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## THE UNKNOTTING NUMBERS OF $10_{139}$ AND $10_{152}$ ARE 4

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### 1. Introduction and the statement of main results

A *link* is a closed oriented 1-manifold smoothly embedded in the 3-sphere  $S^3$  and a *knot* is a link with one component. Let  $K$  be a knot. The *unknotting number*  $u(K)$  of  $K$  is the minimal number of crossing changes needed to create the trivial knot.

In the 1960s, Milnor [5] conjectured that the unknotting number of any algebraic knot would be equal to the genus of the Milnor fiber. In the 1990s, Kronheimer and Mrowka [2] [3] proved this conjecture using gauge theory. In particular, the unknotting number of  $(p, q)$ -torus knot is  $((p-1)(q-1))/2$ . In 1995, Auckly [1] gave an alternative proof of this conjecture for certain torus knots using Seiberg-Witten theory. In this paper we shall compute unknotting numbers of certain knots using Seiberg-Witten theory.

An argument explained in Auckly's lecture notes [1] is based on the following theorem, which is the so-called generalized adjunction formula.

**Theorem 1.1** (Kronheimer-Mrowka [4], Morgan-Szabó-Taubes [7]). *Let  $X$  be a smooth closed oriented four-manifold with  $\dim H_+^2(X, R) > 1$ . If  $F \hookrightarrow X$  is a smoothly embedded closed oriented surface of genus  $g \geq 1$  and  $K$  is a basic class of  $X$ , then*

$$2g - 2 \geq K(F) + F \cdot F$$

where  $A \cdot B$  denotes the intersection number of  $A$  and  $B$ .

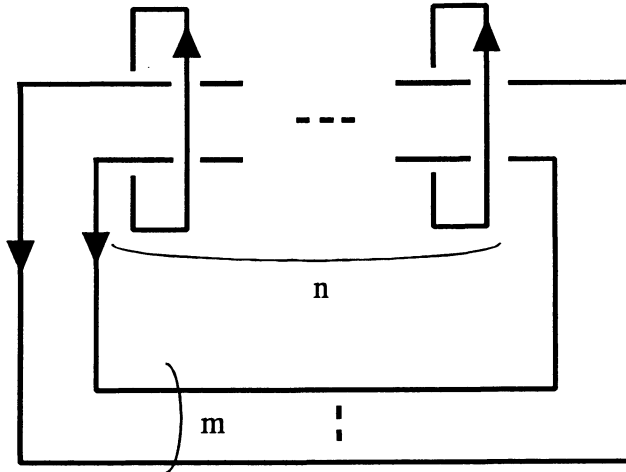
By extending the argument in [1], we obtain the following Theorem.

**Theorem 1.2.** *Let  $K$  be an oriented knot in  $S^3 \times \{1\}$  and  $L_{m,n}$  the link in  $S^3 \times \{0\}$  illustrated in Fig. 1. If there is a compact connected oriented surface  $\hat{F}$  in  $S^3 \times [0, 1]$  such that  $\partial\hat{F} = L_{m,n} \sqcup K$ , then*

$$u(K) \geq (m-1)(n-1) - g(\hat{F})$$

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Fig. 1. Link  $L_{m,n}$ 

where  $g(\hat{F})$  denotes the genus of  $\hat{F}$ .

By means of Theorem 1.2, we can determine the unknotting numbers of certain knots. In particular, in the present paper, we show the following result.

**Theorem 1.3.** (1) *The unknotting number of the knot  $10_{139}$  is 4.*  
 (2) *The unknotting number of the knot  $10_{152}$  is 4.*

The unknotting numbers of  $10_{139}$  and  $10_{152}$  had been known to be either 3 or 4, but they had not been determined. The author was suggested by Kawauchi that this problem might be solved by Theorem 1.2, and Theorem 1.3 is the answer to it.

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## 2. The proof of Theorem 1.2 and its corollary

Before proving Theorem 1.2, we shall review the definition of basic class.

