Hyperellipticity of Offsets to Rational Plane Curves

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Preface

Offset curves are also known as parallel curves. In the field of computer aided geometric design (CAGD), rational curves and piecewise rational curves are used as rational Bézier curves and non-uniform rational B-splines (NURBS), and offsets to rational plane curves are often needed. But, in general, offsets to rational plane curves are not rational curves. Pottmann gave explicit representations of rational plane curves with rational offsets in [7], and Lü studied necessary and sufficient condition for rational plane curves to have rational offsets in [6]. Farouki and Neff analyzed geometric and topological properties of offset curves, and studied implicit representations of offsets to rational plane curves in [3, 4]. Arrondo, Sendra and Sendra introduced the notion of simple and special components of generalized offsets to hypersurfaces, and proved that each component of reducible generalized offsets to rational hypersurfaces is rational and that simple components of reducible generalized offsets to hypersurfaces are birationally equivalent to original hypersurfaces in [1], and gave a genus formula for generalized offset curves in terms of the degree and the genus of original curves with some conditions on singularities in [2]. Furthermore, Sendra and Sendra studied degeneration and existence of simple and special components of generalized offsets to hypersurfaces in [9].

However, the relation between the genus of offsets to rational plane curves and proper parametrizations of original curves has not been studied so much except for the case of rational offsets. In this paper, we construct a birational correspondence between offsets to rational plane curves with no special components and hyperelliptic curves derived from proper parametrizations of original curves, and thus we can compute the genus of offsets to rational plane curves. Though our result is limited to the case when original curves are rational, it does not require conditions on singularities of original curves. Note that offsets to rational curves with special components are reducible from a result in [9], and thus together with the result in [1] mentioned above, each component is rational in this case. We also give a criterion to decide the irreducibility of offsets to rational plane curves.

Acknowledgements

First of all, I wish to express my gratitude to Professor Yusuke Sakane for his thoughtful guidance and encouragement.

I also would like to thank the anonymous referee of the paper [5] for the useful comments and valuable suggestions.

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Contents

1 Offsets to Rational Plane Curves

First we define the offsets to rational plane curves. We say a parametric rational plane curve $r(t)$ is properly parametrized provided that, with a finite number of exceptions, for every point (x_0, y_0) of $r(t)$, there is a unique parameter value t_0 such that $r(t_0) = (x_0, y_0)$. A rational parametric plane curve which is not properly parametrized is said to be improperly parametrized. From Lüroth's theorem, any improperly parametrized rational plane curve can be expressed as a properly parametrized rational plane curve (cf. [8],[10]).

For polynomials $X(t), Y(t), W(t)$ with real coefficients, let

$$r(t) = (X(t)/W(t), Y(t)/W(t)) \text{ for } t \in \mathbb{R} \tag{1}$$

be a properly parametrized rational plane curve. Without loss of generality, we may assume that

$$\mathrm{GCD}(X(t), Y(t), W(t)) = 1. \tag{2}$$

Let d be a non-zero real number and regard d as a signed distance. Then the offset $r_d(t)$ to $r(t)$ at distance d is defined by

$$r_d(t) = \left(\frac{X(t)}{W(t)} + d\frac{V(t)}{\sqrt{U^2(t)+V^2(t)}}, \frac{Y(t)}{W(t)} - d\frac{U(t)}{\sqrt{U^2(t)+V^2(t)}}\right) \text{ for } t \in \mathbb{R}, \tag{3}$$

where

$$U(t) = X'(t)W(t) - X(t)W'(t), \quad V(t) = Y'(t)W(t) - Y(t)W'(t). \tag{4}$$

We call $r(t)$ the generator curve of $r_d(t)$.

To obtain an implicit form $f_{|d|}(x, y) = 0$, dependent on the distance d, which represents the offset curve, we follow a method due to Farouki and Neff [4]. Let

$$x = \frac{X(t)}{W(t)} + d\frac{V(t)}{\sqrt{U^2(t)+V^2(t)}}, \quad y = \frac{Y(t)}{W(t)} - d\frac{U(t)}{\sqrt{U^2(t)+V^2(t)}}. \tag{5}$$

Then we see that

$$\left(x - \frac{X(t)}{W(t)}\right)^2 + \left(y - \frac{Y(t)}{W(t)}\right)^2 - d^2 = 0, \tag{6}$$

$$d\frac{V(t)}{x - \dfrac{X(t)}{W(t)}} = \sqrt{U^2(t)+V^2(t)} = -d\frac{U(t)}{y - \dfrac{Y(t)}{W(t)}}, \tag{7}$$

and by multiplying their denominators, we have

$$W^2(t)(x^2+y^2) - 2W(t)(X(t)x + Y(t)y) + (R(t) - d^2W^2(t)) = 0, \tag{8}$$

$$W(t)(U(t)x + V(t)y) - (U(t)X(t) + V(t)Y(t)) = 0, \tag{9}$$

where $R(t) = X^2(t) + Y^2(t)$. We regard (8), (9) as the polynomial equations in x, y. Note that the intersection of curves (8) and (9) consists of two points $r_d(t)$ and $r_{-d}(t)$ whenever

$$W(t) \neq 0 \text{ and } (U(t), V(t)) \neq 0. \tag{10}$$

Let

$$\theta(t) = \mathrm{GCD}(W^2(t), W(t)X(t), W(t)Y(t), R(t) - d^2W^2(t)), \tag{11}$$
$$\psi(t) = \mathrm{GCD}(W(t)U(t), W(t)V(t), U(t)X(t) + V(t)Y(t)), \tag{12}$$

be the greatest common divisors of the coefficients of (8), (9). Then we have

$$\theta(t) = \mathrm{GCD}(W(t), R(t)), \tag{13}$$
$$\psi(t) = \zeta(t)\theta(t) \quad (\zeta(t) = \mathrm{GCD}(U(t), V(t))) \tag{14}$$

(cf. [4], Lemma 3.1,3.2). Since $X(t), Y(t), W(t), U(t), V(t)$ and $R(t)$ are polynomials with real coefficients, so are $\theta(t)$ and $\zeta(t)$.

To avoid parameter values t which cause either (8) or (9) to vanish identically in x, y, we divide them by these greatest common divisors $\theta(t)$ and $\psi(t)$. Thus we get new equations

$$\begin{aligned} P_{|d|}(t, x, y) &= W(t)W_1(t)(x^2 + y^2) - 2W_1(t)(X(t)x + Y(t)y) \\ &\quad + (R_1(t) - d^2W(t)W_1(t)) = 0, \end{aligned} \tag{15}$$
$$Q(t, x, y) = W_1(t)(U_1(t)x + V_1(t)y) - S(t) = 0, \tag{16}$$

where

$$\begin{aligned} W_1(t) = W(t)/\theta(t), \quad R_1(t) = R(t)/\theta(t), \quad U_1(t) = U(t)/\zeta(t), \\ V_1(t) = V(t)/\zeta(t), \quad S(t) = (U_1(t)X(t) + V_1(t)Y(t))/\theta(t). \end{aligned} \tag{17}$$

Note that $P_{|d|}(t, x, y)$ is a polynomial in t, x, y with real coefficients, only even powers of d appearing in its coefficients, and $Q(t, x, y)$ is a polynomial in t, x, y with real coefficients. From now on, we use the subscript $|d|$ to express the polynomials in which only even powers of d appear in their coefficients.

Let $h_{|d|}(x, y)$ be the resultant of $P_{|d|}(t, x, y)$ and $Q(t, x, y)$ with respect to t. Then $h_{|d|}(x, y)$ is a polynomial in x, y with real coefficients. We consider the algebraic curve $h_{|d|}(x, y) = 0$.

By definition of resultant, for any point (x_0, y_0) of the curve $h_{|d|}(x, y) = 0$, there exists t_0 such that $P_{|d|}(t_0, x_0, y_0) = Q(t_0, x_0, y_0) = 0$. Since x_0, y_0 and t_0 are complex numbers in general, we extend the parameter value t of $r_d(t)$ from real numbers to complex numbers.

Note that a point (x_0, y_0) of $h_{|d|}(x, y) = 0$ is a point of the offset $r_d(t)$ or $r_{-d}(t)$ if and only if there exists $t_0 \in \mathbb{C}$ such that $P_{|d|}(t_0, x_0, y_0) = Q(t_0, x_0, y_0) = 0$, and $P_{|d|}(t_0, x, y) = 0$ is an irreducible circle and $Q(t_0, x, y) = 0$ is a line. Since $P_{|d|}(t_0, x, y) = 0$ and $Q(t_0, x, y) = 0$ do not vanish identically in x, y, we consider degenerate cases of the equation $P_{|d|}(t_0, x, y) = 0$ to extract extraneous components of $h_{|d|}(x, y) = 0$.

We compute the discriminant D of the quadratic $P_{|d|}(t_0, x, y) = 0$. Since $R_1(t)W(t) = R(t)W_1(t)$, we have

$$\begin{aligned} D &= \begin{vmatrix} W(t_0)W_1(t_0) & 0 & -W_1(t_0)X(t_0) \\ 0 & W(t_0)W_1(t_0) & -W_1(t_0)Y(t_0) \\ -W_1(t_0)X(t_0) & -W_1(t_0)Y(t_0) & R_1(t_0) - d^2W(t_0)W_1(t_0) \end{vmatrix} \\ &= -d^2W^3(t_0)W_1^3(t_0). \end{aligned} \tag{18}$$

Thus $P_{|d|}(t_0, x, y) = 0$ is irreducible, if it is quadratic. We consider the cases of $P_{|d|}(t_0, x, y) = 0$ not being quadratic, that is, the case of $W_1(t_0) = 0$ and the case of $W(t_0) = 0$ and $W_1(t_0) \neq 0$.

