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Boolean Algebras and Fields of Sets

By Shizu ENOMOTO

An arbitrary Boolean algebra is isomorphic with a field of sets.[1] However, a σ-complete Boolean algebra can be no isomorph of a σ-additive field of sets: for example, the complete quotient algebra B/I where B is the family of all Borel sets of the set $[0, 1]$ and I is the family of all elements of B which are of Lebesgue measure 0. In general, such a problem of the representation of an $\mathfrak{n}$-complete Boolean algebra as an $\mathfrak{n}$-additive field of sets, has been studied by a number of authors.[2]

In this paper, in relation to such a problems, we shall chiefly investigate an arbitrary Boolean algebra which is not always complete, in connection with the structure of a field of sets on which it is represented or with the existence of special measures on it. In order to investigate such a problem as clearly as possible, we introduce in §1 the conception of a ramification set and in §2 we consider a representation of a Boolean algebra on a field of sets by using ramification sets in it. The results given in §3 contain the fact that the problem already posed by A. Horn-A. Tarski in their paper [1], i.e. the problem whether for an arbitrary Boolean algebra it is atomic if and only if it is distributive in the wider sense, can be answered in the positive.

Let us notice here that it is entirely due to Theorems 1.2 and 1.4 that the theorems in §3 and §4 hold without any condition of completeness properties of a Boolean algebra.[3]

§1. Ramification set.

Throughout the present paper, the symbol A designates a Boolean algebra. In this section, we shall introduce the conception of a

1) See M. H. Stone [1], [2].

2) Among the authors, we may mention A. Tarski [1]-[5], A. Horn — A. Tarski [1], L. H. Loomis [1], [2], R. Sikorski [1], [2], L. Rieger [1] etc.

3) For notions usually used in lattice theory and for theorems, we refer to G. Birkhoff [1].

ramification set and consider a number of properties which are necessary in §2 and §3.

DEFINITION 1.1. If a subset R of a Boolean algebra A which does not contain 0 satisfies the following conditions, then it is called a *ramification set.*[4]

1) For any elements $x, y \in R$, either $x \wedge y = 0$ or $x \geqq y$ or $x \leqq y$.

2) For every element $x \in R$, the set $\{y \mid y \in R$ and $y \geqq x\}$ is totally ordered and any subset of it has a greatest element.

DEFINITION 1.2. For ramification sets R_1 and R_2, we define $R_1 < R_2$ if $R_1 \subseteqq R_2$ and if, whenever $x \in R_1$ and $y \in R_2 - R_1$, we have $x \wedge y = 0$ or $x > y$. The relation $<$ thus defined establishes a partial ordering in the family of all ramification sets in A. We say that R is a *maximal ramification set* if it is maximal by the relation $<$.

Unless otherwise stated the symbols R, R_α, R' etc. will be used to denote maximal ramification sets.

Theorem 1.1. *For an arbitrary ramification set R in A, there is a maximal ramification set which contains R.*[5]

Theorem 1.2. *For any elements $a, a_1 \in R$ such that $a \geqq a_1$, there exists a subset $M_0 \subseteqq R$ such that $x_1 \wedge x_2 = 0$ whenever $x_1, x_2 \in M_0$ and $x_1 \neq x_2$, and such that $a_1 \in M_0$ and $a = \bigvee_{x \in M_0} x$.*

Proof. Let $\mathfrak{M}$ be a family of all subsets $M \subseteqq R$ such that $x_1 \wedge x_2 = 0$ whenever $x_1, x_2 \in M$ and $x_1 \wedge x_2 = 0$, and such that $M \ni a_1$ and $x \leqq a$ for every $x \in M$, and let us define $M_1 \leqq M_2$ for $M_1, M_2 \in \mathfrak{M}$ if M_1 is a refinement of M_2, then $\mathfrak{M}$ is the partially ordered set, and moreover inductive with respect to the relation. To see this, it is enough to show that a totally ordered subset $\mathfrak{M}_1$ of $\mathfrak{M}$ has a least upper bound. Let

$$M' = \{x' \mid x' \in \textstyle\sum_{M \in \mathfrak{M}_1} M^{6)} \text{ and there is no element } x'' \text{ of } \textstyle\sum_{M \in \mathfrak{M}_1} M \text{ such that } x'' > x'\},$$

then this is the required set. For, if we put $K = \{x \mid x \in \sum_{M \in \mathfrak{M}_1} M$ and $x \geqq x_0\}$ for $x_0 \in M$ $(M \in \mathfrak{M}_1)$, then K has a greatest element x_0', since $x_1 \wedge x_2 \geqq x_0$ whenever $x_1, x_2 \in K$, and so $x_1 \geqq x_2$ or $x_2 \geqq x_1$ by the

4) See A. Horn—A. Tarski [1]. As we need for the applications, we here defined a ramification set for a subset R which does not contain 0.

5) We can prove easily, by using Zorn's Lemma.

6) By "$\sum$", "$+$", we understand the union of sets, and by "$\sum$", "$.$", the common part of sets.