Let  $X$  be a smooth closed oriented 4-manifold with  $b_2^+(X) > 1$ . The Seiberg-Witten invariant of  $X$  is an integer valued function which is defined on the set of  $Spin^c$ -structures over  $X$ , (cf. for example [4] [6] [10]). This invariant is considered

as a map

$$n : H^2(X; Z)/\text{Torsion} \rightarrow Z,$$

(cf. [1]). If  $n(K) \neq 0$ , then  $K$  is called a *basic class*.

To prove Theorem 1.2, we will use the following lemma;

**Lemma 2.1** (Auckly [1]). *0 is a basic class for  $T^4$ .*

More generally, the work by Witten [10, pp. 786–789] implies that if  $X$  is a Kähler-Einstein manifold, the canonical class of  $X$  is a basic class. Lemma 2.1 can be also proved by this fact.

Auckly [1] presented a way of computing the unknotting number of  $(2, 5)$ -torus knot applying Theorem 1.1 and Lemma 2.1 to a suitable surface in  $T^4$ . Theorem 1.2 is an extension of result in [1], and the argument in proof of Theorem 1.2 is almost same as his argument.

**Proof of Theorem 1.2.** We consider  $T^2 = [0, 1]^2 / \sim$  and  $T^4 = T^2 \times T^2$ , where  $\sim$  is the equivalent relation defined by  $(0, t) \sim (1, t)$  and  $(s, 0) \sim (s, 1)$ . We will construct a surface  $F$  embedded in  $T^4$ .

We define  $E$  and  $J$  by

$$E = \left( \bigcup_{k=1, \dots, m} \left\{ \left( \frac{k}{m+1}, \frac{k}{m+1} \right) \right\} \times T^2 \right) \cup \left( \bigcup_{k=1, \dots, n} T^2 \times \left\{ \left( \frac{k}{n+1}, \frac{k}{n+1} \right) \right\} \right)$$

$$J = \left[ \frac{1}{m+2}, \frac{m+1}{m+2} \right]^2 \times \left[ \frac{1}{n+2}, \frac{n+1}{n+2} \right]^2.$$

The four-disk  $J$  includes all self-intersections of  $E$ . Then  $\partial(E - J) = E \cap \partial J \subset \partial J$  is equivalent to  $L_{m,n} \subset S^3$ . We suppose that  $K$  can be unknotted by  $u$ -crossing changes. Then there is a surface  $F' \subset D^4$  with genus  $u$  such that  $\partial F' = K \subset S^3 \cong \partial D^4$ .

We can regard  $\partial J \times [0, 1]$  as a collar of  $\partial J$  in  $J$  and  $\partial J \times \{0\}$  as  $\partial J$ . We can identify  $\partial J \times [0, 1]$  with  $S^3 \times [0, 1]$  since  $\partial J \cong S^3$ . We can consider that  $\hat{F}$  lies in  $\partial J \times [0, 1]$  and  $\hat{F} \cap \partial J = E \cap \partial J$ . We can identify the closure of  $J - (\partial J \times [0, 1])$  with  $D^4$ . We can consider that  $F'$  lies in the closure of  $J - (\partial J \times [0, 1])$  and  $\hat{F} \cap \partial J \times \{1\} = \partial F'$ . We define a surface  $F$  by

$$F = (E - J) \cup \hat{F} \cup F'.$$

The genus of  $F$  is equal to  $m + n + g(\hat{F}) + u$ . The self-intersection number  $F \cdot F$  is the same as  $E \cdot E$ , because  $F$  is homologous to  $E$ . So  $F \cdot F = E \cdot E = 2mn$ .