If $W_1(t_0) = 0$, we have $P_{|d|}(t_0, x, y) = R_1(t_0)$. Since $W_1(t)$ and $R_1(t)$ are relatively prime, there are no points which satisfy the equation $P_{|d|}(t_0, x, y) = 0$.

If $W(t_0) = 0$ and $W_1(t_0) \neq 0$, $P_{|d|}(t_0, x, y) = 0$ represents the same line

$$X(t_0)x + Y(t_0)y - \frac{R_1(t_0)}{2W_1(t_0)} = 0 \tag{19}$$

as $Q(t_0, x, y) = 0$ (cf. [4], Lemma 3.4). Note that, by definition of resultant, there exist polynomials $K(t, x, y), L(t, x, y) \in \mathbb{R}[t, x, y]$ such that

$$h_{|d|}(x, y) = K(t, x, y)P_{|d|}(t, x, y) + L(t, x, y)Q(t, x, y) \tag{20}$$

(cf. [10]). Let

$$E = \{t_0 \in \mathbb{C} \,|\, W(t_0) = 0, W_1(t_0) \neq 0\}. \tag{21}$$

Then, from (20), for any $t_i \in E$, there exists $r_i \in \mathbb{N}$ such that

$$P_{|d|}(t_i, x, y)^{r_i} \mid h_{|d|}(x, y) \text{ and } P_{|d|}(t_i, x, y)^{r_i+1} \nmid h_{|d|}(x, y) \tag{22}$$

and $h_{|d|}(x, y) = 0$ has the lines $\prod_{t_i \in E} P_{|d|}(t_i, x, y)^{r_i} = 0$ as extraneous components. Note that if t_i is a complex number, then the complex conjugate $\bar{t}_i$ of t_i belongs to E and satisfies

$$P_{|d|}(\bar{t}_i, x, y)^{r_i} \mid h_{|d|}(x, y) \text{ and } P_{|d|}(\bar{t}_i, x, y)^{r_i+1} \nmid h_{|d|}(x, y). \tag{23}$$

Thus $\prod_{t_i \in E} P_{|d|}(t_i, x, y)^{r_i}$ is a polynomial in x, y with real coefficients.

From the observation above, if we put

$$f_{|d|}(x, y) = \frac{h_{|d|}(x, y)}{\displaystyle\prod_{t_i \in E} P_{|d|}(t_i, x, y)^{r_i}}, \tag{24}$$

then $f_{|d|}(x,y)$ is a polynomial in x,y with real coefficients, only even powers of d appearing in its coefficients, and the equation $f_{|d|}(x,y) = 0$ is the implicit representation of the offsets at distance $\pm d$ with no extraneous components (cf. [4], Theorem 3.6), except for the case when the generator curve is a circle of radius $|d|$ (see Example 2).

We shall give two examples of an algebraic offset $f_{|d|}(x,y) = 0$.

Example 1. We consider the algebraic offset $f_{|d|}(x,y) = 0$ to the properly parametrized rational curve

$$r(t) = \left(\frac{2(5t^4-10t^2+1)}{(t^2+1)^3}, \frac{2t(t^4-10t^2+5)}{(t^2+1)^3}\right). \tag{25}$$

In this case, we have $X(t) = 2(5t^4-10t^2+1), Y(t) = 2t(t^4-10t^2+5), W(t) = (t^2+1)^3$ and $U(t) = -4t(t^2+1)^2(5t^4-30t^2+13), V(t) = -2(t^2+1)^2(t^6-35t^4+55t^2-5), R(t) = 4(t^2+1)^5$. Noting that $\theta(t) = (t^2+1)^3$ and $\zeta(t) = 2(t^2+1)^2$, we see that

$$\begin{aligned}
r_d(t) &= \left(\frac{2(t^4-10t^2+1)}{(t^2+1)^3} - \frac{d(t^6-35t^4+55t^2-5)}{\sqrt{(t^2+1)^5(t^2+25)}},\right. \\
&\qquad \left.\frac{2t(t^4-10t^2+5)}{(t^2+1)^3} + \frac{2dt(5t^4-30t^2+13)}{\sqrt{(t^2+1)^5(t^2+25)}}\right), && (26)\\
P_{|d|}(t,x,y) &= (x^2+y^2-d^2)t^6 - 4yt^5 + (3(x^2+y^2-d^2) - 20x + 4)t^4 \\
&\quad +40yt^3 + (3(x^2+y^2-d^2) + 40x + 8)t^2 - 20yt && (27)\\
&\quad +(x^2+y^2-d^2-4x+4), \\
Q(t,x,y) &= -yt^6 - (10x-2)t^5 + 35yt^4 + (60x+4)t^3 - 55yt^2 \\
&\quad -(26x-2)t + 5y, && (28)
\end{aligned}$$

and by computing the resultant of $P_{|d|}(t,x,y)$ and $Q(t,x,y)$ with respect to t, we have

$$\begin{aligned}
h_{|d|}(x,y) &= 1073741824(x^2+y^2)^2(-64d^2 + 16d^4 + 1184d^2x - 3160d^4x \\
&\quad +1584d^6x - 216d^8x + 64x^2 - 7652d^2x^2 + 28105d^4x^2 \\
&\quad -46724d^6x^2 + 29430d^8x^2 - 7776d^{10}x^2 + 729d^{12}x^2 - 1184x^3 \\
&\quad +22820d^2x^3 - 52852d^4x^3 + 21124d^6x^3 - 2340d^8x^3 \\
&\quad +216d^{10}x^3 + 7636x^4 - 41210d^2x^4 + 45922d^4x^4 - 48020d^6x^4 \\
&\quad +23830d^8x^4 - 4374d^{10}x^4 - 19660x^5 + 25952d^2x^5 \\
&\quad +10424d^4x^5 + 4860d^6x^5 - 1080d^8x^5 + 13105x^6 + 35828d^2x^6 \\
&\quad +33105d^4x^6 - 17560d^6x^6 + 10935d^8x^6 + 25316x^7 \\
&\quad -20856d^2x^7 - 540d^4x^7 + 2160d^6x^7 - 35026x^8 - 39870d^2x^8
\end{aligned}$$

$$
\begin{aligned}
&-12540d^4x^8 - 14580d^6x^8 - 10476x^9 - 4140d^2x^9 \\
&-2160d^4x^9 + 25355x^{10} + 21320d^2x^{10} + 10935d^4x^{10} \\
&+2160x^{11} + 1080d^2x^{11} - 7274x^{12} - 4374d^2x^{12} - 216x^{13} \\
&+729x^{14} - 3716d^2y^2 + 27625d^4y^2 - 46660d^6y^2 + 29430d^8y^2 \\
&-7776d^{10}y^2 + 729d^{12}y^2 - 320xy^2 + 22820d^2xy^2 \\
&-52852d^4xy^2 + 21124d^6xy^2 - 2340d^8xy^2 + 216d^{10}xy^2 \\
&+8036x^2y^2 - 71260d^2x^2y^2 + 88352d^4x^2y^2 - 96040d^6x^2y^2 \\
&+47660d^8x^2y^2 - 8748d^{10}x^2y^2 - 39320x^3y^2 + 51904d^2x^3y^2 \\
&+20848d^4x^3y^2 + 9720d^6x^3y^2 - 2160d^8x^3y^2 + 43635x^4y^2 \\
&+99276d^2x^4y^2 + 99315d^4x^4y^2 - 52680d^6x^4y^2 \\
&+32805d^8x^4y^2 + 75948x^5y^2 - 62568d^2x^5y^2 - 1620d^4x^5y^2 \\
&+6480d^6x^5y^2 - 140968x^6y^2 - 159480d^2x^6y^2 - 50160d^4x^6y^2 \\
&-58320d^6x^6y^2 - 41904x^7y^2 - 16560d^2x^7y^2 - 8640d^4x^7y^2 \\
&+126775x^8y^2 + 106600d^2x^8y^2 + 54675d^4x^8y^2 + 10800x^9y^2 \\
&+5400d^2x^9y^2 - 43644x^{10}y^2 - 26244d^2x^{10}y^2 - 1296x^{11}y^2 \\
&+5103x^{12}y^2 + 400y^4 - 30050d^2y^4 + 42430d^4y^4 - 48020d^6y^4 \\
&+23830d^8y^4 - 4374d^{10}y^4 - 19660xy^4 + 25952d^2xy^4 \\
&+10424d^4xy^4 + 4860d^6xy^4 - 1080d^8xy^4 + 47955x^2y^4 \qquad (29) \\
&+91068d^2x^2y^4 + 99315d^4x^2y^4 - 52680d^6x^2y^4 + 32805d^8x^2y^4 \\
&+75948x^3y^4 - 62568d^2x^3y^4 - 1620d^4x^3y^4 + 6480d^6x^3y^4 \\
&-212748x^4y^4 - 239220d^2x^4y^4 - 75240d^4x^4y^4 - 87480d^6x^4y^4 \\
&-62856x^5y^4 - 24840d^2x^5y^4 - 12960d^4x^5y^4 + 253550x^6y^4 \\
&+213200d^2x^6y^4 + 109350d^4x^6y^4 + 21600x^7y^4 + 10800d^2x^7y^4 \\
&-109110x^8y^4 - 65610d^2x^8y^4 - 3240x^9y^4 + 15309x^{10}y^4 \\
&+17425y^6 + 27620d^2y^6 + 33105d^4y^6 - 17560d^6y^6 \\
&+10935d^8y^6 + 25316xy^6 - 20856d^2xy^6 - 540d^4xy^6 \\
&+2160d^6xy^6 - 142696x^2y^6 - 159480d^2x^2y^6 - 50160d^4x^2y^6 \\
&-58320d^6x^2y^6 - 41904x^3y^6 - 16560d^2x^3y^6 - 8640d^4x^3y^6 \\
&+253550x^4y^6 + 213200d^2x^4y^6 + 109350d^4x^4y^6 + 21600x^5y^6 \\
&+10800d^2x^5y^6 - 145480x^6y^6 - 87480d^2x^6y^6 - 4320x^7y^6 \\
&+25515x^8y^6 - 35890y^8 - 39870d^2y^8 - 12540d^4y^8 \\
&-14580d^6y^8 - 10476xy^8 - 4140d^2xy^8 - 2160d^4xy^8 \\
&+126775x^2y^8 + 106600d^2x^2y^8 + 54675d^4x^2y^8 + 10800x^3y^8 \\
&+5400d^2x^3y^8 - 109110x^4y^8 - 65610d^2x^4y^8 - 3240x^5y^8 \\
&+25515x^6y^8 + 25355y^{10} + 21320d^2y^{10} + 10935d^4y^{10}
\end{aligned}
$$