property of R. Then evidently, $x_0' \in M_0$ and $x_0 \leqq x_0'$. Therefore, M' is an upper bound of $\mathfrak{M}$. Moreover we see easily that M' is the least upper bound. Hence, there is a maximal element M_0 of $\mathfrak{M}$ by Zorn's lemma. Let us show that the subset M_0 of R thus defined is the set required in this theorem. To see this, it is enough to show that $a = \bigvee_{x \in M_0} x$. Suppose on the contrary that M_0 does not satisfy $a = \bigvee_{x \in M_0} x$, then there exists an element $a' \in A$ such that $a > a'$ and $a' \geqq x$ for every $x \in M_0$. By the property of R there is an element x of R such that $a \geqq x$ and $x \bigwedge (a-a') \neq 0$,[7] since $a-a' \in A$ and $a-a' \neq 0$. Let B be the set of such elements x, and suppose for a moment that for every $x \in B$, $x \bigwedge a_1 = 0$ does not hold, i.e. $x > a_1$. Then, first, if we suppose that $x > a-a'$ for every $x \in B$, then for $R' = R + \{(a-a')\}$, $R' > R$ must hold. This contradicts the property of R. Second, if we suppose that there is an $x \in B$ such that $x = a-a'$, then $x \bigwedge a_1 = 0$ since $a \in M_0$ and $a' > a_1$. This contradicts $x > a_1$ for every $x \in B$. Therefore, there must be an $x \in B$ such that $(a-a')-x \neq 0$. Let B_0 be the set of such elements x. For any elements $x_1, x_2 \in B$, $x_1 > a_1$ and $x_1 > a_2$ hold since x_1 and x_2 are the elements of B, and so $x_1 \bigwedge x_2 \neq 0$. Consequently, since B_2 is a tatally ordered set, there exists the greatest element x_0 of B_0. Now, if we put $a_0 = (a-a')-x$ and $R' = R + \{a_0\}$ for such x_0, it is easy to see $R' > R$. This contradicts the property of R. Therefore there exists $x_0 \in B$ such that $x_0 \bigwedge a_1 = 0$, i.e. there exists $x_0 \in R$ such that $a \geqq x_0$, $x_0 \bigwedge (a-a') \neq 0$ and $x_0 \bigwedge a = 0$. For such x_0, let us put $M_0' = \{x_0\} + \{x \mid x \in M_0 \text{ and } x \bigwedge x_0 = 0\}$. Since $a_1 \in M_0$ and $a_1 \bigwedge x_0 = 0$, $a_1 \in M'$, and also, since $x_0 \leqq a$ and M_0' is the subset of R such that $x_1 \bigwedge x_2 = 0$ whenever $x_1, x_2 \in M_0'$ and $x_1 \neq x_2$, we have $M_0' \in \mathfrak{M}$. For an $x \in M_0$ such that $x \bigwedge x_0 = 0$, $x \in M_0'$ is evident. For an $x \in M_0$ such that $x \bigwedge x_0 \neq 0$, either $x \geqq x_0$ or $x < x_0$. If we suppose here $x \geqq x_0$, $x \bigwedge (a-a') = 0$ since $a' \geqq x$, and so $x_0 \bigwedge (a-a') = 0$. This contradicts $x_0 \bigwedge (a-a') \neq 0$. Therefore $M_0' \geqq M_0$, and moreover since $x_0 \bigwedge (a-a') \neq 0$, $x_0 \,\bar{\in}\, M_0$ and $M_0' \neq M_0$. This contradiction for the definition of M_0 proves the present theorem.

Evidently

Theorem 1.3. *For every maximal ramification set $R \subseteqq A$, an atom $a_0 \in A$ is an element of R.*

DEFINITION 1.3. A subset I of a Boolean algebra A which does not contain 0, is called a *point component* if it satisfies the following conditions.

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7) We denote by $a-b$, the meet $a \frown b'$, where b' is the complement of b.

1) I is a totally ordered set, and any subset of I has a greatest element.

2) $\{y \mid y \in A$ and $y < x$ for every $x \in I\} = \{0\}$.

In particular, when I is a subset of R, it is called *point component in R.*

Unless otherwise stated I, I_α, I' etc. will be used to denote point components. Clearly a point is a ramification set.

Now we obtain easily the following theorem.

Theorem 1.4. *For a point component $I \subseteqq A$, there exists $\bigwedge_{x \in I} x$, and we have either*

1) $\bigwedge_{x \in I} x = 0$

or

2) $\bigwedge_{x \in I} x = x_0$, $x_0 \in I$ *and* x_0 *is an atom.*

Theorem 1.5. *For any totally ordered subset J of R, there exists a point component in R which contains J.*[5]

§ 2. Representations of Boolean algebras.

It is well known that a Boolean algebra is isomorphic with a field of sets.[1] But when, for example, we consider the problem under which conditions a Boolean algebra has a special kind of measure, it may be hard to see the structure of a Boolean algebra by representations given until now, because a measure gives the connection between elements x_1 and x_2 such that $x_1 \wedge x_2 = 0$. Therefore, by using a ramification set which is already used by Horn-Tarski in order to investigate the problem of the existence of a strictly positive measure in a Boolean algebra,[8] and by using a point component in order to investigate the connections among ramification sets in it, we shall represent here a Boolean algebra on a field of sets.

DEFINITION 2.1. Let I_1 and I_2 be point components in a Boolean algebra A. Then if, for every element $x \in I_1$, there is $x' \in I_2$ such that $x \geqq x'$, we define $I_1 \leqq I_2$. If $I_1 \geqq I_2$ and $I_1 \leqq I_2$, we define $I_1 \sim I_2$.

Clearly the relation "$\sim$" satisfies the equivalence relation.

DEFINITION 2.2. Let $R_\lambda (\lambda \in \Lambda)$ be the family of all maximal ramification sets in A. Let $\Lambda' \subseteqq \Lambda$ and take an arbitrary point component I_λ in R_λ and let $(I_\lambda; \lambda \in \Lambda')$ be the class of such point components. By $S_{\Lambda'}$, we shall denote the family of all such classes $(I_\lambda; \lambda \in \Lambda')$ satisfying the following three conditions.

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8) See A. Horn—A. Tarski [1], [2].

1) $(I_\lambda; \lambda \in \Lambda')$ satisfies the finite intersection property, i.e. $\bigwedge_{i=1}^{n} x_i = 0$ for an arbitrary finite elements $x_i \in \sum_{\lambda \in \Lambda'} I_\lambda$ $(i = 1, 2, \cdots, n)$.

2) For any point component I_{λ_0} in R_{λ_0} corresponding to a suffix λ_0 which is an element of $\Lambda - \Lambda'$, the class $(I_\lambda; \lambda \in \Lambda' + \{\lambda_0\})$ consisting of such I_0 and the point components I_λ belonging to the class $(I_\lambda; \lambda \in \Lambda')$ does not satisfy the finite intersection property.

3) $p_1 = (I_\lambda; \lambda \in \Lambda')$ and $p_2 = (I_\lambda'; \lambda \in \Lambda')$ satisfying 1) and 2), we regard as same one if $I_\lambda \sim I_\lambda'$ for every $\lambda \in \Lambda'$.

An element of $S_{\Lambda'}$ so defined is called a *point*. Of course, it may happen that $S_{\Lambda'}$ is void for some $\Lambda' \subseteqq \Lambda$. Any of the point components I_λ which constitute a point p of $S_{\Lambda'}$, will be called a *component* of p.

