* $2\leqq p\leqq\infty$.

Corollary 3.7. *Let* $0\leqq\rho<1$, $2\leqq p\leqq\infty$ *and* $m>m_p$. *If a symbol* $p(x, \xi)$ *satisfies*

$$N=\sup_{|\alpha|\leqq\kappa,\, |\beta|\leqq\kappa,\, (x,\xi)} |p_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle^{m+\rho(|\alpha|-|\beta|)}<\infty\,, \tag{3.17}$$

then $p(X, D_x)$ *belongs to* $L(L^p)$ *and we have*

$$(3.18) \qquad \|p(X, D_x)u\|_p \leqq C\, N\|u\|_p\,,$$

where the constant C is independent of $2 \leqq p \leqq \infty$.

We can prove Corollary 3.6 directly from Theorem 3.3 by taking $\omega(t)=t^\tau$, $\tau=m-n(1-\rho)/2$. Corollary 3.7 can be proved from Corollary 3.5 by taking $\omega(t)=t^\tau$, $\tau=m-m_p$.

If $\omega(t)$ satisfies (3.7) then we have

$$(3.7)' \qquad \int_0^1 \frac{\omega(t^\tau)}{t}\, dt = \frac{1}{\tau} M_1 < \infty$$

for any positive τ. Hence we have

Corollary 3.8. *Let* ρ *and* $\omega(t)$ *be the same as in Theorem* 3.3. *If a symbol* $p(x, \xi)$ *satisfies*

$$(3.8)' \qquad N = \sup_{|\alpha|\leqq\kappa,\, (x,\xi)} |p^{(\alpha)}(x, \xi)|\, \omega(\langle\xi\rangle^{-\tau})^{-1} \langle\xi\rangle^{n(1-\rho)/2+\rho|\alpha|} < \infty$$

for some positive τ, *then* $p(X, D_x)$ *is* L^p*-bounded for* $2 \leqq p \leqq \infty$ *and the inequality* (3.9) *holds.*

We use Corollary 3.8 in the proof of Theorem 4.4.

4. L^p-boundedness of operators of the critical decreasing order

In this section we show L^p-boundedness theorems for operators of symbols which have the critical decreasing order as $|\xi| \to \infty$.

We denote the norm of bounded mean oscillation for a function $f(x)$ on R^n by $\|f\|_* = \|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx$, where Q denotes an arbitrary cube in R^n, $|Q|$ is the volume of the cube Q and $f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx$. The following theorem has already been proved in [13] and [16]. However we give here a slightly different proof, in which we use a continuous decomposition of the operators.

Theorem 4.1. *We assume that a symbol* $p(x, \xi)$ *satisfies one of the following two conditions.*

$$\text{(i)} \qquad N = \sup_{|\alpha|\leqq\kappa,\, |\beta|\leqq 1,\, (x,\xi)} |p^{(\alpha)}_{(\beta)}(x, \xi)|\, \langle\xi\rangle^{|\alpha|-\delta|\beta|} < \infty\,,$$

where δ *is a positive constant with* $\delta < 1$.

$$\text{(ii)} \qquad N = \sup_{|\alpha|\leqq\kappa,\, |\beta|\leqq\kappa,\, (x,\xi)} |p^{(\alpha)}_{(\beta)}(x, \xi)|\, \langle\xi\rangle^{n(1-\rho)/2+\rho(|\alpha|-|\beta|)} < \infty\,,$$

where ρ *is a positive constant with* $\rho < 1$.

Then the operator $p(X, D_x)$ is bounded from L^∞ to BMO and we have

$$\|p(X, D_x)\,u\|_* \leqq C_n\, N\|u\|_\infty\,. \tag{4.1}$$

Proof. We note that, by Lemma 2.1, if $p(x, \xi)$ satisfies the condition (i) then $p(X, D_x)$ is L^2-bounded and we have

$$\|p(X, D_x)\,u\|_2 \leqq C\,N\|u\|_2\,. \tag{4.2}$$

Moreover if $p(x, \xi)$ satisfies the condition (ii) then, by Lemma 2.2, the operator $p(X, D_x)\langle D_x\rangle^{n(1-\rho)/2}$ is L^2-bounded and we have the similar estimate to (4.2).

