

Title	The unknotting numbers of 10_{-139} and 10_{-152} are 4
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Citation	Osaka Journal of Mathematics. 1998, 35(3), p. 539-546
Version Type	VoR
URL	https://doi.org/10.18910/9118
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THE UNKNOTTING NUMBERS OF 10_{139} AND 10_{152} ARE 4

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(Received May 6, 1997)

1. Introduction and the statement of main results

A *link* is a closed oriented 1-manifold smoothly embedded in the 3-sphere S^3 and a *knot* is a link with one component. Let K be a knot. The *unknotting number* $u(K)$ of K is the minimal number of crossing changes needed to create the trivial knot.

In the 1960s, Milnor [5] conjectured that the unknotting number of any algebraic knot would be equal to the genus of the Milnor fiber. In the 1990s, Kronheimer and Mrowka [2] [3] proved this conjecture using gauge theory. In particular, the unknotting number of (p, q) -torus knot is $((p-1)(q-1))/2$. In 1995, Auckly [1] gave an alternative proof of this conjecture for certain torus knots using Seiberg-Witten theory. In this paper we shall compute unknotting numbers of certain knots using Seiberg-Witten theory.

An argument explained in Auckly's lecture notes [1] is based on the following theorem, which is the so-called generalized adjunction formula.

Theorem 1.1 (Kronheimer-Mrowka [4], Morgan-Szabó-Taubes [7]). *Let X be a smooth closed oriented four-manifold with $\dim H_+^2(X, R) > 1$. If $F \hookrightarrow X$ is a smoothly embedded closed oriented surface of genus $g \geq 1$ and K is a basic class of X , then*

$$2g - 2 \geq K(F) + F \cdot F$$

where $A \cdot B$ denotes the intersection number of A and B .

By extending the argument in [1], we obtain the following Theorem.

Theorem 1.2. *Let K be an oriented knot in $S^3 \times \{1\}$ and $L_{m,n}$ the link in $S^3 \times \{0\}$ illustrated in Fig. 1. If there is a compact connected oriented surface \hat{F} in $S^3 \times [0, 1]$ such that $\partial \hat{F} = L_{m,n} \sqcup K$, then*

$$u(K) \geq (m-1)(n-1) - g(\hat{F})$$

*The author is partially supported by JSPS Research Fellowships for Young Scientists.

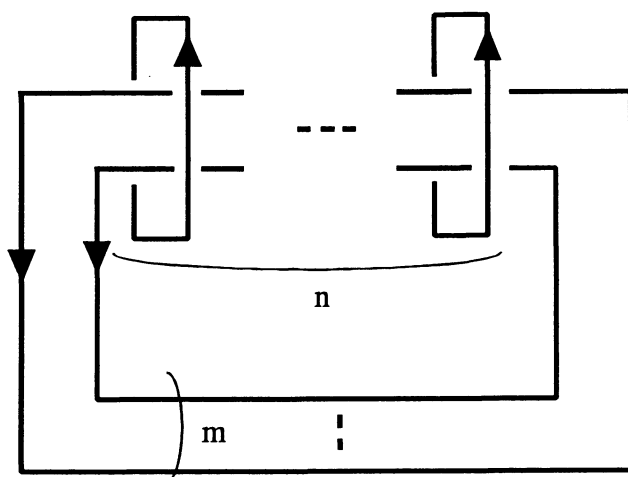


Fig. 1. Link $L_{m,n}$

where $q(\hat{F})$ denotes the genus of \hat{F} .

Theorem 1.3. (1) *The unknotting number of the knot 10_{139} is 4.*
(2) *The unknotting number of the knot 10_{152} is 4.*

The unknotting numbers of 10_{139} and 10_{152} had been known to be either 3 or 4, but they had not been determined. The author was suggested by Kawauchi that this problem might be solved by Theorem 1.2, and Theorem 1.3 is the answer to it.

ACKNOWLEDGEMENT. The author would like to thank Professor Akio Kawauchi and Professor Toshitake Kohno for their useful advice. She also would like to thank Professor Dave Auckly and Professor Lee Rudolph for helpful communications.

