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Random Ergodic Theorem with Finite Possible States

By Hirotada Anzai

The purpose of this note is to give a special model of random ergodic theorem.¹

Let $X$ be the infinite direct product measure space:

$$X = \prod_{k=-\infty}^{\infty} H_k, \quad x \in X, \quad x = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots), \quad x_k \in H_k,$$

$$k = 0, \pm 1, \pm 2, \ldots.$$

We assume that each component space $H_k$ consists of $p$ points, which are described by $p$ figures; 1, 2, ..., $p$, each having the same probability (measure) $1/p$. We denote the $k$-component $x_k$ of a point $x$ of $X$ by $\eta_k(x)$. The measure on $X$ is denoted by $m$. Let $\sigma$ be the shift transformation of $X$:

$$\eta_k (\sigma x) = \eta_{k+1} (x), \quad k = 0, \pm 1, \pm 2, \ldots.$$

It is well-known that $\sigma$ is an ergodic transformation of strongly mixing type. Let $\Omega$ be another probability field (i.e. measure space). In this note we restrict ourselves to the case in which $\Omega$ consists of $q$ points; $\Omega = (\omega_1, \omega_2, \ldots, \omega_q)$, each having the same a priori probability $1/q$.

Suppose that it is given a family $\Phi$ of permutations $T_1, T_2, \ldots, T_p$ of $\Omega$. Starting from any point $\omega_i$ of $\Omega$, we take up at random a point from $H_i$, if it is $x_i$, we operate $T_{x_i}$ to $\omega_i$, then $\omega_i$ is transferred to $T_{x_i} \omega_i$, at the second step we take up at random a point from $H_2$, if it is $x_2$, we operate $T_{x_2}$ to $T_{x_1} \omega_i$, then we arrive at $T_{x_2} T_{x_1} \omega_i$, and so on.

Continuing this process, the transition probability that $\omega_i$ is transferred to $\omega_2$ after the elapse of $n$ units of time is given by

$$m \left\{ x \mid \omega_2 = T_{\eta_n(x)} T_{\eta_{n-1}(x)} \ldots T_{\eta_1(x)} \omega_i \right\} :$$

We can represent any permutation $T$ of $\Omega$ in a matrix form of degree $q$; $T = (\tau_{ij})$, $1 \leq i, j \leq q$. The $i$-$j$ element $\tau_{ij}$ of $T$ is equal to 1 if $\omega_i = T \omega_j$, $\tau_{ij} = 0$ if $\omega_i = T \omega_j$.

Set

\[ T_0 = 1/p (T_1 + T_2 + \ldots + T_p). \]

\( T_0 \) is a Markov matrix. It is easy to verify that the \( i-j \) element \( \tau_{ij} \) of \( T_0 \) is equal to \( m \left\{ x \mid T_\eta_{\eta_i(x)} \ldots T_\eta_{\eta_j(x)} \omega_j = \omega_i \right\} \), that is the transition probability that \( \omega_j \) is transferred to \( \omega_i \) after the elapse of \( n \) units of time. It is a well-known fact that if for some integer \( n \), \( \tau_{ij}^{(n)} > 0 \) for all \( i, j \), then \( \lim_{n \to \infty} T_0 = Q \) exists and all \( i-j \) elements of \( Q \) are equal to \( 1/q \). In this case the family \( \Phi \) is said to be strongly mixing.

Let \( \Xi \) be the direct product measure space of \( X \) and \( \Omega \):
\[ \Xi = X \times \Omega, \quad \xi \in \Xi, \quad \xi = (x, \omega), \quad x \in X, \quad \omega \in \Omega. \]

Let \( \varphi \) be the measure preserving transformation of \( \Xi \) defined by
\[ \varphi(x, \omega) = (\sigma x, T_{\eta_i(x)} \omega). \]

**Theorem 1.** \( \varphi \) is strongly mixing if and only if \( \Phi \) is strongly mixing.

**Proof:** Define the functions \( f_i(\omega), i = 1, 2, \ldots, q \), as follows.
\[ f_i(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_i, \\ 0 & \text{if } \omega \neq \omega_i. \end{cases} \]

Set
\[ F(x, \omega) = f_1(\omega), \quad G(x, \omega) = f_j(\omega). \]

Then we have
\[ F(\varphi^n(x, \omega)) = f_1(T_{\eta_{n}(x)} T_{\eta_{n-1}(x)} \ldots T_{\eta_1(x)} \omega). \]