As in the proof of Theorem 3.3, we take a smooth function $f(t)$ so that the support is contained in the interval $[1/2,1]$ and

$$\int_{-\infty}^{\infty} \frac{f(t)}{t}\,dt = \int_0^{\infty} \frac{f(t)}{t}\,dt = 1\,.$$

Let Q be an arbitrary cube with side d and center x^0. Then we note $|Q|=d^n$. We may assume without loss of generality that the sides of the cube are parallel to the coordinate axis and $d<1$. Hence we can write $Q=\{x=(x_1, \cdots, x_n);\ |x_j-x_j^0|\leqq d/2, j=1, \cdots, n\}$. We take a $C_0^\infty(R^1)$ and even function $\phi(t)$ so that the support is contained in the interval $[-2, 2]$, $\phi(t)=1$ for $|t|\leqq 1$ and $\phi(t)\geqq 0$. We set $\psi_d(\xi)=\phi(d|\xi|)$. By Corollary 3.2, we may assume that the support of $p(x, \xi)$ is contained in $\{\xi;\ |\xi|\geqq 2\}$ and $p(x, \xi)$ satisfies

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leqq c_n\, N|\xi|^{-\rho|\alpha|+\delta|\beta|-n(1-\rho)/2} \qquad (|\xi|\geqq 2)$$

for $|\alpha|\leqq\kappa$ and $|\beta|\leqq 1$ in the case $\rho=1$ and $|\beta|\leqq\kappa$ in the case $0<\delta=\rho<1$. Then we split the symbol $p(x, \xi)$ as

$$p(x, \xi) = p(x, \xi)\,\psi_d(\xi)+p(x, \xi)\,(1-\psi_d(\xi)) = p_0(x, \xi)+p_1(x, \xi)\,.$$

Then we see that

$$|p_{j(\beta)}^{(\alpha)}(x, \xi)| \leqq c_n\, N|\xi|^{-n(1-\rho)/2-\rho|\alpha|+\delta|\beta|} \qquad (j=0, 1) \tag{4.3}$$

for $|\alpha|\leqq\kappa$ and $|\beta|\leqq 1$ in the case $\rho=1$ and $|\beta|\leqq\kappa$ in the case $0<\delta=\rho<1$, where the constant c_n is independent of the length d of the cube.

First we consider the operator $p_0(X, D_x)$. Since the support of $p_0(x, \xi)f(t|\xi|)$ is contained in the set

$$\{\xi;\ 1/(2t)\leqq|\xi|\leqq 1/t,\ 2\leqq|\xi|\leqq 2/d\}\ ,$$

we have

$$D_{x_j}\,p_0(X, D_x)\,u(x) = \int_{d/4}^{1/2} D_{x_j}\,p_0(t, X, D_x)\,u(x)\,\frac{dt}{t}\,,$$

where $p_0(t, x, \xi)=p_0(x, \xi)f(t|\xi|)$. The symbol of $D_{x_j}\,p_0(t, X, D_x)$ is equal to

$$p_{0,j}(t, x, \xi) = \{p_{0,(e_j)}(x, \xi)+\xi_j p_0(x, \xi)\} f(t|\xi|)\,.$$

Hence by (4.3) we have

$$||p_{0,j}(t, x, \cdot)||^2_{H^\kappa(t^{-\rho})} \leqq C^2 N^2 t^{-2-n\rho}\,,$$

which gives with the aid of Proposition 3.1

$$\begin{aligned}||D_{x_j} p_0(X, D_x)\, u||_\infty &\leqq \int_{d/4}^{1/2} ||D_{x_j} p_0(t, X, D_x)\, u||_\infty \frac{dt}{t} \\ &\leqq C N \int_{d/4}^{1/2} \frac{dt}{t^2} ||u||_\infty \leqq 4\, C N d^{-1} ||u||_\infty\,.\end{aligned}$$