2. The proof of Theorem 1.2 and its corollary

Before proving Theorem 1.2, we shall review the definition of basic class.

Let X be a smooth closed oriented 4-manifold with $b_2^+(X) > 1$. The Seiberg-Witten invariant of X is an integer valued function which is defined on the set of $Spin^c$ -structures over X , (cf. for example [4] [6] [10]). This invariant is considered

as a map

$$n : H^2(X; Z)/\text{Torsion} \rightarrow Z,$$

(cf. [1]). If $n(K) \neq 0$, then K is called a *basic class*.

To prove Theorem 1.2, we will use the following lemma;

Lemma 2.1 (Auckly [1]). *0 is a basic class for T^4 .*

More generally, the work by Witten [10, pp. 786–789] implies that if X is a Kähler-Einstein manifold, the canonical class of X is a basic class. Lemma 2.1 can be also proved by this fact.

Auckly [1] presented a way of computing the unknotting number of $(2, 5)$ -torus knot applying Theorem 1.1 and Lemma 2.1 to a suitable surface in T^4 . Theorem 1.2 is an extension of result in [1], and the argument in proof of Theorem 1.2 is almost same as his argument.

Proof of Theorem 1.2. We consider $T^2 = [0, 1]^2 / \sim$ and $T^4 = T^2 \times T^2$, where \sim is the equivalent relation defined by $(0, t) \sim (1, t)$ and $(s, 0) \sim (s, 1)$. We will construct a surface F embedded in T^4 .

We define E and J by

$$E = \left(\bigcup_{k=1, \dots, m} \left\{ \left(\frac{k}{m+1}, \frac{k}{m+1} \right) \right\} \times T^2 \right) \cup \left(\bigcup_{k=1, \dots, n} T^2 \times \left\{ \left(\frac{k}{n+1}, \frac{k}{n+1} \right) \right\} \right)$$

$$J = \left[\frac{1}{m+2}, \frac{m+1}{m+2} \right]^2 \times \left[\frac{1}{n+2}, \frac{n+1}{n+2} \right]^2.$$

The four-disk J includes all self-intersections of E . Then $\partial(E - J) = E \cap \partial J \subset \partial J$ is equivalent to $L_{m,n} \subset S^3$. We suppose that K can be unknotted by u -crossing changes. Then there is a surface $F' \subset D^4$ with genus u such that $\partial F' = K \subset S^3 \cong \partial D^4$.

We can regard $\partial J \times [0, 1]$ as a collar of ∂J in J and $\partial J \times \{0\}$ as ∂J . We can identify $\partial J \times [0, 1]$ with $S^3 \times [0, 1]$ since $\partial J \cong S^3$. We can consider that \hat{F} lies in $\partial J \times [0, 1]$ and $\hat{F} \cap \partial J = E \cap \partial J$. We can identify the closure of $J - (\partial J \times [0, 1])$ with D^4 . We can consider that F' lies in the closure of $J - (\partial J \times [0, 1])$ and $\hat{F} \cap \partial J \times \{1\} = \partial F'$. We define a surface F by

$$F = (E - J) \cup \hat{F} \cup F'.$$

The genus of F is equal to $m + n + g(\hat{F}) + u$. The self-intersection number $F \cdot F$ is the same as $E \cdot E$, because F is homologous to E . So $F \cdot F = E \cdot E = 2mn$.