We can apply Theorem 1.1 and Lemma 2.1 for  $F$ . Then we obtain the inequality

$$2(m + n + g(\hat{F}) + u) - 2 \geq 0 + 2mn.$$

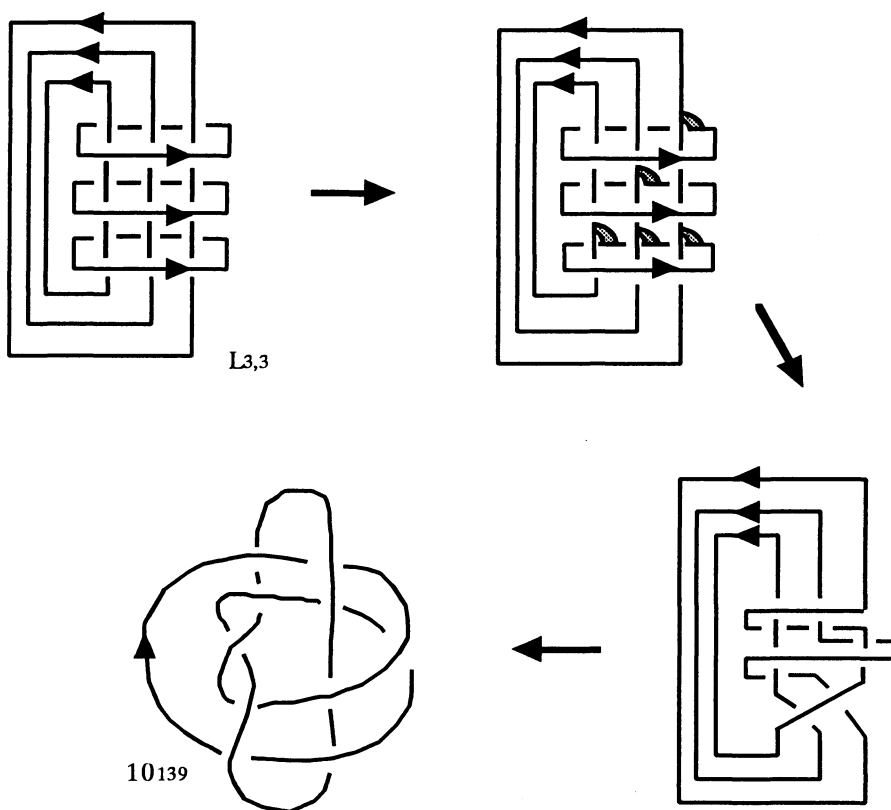


Fig. 2. Fusion procedure to obtain  $10_{139}$  from  $L_{3,3}$

Thus, we obtain the desired inequality  $u \geq (m - 1)(n - 1) - g(\hat{F})$ . □

To prove Theorem 1.3, we show a corollary of Theorem 1.2. In order to state it, we shall review the definition of fusion procedure.

Let  $L$  be a  $\mu$ -component oriented link. Let  $B_1, \dots, B_\nu$  be mutually disjoint oriented bands in  $S^3$  such that  $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha'_i$ , where  $\alpha_1, \alpha'_1, \dots, \alpha_\nu, \alpha'_\nu$  are disjoint connected arcs. The closure of  $L \cup \partial B_1 \cup \dots \cup \partial B_\nu - \alpha_1 \cup \alpha'_1 \cup \dots \cup \alpha_\nu \cup \alpha'_\nu$  is also the link. We will write it by  $L'$ .

DEFINITION. If  $L'$  has the orientation compatible with the orientation of  $L - \bigcup_{i=1, \dots, \nu} \alpha_i \cup \alpha'_i$  and  $\bigcup_{i=1, \dots, \nu} (\partial B_i - \alpha_i \cup \alpha'_i)$ ,  $L'$  is called the link obtained from  $L$  by the *band surgery* along the bands  $B_1, \dots, B_\nu$ . Moreover if  $L'$  has  $(\mu - \nu)$ -components, this transformation is called a *fusion*.