Therefore, for x' in Q we have

$$\begin{aligned}&\left|\frac{1}{|Q|}\int_Q p_0(X, D_x)\, u(x)\, dx - p_0(X, D_x)\, u(x')\right| \\ &\leqq \frac{1}{|Q|}\int_Q |p_0(X, D_x)\, u(x) - p_0(X, D_x)\, u(x')|\, dx \\ &\leqq C N ||u||_\infty\,.\end{aligned}$$

This implies

$$||p_0(X, D_x)\, u||_* \leqq CN ||u||_\infty\,. \tag{4.4}$$

Next we show the boundedness of the operator $p_1(X, D_x)$. Let $\chi(x)$ be a $C_0^\infty(R^n)$ function which satisfies $\chi(x)=1$ for any $x=(x_1, \cdots, x_n)$ with $|x_j| \leqq 2$ $(j=1, \cdots, n)$ and $\chi(x)=0$ for any $x=(x_1, \cdots, x_n)$ with $|x_{j_0}| \geqq 4$ for some j_0. We set $\chi_d(x)=\chi(d^{-\rho}(x-x^0))$, and we write

$$\begin{aligned}p_1(X, D_x)\, u(x) &= p_1(X, D_x)\, (\chi_d\, u)\, (x) + p_1(X, D_x)\, (u - \chi_d\, u)\, (x) \\ &= I\, u(x) + II\, u(x)\,.\end{aligned} \tag{4.5}$$

Then, we see

$$II\, u(x) = \int_0^d \frac{dt}{t} \int K_1(t, x, z)\, (u - \chi_d\, u)\, (x - tz)\, dz\,,$$

where $K_1(t, x, z)$ is defined by

$$K_1(t, x, z) = \int e^{iz\cdot\xi}\, p(x, \frac{1}{t}\xi)\, (1 - \psi_d\left(\frac{1}{t}\xi\right))\, f(|\xi|)\, d\xi\,.$$

Since $|x_j - x_j^0| \geqq 2d^\rho$ for some $j \in \{1, \cdots, n\}$ in the support of $u(x) - \chi_d(x)\, u(x)$, for any x in Q we have

$$|tz_j| \geqq |x_j - x_j^0 - tz_j| - |x_j - x_j^0| \geqq 2d^\rho - d/2 \geqq d^\rho\,.$$

Hence if x belongs to Q, then $|z| \geqq t^{-1} d^\rho$ in the integrand of $II\, u(x)$. Then

$$\int_{|z|\geqq t^{-1}d^{\rho}} |K_1(t, x, z)|\,dz$$
$$\leqq \{\int_{|z|\geqq t^{-1}d^{\rho}} |z|^{-2\kappa}\,dz\}^{1/2}\,\{\int |z|^{2\kappa}|K_1(t, x, z)|^2\,dz\}^{1/2}$$
$$\leqq c_n\left(\frac{d^{\rho}}{t}\right)^{-\kappa+n/2}\{\sum_{|\alpha|=\kappa}\int|\partial_\xi^\alpha\{p\left(x, \frac{1}{t}\xi\right)(1-\psi_d\left(\frac{1}{t}\xi\right))f(|\xi|)\}|^2\,d\xi\}^{1/2}$$
$$\leqq c_n N\left(\frac{d^{\rho}}{t}\right)^{-\kappa+n/2} t^{n(1-\rho)/2-\kappa(1-\rho)} = C\,N\,t^{\rho(\kappa-n/2)}\,d^{\rho(n/2-\kappa)}\,.$$

Therefore we have

$$|II\,u(x)|\leqq C\,N\,d^{\rho(n/2-\kappa)}\int_0^d t^{-1+\rho(\kappa-n/2)}\,dt||u||_\infty\leqq C\,N\,||u||_\infty$$

for x in Q. This implies

$$\frac{1}{|Q|}\int_Q |II\,u(x)|\,dx\leqq C\,N\,||u||_\infty\,. \tag{4.6}$$

In order to estimate $I\,u(x)$ we use the L^2-boundedness of the operator $p(X, D_x)\langle D_x\rangle^{n(1-\rho)/2}$ under one of the two conditions (i) and (ii). Since