We can apply Theorem 1.1 and Lemma 2.1 for F . Then we obtain the inequality

$$2(m + n + g(\hat{F}) + u) - 2 \geq 0 + 2mn.$$

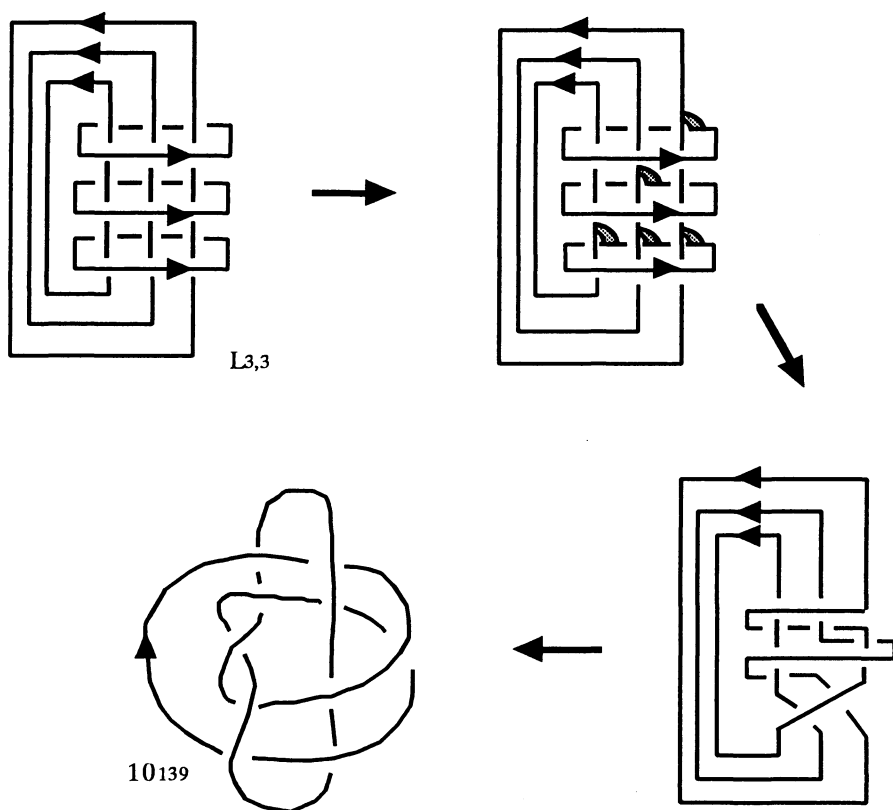


Fig. 2. Fusion procedure to obtain 10_{139} from $L_{3,3}$

Thus, we obtain the desired inequality $u \geq (m-1)(n-1) - g(\hat{F})$. \square

To prove Theorem 1.3, we show a corollary of Theorem 1.2. In order to state it, we shall review the definition of fusion procedure.

Let L be a μ -component oriented link. Let B_1, \dots, B_ν be mutually disjoint oriented bands in S^3 such that $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha'_i$, where $\alpha_1, \alpha'_1, \dots, \alpha_\nu, \alpha'_\nu$ are disjoint connected arcs. The closure of $L \cup \partial B_1 \cup \dots \cup \partial B_\nu - \alpha_1 \cup \alpha'_1 \cup \dots \cup \alpha_\nu \cup \alpha'_\nu$ is also the link. We will write it by L' .

DEFINITION. If L' has the orientation compatible with the orientation of $L - \bigcup_{i=1, \dots, \nu} \alpha_i \cup \alpha'_i$ and $\bigcup_{i=1, \dots, \nu} (\partial B_i - \alpha_i \cup \alpha'_i)$, L' is called the link obtained from L by the *band surgery* along the bands B_1, \dots, B_ν . Moreover if L' has $(\mu - \nu)$ -components, this transformation is called a *fusion*.