Assume that \( \varphi \) is strongly mixing, then we have
\[ \lim_{n \to \infty} \int F(\varphi^n \xi) G(\xi) d\xi = \int F(\xi) d\xi \int G(\xi) d\xi = \int f_1(\omega) d\omega \int f_j(\omega) d\omega = 1/q \cdot 1/q = 1/q^2. \]

The integral of the left hand side of the above equality is
\[ \int F(\varphi^n \xi) G(\xi) d\xi = \int \left\{ \int f_1(T_{\eta_{n}(x)} T_{\eta_{n-1}(x)} \ldots T_{\eta_1(x)} \omega) dx \right\} f_j(\omega) d\omega. \]

Therefore we obtain the equality \( \lim_{n \to \infty} 1/q \tau_{ij}^{(n)} = 1/q^2 \), that is, \( \lim_{n \to \infty} \tau_{ij}^{(n)} = 1/q \). This shows that \( \Phi \) is strongly mixing.

Conversely assume that \( \Phi \) is strongly mixing.

2) In \( \Omega \), each point has the positive measure \( 1/q \). Following the usual custom we should replace the integral notation \( \int d\omega \) by the summation notation \( \sum \). But, for the sake of simplicity, we use the integral notation.
Set
\[ \xi_j(\eta) = \exp(2\pi ij\eta/p),\ j = 0, 1, 2, \ldots, p-1, \ \eta \in \Pi. \]
Obviously \( \{\xi_j(\eta)\},\ j = 0, 1, 2, \ldots, p-1, \) are the complete orthonormal system of \( L^2(\Pi) \) for any \( k. \)

Hence \( \{\xi_{k_1}(\eta_{i_1}(x)) \xi_{k_2}(\eta_{i_2}(x)) \ldots \xi_{k_s}(\eta_{i_s}(x))\},\ 0 \leq s < \infty, -\infty < i_1, \ldots, i_s < \infty, 0 \leq k_1, \ldots, k_s \leq p-1, \) are the complete orthonormal system of \( L^2(X). \)

We denote this system by \( \Psi. \)

In order to prove that \( \varphi \) is strongly mixing, it is sufficient to show that

\[
\lim_{n \to \infty} \int g(\sigma^n x) f_1(T_{\eta_n(x)} \ldots T_{\eta_1(x)} \omega) h(\omega) f_0(\omega) d\omega dx = 1/q \int g(x) dx \int h(x) dx
\]
holds for any \( g(x), h(x) \in \Psi. \)

If \( g(x) = 1, \) and \( h(x) = 1, \) then the integral of the left hand side of (1) is equal to \( 1/q \cdot \gamma_j^2, \) which tends to \( 1/q^2 \) as \( n \to \infty, \) therefore the equality (1) holds. In general if

\[
g(x) = \xi_{k_1}(\eta_{i_1}(x)) \ldots \xi_{k_s}(\eta_{i_s}(x))
\]

\[
h(x) = \xi_{i_1}(\eta_{j_1}(x)) \ldots \xi_{i_r}(\eta_{j_r}(x))
\]

then the integral of the left hand side of (1) is

\[
1/q \int \xi_{k_1}(\eta_{i_1+n}(x)) \ldots \xi_{k_s}(\eta_{i_s+n}(x)) f_1(T_{\eta_n(x)} \ldots T_{\eta_1(x)} \omega) h(\omega) f_0(\omega) d\omega dx.
\]

Suppose \( i_0 > i_1 > \ldots > i_s \) and \( j_1 > j_2 > \ldots > j_r. \) There is no loss of generality in assuming that \( i_0 < 0, j_1 > 0, j_r < 0. \) We may consider \( n \) to be sufficiently large that \( i_0 + n > j_1 > 0. \)

Set
\[
E_{j_1, j_1-1, \ldots, j_r+1, j_r} = \{x | \eta_{j_1}(x) = \eta_{j_1}, \eta_{j_1-1}(x) = \eta_{j_1-1}, \ldots, \eta_{j_r+1}(x) = \eta_{j_r+1}, \eta_{j_r}(x) = \eta_{j_r}\}.
\]

Then the sets \( E_{j_1, j_1-1, \ldots, j_r+1, j_r} \) and \( E_{i_1+n, i_1-1+n, \ldots, i_1+n} \) are mutually stochastically independent for any

\[ 1 \leq \eta_{j_1}, \ldots, \eta_{j_r}, \eta_{i_1}, \ldots, \eta_{i_s} \leq p. \]