$$I\,u(x) = p(X, D_x)\,(1-\psi_d(D_x))\,(\chi_d u)\,(x)\,,$$

we can see

$$\frac{1}{|Q|}\int_Q |I\,u(x)|\,dx\leqq \{\frac{1}{|Q|}\int_Q |I\,u(x)|^2\,dx\}^{1/2}$$
$$\leqq \frac{CN}{|Q|^{1/2}}\,||\tilde{\psi}_d(D_x)\,\chi_d\,u||_2\,,$$

where $\tilde{\psi}_d(\xi)=\langle\xi\rangle^{-n(1-\rho)/2}(1-\psi_d(\xi))\,\tilde{\chi}(\xi)$, $\tilde{\chi}(\xi)=1$ for $|\xi|\geqq 2$ and $\tilde{\chi}(\xi)=0$ for $|\xi|\leqq 2$. Since $|\tilde{\psi}_d(\xi)|\leqq c_n\,d^{n(1-\rho)/2}$ and $|Q|=d^n$, it follows from Plancherel's formula that

$$\frac{1}{|Q|}\int_Q |I\,u(x)|\,dx\leqq C\,N\,d^{-n/2}\,d^{n(1-\rho)/2}||\chi_d\,u||_2 \tag{4.7}$$
$$\leqq C\,N\,d^{-\rho n/2}||\chi_d||_2||u||_\infty\leqq C\,N||\chi||_2||u||_\infty\,.$$

From the inequalities (4.4), (4.6) and (4.7) we get

$$||p(X, D_x)\,u||_*\leqq C\,N||u||_\infty\,.$$

Thus we complete the proof of Theorem 4.1. Q.E.D.

Theorem 4.2. *Let* $2\leqq p<\infty$. *Suppose that a symbol* $p(x, \xi)$ *satisfies the condition* (i) *in Theorem* 4.1 *or satisfies* (ii)′ $$N=\sup_{|\alpha|\leqq\kappa,\,|\beta|\leqq\kappa,\,(x,\xi)}|p^{(\alpha)}_{(\beta)}(x,\xi)|\langle\xi\rangle^{m_p+\rho(|\alpha|-|\beta|)}<\infty\,,$$

where m_p is the critical decreasing order $n(1-\rho)(1/2-1/p)$ and $0<\rho<1$. Then $p(X, D_x)$ is L^p-boundeed and we have

$$\|p(X, D_x)\,u\|_p\leqq C_p\,N\|u\|_p\,. \tag{4.8}$$

Proof. When the symbol $p(x, \xi)$ satisfies the condition (i), the operator $p(X, D_x)$ is L^2-bounded by Lemma 2.1 and bounded from L^∞ to BMO by Theorem 4.1. Therefore by the interpolation theorem of Fefferman-Stein in [4] we can obtain the estimate (4.8). In a similar way, we can obtain the estimate (4.8), when $p(x, \xi)$ satisfies (ii)′, from the interpolation theorem of Fefferman-Stein in [4] (see [3] and [16]). Q.E.D.

REMARK 4.3. We note also that Theorem 4.2 has already been proved in [16].

Theorem 4.4. *Let $0\leqq\delta<\rho\leqq1$, $\tau>0$ and let $\omega(t)$ be a non-negative and non-decreasing function which satisfies*

$$\int_0^1\frac{\omega(t)}{t}\,dt=M_1<\infty\,. \tag{4.9}$$

We assume that a symbol $p(x, \xi)$ satisfies

$$\begin{cases} N_0=\sup\limits_{|\alpha|\leqq\kappa,\,(x,\xi)}|p^{(\alpha)}(x,\xi)|\langle\xi\rangle^{n(1-\rho)/2+\rho|\alpha|}<\infty\,,\\ N_1=\sup\limits_{|\alpha|\leqq\kappa,\,(x,y,\xi)}\{|p^{(\alpha)}(x,\xi)-p^{(\alpha)}(y,\xi)|\,\omega(|x-y|^{\tau}\langle\xi\rangle^{\delta\tau})^{-1}\\ \qquad\times\langle\xi\rangle^{n(1-\rho)/2+\rho|\alpha|}\}<\infty\,. \end{cases} \tag{4.10}$$

Then $p(X, D_x)$ is bounded from L^∞ to BMO and is L^p-bounded for $2\leqq p<\infty$, and we have

$$\|p(X, D_x)\,u\|_p\leqq(C_p\,N_0+C_0\,N_1\,M_1)\,\|u\|_p\,, \tag{4.11}$$

$$\|p(X, D_x)\,u\|_*\leqq C_0(N_0+N_1\,M_1)\,\|u\|_\infty\,, \tag{4.12}$$

where the constant C_0 is independent of $2\leqq p<\infty$.