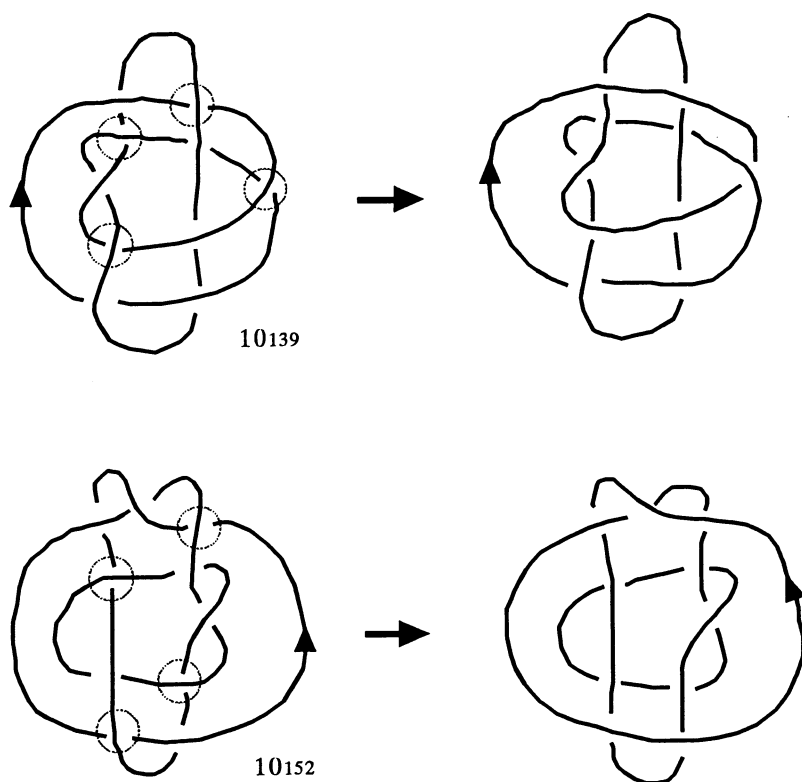


Fig. 3. Crossing changes

Corollary 2.2. *Let $L_{m,n}$ be the link illustrated in Fig. 1. If an oriented knot K in S^3 is obtained from $L_{m,n}$ by the fusion, then*

$$u(K) \geq (m-1)(n-1).$$

Proof of Corollary 2.2. To apply Theorem 1.2, we construct a suitable surface in $S^3 \times [0, 1]$. Let B_1, \dots, B_{m+n-1} be the surgery bands. Identifying $\hat{F} \cap S^3 \times \{t\}$ with this band surgery in S^3 at time t ($t \in [0, 1]$), we get a proper surface $\hat{F} \subset S^3 \times [0, 1]$ such that $\partial \hat{F} = L_{m,n} \sqcup K$. Here we consider that K lies in $S^3 \times \{1\}$ and $L_{m,n}$ in $S^3 \times \{0\}$, and that $L_{m,n} \cup B_1 \cup \dots \cup B_{m+n-1}$ lies in $S^3 \times \{1/2\}$. The surface \hat{F} is homeomorphic to a surface which is obtained as S^2 minus $m+n+1$ disjoint open disks, so $g(\hat{F}) = 0$.