$$+2160xy^{10} + 1080d^2xy^{10} - 43644x^2y^{10} - 26244d^2x^2y^{10}$$
$$-1296x^3y^{10} + 15309x^4y^{10} - 7274y^{12} - 4374d^2y^{12}$$
$$-216xy^{12} + 5103x^2y^{12} + 729y^{14}).$$

Since $E = \{i, -i\}$ and $P_{|d|}(i, x, y)P_{|d|}(-i, x, y) = 4096(x^2 + y^2)$, we have from (29) that

$$f_{|d|}(x, y) = \frac{h_{|d|}(x, y)}{(4096(x^2 + y^2))^2}. \tag{30}$$

In Figure 1, we plot the generator curve (25) and its offsets at distnace ± 1. The black curve represents the generator curve, the blue curve represents offsets at positive distance and the red curve represents offset at negative distance. Note that the blue curve and the red curve together constitute the algebraic offset $f_{|1|}(x, y) = 0$.

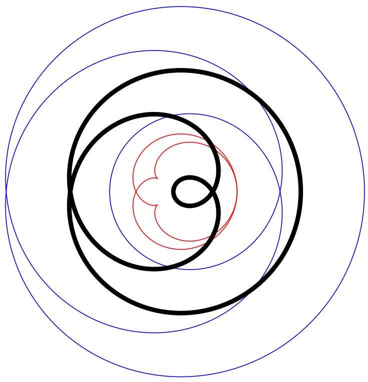

Figure 1: The curve (25) in Example 1 with its offsets at distance ± 1

Example 2. We consider the algebraic offset $f_{|d|}(x, y) = 0$ to the circle

$$x^2 + y^2 = r^2. \tag{31}$$

The circle (31) is properly parametrized by $X(t) = r(1 - t^2), Y(t) = 2rt, W(t) = 1 + t^2$, and we observe that $U(t) = -4rt, V(t) = 2r(1 - t^2), R(t) = r^2(1 + t^2)^2$. Since $\theta(t) = 1 + t^2$ and $\zeta(t) = 2r$, we have

$$r_d(t) = \left(\frac{(r + d)(1 - t^2)}{1 + t^2}, \frac{2(r + d)t}{1 + t^2} \right), \tag{32}$$
$$P_{|d|}(t, x, y) = (x^2 + y^2 + r^2 - d^2)(1 + t^2) - 2r((1 - t^2)x + 2ty), \tag{33}$$
$$Q(t, x, y) = -2tx + (1 - t^2)y, \tag{34}$$

and by computing the resultant of $P_{|d|}(t,x,y)$ and $Q(t,x,y)$ with respect to t, we see that

$$h_{|d|}(x,y) = 4(x^2+y^2)(x^2+y^2-(d+r)^2)(x^2+y^2-(d-r)^2). \tag{35}$$

Note that $E = \{-i, i\}$ and $P_{|d|}(i,x,y)P_{|d|}(-i,x,y) = 16(x^2+y^2)$. Thus, if $d \neq \pm r$, from (35), we have

$$f_{|d|}(x,y) = \frac{1}{4}(x^2+y^2-(d+r)^2)(x^2+y^2-(d-r)^2) \tag{36}$$

and algebraic offsets to circles consist of two circles in general (See Figure 2).

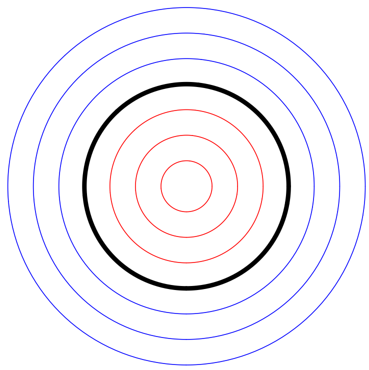

Figure 2: A unit circle with its offsets at distance $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}$

On the other hand, if $d = \pm r$, we have

$$f_{|d|}(x,y) = \frac{1}{64}(x^2+y^2-(2r)^2) \quad (d = \pm r) \tag{37}$$

and $f_{|d|}(x,y) = 0$ is not the implicit representation of the offsets at distance $\pm d$ in this case, since the origin $r_{-d}(t)$ or $r_d(t)$ is omitted.

Note that, since the set E is always finite, an algebraic offset $f_{|d|}(x,y) = 0$ does not contain infinitely many points of $r_d(t)$ or $r_{-d}(t)$ if and only if the generator curve is a circle of radius $|d|$. From now on, we exclude such cases from our consideration.

2 Birational Correspondence

We consider a correspondence between the algebraic offset $f_{|d|}(x,y)=0$ and the algebraic curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$. We assume that algebraic offsets always contain infinitely many points of both $r_d(t)$ and $r_{-d}(t)$.

Lemma 3. *Let $f_{|d|}(x,y)=0$ be an algebraic offset to a properly parametrized rational plane curve $r(t)=(X(t)/W(t), Y(t)/W(t))$. Then the equation*

$$x = \frac{X(\tilde{x})}{W(\tilde{x})} + d\frac{V(\tilde{x})}{\tilde{y}}, \quad y = \frac{Y(\tilde{x})}{W(\tilde{x})} - d\frac{U(\tilde{x})}{\tilde{y}}, \tag{38}$$

defines a rational transformation from the algebraic curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ into the offset $f_{|d|}(x,y)=0$.

Proof. Let $(\tilde{x},\tilde{y})$ be a point of the curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ which satisfies the conditions

$$W(\tilde{x}) \neq 0 \text{ and } \tilde{y} \neq 0. \tag{39}$$

Then we have either $\tilde{y} = \sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}$ or $\tilde{y} = -\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}$. If $\tilde{y} = \sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}$, we see that

$$x = \frac{X(\tilde{x})}{W(\tilde{x})} + d\frac{V(\tilde{x})}{\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}}, \quad y = \frac{Y(\tilde{x})}{W(\tilde{x})} - d\frac{U(\tilde{x})}{\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}} \tag{40}$$

from (38), and this gives the point $r_d(\tilde{x})$ of the offset $f_{|d|}(x,y)=0$. On the other hand, if $\tilde{y} = -\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}$, we see that

$$x = \frac{X(\tilde{x})}{W(\tilde{x})} + (-d)\frac{V(\tilde{x})}{\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}}, \quad y = \frac{Y(\tilde{x})}{W(\tilde{x})} - (-d)\frac{U(\tilde{x})}{\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}} \tag{41}$$

from (38), and this gives the point $r_{-d}(\tilde{x})$ of the offset $f_{|d|}(x,y)=0$. Thus the equation (38) defines a rational transformation from the algebraic curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ into the offset $f_{|d|}(x,y)=0$. □

We shall give an example of a rational transformation in Lemma 3.

Example 4. We consider the algebraic offset $f_{|d|}(x,y)=0$ to the cubic $r(t) = (t,t^3)$ (See Figure 3). In this case, we have $X(t)=t, Y(t)=t^3, W(t)=1$ and $U(t)=1, V(t)=3t^2, R(t)=t^2+t^6$. Noting that $\theta(t)=\zeta(t)=1$, we see that

$$r_d(t) = \left(t + \frac{3dt^2}{\sqrt{1+9t^4}}, t^3 - \frac{d}{\sqrt{1+9t^4}}\right), \tag{42}$$

$$P_{|d|}(t,x,y) = t^6 - 2yt^3 + t^2 - 2xt + (x^2+y^2-d^2), \tag{43}$$

$$Q(t,x,y) = -3t^5 + 3yt^2 - t + x. \tag{44}$$

Since $W(t) = 1$, there is no extraneous factor, and by computing the resultant of $P_{|d|}(t, x, y)$ and $Q(t, x, y)$ with respect to t, we have

$$\begin{aligned} f_{|d|}(x,y) &= -16d^2 - 216d^6 - 729d^{10} - 2052d^4x^2 + 3645d^8x^2 - 873d^2x^4 \\ &\quad -7290d^6x^4 + 16x^6 + 7290d^4x^6 - 3645d^2x^8 + 729x^{10} \\ &\quad +432d^2xy - 4860d^6xy - 32x^3y + 7830d^4x^3y - 1080d^2x^5y \\ &\quad -1890x^7y + 16y^2 + 1188d^4y^2 + 1458d^8y^2 + 594d^2x^2y^2 \\ &\quad -5832d^6x^2y^2 + 1593x^4y^2 + 8748d^4x^4y^2 - 5832d^2x^6y^2 \\ &\quad +1458x^8y^2 - 432xy^3 + 9234d^4xy^3 - 6318d^2x^3y^3 - 2916x^5y^3 \\ &\quad -1701d^2y^4 - 729d^6y^4 + 1458x^2y^4 + 729y^6 + 2187d^4x^2y^4 \\ &\quad -2187d^2x^4y^4 + 729x^6y^4 - 4374d^2xy^5 - 1458x^3y^5. \end{aligned} \tag{45}$$