Proof. We take a $C_0^\infty(R^n)$ function $\phi(y)$ such that the support is contained in $\{y;\,|y|\leqq1\}$ and $\int\phi(y)\,dy=1$. We take a positive constnat δ' so that $\delta'=\rho$ if $\rho<1$ and $\delta<\delta'<1$ if $\rho=1$. Now we define symbols $\tilde{p}(x, \xi)$ and $q(x, \xi)$ by

$$\begin{aligned}\tilde{p}(x,\xi)&=\int\phi(y)\,p(x-\langle\xi\rangle^{-\delta'}y,\xi)\,dy\\ &=\int\phi(\langle\xi\rangle^{\delta'}(x-y))\,p(y,\xi)\,\langle\xi\rangle^{\delta'n}\,dy\,,\end{aligned}$$

and $q(x, \xi)=p(x, \xi)-\tilde{p}(x, \xi)$. Then by Lemma 2.3 we can show that

$$|\tilde{p}_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha,\beta} N_0 \langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|+\delta'|\beta|}$$

for any β and α with $|\alpha| \leqq \kappa$, and

$$|q^{(\alpha)}(x, \xi)| \leqq C_\alpha N_1 \omega(\langle\xi\rangle^{-\tau(\delta'-\delta)}) \langle\xi\rangle^{-n(1-\rho)/2-\rho|\alpha|}$$

for $|\alpha| \leqq \kappa$ (see, for example, [8] or [11]). Therefore it follows from Lemma 2.2 and Theorem 4.1 that $\tilde{p}(X, D_x)$ is L^2-bounded and bounded from L^∞ to BMO, and by the interpolation theorem of Fefferman-Stein we have

$$||\tilde{p}(X, D_x) u||_p \leqq C_p N_0 ||u||_p \qquad (2 \leqq p < \infty)$$
$$||\tilde{p}(X, D_x) u||_* \leqq C_0 N_0 ||u||_\infty .$$

Moreover by Corollary 3.8, we have

$$||q(X, D_x) u||_p \leqq C_0 N_1 M_1 ||u||_p \qquad (2 \leqq p \leqq \infty) .$$

Thus we get the theorem. Q.E.D.

In this theorem, we got L^p-boundedness under a weak continuity condition (4.10) of symbols with respect to the space variables x, however, the decreasing order of symbols as $|\xi| \to \infty$ was the constant $n(1-\rho)/2$. We know that when $\rho<1$ this is not the critical decreasing order for L^p-boundedness except for $p=\infty$. So next we show an L^p-boundedness of operators of the critical decreasing order under some continuity condition in the space variables.

Theorem 4.5. *Let $0<\rho<1$ and $2 \leqq p < \infty$. We denote*

$$m_p = n(1-\rho)(1/2-1/p),\ \mu_p = \frac{\kappa n(1-\rho)}{\kappa p\rho+n(1-\rho)} . \tag{4.13}$$

Let μ be an arbitrary positive number greater than μ_p. We suppose that a symbol $p(x, \xi)$ satisfies

$$N_0 = \sup_{|\alpha| \leqq \kappa,\ |\beta| \leqq \kappa,\ (x, \xi)} |p_{(\beta)}^{(\alpha)}(x, \xi)| \langle\xi\rangle^{m_p+\rho|\alpha|} < \infty . \tag{4.14}$$

Moreover if $\mu_0=\mu-[\mu]>0$, then we assume that

$$N_1 = \sup_{|\alpha| \leqq \kappa,\ |\beta| \leqq [\mu],\ (x, y, \xi)} |p_{(\beta)}^{(\alpha)}(x, \xi)-p_{(\beta)}^{(\alpha)}(y, \xi)|\ |x-y|^{-\mu_0} \langle\xi\rangle^{m_p+\rho|\alpha|} < \infty . \tag{4.15}$$