Theorem 2. 1. *For an arbitrary point component I_λ in R_λ, there is a point p of $S_{\Lambda'}$ for a suitable subset Λ' of Λ. I_λ is a point component of p.*[5]

Theorem 2. 2. *For an arbitrary point $p = (I_\lambda; \lambda \in \Lambda')$ of $S_{\Lambda'}(\Lambda' \subseteqq \Lambda)$, the set $B_p = \{a \mid a \in A, \ \exists x : x \in \sum_{\lambda \in \Lambda'} I_\lambda$ and $a \geqq x\}$ is a maximal dual ideal.*

Proof. 1°) For any finite number of elements $a_i \in B_p$ $(i = 1, 2, \cdots, n)$, $\bigwedge_{i=1}^{n} a_i \neq 0$. For each a_i, there exists x_i such that $a_i \geqq x_i$ and $x_i \in \sum_{\lambda \in \Lambda} I_\lambda$. $(I_\lambda; \lambda \in \Lambda')$ having the property $\bigwedge_{i=1}^{n} x_i \neq 0$ and so $\bigwedge_{i=1}^{n} a_i \neq 0$.

2°) For an arbitrary element a which is not an element of B_p, there is a finite number of elements $a_i \in B_p$ $(i = 1, 2, \cdots, n)$ such that $(\bigwedge_{i=1}^{n} a_i) \wedge a = 0$. Suppose this is not the case. Then since $\sum_{\lambda \in \Lambda'} I_\lambda \subseteqq B_p$, $a \wedge (\bigwedge_{i=1}^{m} x_i) \neq 0$ for any finite number of elements $x_i \in \sum_{\lambda \in \Lambda} I_\lambda$ $(i = 1, 2, \cdots, m)$. Therefore, $I'_{\lambda_0} = \{a \wedge x \mid x \in I_0\}$ is a point component for any suffix $\lambda_0 \in \Lambda'$. We see easily, that the class of point components consisting of I'_{λ_0} and of all the components $I_\lambda (\lambda \in \Lambda')$ of p has the finite intersection property. Therefore, since λ_0 must belong to Λ' by the definition of a point p of $S_{\Lambda'}$, $I'_{\lambda_0} \sim I_{\lambda_0}$ and so $a \in B_p$. This is obviously a contradiction.

Theorem 2. 3. *For any points $p_1, p_2 \in \sum_{\Lambda' \subseteqq \Lambda} S_{\Lambda'}$, $B_{p_1} \neq B_{p_2}$ if $p_1 \neq p_2$.*

Proof. 1°) When $p_1 \in S_{\Lambda'}$, $p_2 \in S_{\Lambda''}$ and $\Lambda' \neq \Lambda''$. We can suppose $\Lambda' - \Lambda'' \neq \phi$ since $\Lambda' \neq \Lambda''$. Let $p_1 = (I_\lambda'; \lambda \in \Lambda')$ and $p_2 = (I_\lambda''; \lambda \in \Lambda'')$. Since a class consisting of all components of p_2 and a component I'_{λ_0} of p_1 corresponding to $\lambda_0 \in \Lambda' - \Lambda''$ has not the finite intersection property, there are an element x_0' of I'_{λ_0} and a finite number of elements x'' $(i = 1, 2, \cdots, m)$ of $\sum_{\lambda \in \Lambda''} I_\lambda''$ such that $(\bigwedge_{i=1}^{m} x'') \wedge x_0' = 0$. If we suppose that this x_0' is an element of B_{p_2}, there is an element x'' of $\sum_{\lambda\in\Lambda''} I_\lambda''$ such that $x_0' \geqq x''$. And so $(\bigwedge_{i=1}^m x'') \wedge x_0' \wedge x'' = (\bigwedge_{i=1}^m x_i'') \wedge x'' \neq 0$, and $(\bigwedge_{i=1}^m x_i'') \wedge x_0' \neq 0$. This is an obvious contradition, and therefore $x_0' \bar{\in} B_p$. On the other hand, $x_0' \in B_{p_1}$. Hence $B_{p_1} \neq B_{p_2}$.

2°) When $p_1, p_2 \in S_{\Lambda'}$ and $p_1 \neq p_2$: Let $p_1 = (I_\lambda'; \lambda \in \Lambda')$ and $p_2 = (I_\lambda''; \lambda \in \Lambda'')$, then since $p_1 \neq p_2$, there is a suffix and we can not have the relation $I'_{\lambda_0} \sim I''_{\lambda_0}$ for the λ_0. Hence, it is easy to see $B_{p_1} \neq B_{p_2}$.

DEFINITION 2.2. For an arbitrary element a of A, we define $\varphi_{\Lambda'}(a) = \{p \mid p = (I_\lambda; \lambda \in \Lambda') \in S_{\Lambda'}, \ \exists x : x \in \sum_{\lambda\in\Lambda'} I_\lambda$ and $a \geqq x\}$ and $\varphi(a) = \sum_{\Lambda' \subseteqq \Lambda} \varphi_{\Lambda'}(a)$.

$\varphi_{\Lambda'}(a)$ and $\varphi_\Lambda(a)$ thus defined give the mappings from A on the respective families of subsets of $S_{\Lambda'}$ and of $S = \sum_{\Lambda' \subseteqq \Lambda} S_{\Lambda'}$.

Theorem 2. 4. *A Boolean algebra A is isomorphic with the field of sets given by the mapping φ. $\varphi(e) = S$ and $\varphi(0) = \phi$ where e designate the unit element of A.*