Now a rational transformation from the curve $\tilde{y}^2 = 1 + 9\tilde{x}^4$ into the offset $f_{|d|}(x, y) = 0$ is given by

$$x = \tilde{x} + \frac{3d\tilde{x}^2}{\tilde{y}}, \quad y = \tilde{x}^3 - \frac{d}{\tilde{y}}. \tag{46}$$

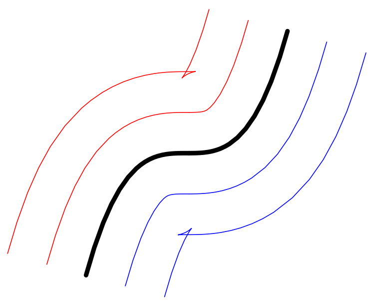

Figure 3: The cubic (t, t^3) with its offsets at distance $\pm\frac{1}{2}, \pm 1$

We shall construct a rational transformation from an irreducible component of the offset $f_{|d|}(x, y) = 0$ into the curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$.

By decomposing the polynomial $f_{|d|}(x, y)$ into irreducible factors, we have

$$f_{|d|}(x,y) = \bar{f}_1(x,y)\bar{f}_2(x,y)\cdots\bar{f}_j(x,y)\cdots\bar{f}_l(x,y), \tag{47}$$

where $\bar{f}_j(x, y) = f_j^{n_j}(x, y)$ for some irreducible polynomial $f_j(x, y) \in \mathbb{C}[x, y]$ and for some $n_j \in \mathbb{N}$, and $f_i(x, y) \neq f_j(x, y)$ if $i \neq j$.

Now we recall the notion of simple and special components of $f_{|d|}(x, y) = 0$ due to Arrondo, Sendra and Sendra (cf. [1], Definition 2.2). An irreducible component $f_j(x, y) = 0$ of $f_{|d|}(x, y) = 0$ is called simple provided that without a finite number of exceptions, for every point (x, y) of $f_j(x, y) = 0$, there exists a unique parameter value t such that $P_{|d|}(t,x,y) = Q(t,x,y) = 0$. Otherwise $f_j(x,y) = 0$ is called special.

We also recall the results due to Sendra and Sendra [9] in our case, although the cases of generalized offsets to hypersurfaces are discussed in [9]. If $f_j(x,y) = 0$ is special, there exist infinitely many points of self-intersection of the offset $f_{|d|}(x,y) = 0$. Thus, from Bézout's theorem (cf. [10]), we see that the multiplicity $n_j \geq 2$. In particular, if the polynomial $f_{|d|}(x,y)$ is irreducible then the offset $f_{|d|}(x,y) = 0$ is simple. Furthermore, since the generator curve $r(t)$ is properly parametrized, by definition of offset curves, the offset $f_{|d|}(x,y) = 0$ contains a special component if and only if the generator curve $r(t)$ is the rational offset curve at distance d of some curve (cf. [9], Theorem 7).

We shall give an example of a special component. This example is due to the referee of the paper [5].

Example 5. The offset curve $r_1(t)$ to the parabola $r(t) = (t,t^2)$ at distance 1 is given by

$$r_1(t) = \left(t + \frac{2t}{\sqrt{1+4t^2}}, t^2 - \frac{1}{\sqrt{1+4t^2}}\right). \tag{48}$$

If we reparametrize $r_1(t)$ by

$$t = \frac{u^2-1}{4u}, \tag{49}$$

we have a properly parametrized rational curve

$$\begin{aligned} s(u) &= r_1\left(\frac{u^2-1}{4u}\right) \\ &= \left(\frac{4u(u^2-1)(u^2+4u+1)}{16u^2(u^2+1)}, \frac{u^6-u^4-32u^3-u^2+1}{16u^2(u^2+1)}\right). \end{aligned} \tag{50}$$

We study components of the algebraic offset $f_{|d|}(x,y) = 0$ to the curve $s(u)$. By computing the resultant of $P_{|d|}(u,x,y)$ and $Q(u,x,y)$ with respect to u, we have

$$\begin{aligned} h_{|d|}(x,y) = \; & 70368744177664(1+16x^2-8y+16y^2) \\ & ((-1156+688x^2-191x^4+16x^6+544y+30x^2y-40x^4y \\ & +225y^2-96x^2y^2+16x^4y^2-136y^3-32x^2y^3+16y^4) \\ & +(1-d)(3+d)(-220x^2+208d+168d^2+64d^3+16d^4 \\ & +377-96dx^2-48d^2x^2+48x^4-72y^2-168y-64dy \\ & -32d^2y-8x^2y-32dy^2-16d^2y^2+32x^2y^2+32y^3)) \\ & ((x^2-y)^2(1+16x^2-8y+16y^2) \\ & -(1-d)^2(25-80d+104d^2-64d^3+16d^4-28x^2 \\ & +96dx^2-48d^2x^2+48x^4-40y+64dy-8y^2-32d^2y \\ & -8x^2y+32dy^2-16d^2y^2+32x^2y^2+32y^3)). \end{aligned} \tag{51}$$

Note that $E = \{i, -i\}$ and $P_{|d|}(i, x, y)P_{|d|}(-i, x, y) = 65536(1+16x^2-8y+16y^2)$. Thus we have

$$f_{|d|}(x, y) = \frac{h_{|d|}(x, y)}{65536(1+16x^2-8y+16y^2)} \quad (|d| \neq 1) \tag{52}$$

and

$$\begin{aligned} f_{|1|}(x, y) &= \frac{h_{|1|}(x, y)}{(65536(1+16x^2-8y+16y^2))^2} \\ &= 16384(x^2-y)^2(-1156+688x^2-191x^4+16x^6 \\ &\quad +544y+30x^2y-40x^4y+225y^2-96x^2y^2 \\ &\quad +16x^4y^2-136y^3-32x^2y^3+16y^4). \end{aligned} \tag{53}$$

Though algebraic offset $f_{|d|}(x, y) = 0$ has no special components if $|d| \neq 1$, the algebraic offset $f_{|1|}(x, y) = 0$ has a special component $x^2 - y = 0$ (See Figure 4).

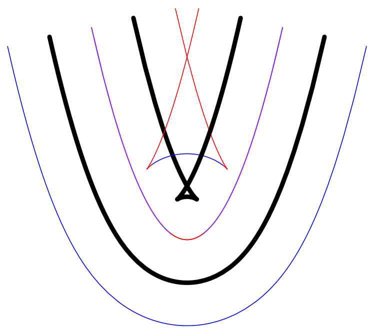

Figure 4: The curve (50) in Example 5 with its offsets at distance ± 1

It is also known that for almost every distance d all the components of the algebraic offset $f_{|d|}(x, y) = 0$ are simple (cf. [9], Theorem 8). From now on, we only deal with the offsets with no special components. We shall construct a rational transformation from a simple component of the offset $f_{|d|}(x, y) = 0$ into the curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$.

Let

$$f_{|d|}(x, y) = f_1(x, y)f_2(x, y) \cdots f_j(x, y) \cdots f_l(x, y) \tag{54}$$

be a decomposition of $f_{|d|}(x, y)$ into irreducible factors in $\mathbb{C}[x, y]$.

We regard $P_{|d|}(t, x, y)$ and $Q(t, x, y)$ as polynomials in t over the rational function field $\mathbb{R}(x, y)$ and apply the Euclidean algorithm to them. Then we have

$$P_{|d|}(t, x, y) = Q(t, x, y)\frac{q^1_{|d|}(t, x, y)}{s^1_{|d|}(x, y)} + \frac{r^1_{|d|}(t, x, y)}{s^1_{|d|}(x, y)},$$

$$
\begin{aligned}
Q(t,x,y) &= r^1_{|d|}(t,x,y)\frac{q^2_{|d|}(t,x,y)}{s^2_{|d|}(x,y)} + \frac{r^2_{|d|}(t,x,y)}{s^2_{|d|}(x,y)},\\
r^1_{|d|}(t,x,y) &= r^2_{|d|}(t,x,y)\frac{q^3_{|d|}(t,x,y)}{s^3_{|d|}(x,y)} + \frac{r^3_{|d|}(t,x,y)}{s^3_{|d|}(x,y)},\\
&\vdots \\
r^{p-3}_{|d|}(t,x,y) &= r^{p-2}_{|d|}(t,x,y)\frac{q^{p-1}_{|d|}(t,x,y)}{s^{p-1}_{|d|}(x,y)} + \frac{r^{p-1}_{|d|}(t,x,y)}{s^{p-1}_{|d|}(x,y)},\\
r^{p-2}_{|d|}(t,x,y) &= r^{p-1}_{|d|}(t,x,y)\frac{q^{p}_{|d|}(t,x,y)}{s^{p}_{|d|}(x,y)} + \frac{r^{p}_{|d|}(x,y)}{s^{p}_{|d|}(x,y)},
\end{aligned}
\tag{55}
$$

where $q^i_{|d|}(t,x,y), r^i_{|d|}(t,x,y), s^i_{|d|}(x,y), r^p_{|d|}(x,y)$ are polynomials with real coefficients and

$$
\deg_t Q > \deg_t r^1_{|d|} > \deg_t r^2_{|d|} > \cdots > \deg_t r^{p-1}_{|d|} > \deg_t r^p_{|d|} = 0. \tag{56}
$$