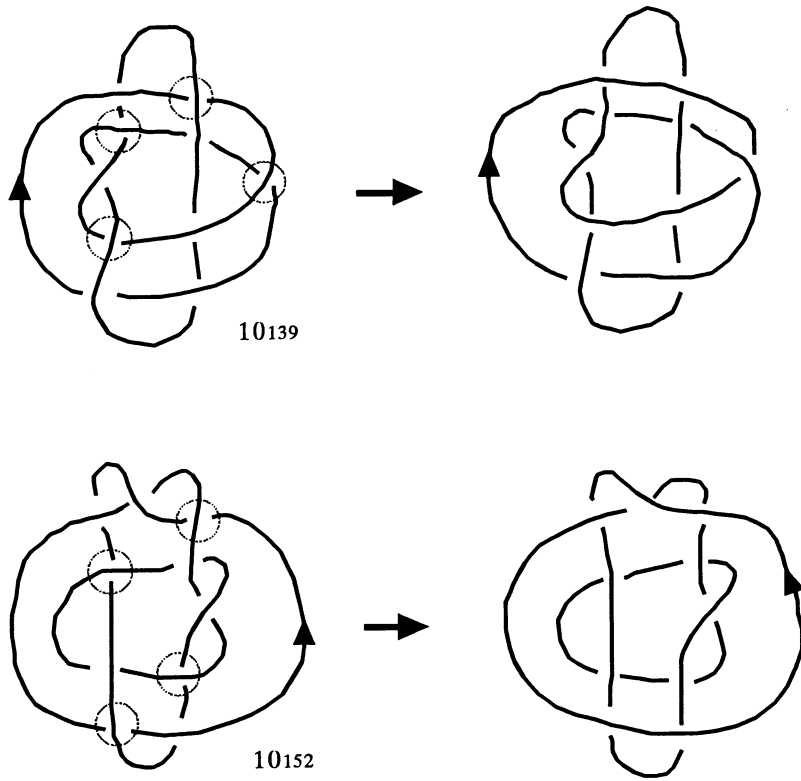


Fig. 3. Crossing changes

**Corollary 2.2.** *Let  $L_{m,n}$  be the link illustrated in Fig. 1. If an oriented knot  $K$  in  $S^3$  is obtained from  $L_{m,n}$  by the fusion, then*

$$u(K) \geq (m - 1)(n - 1).$$

**Proof of Corollary 2.2.** To apply Theorem 1.2, we construct a suitable surface in  $S^3 \times [0, 1]$ . Let  $B_1, \dots, B_{m+n-1}$  be the surgery bands. Identifying  $\hat{F} \cap S^3 \times \{t\}$  with this band surgery in  $S^3$  at time  $t$  ( $t \in [0, 1]$ ), we get a proper surface  $\hat{F} \subset S^3 \times [0, 1]$  such that  $\partial\hat{F} = L_{m,n} \sqcup K$ . Here we consider that  $K$  lies in  $S^3 \times \{1\}$  and  $L_{m,n}$  in  $S^3 \times \{0\}$ , and that  $L_{m,n} \cup B_1 \cup \dots \cup B_{m+n-1}$  lies in  $S^3 \times \{1/2\}$ . The surface  $\hat{F}$  is homeomorphic to a surface which is obtained as  $S^2$  minus  $m + n + 1$  disjoint open disks, so  $g(\hat{F}) = 0$ .