By Theorem 1.2, we have

$$u(K) \geq (m-1)(n-1).$$

□

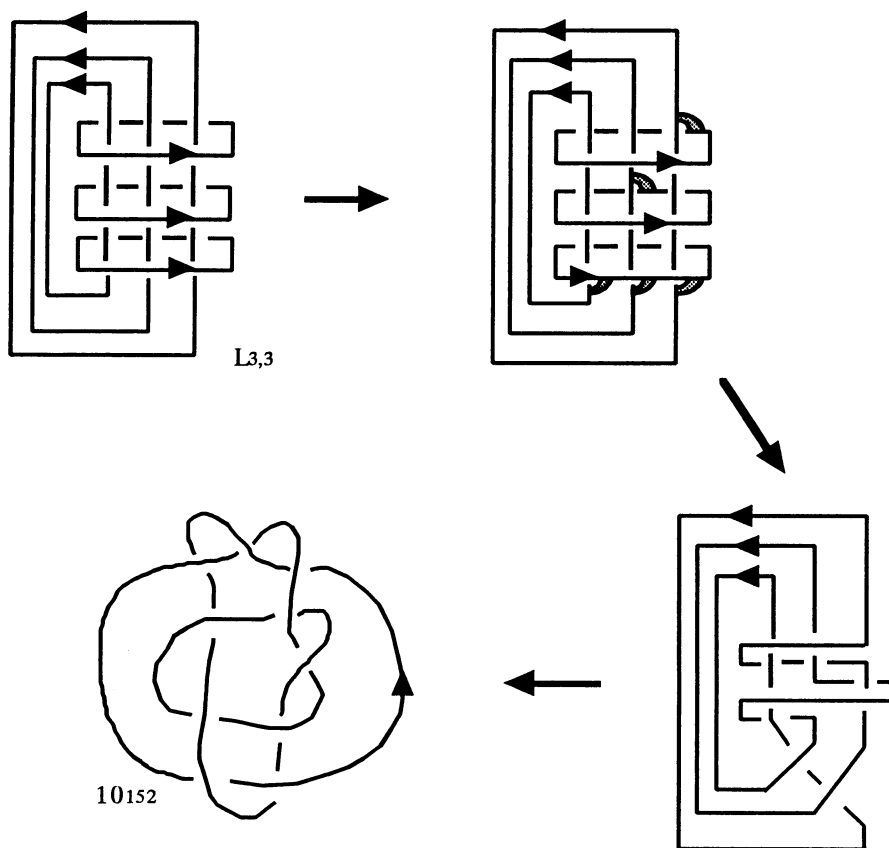


Fig. 4. Fusion procedure to obtain 10_{152} from $L_{3,3}$

3. The proof of Theorem 1.3

In this section, we prove Theorem 1.3 by using Corollary 2.2.

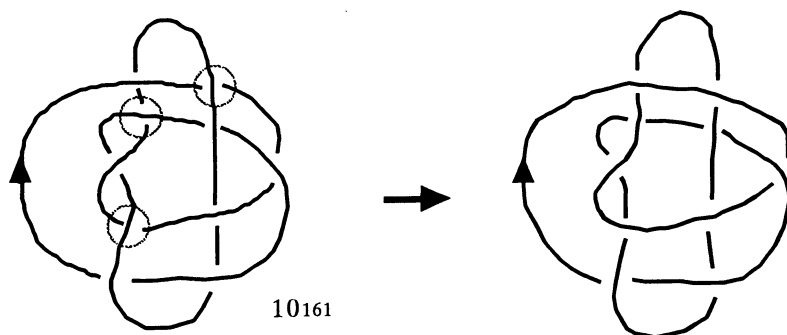
Proof of Theorem 1.3. (1) Fig. 2 shows that the knot 10_{139} is obtained from $L_{3,3}$ by connecting the components with 5 bands. It implies that 10_{139} is obtained from $L_{3,3}$ by a fusion.

By Corollary 2.2, we have

$$u(10_{139}) \geq (3-1)(3-1) = 4.$$

On the other hand, we observe that the knot 10_{139} can be unknotted by 4-crossing changes. By changing the crossings which are marked as in Fig. 3, we obtain the trivial knot. Therefore we conclude that $u(10_{139}) = 4$.

(2) Similarly to (1), we can show that the knot 10_{152} is obtained from $L_{3,3}$ by

Fig. 5. Changing the crossings of 10_{161}

a fusion as in Fig. 4. By Corollary 2.2, we have

$$u(10_{152}) \geq (3 - 1)(3 - 1) = 4.$$

On the other hand, we observe that the knot 10_{152} can be unknotted by 4-crossing changes. By changing the crossings which are marked as in Fig. 3, we obtain the trivial knot. Therefore we conclude that $u(10_{152}) = 4$. \square

REMARK. By Theorem 1.3(1), it can be shown that the unknotting number of 10_{161} is 3. 10_{161} is illustrated in Fig. 5, and can be obtained from 10_{139} by 1-crossing change. We see that 10_{161} cannot be unknotted by 2-crossing changes, because the unknotting number of 10_{139} is 4. On the other hand, we observe that the knot 10_{161} can be unknotted by 3-crossing changes. By changing the crossings which are marked as in Fig. 5, we obtain the trivial knot. Therefore we conclude that $u(10_{161}) = 3$. The author was informed by Shimokawa that Tanaka [9] proved the result $u(10_{161}) = 3$ using a result of Rudolph [8, pp. 56, Corollary] on a quasipositive link. That is, our arguments also give an alternative proof of a result of Tanaka. The author would like to thank Doctor Koya Shimokawa for his interest in this work.

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