Note that for each i, there exist polynomials $A^i_{|d|}(t,x,y), B^i_{|d|}(t,x,y) \in \mathbb{R}[t,x,y]$ such that

$$
\begin{aligned}
r^i_{|d|}(t,x,y) &= A^i_{|d|}(t,x,y)P_{|d|}(t,x,y) \\
&\quad + B^i_{|d|}(t,x,y)Q(t,x,y) \quad (i = 1,\cdots,p-1), \qquad (57)\\
r^p_{|d|}(x,y) &= A^p_{|d|}(t,x,y)P_{|d|}(t,x,y) + B^p_{|d|}(t,x,y)Q(t,x,y). \qquad (58)
\end{aligned}
$$

In our discussion, the polynomial $r^{p-1}_{|d|}(t,x,y)$ plays an important role. Let

$$
\begin{aligned}
r^{p-1}_{|d|}(t,x,y) &= a^{m_{p-1}}_{|d|}(x,y)t^{m_{p-1}} + a^{m_{p-1}-1}_{|d|}(x,y)t^{m_{p-1}-1} \\
&\quad + \cdots + a^1_{|d|}(x,y)t + a^0_{|d|}(x,y),
\end{aligned}
\tag{59}
$$

where $a^i_{|d|}(x,y)$ are polynomials in x, y. From Bézout's theorem, we see that if $a^i_{|d|}(x,y)$ vanishes for infinitely many points of the curve $f_j(x,y) = 0$, then $a^i_{|d|}(x,y)$ vanishes identically on $f_j(x,y) = 0$. Now, suppose that all coefficients $a^i_{|d|}(x,y)\,(i = 0,\cdots,m_{p-1})$ vanish for infinitely many points of $f_j(x,y) = 0$. Then for any $t \in \mathbb{C}$, the polynomial $r^{p-1}_{|d|}(t,x,y)$ in x, y vanishes identically on $f_j(x,y) = 0$. Note that, with a finite number of exceptions, for every point (x,y) of $f_j(x,y) = 0$, there exists a unique parameter value t such that $P_{|d|}(t,x,y) = Q(t,x,y) = 0$, since the component $f_j(x,y) = 0$ is simple. Since $\deg_t r_p = 0$, we observe, from (58), that the polynomial $r^p_{|d|}(x,y)$ also vanishes identically on the curve $f_j(x,y) = 0$. Then, from (55), for any point (x,y) of $f_j(x,y) = 0$ and for any $t \in \mathbb{C}$, we have $P_{|d|}(t,x,y) = Q(t,x,y) = 0$. This contradicts the hypothesis that the component $f_j(x,y) = 0$ is simple. Thus there exists an $i \in \mathbb{N}$ such that $a^i_{|d|}(x,y)$ does not vanish identically on the curve $f_j(x,y)=0$. Let m_j be the maximum of such i. If $m_j=0$, with a finite number of exceptions, for every point (x,y) of $f_j(x,y)=0$ and for any $t\in\mathbb{C}$ satisfying $P_{|d|}(t,x,y)=Q(t,x,y)=0$, we have $r_{|d|}^{p-1}(t,x,y)=a_{|d|}^{0}(x,y)\neq 0$. However, this is impossible, since $r_{|d|}^{p-1}(t,x,y)=0$ from (57). Thus we have $1\leq m_j\leq \deg_t r_{|d|}^{p-1}$.

Let (x_0,y_0) be a generic point of $f_j(x,y)=0$. Then we have $a_{|d|}^{m_j}(x_0,y_0)\neq 0$ and there exists a unique parameter value t_0 such that $P_{|d|}(t_0,x_0,y_0)=Q(t_0,x_0,y_0)=0$. Since $r_{|d|}^{p}(x_0,y_0)=0$, if t_1 is a parameter value satisfying $r_{|d|}^{p-1}(t_1,x_0,y_0)=0$, we also have $P_{|d|}(t_1,x_0,y_0)=Q(t_1,x_0,y_0)=0$ from (55). Because of the uniqueness of t_0, this implies that $t_1=t_0$ and the polynomial $r_{|d|}^{p-1}(t,x_0,y_0)$ must have the factorization

$$r_{|d|}^{p-1}(t,x_0,y_0)=a_{|d|}^{m_j}(x_0,y_0)(t-t_0)^{m_j}. \tag{60}$$

Comparing this expression with (59), we obtain

$$t_0=-\frac{1}{m_j}\frac{a_{|d|}^{m_j-1}(x_0,y_0)}{a_{|d|}^{m_j}(x_0,y_0)}. \tag{61}$$

Now we put

$$t_{|d|}^{j}(x,y)=-\frac{1}{m_j}\frac{a_{|d|}^{m_j-1}(x,y)}{a_{|d|}^{m_j}(x,y)}. \tag{62}$$

Then we have the following.

Lemma 6. *Let $f_{|d|}(x,y)=0$ be an algebraic offset to a properly parametrized rational plane curve $r(t)=(X(t)/W(t),Y(t)/W(t))$ with no special components, and let $f_j(x,y)=0$ be an irreducible component of $f_{|d|}(x,y)=0$. Then, with a finite number of exceptions, for every point (x,y) of $f_j(x,y)=0$, the value given by (62) satisfies the equations*

$$P_{|d|}(t_{|d|}^{j}(x,y),x,y)=Q(t_{|d|}^{j}(x,y),x,y)=0. \tag{63}$$

Moreover, the equation

$$\tilde{x}=t_{|d|}^{j}(x,y),\quad \tilde{y}=-d\frac{U(t_{|d|}^{j}(x,y))}{y-\dfrac{Y(t_{|d|}^{j}(x,y))}{W(t_{|d|}^{j}(x,y))}}, \tag{64}$$

defines a rational transformation from $f_j(x,y)=0$ into the curve $\tilde{y}^2=U^2(\tilde{x})+V^2(\tilde{x})$.

Proof. From the discussion above, with a finite number of exceptions, for every point (x, y) of $f_j(x, y) = 0$, $r^p_{|d|}(x, y) = 0$ and the value $t^j_{|d|}(x, y)$ satisfies $r^{p-1}_{|d|}(t^j_{|d|}(x, y), x, y)) = 0$, and hence from (55) we get the first part.

To prove the second part, consider a generic point of the curve $f_j(x, y) = 0$ satisfying (63) and the conditions

$$W(t^j_{|d|}(x, y)) \neq 0 \text{ and } W(t^j_{|d|}(x, y))y - Y(t^j_{|d|}(x, y)) \neq 0. \tag{65}$$

Note that (x, y) is a point of $r_d(t)$ or a point of $r_{-d}(t)$. If (x, y) is a point of $r_d(t)$, then we see from (7) that

$$\begin{aligned} \tilde{y} &= -d\frac{U(t^j_{|d|}(x, y))}{y - \dfrac{Y(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))}} = \sqrt{U^2(t^j_{|d|}(x, y)) + V^2(t^j_{|d|}(x, y))} \\ &= \sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}, \end{aligned} \tag{66}$$

and hence (x, y) is transformed into a point $(\tilde{x}, \tilde{y})$ of $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$. On the other hand, if (x, y) is a point of $r_{-d}(t)$, then we see from (7) that

$$\begin{aligned} \tilde{y} &= -1 \times -(-d)\frac{U(t^j_{|d|}(x, y))}{y - \dfrac{Y(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))}} = -\sqrt{U^2(t^j_{|d|}(x, y)) + V^2(t^j_{|d|}(x, y))} \\ &= -\sqrt{U^2(\tilde{x}) + V^2(\tilde{x})}, \end{aligned} \tag{67}$$

and hence (x, y) is transformed into a point $(\tilde{x}, \tilde{y})$ of $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$. Thus the equation (64) defines a rational transformation from $f_j(x, y) = 0$ into $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$. □

We denote the rational transformation (38) in Lemma 3 from the curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ into the algebraic offset $f_{|d|}(x, y) = 0$ by

$$\Phi(\tilde{x}, \tilde{y}) = (\Phi_1(\tilde{x}, \tilde{y}), \Phi_2(\tilde{x}, \tilde{y})) = \left(\frac{X(\tilde{x})}{W(\tilde{x})} + d\frac{V(\tilde{x})}{\tilde{y}}, \frac{Y(\tilde{x})}{W(\tilde{x})} - d\frac{U(\tilde{x})}{\tilde{y}}\right), \tag{68}$$

and for each $j = 1, \cdots, l$, the transformation (64) in Lemma 6 from the irreducible component $f_j(x, y) = 0$ of the offset $f_{|d|}(x, y) = 0$ into the curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ by

$$\Psi^j(x, y) = (\Psi^j_1(x, y), \Psi^j_2(x, y)) = \left(t^j_{|d|}(x, y), -d\frac{U(t^j_{|d|}(x, y))}{y - \dfrac{Y(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))}} \right). \tag{69}$$

Now we have the following.