Proof. It is evident that $\varphi(e) = S$ and $\varphi(0) = \phi$. Let $\mathfrak{A} = \{B_p \mid p \in S\}$, then by the last theorem, the mapping $\psi(p) = B_p$ gives the one-to-one correspondence between S and $\mathfrak{A}$. Let $\Phi(a)$ be the family of all maximal dual ideals containing $a \in A$, and let Ω be the family of all maximal dual ideals, then it is well known that the mapping $\Phi(a)$ gives the isomorphic mapping between A and the family of subsets of Ω.[9] 1°) $\psi(\varphi(a)) = \Phi(a) \cdot \mathfrak{A}$: If p is a point of $\varphi(a)$, $a \in \varphi(p) = B_p$. And since $\psi(p)$ is a maximal dual ideal, $\psi(p) \in \Phi(a)$ and so $\psi(\varphi(a)) \subseteqq \Phi(a) \cdot \mathfrak{A}$. On the other hand, if B_p is an element of $\Phi(a) \cdot \mathfrak{A}$, then $a \in B_p$. Therefore, if we put $p = (I_\lambda; \lambda \in \Lambda')$, there is an element x of $\sum_{\lambda\in\Lambda'} I_\lambda$ such that $a \geqq x$. And so, $p \in \varphi_{\Lambda'}(a) \subseteqq \varphi(a)$. Hence $\psi(\varphi(a)) \supseteqq \Phi(a) \cdot \mathfrak{A}$, since $p = \psi^{-1}(B_p)$. 2°) If $a, b \in A$, then $\varphi(a \vee b) = \varphi(a) + \varphi(b)$ and $\varphi(a \wedge b) = \varphi(a) \cdot \varphi(b)$: $\varphi(a \vee b) = \psi^{-1}(\Phi(a \vee b) \cdot \mathfrak{A}) = \psi^{-1}((\Phi(a) \cdot \mathfrak{A}) + (\Phi(b) \cdot \mathfrak{A})) = \psi^{-1}(\Phi(a) \cdot \mathfrak{A}) + \psi^{-1}(\Phi(b) \cdot \mathfrak{A}) = \varphi(a) + \varphi(b)$. Analogously, $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. 3°) For an element a of A which is not $0, \varphi(a) \neq \phi$; There is a maximal ramification set R_{λ_0} containing the element a, and there is a point component I_0 in R_{λ_0} containing a. And, by Theorem 2.1, there is a point p of $S_{\Lambda'}(\Lambda' \subseteqq \Lambda)$ such that I_{λ_0} is the component of the point p. Hence, $\varphi(a) \neq \phi$.

§ 3. Atomic Boolean algebras.

If a Boolean algebra has the property that for every $a \in A$, there

9) See, for example, G. Birkhoff [1].

Theorem 3. 1. *For a Boolean algebra A, the following 4 conditions 1)-4) are equivalent with each other.*

1) *A is isomorphic with a field of sets which is completely additive in the wider sense.*

2) *For every $a \in A$ which is not 0, there is a two-valued measure f which is strongly completely additive in the wider sense and for which $f(a)=1$.*

3) *A is atomic.*

4) *A is completely distributive in the wider sense.*

Proof. 1)→4): By the assumption, there is an isomorphic mapping φ' on a field of sets such that $\varphi'(\bigvee_{\mu\in\Delta} a_\mu)=\sum_{\mu\in\Delta}\varphi'(a_\mu)$ for arbitrary elements $a_\mu \in A(\mu\in\Delta)$ for which $\bigvee_{\mu\in\Delta} a_\mu$ exists. We see easily from such properties, that $\varphi'(\bigwedge_{\mu\in\Delta} a_\mu)=\prod_{\mu\in\Delta}\varphi'(a_\mu)$ if there is $\bigwedge_{\mu\in\Delta} a_\mu$ for arbitrary elements $a_\mu \in A(\mu\in\Delta)$. By these two properties, it is easy to see that A is completely distributive in the wider sense.

4)→3): Let a be an arbitrary element of A and let R be a maximal ramification set containing a. Let $R'=\{x_\nu \mid \nu\in\Delta\}$ be the subset of A consisting of all x such that $x\in R$ and $x\leqq a$. By Theorem 1. 2, for each $x_\nu\in R'$ there is a subset M_ν of R such that $x_\nu\in M_\nu$, $a=\bigvee_{M_\nu} x_\mu^\nu$[11] and $x\wedge x'=0$ whenever $x, x'\in M_\nu$ and $x\neq x'$. If we can prove that there is $\bigwedge_\nu x_{f(\nu)}^\nu$ for each $f(\nu)$ and it is 0 or an atom, then we can show easily that A is atomic, because $a=\bigwedge_{\nu\in\Delta}(\bigwedge_{M_\nu} x_\mu^\nu)$ $=\bigvee_{f(\nu)}(\bigwedge_\nu x_{f(\nu)}^\nu)$ by 4). Let us show that $\bigwedge_\nu x_{f(\nu)}^\nu=0$ or an atom for each $f(\nu)$. Let $J=\{x_{f(\nu)}^\nu \mid \nu\in\Delta\}$. If there are $x_1, x_2\in J$ such that $x_1\wedge x_2=0$, $\bigwedge_\nu x_{f(\nu)}^\nu=0$. Therefore, it is enough to show that $\bigwedge_\nu x_{f(\nu)}^\nu=0$ or an atom in the case in which $x_1\wedge x_2\neq 0$ for any $x_1, x_2\in J$. Since $J\subseteqq R$, $0\bar{\in} J$, J is totally ordered and its arbitrary subset has a greatest element. Let I be a point component in R containing J. By Theorem 1. 5, such point component I exists in R. Now, suppose there is $x_0\in I$ such that $x_0\geqq x'$ does not hold for every $x'\in J$. Since $x_0, x'\in I$, $x_0\wedge x'\neq 0$ and so $x_0<x'$. Since $x_0<x'\leqq a$ and $x_0\in R$, $x_0\in R'$ and therefore there is $M_{\nu_0}(\nu_0\in\Delta)$ such that $x_0=x_{\nu_0}\in M_{\nu_0}$. Also, since $x_{f(\nu_0)}^{\nu_0}\in M_{\nu_0}$, $x_{\nu_0}\wedge x_{f(\nu_0)}^{\nu_0}=0$ or $x_{\nu_0}=x_{f(\nu_0)}^{\nu_0}$. On the other hand, $x_{\nu_0}<x_{f(\nu_0)}^{\nu_0}$ from the assumption. This is an abvious contradiction.

3)→2): Let a be an element of A which is not 0, and let a_0 be an atom of A such that $a_0\leqq a$. Then, the two-valued function f such that $f(x)=1$ if $x\geqq a_0$ and $f(x)=0$ if $x\wedge a_0=0$ for $x\in A$, is the two-

11) We denote $\bigcup_{x_\mu^\nu\in M_\nu} x_\mu^\nu$, by $\bigcup_{M_\nu} x_\mu^\nu$.

is an atom $a_0 \in A$ such that $a \geqq a_0$, then it is called atomic. For a complete Boolean algebra A, the following 3 conditions 1)-3) are as is well known equivalent with each other.[10]

1) A is atomic.

2) A is isomorphic with the family of all subsets of a set.

3) A is completely distributive.