Therefore we have
\[
m \left( E_{j_1, j_1-1, \ldots, j_r+1, j_r} \cap E_{i_1+n, i_1-1+n, \ldots, i_1+n} \right) = m \left( E_{i_1+n, i_1-1+n, \ldots, i_1+n} \right) m \left( E_{j_1, j_1-1, \ldots, j_r} \right)
\]
The value of the integral (2) on the set
\[ E_{\eta_{i_1}, \eta_{i_2}^{-1}} \cap E_{\eta_{j_1}, \eta_{j_2}^{-1}} \cap \ldots \cap E_{\eta_{i_n}, \eta_{j_n}^{-1}} \]
is
\[ \frac{1}{q^2} \sum_{i} \sum_{j} \frac{m(E_{\eta_{i_1}^{-1} \eta_{i_2}^{-1} \ldots \eta_{i_n}^{-1}}, \ldots, \eta_{j_n}^{-1}) m(E_{\eta_{j_1}, \eta_{j_2}^{-1}}, \ldots, \eta_{j_n}^{-1})}{m(E_{\eta_{i_1}^{-1} \eta_{i_2}^{-1} \ldots \eta_{i_n}^{-1}, \ldots, \eta_{j_n}^{-1})} \]
\[ \xi_{\eta_{i_1}}(\eta_{j_1}) \ldots \xi_{\eta_{i_n}}(\eta_{j_n}) \xi_{\eta_{j_1}}(\eta_{j_2}) \ldots \xi_{\eta_{j_n}}(\eta_{j_n}) \]
\[ \cdot \int f_x(T_{\eta_0} T_{\eta_1} \ldots T_{\eta_{i_n}} T_{\eta_{i_n-1} + n} \ldots T_{\eta_{j_1}} T_{\eta_{j_2}} \ldots T_{\eta_0} \omega_n) \, dx \]
The value of the integral in (3) indicates the transition probability
that the point \( T_{\eta_1} \ldots T_{\eta_{j_2}} \omega = \omega \)
after the elapse of \( n-1+i_2-\ldots-j_n \) units of time, which tends to \( 1/q \) as \( n \to \infty \) by our assumption. Therefore the left hand side of (1) exists
and is equal to
\[ \frac{1}{q^2} \sum_{i} \sum_{j} \frac{m(E_{\eta_{i_1}^{-1} \eta_{i_2}^{-1} \ldots \eta_{i_n}^{-1}}, \ldots, \eta_{j_n}^{-1}) m(E_{\eta_{j_1}, \eta_{j_2}^{-1}}, \ldots, \eta_{j_n}^{-1})}{m(E_{\eta_{i_1}^{-1} \eta_{i_2}^{-1} \ldots \eta_{i_n}^{-1}, \ldots, \eta_{j_n}^{-1})} \]
\[ \xi_{\eta_{i_1}}(\eta_{j_1}) \ldots \xi_{\eta_{i_n}}(\eta_{j_n}) \xi_{\eta_{j_1}}(\eta_{j_2}) \ldots \xi_{\eta_{j_n}}(\eta_{j_n}) \]
\[ = \frac{1}{q^2} \int g(\sigma^n x)dx \int h(x)dx = \frac{1}{q^2} \int g(x)dx \int h(x)dx . \]
This is the required result.

**THEOREM 2.** \( \phi \) is ergodic if and only if \( \Omega \) contains no \( \Phi \)-invariant
subset except \( \Omega \) and the empty set.

**Proof:** If \( \Omega \) contains a non-trivial \( \Phi \)-invariant subset \( A \), then
\( X \times A \) is a non-trivial \( \phi \)-invariant subset of \( \Xi \), therefore \( \phi \) is not
ergodic.