Theorem 7. *With a finite number of exceptions, there is a one-to-one correspondence between the points of the curve $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ and those of the algebraic offset $f_{|d|}(x, y) = 0$ with no special components via the rational transformations Φ and Ψ^j $(j = 1, \cdots, l)$. In particular, the irreducible components of $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ and those of $f_{|d|}(x, y) = 0$ correspond each other, and their corresponding irreducible components are birationally equivalent.*

Proof. Let $(\tilde{x}, \tilde{y})$ be a point of $\tilde{y}^2 = U^2(\tilde{x}) + V^2(\tilde{x})$ satisfying the conditions (39). By construction of Φ, with a finite number of exceptions, we may assume that $\Phi(\tilde{x}, \tilde{y})$ is not a point of the self-intersections of $f_{|d|}(x, y) = 0$. Then there exists a unique j such that $\Phi(\tilde{x}, \tilde{y})$ is a point of $f_j(x, y) = 0$. Furthermore, we assume that $\Phi(\tilde{x}, \tilde{y})$ satisfies the conditions (63), (65). Now, by definition of $t^j_{|d|}(x, y)$, we see that

$$\begin{aligned}
(\Psi^j_1 \circ \Phi)(\tilde{x}, \tilde{y}) &= t^j_{|d|}(\Phi_1(\tilde{x}, \tilde{y}), \Phi_2(\tilde{x}, \tilde{y})) \\
&= t^j_{|d|}\left(\frac{X(\tilde{x})}{W(\tilde{x})} + d\frac{V(\tilde{x})}{\tilde{y}}, \frac{Y(\tilde{x})}{W(\tilde{x})} - d\frac{U(\tilde{x})}{\tilde{y}}\right) = \tilde{x}, \qquad (70) \\
(\Psi^j_2 \circ \Phi)(\tilde{x}, \tilde{y}) &= \frac{-d \cdot U(t^j_{|d|}(\Phi_1(\tilde{x}, \tilde{y}), \Phi_2(\tilde{x}, \tilde{y})))}{\Phi_2(\tilde{x}, \tilde{y}) - \dfrac{Y(t^j_{|d|}(\Phi_1(\tilde{x}, \tilde{y}), \Phi_2(\tilde{x}, \tilde{y})))}{W(t^j_{|d|}(\Phi_1(\tilde{x}, \tilde{y}), \Phi_2(\tilde{x}, \tilde{y})))}} \\
&= \frac{-d \cdot U(\tilde{x})}{\left(\dfrac{Y(\tilde{x})}{W(\tilde{x})} - \dfrac{d \cdot U(\tilde{x})}{\tilde{y}}\right) - \dfrac{Y(\tilde{x})}{W(\tilde{x})}} = \tilde{y}, \qquad (71)
\end{aligned}$$

and hence we have $(\Psi^j \circ \Phi)(\tilde{x}, \tilde{y}) = (\tilde{x}, \tilde{y})$.

Conversely, let (x, y) be a point of $f_{|d|}(x, y) = 0$ satisfying the conditions (63), (65), and not being a point of the self-intersections. Then there exists a unique j such that (x, y) is a point of $f_j(x, y) = 0$. By construction of Ψ^j, with a finite number of exceptions, we may assume that $\Psi^j(x, y)$ satisfies the conditions (39). Now, by definition of $t^j_{|d|}(x, y)$, we see that

$$\begin{aligned}
(\Phi_1 \circ \Psi^j)(x, y) &= \frac{X(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))} + \frac{d \cdot V(t^j_{|d|}(x, y))}{-d \cdot U(t^j_{|d|}(x, y))}\left(y - \frac{Y(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))}\right) \\
&= \frac{X(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))} + \frac{d \cdot V(t^j_{|d|}(x, y))}{\sqrt{U^2(t^j_{|d|}(x, y)) + V^2(t^j_{|d|}(x, y))}} = x, \qquad (72)
\end{aligned}$$

$$\begin{aligned}
(\Phi_2 \circ \Psi^j)(x, y) &= \frac{Y(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))} - \frac{d \cdot U(t^j_{|d|}(x, y))}{-d \cdot U(t^j_{|d|}(x, y))}\left(y - \frac{Y(t^j_{|d|}(x, y))}{W(t^j_{|d|}(x, y))}\right) \\
&= y, \qquad (73)
\end{aligned}$$

and hence we have $(\Phi \circ \Psi^j)(x,y) = (x,y)$.

The second part of the theorem follows immediately from the first part. □

We shall give an example of a birational correspondence in Theorem 7.

Example 8. In Example 4, we considered the algebraic offset $f_{|d|}(x,y) = 0$ to the cubic $r(t) = (t, t^3)$ and constructed a rational transformation from the curve $\tilde{y}^2 = 1 + 9\tilde{x}^4$ into the offset $f_{|d|}(x,y) = 0$. Here we construct a rational transformation from the offset $f_{|d|}(x,y) = 0$ into the curve $\tilde{y}^2 = 1 + 9\tilde{x}^4$.

Since the curve $\tilde{y}^2 = 1 + 9\tilde{x}^4$ is irreducible, so is the offset $f_{|d|}(x,y) = 0$, and hence $l = 1$ (cf. Lemma 9 and Theorem 10). By applying the Euclidean algorithm (55) to the polynomials

$$\begin{aligned} P_{|d|}(t,x,y) &= t^6 - 2yt^3 + t^2 - 2xt + (x^2 + y^2 - d^2), \\ Q(t,x,y) &= -3t^5 + 3yt^2 - t + x, \end{aligned}$$

which we computed in Example 4, we have

$$r^1_{|d|}(t,x,y) = -3yt^3 + 2t^2 - 5xt + 3(x^2 + y^2 - d^2), \tag{74}$$

$$\begin{aligned} r^2_{|d|}(t,x,y) = {} & (-8 + 60xy + 27d^2y^2 - 27x^2y^2)t^2 \\ & +(20x + 18d^2y - 93x^2y - 27y^3)t \\ & +(12d^2 - 12x^2 - 45d^2xy + 45x^3y - 12y^2 + 54xy^3), \end{aligned} \tag{75}$$

$$\begin{aligned} r^3_{|d|}(t,x,y) = {} & -9y^3(16 + 108d^4 + 1134d^2x^2 + 633x^4 - 336xy \\ & +810d^4xy - 1620d^2x^3y + 810x^5y - 378d^2y^2 \\ & +648x^2y^2 - 486d^2xy^3 + 486x^3y^3 + 243y^4)t \\ & +9y^3(16x - 540d^4x - 45d^2x^3 + 585x^5 + 108d^2y \\ & -243d^6y - 288x^2y + 729d^4x^2y - 729d^2x^4y \\ & +243x^6y + 27d^2xy^2 + 1323x^3y^2 - 108y^3 \\ & +243d^4y^3 - 486d^2x^2y^3 + 243x^4y^3 + 486xy^4). \end{aligned} \tag{76}$$

Thus from Lemma 6, if we put

$$\begin{aligned} t^1_{|d|}(x,y) = {} & (16x - 540d^4x - 45d^2x^3 + 585x^5 + 108d^2y \\ & -243d^6y - 288x^2y + 729d^4x^2y - 729d^2x^4y \\ & +243x^6y + 27d^2xy^2 + 1323x^3y^2 - 108y^3 \\ & +243d^4y^3 - 486d^2x^2y^3 + 243x^4y^3 + 486xy^4)/ \\ & (16 + 108d^4 + 1134d^2x^2 + 633x^4 - 336xy \\ & +810d^4xy - 1620d^2x^3y + 810x^5y - 378d^2y^2 \\ & +648x^2y^2 - 486d^2xy^3 + 486x^3y^3 + 243y^4), \end{aligned} \tag{77}$$

we have a rational transformation

$$\tilde{x} = t^1_{|d|}(x, y), \quad \tilde{y} = -d \frac{U(t^1_{|d|}(x, y))}{y - \dfrac{Y(t^1_{|d|}(x, y))}{W(t^1_{|d|}(x, y))}} \tag{78}$$

from the offset $f_{|d|}(x, y) = 0$ into the curve $\tilde{y}^2 = 1 + 9\tilde{x}^4$.

From Example 4 and Theorem 7, the algebraic offset $f_{|d|}(x, y) = 0$ to the cubic $r(t) = (t, t^3)$ is birationally equivalent to the curve $\tilde{y}^2 = 1 + 9\tilde{x}^4$ via the rational transformations (46) and (78).

We need the following elementary lemma.

Lemma 9. *Let $w(x) \in \mathbb{R}[x]$ be a polynomial with positive leading coefficient. Then the curve $y^2 - w(x) = 0$ is reducible if and only if there exists a polynomial $w_*(x) \in \mathbb{R}[x]$ such that $w(x) = w_*^2(x)$.*

Proof. If $w(x) = w_*^2(x)$, then we have $y^2 - w(x) = (y + w_*(x))(y - w_*(x))$, and hence the curve $y^2 - w(x) = 0$ is reducible.

Conversely, if the curve $y^2 - w(x) = 0$ is reducible, then there exist $b(x), c(x) \in \mathbb{C}[x]$ such that

$$y^2 - w(x) = (y + b(x))(y - c(x)). \tag{79}$$

By comparing the coefficients with respect to y, we see that $b(x) = c(x)$ and $w(x) = b(x)c(x)$. Thus, if we put $w_*(x) = b(x)$, we have $w(x) = w_*^2(x)$. We shall prove $w_*(x) \in \mathbb{R}[x]$. Since the leading coefficient of $w(x)$ is positive, the leading coefficient of $w_*(x)$ is a real number. Note that $w(\alpha) = 0$ implies $w_*(\alpha) = 0$. Thus, if all solutions of the equation $w(x) = 0$ are real numbers, then we have $w_*(x) \in \mathbb{R}[x]$. Suppose a solution α of $w(x) = 0$ is not real. Then the complex conjugate $\bar{\alpha}$ of α is also a solution of $w(x) = 0$, and hence $w_*(x)$ is divisible by $(x - \alpha)(x - \bar{\alpha}) \in \mathbb{R}[x]$. Thus, even if $w(x) = 0$ has complex solutions, we conclude that $w_*(x) \in \mathbb{R}[x]$. □

From Theorem 7 and Lemma 9 we have the following.