But, in the case in which a Boolean algebra is not always completely additive, no results seem to be known. A. Horn—A. Tarski, in their paper [1], have posed the problem whether a Boolean algebra is atomic if and only if it is completely distributive in the wider sense. Here, we shall show that the above results for a complete Boolean algebra can be extended to an arbitrary Boolean algebra, and as one of the results the problem of A. Horn—A. Tarski will be answered in the positive.

DEFINITION 3.1. A field of sets F is called *completely additive in the wider sense* if it satisfies the following condition: whenever for an arbitrary family of sets $S_\mu (\mu \in \Delta)$ which are elements of F there is a smallest set S in F including all of them, then the set S coincides with the sum of the sets $S_\mu (\mu \in \Delta)$.

We see easily, that a Boolean algebra A is isomorphic with a field of sets F which is completely additive in the wider sense, if and only if there exists an isomorphic mapping φ' on a field of sets such that $\varphi'(\bigvee_{\mu \in \Delta} a_\mu) = \sum_{\mu \in \Delta} \varphi'(a_\mu)$ for any elements $a_\mu \in A (\mu \in \Delta)$ for which $\bigvee_{\mu \in \Delta} a_\mu$ exists.

DEFINITION 3.2. A two-valued measure f defined on a Boolean algebra A, i.e. a finitely additive function which is not identically 0 and assumes only the values 0 and 1, is called *strongly completely additive in the wider sense* if $f(\bigvee_{\mu \in \Delta} a_\mu) = 0$ for any $a_\mu (\mu \in \Delta)$ such that $f(a_\mu) = 0$ for every $\mu \in \Delta$ and for which $\bigvee_{\mu \in \Lambda} a_\mu$ exists.

DEFINITION 3.3. A Boolean algebra A is called *completely distributive in the wider sense*, if, whenever Θ_ν is a non-void set corresponding to a suffix ν which is an element of a non-void set Δ, a_μ^ν is an element of A corresponding to $\mu \in \Theta_\nu$ and whenever there are $\bigvee_{\mu \in \Theta_\nu} a_\mu^\nu$, $\bigwedge_{\nu \in \Lambda} (\bigvee_{\mu \in \Theta_\nu} a_\mu^\nu)$ and $\bigwedge_{\nu \in \Delta} a_{f(\nu)}^\nu$, then there exists $\bigvee_{f(\nu) \in [\Theta_\nu]^\Delta} (\bigwedge_{\nu \in \Delta} a_{f(\nu)}^\nu)$ and $\bigwedge_\nu \{\bigvee_{\mu \in \Theta_\nu} a_\mu^\nu\} = \bigvee_{f(\nu) \in [\Theta_\nu]^\Delta} (\bigwedge_\nu a_{f(\nu)}^\nu)$ holds. Here $[\Theta_\nu]^\Delta$ is the family of all one valued functions f which are defined on Δ and take one $\mu \in \Theta_\nu$ for every $\nu \in \Delta$ such that $f(\nu) = \mu$.

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10) See, A. Tarski [1], [2].

valued measure required in 2).

2)→1): For $a \in A$ which is not 0,

$\varphi'(a) = \{f \mid f$ is a two-valued measure which additive in the wider sense and for which $f(a) = 1\}$
is a required is strongly completely isomorphic mapping.

In particular, when A is complete, this theorem contains the results stated at the beginning of the chapter.

In order to consider the structure of a Boolean algebra in connection with powers of ramification sets in it, let us give the following definitions.

DEFINITION 3.4. In the definitions 3.1 and 3.2, let us give the additional condition $\overline{\overline{\Delta}} \leqq \mathfrak{n}$, where $\mathfrak{n}$ is a power, and in the definition 3.3, let us give the additional condition $\overline{\overline{\Theta}}_\nu \leqq \mathfrak{n}$ besides $\overline{\overline{\Delta}} \leqq \mathfrak{n}$, then, we say respectively, that F is $\mathfrak{n}$-*additive in the wider sense*, two-valued measure f is *strongly* $\mathfrak{n}$-*additive in the weder sense* and A is $\mathfrak{n}$-*additive in the wider sense.* In particular, in the case in which $\overline{\overline{F}} \leqq \mathfrak{n}$ and $\overline{\overline{A}} \leqq \mathfrak{n}$, they coincide with the respective definitions 3.1, 3.2 and 3.3.[12]

By the proofs which are almost identical with the ones given in Theorem 3.1, we can show the following theorem.

Theorem 3.2. *For a Boolean algebra A, the following conditions* 1) *and* 2) *are equivalent.*

1) *A is isomorphic with a field of sets which is* $\mathfrak{n}$-*additive in the wider sense.*

2) *For every element* $a \in A$ *which is not* 0, *there is a two-valued measure f which is strongly* $\mathfrak{n}$-*additive in the wider sense and for which* $f(a) = 1$.

If A satisfies the condition 1) *or* 2), *then*

3) *A is* $\mathfrak{n}$-*distributive in the wider sense.*

In particular, if powers of ramification sets in A are at most $\mathfrak{n}$, *then the conditions* 1)-3) *and the following condition* 4) *are equivalent with each other.*

4) *A is atomic.*

Therefore, if powers of ramification sets in A are at most $\mathfrak{n}$, the conditions 1)-4) are also equivalent with the conditions 1), 2) and 4) in Theorem 3.1. Considering the case in which $\overline{\overline{A}} = \mathfrak{n}$, we see that Theorem 3.1 is a particular case of the above theorem.

By the following example, we can see that even if a Boolean

12) When $\mathfrak{n} = \aleph_0$, we denote "σ-" instead of "$\aleph_0$-".

algebra A satisfies the condition 3), it can not satisfy the condition 1) or 2).

EXAMPLE 1. Let U be the family of all sets of real numbers, let B be the Boolean algebra of all subsets of U and let I be the ideal of all subsets of U of power $\mathfrak{c}$. Then, since $\bar{\bar{B}}$ is weakly accessible from $\aleph_0$ and since I is σ-additive, but is not completely additive, the σ-complete Boolean algebra B/I is not isomorphic with any σ-additive field of sets. We see easily that there is a ramification set which is not countable, since it is σ-distributive.[13]

§ 4. Boolean algebras represented by $\varphi_\Lambda(a)$.