Conversely assume that \( F(x, \omega) \) is a \( \phi \)-invariant function, which
is not a constant:
\[ F(x, \omega) = F(\sigma x, T_{\eta_j(x)} \omega) . \]

In order to conclude the existence of a non-trivial \( \Phi \)-invariant subset of \( \Omega \), it is sufficient to show that \( F(x, \omega) \) is a function depending
only on the variable \( \omega \). If \( F(x, \omega) \) depends only on the variable \( x \), then
we may conclude from (4) immediately that \( F(x, \omega) \) is a constant.
Let \( \delta \) be the least positive value of
\[ \int \left| F(x, \omega_i) - F(x, \omega_j) \right|^2 \, dx, \quad 1 \leq i, j \leq q . \]

Let \( h \) be the order of the permutation group \( [\Phi] \) of \( \Omega \) generated by
\( \Phi \), \( h \) is at most \( q! \). Let \( \varepsilon \) be a positive number such that
\[ 6h(1+9ph) \varepsilon < \delta . \]
By the definition of $L^2(X)$, it is easy to conclude the existence of a function $G(x, \omega)$ and a positive number $n$ such that

\[(6) \quad \int |F(x, \omega) - G(x, \omega)|^2 \, dx < \varepsilon \quad \text{for all } \omega \in \Omega,\]

\[(7) \quad G(\sigma^k x, \omega) \text{ does not depend on the value of } \eta_k(x) \text{ for } |k| > n,\]

\[(8) \quad \int |F(x, \omega) - F(\sigma^k x, \omega)|^2 \, dx < \varepsilon \quad \text{for all } \omega \in \Omega,\]

where $\sigma$ is any measure preserving transformation of $X$ satisfying the equalities $\eta_k(x) = \eta_k(\sigma x)$ for all $|k| \leq n$.

Let $S_1$ and $S_2$ be elements of $[\Phi]$.

Set

\[A(S_1) = \{x | T_{\eta_1(x)}T_{\eta_{n-1}(x)} \cdots T_{\eta_1(x)} = S_1\},\]

\[A'(S_2) = \{x | T_{\eta_{n+1}(x)}T_{\eta_{2n}(x)} \cdots T_{\eta_{n+2}(x)} = S_2\} = \sigma^{n+1}(A(S_2))\]

\[A(S_1, S_2) = A(S_1) \cap A'(S_2), \quad B_\eta = \{x | \eta_{n+1}(x) = \eta \}.\]

Let $\eta'$ and $\eta''$ be any mutually different integers between 1 and $p$.

Let $\nu$ be a measure preserving transformation of $X$ such that

\[V \{x | \eta_{n+1}(x) = \eta' \} = \{x | \eta_{n+1}(x) = \eta'' \},\]

\[V \{x | \eta_{n+1}(x) = \eta'' \} = \{x | \eta_{n+1}(x) = \eta' \},\]

and $\eta_k(x) = \eta_k(\sigma x)$ for all $k \neq n+1$.

$Vx \in A(S_1, S_2)$ if and only if $x \in A(S_1, S_2)$.

From (4) we obtain

\[(9) \quad F(x, \omega) = F(\sigma^{2n+1} x, T_{\eta_{2n+1}(x)} \cdots T_{\eta_{n+1}(x)} T_{\eta_n(x)} \cdots T_{\eta_1(x)} \omega).\]

If $x \in A(S_1, S_2)$, then from (9) we have

\[(10) \quad F(x, \omega) = F(\sigma^{2n+1} x, S_2 T_{\eta_{n+1}(x)} S_1 \omega)\]

If $x \in A(S_1, S_2) \cap B_\eta'$, then from (10) we have

\[(11) \quad F(Vx, \omega) = F(\sigma^{2n+1} Vx, S_2 T_{\eta_0 S_1} \omega).\]

From (11) and (8) we have

\[(12) \quad \int_{A(S_1, S_2) \cap B_\eta'} \left| F(\sigma^{2n+1} x, S_2 T_{\eta_0 S_1} \omega) - F(\sigma^{2n+1} Vx, S_2 T_{\eta_0 S_1} \omega) \right|^2 \, dx \leq 2 \int_{A(S_1, S_2) \cap B_\eta'} \left| F(\sigma^{2n+1} x, S_2 T_{\eta_0 S_1} \omega) - F(\sigma^{2n+1} Vx, S_2 T_{\eta_0 S_1} \omega) \right|^2 \, dx\]
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+ 2 \int_{\mathcal{A}(S_1, S_2) \cap B_{\eta'}} |G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx

\leq 2 \int_{\mathcal{A}(S_1, S_2) \cap B_{\eta'}} |F(x, \omega) - F(Vx, \omega)|^2 \, dx + 2 \varepsilon < 4 \varepsilon .

From (6) and (12) we have

(13) \int_{\mathcal{A}(S_1, S_2) \cap B_{\eta'}} |G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx

\leq 3 \int_{\mathcal{A}(S_1, S_2) \cap B_{\eta'}} |F(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - F(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx

+ 3 \int_{\mathcal{A}(S_1, S_2) \cap B_{\eta'}} |F(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - F(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx

\leq 3 (\varepsilon + 4 \varepsilon + \varepsilon) = 18 \varepsilon .