Theorem 10. *Let $f_{|d|}(x, y) = 0$ be an algebraic offset with no special components. Then $f_{|d|}(x, y) = 0$ is reducible if and only if there exists a polynomial $w_*(\tilde{x}) \in \mathbb{R}[\tilde{x}]$ such that $U^2(\tilde{x}) + V^2(\tilde{x}) = w_*^2(\tilde{x})$. In particular, if $f_{|d|}(x, y) = 0$ is reducible, it consists of two rational curves.*

We shall apply Theorem 10 to the algebraic offsets to an ellipse and an asteroid.

Example 11. We consider the algebraic offset $f_{|d|}(x,y)=0$ to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a,b>0) \tag{80}$$

(See Figure 5). The curve (80) is properly parametrized by $X(t)=a(1-t^2), Y(t)=2bt, W(t)=1+t^2$. We observe that $U(t)=-4at, V(t)=2b(1-t^2)$, and the offset $f_{|d|}(x,y)=0$ is birationally equivalent to the curve

$$\tilde{y}^2=16a^2\tilde{x}^2+4b^2(1-\tilde{x}^2)^2. \tag{81}$$

Then the discriminant D of the polynomial $16a^2\tilde{x}^2+4b^2(1-\tilde{x}^2)^2$ with respect to $\tilde{x}$ is

$$D=67108864a^4b^6(a+b)^2(a-b)^2, \tag{82}$$

and the polynomial $16a^2\tilde{x}^2+4b^2(1-\tilde{x}^2)^2$ has a multiple root if and only if $a=b$. Thus, from Theorem 10, we get the following results:

1. Case of $a\neq b$.
   The offset $f_{|d|}(x,y)=0$ is irreducible.

2. Case of $a=b$.
   The curve (80) is a circle of radius a. The offset $f_{|d|}(x,y)=0$ is birationally equivalent to the curve $\tilde{y}^2=\left(2a(1+\tilde{x}^2)\right)^2$, and hence the offset $f_{|d|}(x,y)=0$ is reducible. Furthermore, the offset $f_{|d|}(x,y)=0$ consists of two rational curves, as we have seen in Example 2.

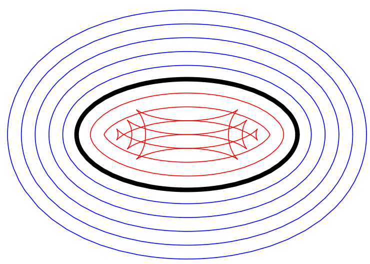

Figure 5: The ellipse $\frac{1}{4}x^2+y^2=1$ with its offsets at distance $\pm\frac{1}{4},\pm\frac{1}{2},\pm\frac{3}{4},\pm 1,\pm\frac{5}{4}$

Example 12. We consider the algebraic offset $f_{|d|}(x,y)=0$ to the asteroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=1 \tag{83}$$

(See Figure 6). The curve (83) is properly prametrized by $X(t)=(1-t^2)^3, Y(t)=8t^3, W(t)=(1+t^2)^3$. Noting that $U(t)=-12t(t^4-1)^2, V(t)=24t^2(t^4-1)(t^2+1)$, we see that the offset $f_{|d|}(x,y)=0$ is birationally equivalent to the curve

$$\tilde{y}^2=\left(12\tilde{x}^2(\tilde{x}-1)(\tilde{x}+1)(\tilde{x}^2+1)^3\right)^2. \tag{84}$$

Thus the offset $f_{|d|}(x,y)=0$ is reducible and consists of two rational curves. In fact, if we compute the polynomial $f_{|d|}(x,y)$, we obtain

$$\begin{aligned} f_{|d|}(x,y) \;=\;& 65536(1-8d^2+16d^4-3x^2-20d^2x^2+3x^4+d^2x^4-x^6\\ &-36dxy+16d^3xy-18dx^3y-3y^2-20d^2y^2-21x^2y^2\\ &+2d^2x^2y^2-3x^4y^2-18dxy^3+3y^4+d^2y^4-3x^2y^4-y^6)\\ &(1-8d^2+16d^4-3x^2-20d^2x^2+3x^4+d^2x^4-x^6\\ &+36dxy-16d^3xy+18dx^3y-3y^2-20d^2y^2-21x^2y^2\\ &+2d^2x^2y^2-3x^4y^2+18dxy^3+3y^4+d^2y^4-3x^2y^4-y^6). \end{aligned} \tag{85}$$

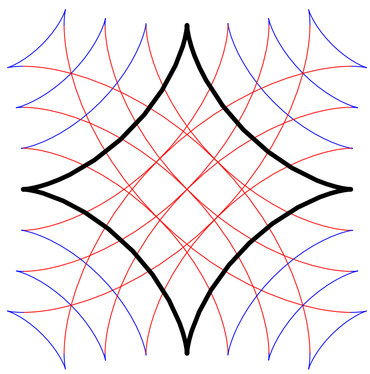

Figure 6: The asteroid (83) with its offsets at distance $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}$

3 Hyperellipticity

In this section, we study the genus of offsets to rational plane curves. Let $w(x)$ be a polynomial of degree $2g+1$ or $2g+2$ with real coefficients and distinct roots. We consider an irreducible plane curve C which is birationally equivalent to the curve $y^2 = w(x)$. Note that the curve $y^2 = w(x)$ has genus g. Since the genus is a birational invariant, the curve C has genus g. If $g = 0$, then $y^2 = w(x)$ is a conic and C is a rational curve. If $g = 1$, C is called an elliptic curve and if $g \geq 2$, C is called a hyperelliptic curve. It is known that curves of genus $0, 1$ and 2 are always rational, elliptic and hyperelliptic, respectively, but curves of genus greater than 2 are not always hyperelliptic.

Let $f(x)$ be a polynomial with real coefficients and let

$$f(x) = c(x-a_1)^{2p_1+1} \cdots (x-a_m)^{2p_m+1}(x-b_1)^{2q_1} \cdots (x-b_n)^{2q_n} \tag{86}$$

be the decomposition of $f(x)$ into linear factors with distinct complex numbers $a_1, \cdots, a_m$, $b_1, \cdots, b_n$. Then we see that the curves $y^2 = f(x)$ and $Y^2 = c(X - a_1) \cdots (X - a_m)$ are birationally equivalent via the rational transformations

$$X = x, \quad Y = \frac{y}{(x-a_1)^{p_1} \cdots (x-a_m)^{p_m}(x-b_1)^{q_1} \cdots (x-b_n)^{q_n}}, \tag{87}$$

$$x = X, \quad y = (x-a_1)^{p_1} \cdots (x-a_m)^{p_m}(x-b_1)^{q_1} \cdots (x-b_n)^{q_n} Y, \tag{88}$$

and thus the curve $y^2 = f(x)$ has genus $[(m-1)/2]$, where the symbol $[\alpha]$ denotes the largest integer $\leq \alpha$. Note that all coefficients appearing in the rational transformations (87), (88) and of the polynomial $c(X-a_1)\cdots(X-a_m)$ are real numbers since $f(x) \in \mathbb{R}[x]$.

Noting that the degree of the polynomial $U^2(\tilde{x}) + V^2(\tilde{x})$ is even, we have the following characterization of the offsets to rational curves from Theorem 7 and Lemma 9.

Theorem 13. (Hyperellipticity) *Let $f_{|d|}(x, y) = 0$ be an irreducible algebraic offset to a properly parametrized rational plane curve $r(t) = (X(t)/W(t), Y(t)/W(t))$ with no special components, and let*

$$U^2(\tilde{x}) + V^2(\tilde{x}) = c(\tilde{x}-a_1)^{2p_1+1} \cdots (\tilde{x}-a_m)^{2p_m+1}(\tilde{x}-b_1)^{2q_1} \cdots (\tilde{x}-b_n)^{2q_n} \tag{89}$$

be the decomposition of the polynomial $U^2(\tilde{x}) + V^2(\tilde{x})$ into linear factors with distinct complex numbers $a_1, \cdots, a_m, b_1, \cdots, b_n$. Then m is a positive even integer and

1) *if $m = 2$, the offset $f_{|d|}(x, y) = 0$ is a rational curve,*

2) *if* $m = 4$*, the offset* $f_{|d|}(x, y) = 0$ *is an elliptic curve,*

3) *if* $m \geq 6$*, the offset* $f_{|d|}(x, y) = 0$ *is a hyperelliptic curve of genus* $(m-2)/2$*.*

In particular, the genus of the offset $f_{|d|}(x, y) = 0$ *with no special components is independent of the distance* d*.*

We shall compute genus of some algebraic offsets by Theorem 13

Example 14. We compute genus of the algebraic offset $f_{|d|}(x, y) = 0$ to the properly parametrized curve

$$r(t) = \left(-2t^3 + 2t, 16t^6 - 24t^4 + 8t^2\right) \tag{90}$$

(See Figure 7). In this case, we have $X(t) = -2t^3 + 2t, Y(t) = 16t^6 - 24t^4 + 8t^2, W(t) = 1$ and $U(t) = -6t^2 + 2, V(t) = 96t^5 - 96t^3 + 16t$. Thus the offset $f_{|d|}(x, y) = 0$ is birationally equivalent to the curve

$$\tilde{y}^2 = 4(2304\tilde{x}^{10} - 4608\tilde{x}^8 + 3072\tilde{x}^6 - 759\tilde{x}^4 + 58\tilde{x}^2 + 1). \tag{91}$$

Noting that the discriminant of the polynomial $2304\tilde{x}^{10} - 4608\tilde{x}^8 + 3072\tilde{x}^6 - 759\tilde{x}^4 + 58\tilde{x}^2 + 1$ is not 0, we see, from Theorem 13, that the offset $f_{|d|}(x, y) = 0$ is a hyperelliptic curve of genus 4.