As a little more complicated case than the case in which a Boolean algebra is atomic, we can consider the case in which it is represented by $\varphi_\Lambda(a)$, i.e. by the representation $\varphi(a)=\sum_{\Lambda'\leqq\Lambda}\varphi_{\Lambda'}(a)$, without $\varphi_{\Lambda'}(a)$ such that $\Lambda'\neq\Lambda$. In concrete, it is the case in which for every $a\in A$, there is a point component $(I_\lambda;\lambda\in\Lambda)$ such that $I_\lambda\in R_\lambda(\lambda\in\Lambda)$ and such that there is an element x with $x\in\sum_{\lambda\in\Lambda}I_\lambda$ and $a\geqq x$. In particular, if we can always take an atom a_0 as such an element x, then A is atomic. Then, we can take as I_λ for every $\lambda\in\Lambda$ the point component $\{a_0\}$ consisting of the single atom a_0, because an atom belongs to every $R_\lambda(\lambda\in\Lambda)$ by Theorem 1.3. For such a case, let us give interesting results which are analogous to Theorems 3.1 and 3.2.

DEFINITION 4.1. In the definitions 3.1 and 3.2, let us give respectively the additional conditions $S_\mu\bigwedge S_{\mu'}=0$ $(\mu\neq\mu')$ and $a_\mu\bigwedge a_{\mu'}=0$ $(\mu\neq\mu')$, then we say that F is *weakly completely additive in the wider sense* and the two-valued measure f is *completely additive in the wider sense* respectively.

Theorem 4.1. *For a Boolean algebra A, the following 3 conditions 1)-3) are equivalent with each other.*

1) *A is isomorphic with a field of sets which is weakly completely additive in the wider sense.*

2) *For every element $a\in A$ which is not 0, there is a two-valued measure f which is completely additive in the wider sense.*

3) *$\varphi_\Lambda(a)$ gives the isomorphic mapping from A on the field of sets.*

Proof. 1)→2): Let φ' be an isomorphic mapping from A on a field

13) See A. Tarski [4], A. Horn—A. Tarski [1].

of sets. For an element $a \in A$ which is not 0, there is a point p of $\varphi'(a)$ since $\varphi'(a) \neq \phi$. Let us fix such one point p. Then, the two-valued function f such that for $x \in A$ $f(x)=1$ if $\varphi'(x) \ni p$ and $f(x)=0$ if $\varphi'(x) \not\ni p$ is the required two-valued measure which is completely additive in the wider sense and $f(a)=1$. 2)→3): By the assumption, for an element $a \in A$ which is not 0, there is a two-valued measure f which is completely additive in the wider sense and for which $f(a)=1$. Using such f, let us show the condition 3). First, let us show the existence of a point component I_0 in a suitable R_{λ_0}, for which $f(x)=1$ and $a \geqq x$ for every $x \in I_0$. By Theorem 1.1, there is a maximal ramification set R_{λ_0} containing the element a. Let us denote by $R'_{\lambda_0} = \{x_\nu \mid \nu \in \Delta\}$, the subset of A consisting of all elements x of A such that $x \in R_{\lambda_0}$ and $x \leqq a$. From Theorem 1.3, for every $x_\nu \in R'_{\lambda_0}$ there is a subset M_ν of R_{λ_0} such that $x_\nu \in M_\nu$, $a = \bigcup_{M_\nu} x_\mu^{\nu}$ [11] and $x \wedge x' = 0$ whenever $x, x' \in M_\nu$ and $x \wedge x' = 0$. And for every $x_\nu \in R'_{\lambda_0}$, there is one element x_ν' which is an element of such M_ν and for which $f(x_\nu')=1$, because f is the two-valued measure which is completely additive in the wider sense and for which $f(a)=1$. Let us put $I = \{x_\nu' \mid \nu \in \Delta\}$, and let us show that it is a required point component. To see this, it is enough to show that I_0 is a point component in R_{λ_0} since $f(x)=1$ and $a \geqq x$ for every $x \in I_0$. $0 \bar{\in} I$ is evident. Since $f(x_1) = f(x_2) = 1$ for arbitrary elements $x_1, x_2 \in I$, $x_1 \wedge x_2 \neq 0$ and so we have $x_1 \geqq x_2$ or $x_2 \geqq x_1$. Therefore I_0 is a totally ordered set. Hence, it is enough to show that for a point component I in R_{λ_0} containing I_0, if $x \in I$, then there is an element x_0 of I such that $x \geqq x_0$. Suppose now that it is not the case, i.e. there is an element x of I such that $x < x_0$ for every element $x_0 \in I_0$. Since $a \geqq x_0 > x$ and $x \in R_{\lambda_0}$, $x = x_{\nu_0}$ for some $\nu_0 \in \Delta$. Therefore, $x'_{\nu_0} > x_{\nu_0}$. On the other hand, we have $x'_{\nu_0} \wedge x_{\nu_0} = 0$ or $x_{\nu_0} = x'_{\nu_0}$ since $x'_{\nu_0}, x_{\nu_0} \in M_{\nu_0}$. This is an obvious contradiction. Analogously, using $f(e)=1$, for every $R_\lambda(\lambda \in \Lambda, \lambda \neq \lambda_0)$, there is a point component I_λ in R_λ such that $f(x)=1$ for every element $x \in I_\lambda$. Let $(I_\lambda; \lambda \in \Lambda)$ be the class consisting of such $I_\lambda(\lambda \in \Lambda, \lambda \neq \lambda')$ and of such I_{λ_0}, then it is easy to see that the class $(I_\lambda; \lambda \in \Lambda)$ has the finite intersection property since $f(x)=1$ for every element x of $\sum_{\lambda \in \Lambda} I_\lambda$. Evidently, this point $(I_\lambda; \lambda \in \Lambda)$ of S_Λ is a point of $\varphi_\Lambda(a)$, and therefore $\varphi_\Lambda(a) \neq \phi$. This shows that $\varphi_\Lambda(a)$ gives the expression of A on a field of sets. 3)→1): In order to prove that $\varphi_\Lambda(a)$ is the mapping required in 1), it is enough to prove that $\varphi_\Lambda(\bigcup a_\mu) \subseteqq \sum \varphi_\Lambda(a_\mu)$ if for an arbitrary $a_\mu(\mu \in \Delta)$ such that $a_\mu \wedge a_{\mu'} = 0$ whenever $a_\mu \neq a_{\mu'}$, there is $\bigwedge_{\mu \in \Delta} a_\mu$. Let $p = (I_\lambda; \lambda \in \Lambda)$ be a point of $\varphi_\Lambda(\bigcup a_\mu)$. Then there is an element x' such that $x' \in \sum_{\lambda \in \Lambda} I$ and