The set $B_{\eta'}$ is stochastically independent of the set $A(S_1, S_2)$ and of the functions appearing in the left hand side of (13), therefore the left hand side of (13) is equal to

(14) \frac{1}{\rho} m(B_{\eta'}) \int_{\mathcal{A}(S_1, S_2)} |G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx

= \frac{1}{\rho} \int_{\mathcal{A}(S_1) \cap A'(S_2)} |G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx .

The set $A(S_1)$ is stochastically independent of the set $A'(S_2)$ and of the functions in (14), therefore (14) is equal to

(15) \frac{1}{\rho} m(A(S_1)) \int_{\mathcal{A}'(S_2)} |G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx .

Let $S_1$ be an element of $[\Phi]$ such that $m(A(S_1)) \geq 1/\rho$, then we have from the inequality (13) and (15),

(16) \int_{\mathcal{A}'(S_2)} |G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - G(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx \leq 18 \rho^2 \varepsilon .

From (6) and (16) we have

(17) \int_{\mathcal{A}'(S_2)} |F(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega) - F(\sigma^{2n+1} x, S_2 T \eta_s S_1 \omega)|^2 \, dx

\leq 3 (\varepsilon + 18 \rho^2 \varepsilon + \varepsilon) = 6 (1 + 9 \rho^2 \varepsilon) \varepsilon .
Summing up (17) over all $S_2 \in [\Phi]$, we obtain

\begin{equation}
\int |F(\sigma^{n+1} x, T_\eta_{n+1}(x) \ldots T_\eta_{n+2}(x) T_\eta S_1 \omega) - F(\sigma^{n+1} x, T_\eta_{n+1}(x) \ldots T_\eta_{n+2}(x) T_\eta^n S_1 \omega)|^2 \, dx < 6h (1 + 9p\delta) \epsilon .
\end{equation}

Since $F(\sigma^{n+1} x, T_\eta_{n+1}(x) \ldots T_\eta_{n+2}(x) \omega) = F(\sigma^n (\sigma^{n+1} x), T_\eta_{n+1}(x) \ldots T_\eta_{n+2}(x) \omega)$, by replacing the variable $\sigma^{n+1} x$ in the right hand side of (18) by $x$, and by making use of (5), we obtain

\begin{equation}
\int |F(\sigma^n x, T_\eta_n(x) T_\eta_{n-1}(x) \ldots T_\eta_1(x) T_\eta^n S_1 \omega) - F(\sigma^n x, T_\eta_n(x) T_\eta_{n-1}(x) \ldots T_\eta_1(x) T_\eta^n S_1 \omega)|^2 \, dx < \delta .
\end{equation}

Since $F(x, \omega)$ is a $\phi$-invariant function,

$F(\sigma^n x, T_\eta_n(x) T_\eta_{n-1}(x) \ldots T_\eta_1(x) \omega) = F(x, \omega)$.

Therefore

\begin{equation}
\int |F(x, T_\eta S_1 \omega) - F(x, T_\eta^n S_1 \omega)|^2 \, dx < \delta .
\end{equation}

By the definition of $\delta$, we have

$F(x, T_\eta S_1 \omega) \equiv F(x, T_\eta^n S_1 \omega)$.

Since $\eta_n$ and $\eta^n$ are arbitrary, we obtain

$F(x, T_1 \omega) \equiv F(x, T_2 \omega) \equiv \ldots \equiv F(x, T_p \omega)$.

Therefore

$F(\sigma x, T_1 \omega) \equiv F(\sigma x, T_2 \omega) \equiv \ldots \equiv F(\sigma x, T_p \omega) \equiv F(x, \omega)$.

Let $r$ be the order of $T_1$, then we have

$F(\sigma^r x, T_1^r \omega) \equiv F(\sigma^r x, \omega) \equiv F(x, \omega)$.

From the ergodicity of $\sigma^r$, we can conclude that $F(x, \omega)$ depende only on the variable $\omega$. This completes the proof of the theorem.

The extension of our results to the general case in which each component space $H_\ast$ is the continuum of $[0, 1]$-interval with the usual Lebesgue measure and $\Phi$ is a family of measure preserving transformations of an arbitrary measure space $\Omega$ was made by S. Kakutani.

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