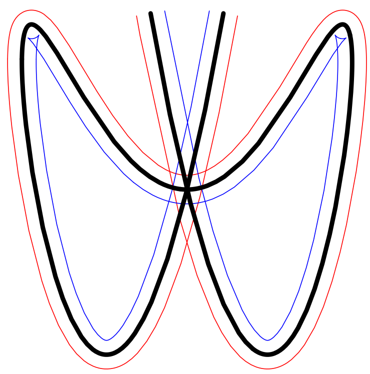

Figure 7: The curve (90) in Example 14 with its offsets at distance $\pm\frac{1}{15}$

Example 15. We compute genus of the algebraic offset $f_{|d|}(x, y) = 0$ to the folium of Descartes

$$x^3 + y^3 - 3xy = 0 \tag{92}$$

(See Figure 8). The curve (92) is properly parametrized by $X(t) = 3t, Y(t) = 3t^2, W(t) = 1 + t^3$ and we see that $U(t) = -6t^3 + 3, V(t) = -3t^4 + 6t$. Thus we observe that the offset $f_{|d|}(x, y) = 0$ is birationally equivalent to the irreducible curve

$$\tilde{y}^2 = 9(\tilde{x}^8 + 4\tilde{x}^6 - 4\tilde{x}^5 - 4\tilde{x}^3 + 4\tilde{x}^2 + 1). \tag{93}$$

Since the discriminant of the polynomial $\tilde{x}^8 + 4\tilde{x}^6 - 4\tilde{x}^5 - 4\tilde{x}^3 + 4\tilde{x}^2 + 1$ is not 0, we observe, form Theorem 13, that the offset $f_{|d|}(x, y) = 0$ is a hyperelliptic curve of genus 3.

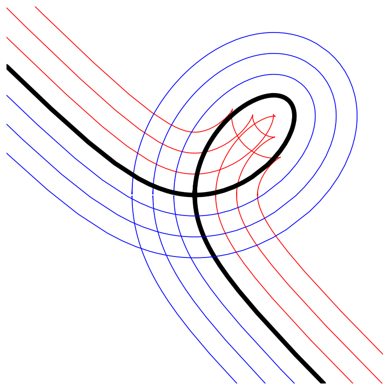

Figure 8: The folium of Descartes (92) with its offsets at distance $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1$

Example 16. We consider the offset $f_{|d|}(x, y) = 0$ to the lemniscate of Bernoulli

$$(x^2 + y^2)^2 - (x^2 - y^2) = 0 \tag{94}$$

(See Figure 9). The curve (94) is properly parametrized by $X(t) = -t^4+1, Y(t) = -2t^3+2t, W(t) = t^4+6t^2+1$, and hence we have $U(t) = -4t(3t^4+2t^2+3), V(t) = 2(t^6 - 9t^4 - 9t^2 + 1)$. Thus the offset $f_{|d|}(x, y) = 0$ is birationally equivalent to the curve

$$\tilde{y}^2 = 4(\tilde{x}^4 + 6\tilde{x}^2 + 1)^3, \tag{95}$$

and we see, from Theorem 13, that the offset $f_{|d|}(x, y) = 0$ is an elliptic curve.

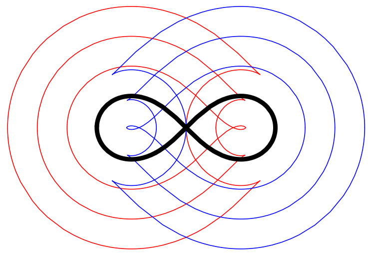

Figure 9: The lemniscate of Bernoulli (94) with its offsets at distance $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1$

From Theorem 13, we have the following propositions.

Proposition 17. *Let n and m be relatively prime natural numbers. Then the algebraic offset to $y^n = x^{n+m}$ has genus $m-1$. In particular, for $m \geq 2$, the algebraic offset to $y = x^m$ has genus $m-2$.*

Proof. The curve $y^n = x^{n+m}$ is properly parametrized by $X(t) = t^n, Y(t) = t^{n+m}, W(t) = 1$. Thus we have $U(t) = nt^{n-1}, V(t) = (n+m)t^{n+m-1}$, and hence the offset to $y^n = x^{n+m}$ is birationally equivalent to the curve

$$\tilde{y}^2 = n^2\tilde{x}^{2n-2} + (n+m)^2\tilde{x}^{2n+2m-2} = \tilde{x}^{2n-2}\left(n^2 + (n+m)^2\tilde{x}^{2m}\right). \tag{96}$$

Since the polynomial $n^2+(n+m)^2\tilde{x}^{2m}$ has no multiple roots, we get our claim. □

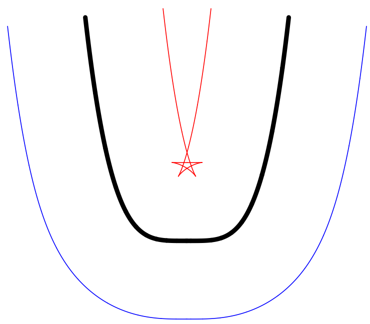

Figure 10: The quartic curve $y = x^4$ with its offsets at distance ± 1

Proposition 18. *Let n and $n+m$ be relatively prime natural numbers. Then the algebraic offset to $y^n = 1/x^{n+m}$ has genus $2n+m-1$. In particular, for $m \geq 1$, the algebraic offset to $y = 1/x^m$ has genus m.*

Proof. The curve $y^n = 1/x^{n+m}$ is properly parametrized by $X(t) = t^{2n+m}, Y(t) = 1, W(t) = t^{n+m}$. Thus we have $U(t) = nt^{3n+2m-1}, V(t) = -(n+m)t^{n+m-1}$, and hence the offset to $y^n = 1/x^{n+m}$ is birationally equivalent to the curve

$$\tilde{y}^2 = n^2\tilde{x}^{6n+4m-2} + (n+m)^2\tilde{x}^{2n+2m-2} = \tilde{x}^{2n+2m-2}\left(n^2\tilde{x}^{4n+2m} + (n+m)^2\right). \quad (97)$$

Since the polynomial $n^2\tilde{x}^{4n+2m} + (n+m)^2$ has no multiple roots, we get our claim. □

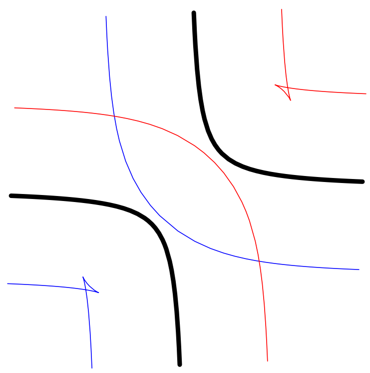

Figure 11: The curve $y = \dfrac{1}{x}$ with its offsets at distance $\pm\dfrac{5}{2}$

In particular,

Proposition 19. *Offsets to conics can be classified in the following.*

1) *The algebraic offsets to parabolas are rational curves.*

2) *The algebraic offsets to circles are reducible and consist of two circles.*

3) *The algebraic offsets to ellipses are elliptic curves.*

4) *The algebraic offsets to hyperbolas are elliptic curves.*

Proof. It is enough to prove the case of hyperbolas. We may assume that a hyperbola is given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a, b > 0). \tag{98}$$

Note that the hyperbola (98) is properly parametrized by $X(t) = a(1+t^2), Y(t) = 2bt, W(t) = 1 - t^2$. Thus we have $U(t) = 4at, V(t) = 2b(1 + t^2)$, and hence the algebraic offset to the hyperbola (98) is birationally equivalent to the curve

$$\tilde{y}^2 = 16a^2\tilde{x}^2 + 4b^2(1 + \tilde{x}^2)^2. \tag{99}$$

Since the discriminant D of the polynomial $16a^2\tilde{x}^2 + 4b^2(1 + \tilde{x}^2)^2$ with respect to $\tilde{x}$ is

$$D = 67108864a^4b^6(a^2 + b^2)^2, \tag{100}$$

the polynomial $16a^2\tilde{x}^2 + 4b^2(1+\tilde{x}^2)^2$ has no multiple roots, and we get our claim. □

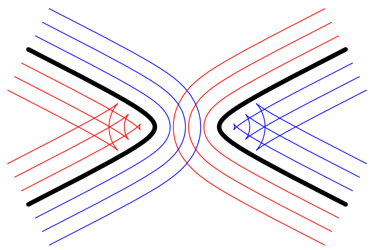

Figure 12: The hyperbola $\frac{1}{4}x^2 - y^2 = 1$ with its offsets at distance $\pm\frac{19}{10}, \pm\frac{19}{5}, \pm\frac{57}{10}$

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