$x' \leqq \bigvee a_\mu$. Since $\{a_\mu \mid \mu \in \Delta\}$ is the ramification set, from Theorem 1.1 there is a maximal ramification set R_{λ_0} such that $\{a_\mu \mid \mu \in \Delta\} \subseteqq R_{\lambda_0}$ and such that if x is an element of R_{λ_0}, then $x > a_\mu$ does not hold for every a_μ. Let I_0 be a component of the point p corresponding to the suffix $\lambda_0 \in \Lambda$, and let x_0 be a greatest element of I_0. From the construction we have $x_0 \leqq a_\mu$ or $x_0 \bigwedge a_\mu = 0$ for every a_μ, but we have not here $x_0 \bigwedge a_\mu = 0$ for every a_μ. Because, in that case $x_0 \bigwedge x' = 0$ since $x_0 \bigwedge (\bigvee_{\mu \in \Delta} a_\mu) = \bigvee_{\mu \in \Delta}(x_0 \bigwedge a_\mu) = 0$, and this contradicts $x_0 \bigwedge x' = 0$ since $x_0, x' \in \sum_{\lambda \in \Lambda} I_\lambda$.

DEFINITION 4.2. In the definitions 3.1 and 3.2, let us give the additional condition $\bar{\bar{\Delta}} \leqq \mathfrak{n}$ and besides, the respective additional conditions $S_\mu \bigwedge S_{\mu'} = 0$ $(\mu \neq \mu')$ and $a_\mu \bigwedge a_{\mu'} = 0$ $(\mu \neq \mu')$, then we say respectively that F is *weakly* $\mathfrak{n}$*-additive in the wider sense* and the two-valued measure f is $\mathfrak{n}$*-additive in the wider sense.* In particular, in the case in which $\bar{\bar{F}} \leqq \mathfrak{n}$ and $\bar{\bar{A}} \leqq \mathfrak{n}$, they coincide with the respective definitions given in Theorem 4.1.[12]

By the proofs which are almost identical with the ones given in Theorem 4.1, we can show the following theorem.

Theorem 4.2. *For a Boolean algebra A, the following conditions* 1) *and* 2) *are equivalent.*

1) *A is isomorphic with a field of sets which is weakly* $\mathfrak{n}$*-additive in the wider sense.*

2) *For every element* $a \in A$ *which is not* 0, *there is a two-valued measure f which is* $\mathfrak{n}$*-additive in the wider sense and for which* $f(a) = 1$. *In particular, if powers of ramification sets in A are at most* $\mathfrak{n}$, *then the conditions* 1), 2) *and the following condition* 3) *are equivalent with each other.*

3) $\varphi_\Lambda(a)$ *gives the isomorphic mapping from A on the field of sets.*

Therefore, if powers of ramification sets in A are at most $\mathfrak{n}$, the conditions 1)—3) are also equivalent with the conditions 1) and 2) in Theorem 4.1. Considering the case in which $\bar{\bar{A}} = \mathfrak{n}$, we see that Theorem 4.1 is a particular case of this theorem.

§ 5. Complete Boolean algebras.

DEFINITION 5.1. A Boolean algebra A is called $\mathfrak{n}$*-complete,* if there is $\bigvee_{x \in X} x$ for every subset X of A such that $\bar{\bar{X}} \leqq \mathfrak{n}$, and *complete* if A is $\mathfrak{n}$-complete for every power $\mathfrak{n}$. Since an element of a field of sets F is a subset of a set, we can consider the sum (union) of all elements of a subset X of F. If for every subset $X(\subseteqq F)$ of power

$\leq \mathfrak{n}$, such a sum belongs to F, then F is called $\mathfrak{n}$-*additive*. If a field of sets is $\mathfrak{n}$-additive for an arbitrary power $\mathfrak{n}$, then it is called *completely additive*.

From the definitions if a $\mathfrak{n}$-complete field of sets is $\mathfrak{n}$-additive in the wider sense, then it is $\mathfrak{n}$-additive, and if a complete field of sets is completely additive in the wider sense, then it is completely additive.

As we stated at the beginning of § 3, a complete Boolean algebra is not always isomorphic with any field of sets which is completely additive. Therefore, the question arises whether a $\mathfrak{n}$-complete Boolean algebra is isomorphic with a $\mathfrak{n}$-additive field of sets. Such a problem, have been studied by a number of authors. The results which are given in this section contain some of the results given in Sikorski [1]. As we can easily prove, for a $\mathfrak{n}$-complete Boolean algebra, it is isomorphic with a field of sets which is $\mathfrak{n}$-additive in the wider sense, if and only if it is isomorphic with a field of sets which is weakly $\mathfrak{n}$-additive in the wider sense. Therefore, by theorems 3.2 and 4.2 the following theorem holds.

Theorem 5.1. *For a* $\mathfrak{n}$-*complete Boolean algebra* A, *the following conditions* 1) *and* 2) *are equivalent.*

1) *A is isomorphic with a field of sets which is* $\mathfrak{n}$-*additive in the wider sense.*

2) *For an arbitrary element $a \in A$ which is not* 0, *there is an* $\mathfrak{n}$-*additive two-valued measure f for which* $f(a)=1$.

Therefore, for an $\mathfrak{n}$-complete Boolean algebra A, in the case in which powers of ramification sets in it are at most $\mathfrak{n}$, if A is not atomic, then it can not be represented by $\varphi_{\Lambda}(a)$.

In particular in the case in which $\mathfrak{n}=\aleph_0$, since a Boolean algebra A is closed with respect to the finite lattice operations, we can easily prove that A is isomorphic with a field of sets which is σ-additive in the wider sense, if and only if A is isomorphic with a field of sets which is weakly σ-additive in the wider sense. Therefore, similarly as the above theorem, the following theorem will be proved. We see that the results contain those given in A. Horn—A. Tarski [1].