Then $p(X, D_x)$ is L^p-bounded and we have

$$||p(X, D_x) u||_p \leqq C_p(N_0+N_1) ||u||_p . \tag{4.16}$$

Proof. We set $\rho'=\rho+n(1-\rho)/(p\kappa)$. Then we see easily $\rho<\rho'<1$. We take a Schwartz rapidly decreasing function $\phi(z)$ such that $\int \phi(z)\, dz=1$ and

$\int z^\alpha \phi(z)\,dz=0$ for any $\alpha \neq 0$ (see [8]). We define new symbols $\tilde{p}(x, \xi)$ and $q(x, \xi)$, as in the proof of Theorem 4.4, by

$$(4.17)\qquad \tilde{p}(x,\xi)=\int \phi(y)\,p(x-\langle\xi\rangle^{-\rho'}y,\xi)\,dy = \int \phi(\langle\xi\rangle^{\rho'}(x-y))\,p(y,\xi)\langle\xi\rangle^{\rho' n}\,dy\,,$$

and $q(x, \xi)=p(x, \xi)-\tilde{p}(x, \xi)$. Then setting $\nu=[\mu]$, we have

$$\tilde{p}(x,\xi)=p(x,\xi)+\sum_{0<|\beta|<\nu}\frac{(-i)^{|\beta|}}{\beta\,!}\int y^\beta\phi(y)\,dy\langle\xi\rangle^{-\rho'|\beta|}p_{(\beta)}(x,\xi)$$
$$+\sum_{|\beta|=\nu}\frac{\nu(-i)^\nu}{\beta\,!}\int_0^1(1-t)^{\nu-1}\int y^\beta\phi(y)\,p_{(\beta)}(x-t\langle\xi\rangle^{-\rho'}y,\xi)\langle\xi\rangle^{-\rho'\nu}\,dy\,dt\,.$$

Since $\int y^\beta\phi(y)\,dy=0$ for $\beta\neq 0$, we have

$$q(x,\xi)=-\nu(-i)^\nu\sum_{|\beta|=\nu}\frac{1}{\beta\,!}\int_0^1(1-t)^{\nu-1}\int\phi_\beta\Big(\frac{\langle\xi\rangle^{\rho'}}{t}(x-y)\Big)$$
$$\times t^{-n}p_{(\beta)}(y,\xi)\,\langle\xi\rangle^{-\rho'\nu+\rho'n}\,dydt$$
$$=-\nu(-i)^\nu\sum_{|\beta|=\nu}\frac{1}{\beta\,!}\int_0^1(1-t)^{\nu-1}\int\phi_\beta\Big(\frac{\langle\xi\rangle^{\rho'}}{t}(x-y)\Big)$$
$$\times t^{-n}\{p_{(\beta)}(y,\xi)-p_{(\beta)}(x,\xi)\}\langle\xi\rangle^{\rho'(n-\nu)}\,dydt\,,$$

where $\phi_\beta(z)=z^\beta\phi(z)$. Thus using Lemma 2.3 we can see that

$$|q^{(\alpha)}(x,\xi)|\leqq CN_1\sum_{|\beta|=\nu}\langle\xi\rangle^{-m_p-\rho'\nu-\rho|\alpha|}\int|\tilde{\phi}_{\alpha,\beta}(\langle\xi\rangle^{\rho'}y)|\,|y|^{\mu_0}\langle\xi\rangle^{\rho'n}\,dy$$
$$\leqq C\,N_1\langle\xi\rangle^{-m_p-\rho'\mu-\rho|\alpha|}$$

for $|\alpha|\leqq\kappa$, where $\tilde{\phi}_{\alpha,\beta}(z)$ are linear combinations of Schwartz functions determined from $\phi_\beta(z)$ and its derivatives of order not greater than $|\alpha|$. By the definitions of μ, μ_p, m_p and ρ', we can see easily that

$$\rho'\mu+m_p>\rho'\mu_p+m_p=n(1-\rho)/2\,.$$

Therefore by Corollary 3.6 we have

$$(4.18)\qquad ||q(X,D_x)\,u||_r\leqq C\,N_1||u||_r$$

for $2\leqq r\leqq\infty$.