By Theorem 1.2, we have

$$u(K) \geq (m - 1)(n - 1).$$

□

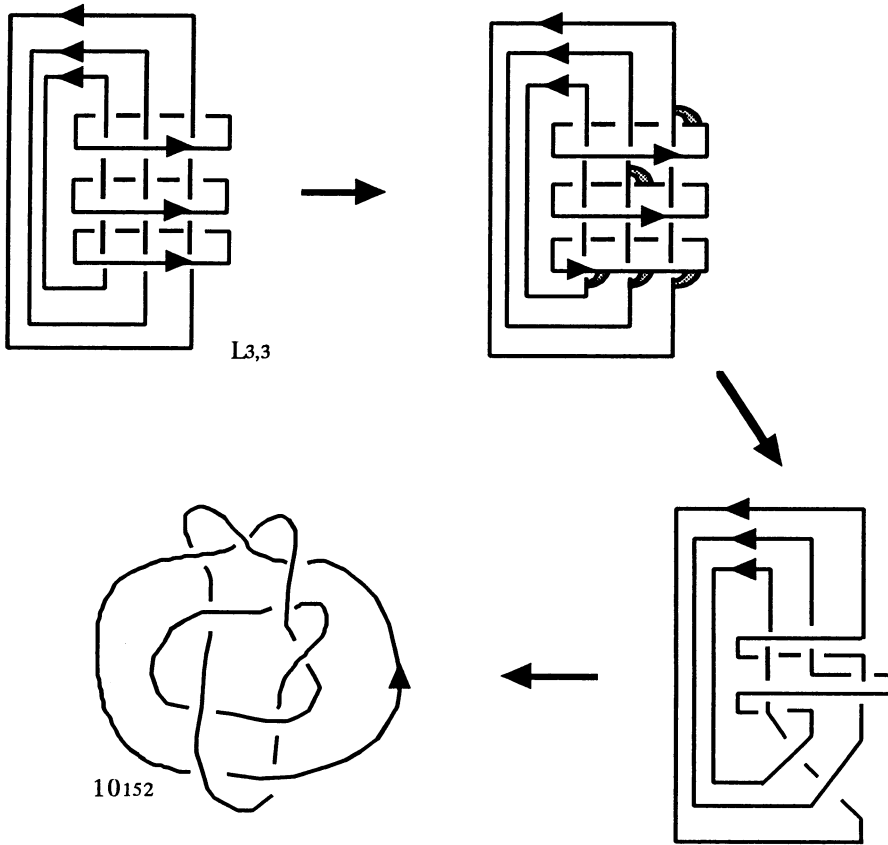


Fig. 4. Fusion procedure to obtain  $10_{152}$  from  $L_{3,3}$

### 3. The proof of Theorem 1.3

In this section, we prove Theorem 1.3 by using Corollary 2.2.

Proof of Theorem 1.3. (1) Fig. 2 shows that the knot  $10_{139}$  is obtained from  $L_{3,3}$  by connecting the components with 5 bands. It implies that  $10_{139}$  is obtained from  $L_{3,3}$  by a fusion.

By Corollary 2.2, we have

$$u(10_{139}) \geq (3 - 1)(3 - 1) = 4.$$

On the other hand, we observe that the knot  $10_{139}$  can be unknotted by 4-crossing changes. By changing the crossings which are marked as in Fig. 3, we obtain the trivial knot. Therefore we conclude that  $u(10_{139}) = 4$ .

(2) Similarly to (1), we can show that the knot  $10_{152}$  is obtained from  $L_{3,3}$  by

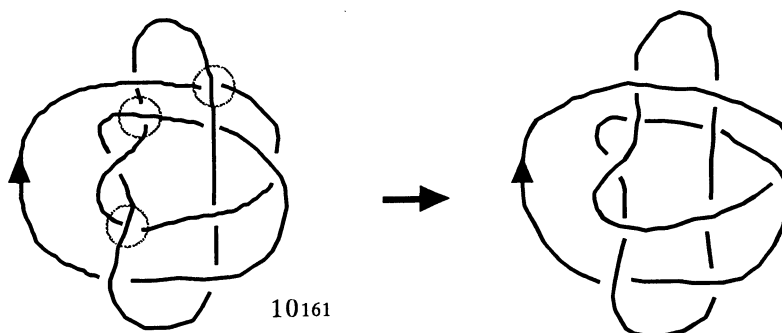


Fig. 5. Changing the crossings of  $10_{161}$

a fusion as in Fig. 4. By Corollary 2.2, we have

$$u(10_{152}) \geq (3 - 1)(3 - 1) = 4.$$

On the other hand, we observe that the knot  $10_{152}$  can be unknotted by 4-crossing changes. By changing the crossings which are marked as in Fig. 3, we obtain the trivial knot. Therefore we conclude that  $u(10_{152}) = 4$ .  $\square$

REMARK. By Theorem 1.3(1), it can be shown that the unknotting number of  $10_{161}$  is 3.  $10_{161}$  is illustrated in Fig. 5, and can be obtained from  $10_{139}$  by 1-crossing change. We see that  $10_{161}$  cannot be unknotted by 2-crossing changes, because the unknotting number of  $10_{139}$  is 4. On the other hand, we observe that the knot  $10_{161}$  can be unknotted by 3-crossing changes. By changing the crossings which are marked as in Fig. 5, we obtain the trivial knot. Therefore we conclude that  $u(10_{161}) = 3$ . The author was informed by Shimokawa that Tanaka [9] proved the result  $u(10_{161}) = 3$  using a result of Rudolph [8, pp. 56, Corollary] on a quasipositive link. That is, our arguments also give an alternative proof of a result of Tanaka. The author would like to thank Doctor Koya Shimokawa for his interest in this work.

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