Theorem 5.2. *For a Boolean algebra A, the following conditions* 1) *and* 2) *are equivalent.*

1) *A is isomorphic with a field of sets which is σ-additive in the wider sense.*

2) *For an arbitrary element $a \in A$ which is not* 0, *there is a two-valued measure f which is σ-additive in the wider sense and for which*

$f(a)=1$.

If a Boolean algebra A satisfies the condition 1) *or* 2), *then*

3) *A is σ-distributive in the wider sense.*

In the case in which powers of ramification sets in a Boolean algebra A are at most $\aleph_0$, if A is not atomic, then it can not be represented by $\varphi_\Delta(a)$.

From Example 1, even if a Boolean algebra satisfies the condition 3), it does not satisfy condition 1) or 2). But as we shall see in § 6, conditions 1), 2) and 3) are for a separable Boolean algebra equivalent with each other.

As we can see from the theorems stated until now, the whole structure of a Boolean algebra can be made clear in a certain case, by powers of its ramification sets.

§ 6. Boolean algebras in which powers of ramification sets are countable.

In connection with the problem of existences of special measures (in particular a strictly positive measure, i.e. a measure such that $f(x)=0$ only for $x=0$) in a Boolean algebra, let us show a number of results.[14]

DEFINITION 6.1. *A* measure f is called *σ-additive in the wider sense*, if $f(\bigvee_{n=1}^{\infty} a_n)=\sum_{n=1}^{\infty} f(a_n)$ whenever for an arbitrary elements $a_n \in A$ $(n=1, 2, \cdots)$ such that $a_n \bigwedge a_m = 0$ $(n \neq m)$, $\bigvee_{n=1}^{\infty} a_n$ exists.[8]

DEFINITION 6.2. A Boolean algebra A is called *separable* if there is a countable non-void subset D such that for an arbitrary element $a \in A$ which is not 0 there is an element x of D such that $a \geqq x$.

DEFINITION 6.3. In definition 3.3, let us give the additional conditions $\bar{\bar{\Delta}} \leqq \aleph_0$, $\bar{\bar{\Theta}}_\nu \leqq \aleph_0$ and $a_\mu^\nu \leqq a_{\mu+1}^\nu$ for every $\nu \in \Delta$, $\mu \in \Theta_\nu$, then A is called *weakly σ-distributive in the wider sense.* In particular, if A is σ-complete, it is simply called *weakly σ-distributive.*

In connection with powers of ramification sets, the following two theorems are already known.

Theorem 6.1. *If a Boolean algebra A has a strictly positive measure, then an arbitrary ramification set in A is countable.*[8]

Theorem 6.2. *A separable Boolean algebra always has a strictly*

14) See D. Maharam [1], A. Horn—A. Tarski [1], R. Sikorski [1].

positive measure.[8]

Therefore, for a Boolean algebra having a strictly positive measure (in particular a separable Boolean algebra), we obtain the following theorems from our results.

Theorem 6. 3. *When a Boollean algebra A is isomorphic with a field of sets which is σ-additive in the uider sense, the following conditions are equivalent.*

1) *A has a strictly positive measure.*

2) *A is atomic, and the number of atoms in it are at most countable.*

Therefore, for a Boolean algebra which is isomorphic with a field of sets which is σ-additive in the wider sense, if it is not atomic or if, even if it is atomic, it has uncountable atoms, then any strictly positive measure can not be defined on it.

In particular, for a separable Boolean algebra, we obtain the following theorems besides Theorem 6. 2.

Theorem 6. 4. *For a separable Boolean algebra, it is atomic if and only if it has a σ-additive measure or a σ-additive two-valued measure.*[8]

Theorem 6. 5. *For a separable Boolean algebra A, it is atomic if and only if it satisfies one of the following conditions.*

1) *For an arbitrary element $a \in A$ which is not* 0, *there is a two-valued σ-additive measure such that $f(a) = 1$.*

2) *A has a σ-additive strictly positive measure.*

3) *A is σ-distributive in the wider sense.*

4) *A is weakly σ-distributive in the wider sense.*[9]

Using the results which are obtained until now, let us investigate the properties of a number of examples.

EXAMPLE 2. Let B be the family of all subsets of the interval $[0,1]$ which is measurable in the sense of Lebesgue, and let I be the family of all elements of B which are of measure 0. Consider the quotient algebra B/I. a) It is not σ-distributive; Let $a^n_m = [1/2^n \times(m-1),\ 1/2^n \times m)$, $m = 1, 2, \cdots, 2^n$, $n = 1, 2, \cdots$, then $\sum_{m=1}^{2^n} a^n_m = [0,1)$ and $\Pi_{n=1}^{\infty}(\sum_{m=1}^{2^n} a^n_m) = [0,1)$. On the other hand, $\Pi a^n_{f(n)}$[15] is void or it consists of a single point. Therefore B/I is not σ-distributive. Therefore, b) it is σ-complete, but it is not isomorphic with a σ-additive field of sets, and c) there is no σ-additive two-valued measure on it. Since d) it has a σ-additive strictly positive measure, e) it is not atomic (indeed, it has no atom), and f) it can not be expressed by $\varphi_\Lambda(a)$. By

15) $f(n)$ corresponds to $f(\nu)$ in theorem 3. 3.

the same reasons, g) it is weakly σ-distributive.[f] It is known that e) it is complete.[16]

EXAMPLE 3. Let B be the family of all Borel sets of the interval $[0, 1]$, and let I be the family consisting of all elements of B of the first category. Consider the quotient algebra B/I. As we know well, it is a) separable, b) it has not an atom and c) it is complete.[8] Therefore, d) it has a strictly positive measure. Since e) it is not σ-distributive, f) it is not weakly σ-distributive. g) There is no σ-additive measure, and of course h) it is not isomorpnic with any σ-additive field, i) it can not be expressed by $\varphi_{\Lambda}(a)$[17]

EXAMPLE 4. Let W be the set of all real-valued functions defined on the set of all real numbers, and for any given real number α let $F(\alpha)$ be the set of all those functions in W which do not assume α as a value. Let A be the smallest a) σ-additive field of sets, which contain all the sets $F(\alpha)$ among it. Evidently, since b) it has not an atom, c) it has no strictly positive measure. But it is easy to see that d) it has a σ-additive two-valued measure and e) it is σ-distributive. We can not determine whether is can be expressed by $\varphi_{\Lambda}(a)$ or not, because f) it is σ-additive, but it is not complete.[f]

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