Next we consider the symbol $\tilde{p}(x, \xi)$. For $|\alpha|\leqq\kappa$ and $|\beta|\leqq\nu=[\mu]$, it follows from Lemma 2.3 that

$$(4.19)\qquad |\tilde{p}^{(\alpha)}_{(\beta)}(x,\xi)|=|\partial_\xi^\alpha\{\int\phi(y)\,p_{(\beta)}(x-\langle\xi\rangle^{-\rho'}y,\xi)\,dy\}\,|$$

$$= |\partial_\xi^\alpha\{\int\phi(\langle\xi\rangle^{\rho'}(x-y))\,p_{(\beta)}(y,\xi)\langle\xi\rangle^{\rho' n}\,dy\}|$$
$$\leqq C\sum_{\alpha^1+\alpha^2+\alpha^3=\alpha}\int|\partial_\xi^{\alpha^1}(\phi(\langle\xi\rangle^{\rho'}(x-y)))\,p_{(\beta)}^{(\alpha^2)}(y,\xi)\,\partial_\xi^{\alpha^3}\langle\xi\rangle^{\rho' n}|\,dy$$
$$\leqq C\,N_0\langle\xi\rangle^{-m_p-\rho|\alpha|}\,.$$

When $|\alpha|\leqq\kappa$ and $\nu<|\beta|\leqq\kappa$, writing $\beta=\beta^1+\beta^2$, $|\beta^1|=\nu$ and $\beta^2\neq 0$, we have

$$|\tilde{p}_{(\beta)}^{(\alpha)}(x,\xi)|=|\partial_\xi^\alpha\partial_x^{\beta^2}\{\int\phi(y)\,p_{(\beta^1)}(x-\langle\xi\rangle^{-\rho'}y,\xi)\,dy\}|$$
$$=|\partial_\beta^\alpha\{\int\phi_{(\beta^2)}(\langle\xi\rangle^{\rho'}(x-y))\,p_{(\beta^1)}(y,\xi)\,\langle\xi\rangle^{\rho'(n+|\beta^2|)}\,dy\}|,$$

where $\phi_{(\beta^2)}(z)=D_x^{\beta^2}\phi(z)$. Since $\int\phi_{(\beta^2)}(z)dz=0$, in a similar way to the estimate for $q(x,\xi)$, we have

$$(4.20)\qquad |\tilde{p}_{(\beta)}^{(\alpha)}(x,\xi)|=|\partial_\xi^\beta\{\int\phi_{(\beta^2)}(\langle\xi\rangle^{\rho'}(x-y))\,\{p_{(\beta^1)}(y,\xi)-p_{(\beta^1)}(x,\xi)\}$$
$$\times\langle\xi\rangle^{\rho'(n+|\beta^2|)}\,dy\}|$$
$$\leqq C\,N_1\langle\xi\rangle^{-m_p-\rho|\alpha|+\rho'(|\beta|-\nu)-\rho'\mu_0}\,.$$

Since

$$\rho'(|\beta|-\nu)-\rho'\mu_0-\rho|\beta|=(\rho'-\rho)|\beta|-\rho'\mu<(\rho'-\rho)\,\kappa-\rho'\mu_p=0$$

for $|\beta|\leqq\kappa$, combining the estimates (4.19) and (4.20), we get

$$|\tilde{p}_{(\beta)}^{(\alpha)}(x,\xi)|\leqq C(N_0+N_1)\langle\xi\rangle^{-m_p-\rho|\alpha|+\rho|\beta|}$$

for $|\alpha|\leqq\kappa$ and $|\beta|\leqq\kappa$. Therefore by Theorem 4.2, we have

$$(4.21)\qquad \|\tilde{p}(X,D_x)\,u\|_p\leqq C(N_0+N_1)\|u\|_p\,.$$

From (4.18) and (4.21) we get (4.16). Q.E.D.

REMARK 4.6. (i) we first note that

$$\mu_p-\kappa(1-\rho)=\kappa\rho(1-\rho)\,(n-p\kappa)/(\kappa p\rho+n(1-\rho))<0$$

for $p\geqq 2$, and therefore $\mu_p<\kappa(1-\rho)$. In the condition (ii)′ of Theorem 4.2, we assumed the κ differentiability of symbols in the space variables x and the covariables ξ, in order to get the L^p-boundedness for the operators of a class which generalizes the Hörmander class $S_{\rho,\rho}^{-m_p}(0<\rho<1)$. However for operators of our class which generalizes the Hörmander class $S_{\rho,0}^{-m_p}\,(0<\rho<1)$, we can obtain the L^p-boundedness under less regularity μ in the space variables x by Theorem 4.5, since $\mu_p<\kappa(1-\rho)<\kappa$.

(ii) It is clear that $\lim_{p\to\infty}\mu_p=0$ and $\lim_{\rho\uparrow 1}\mu_p=0$. This means that if p is sufficiently large or ρ is sufficiently near to 1, then we can obtain the L^p-boundedness under only the Hölder continuity of symbols with respect to the space variables x.

Acknowledgement. The author is heartily grateful to the referee for the improvement of Proposition 3.1 and the simplification of the proofs of Theorem 3.3 and Theorem 4.1.

References

[1] R.R. Coifman and Y. Meyer: *Au dela des opérateurs pseudo-differentiels*, Astérisque **57** (1978), 1–85.

[2] H.O. Cordes: *On compactness of commutators of multiplications and convolutions, and boundedness of pseudo-differential operators*, J. Funct. Anal. **18** (1975), 115–131.

[3] C. Fefferman: *L^p-bounds for pseudo-differential operators*, Israel J. Math. **14** (1973), 413–417.

[4] C. Fefferman and E.M. Stein: *H^p-spaces of several variables*, Acta Math. **129** (1972), 137–193.

[5] L. Hörmander: *Pseudo-differential operators and hypo-elliptic equations*, Proc. Symposium on Singular Integrals, Amer. Math. Soc. **10** (1967), 138–183.

[6] T. Kato: *Boundedness of some pseudo-differential operators*, Osaka J. Math. **13** (1976), 1–9.

[7] H. Kumano-go: Pseudo-differential operators, MIT Press, Cambridge, Mass. and London, England, 1982.

[8] H. Kumano-go and M. Nagase: *Pseudo-differential operators with non-regular symbols and applications*, Funkcial. Ekvac. **22** (1978), 151–192.

[9] T. Muramatu and M. Nagase: *On sufficient conditions for the boundedness of pseudo-differential operators*, Proc. Japan Acad. **55** Ser A (1979), 613–616.

[10] T. Muramatu and M. Nagase: *L^2-boundedness of pseudo-differential operators with non-regular symbols*, Canadian Math. Soc. Conference Proceedings, 1 (1981), 135–144.

[11] M. Nagase: *The L^p-boundedness of pseudo-differential operators with non-regular symbols*, Comm. Partial Differential Equations **2** (1977), 1045–1061.

[12] M. Nagase: *On the boundedness of pseudo-differential operators in L^p-spaces*, Sci. Rep. College Gen. Ed. Osaka Univ. **32** (1983), 9–19.

[13] M. Nagase: *On a class of L^p-bounded pseudo-differential operators*, Sci. Rep. College Gen. Ed. Osaka Univ. **33** (1984), 1–7.

[14] E.M. Stein and G. Weiss: Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, Princeton, NJ, 1971.

[15] M. Taylor: Pseudo-differential operators, Princeton Univ. Press, Princeton, NJ, 1981.

[16] Wang Roughuai and Li Chengzhang: *On the L^p-boundedness of several classes of pseudo-differential operators*, Chinese Ann. Math. **5B**(2) (1984), 193–213.

[17] K. Yabuta: *Calderón-Zygmund operators and pseudo-differential operators*, Comm. Partial Differential Equations **10** (1985), 1005–1022.

[18] A. Zygmund: Trigonometrical series, Cambridge Univ. Press, London, 1968.

Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan