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E_*-INJECTIVE SPECTRA AND INJECTIVE E_*E-COMODULES

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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0. Introduction

In [13] Ohkawa introduced the notion of the injective hull of spaces and spectra with respect to homology and proved the existence theorem [13, Theorem 1]. Following [13, Definition 1 i)] we call a CW-spectrum W *E_*-injective* if any map $f: X \to Y$ induces an epimorphism $f^*: [Y, W] \to [X, W]$ whenever $f_*: E_*X \to E_*Y$ is a monomorphism, for a fixed CW-spectrum E. A CW-spectrum W is E_*-injective if and only if the homomorphism $\kappa_E: [X, W] \to \mathrm{Hom}(E_*X, E_*W)$ defined by $\kappa_E(f) = f_*$ is a monomorphism for any CW-spectrum X (see [13, Proposition 7]). In this note we will be concerned about E_*-injective spectra.

For each CW-spectrum X, E_*X is regarded as a module over the algebra E^*E of cohomology operations. Under the restriction that E is finite, Ohkawa [13, Theorem 3 i) and iii)] gave the following characterization.

Theorem 0. *Assume that a CW-spectrum E is finite. Then the following conditions are equivalent:*
i) *W is an E_*-injective spectrum.*
ii) *W is an E_*-local spectrum such that E_*W is injective as an E^*E-module.*
iii) *$\kappa_E: [X, W] \to \mathrm{Hom}_{E^*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*

According to [2, Proposition III.13.4] (or see [1]), the well known ring spectra $E = S$, HZ/p, MO, MU, MSp, KU and KO satisfy some of nice properties as stated in the beginning of §2. For example, E_*E becomes flat as an E_*-module, and then E_*X may be regarded as a comodule over the coalgebra E_*E. In §2 we will prove the following result (Theorem 2.1) for such a nice ring spectrum E, corresponding to Theorem 0 for a finite spectrum E.

Theorem 1. *Let E be a ring spectrum such that E_*E is flat as an E_*-module. Assume that E satisfies the property* (K″) *stated in the beginning of* §2. *Then the following conditions are equivalent:*

i) *W is an E_*-injective spectrum.*
ii) *W is an E_*-local spectrum such that E_*W is injective as an E_*E-comodule.*
iii) κ_E: $[X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ *is an isomorphism for any CW-spectrum X.*

In §3 we will next study the EG_*-injectivity where EG denotes the CW-spectrum $E \wedge SG$ with coefficients in G. There exists a partial order among CW-spectra by writing $\langle E\rangle_I \leqq \langle F\rangle_I$ when each E_*-injective spectrum is F_*-injective. In order to decide $\langle EG\rangle_I \leqq \langle EG'\rangle_I$ we will find a certain relation between the abelian groups G and G' (Theorem 3.5). Moreover we will prove the following complete result (Theoıem 3.9), especially when $E=H$ "the Eilenberg-MacLane spectrum".

Theorem 2. *A CW-spectrum W is HG_*-injective if and only if it is a generalized Eilenberg-MacLane spectrum $\bigvee_n \Sigma^n HA_n$ in which A_n is a direct summand of* $\mathrm{Hom}(G, D_n)$ *with D_n divisible.*

When $E=KU$ "the complex K-spectrum", we will finally show a partial result (Theorem 3.11) corresponding to Theorem 2.

In this note we will work in the stable homotopy category of CW-spectra. We mean by *a ring spectrum E* an associative ring spectrum with unit, and by *an E-module spectrum F* an associative (left) E-module spectrum. If E or F is not necessarily assumed to be associative, then it is called *a quasi ring spectrum* or *a quasi E-module spectrum.*

1. The Anderson dual spectra $\nabla E(D)$

1.1. Let us fix a CW-spectrum E. Given CW-spectra X and Y a map $f: X \to Y$ is said to be *E_*-monic* if it induces a monomorphism $f_*: E_*X \to E_*Y$. Following [13, Definition 1 i)] (cf. [8, §9]) we call a CW-spectrum W *an E_*-injective spectrum* if any E_*-monic map $f: X \to Y$ induces always an epimorphism f^*: $[Y, W] \to [X, W]$. For any family $\{W_\lambda\}$ of CW-spectra it is obvious by definition that

(1.1) each CW-spectrum W_λ is E_*-injective if and only if the direct product $\prod_\lambda W_\lambda$ is E_*-injective.

Consider the homomorphism κ_E: $[X, W] \to \mathrm{Hom}(E_*X, E_*W)$ assigning to each map $f: X \to W$ its induced homomorphism $f_*: E_*X \to E_*W$. Then the following result involving κ_E can be easily verified.

Proposition 1.1. ([13, Proposition 7]). *A CW-spectrum W is E_*-injective if and only if κ_E: $[X, W] \to \mathrm{Hom}(E_*X, E_*W)$ is a monomorphism for any CW-spectrum X.*

A CW-spectrum W is said to be *E_*-local* (see [6], [7] or [14]) if $[X, W]=0$ for all CW-spectra X with $E_*X=0$. As an immediate result we have

Corollary 1.2. *If a CW-spectrum W is E_*-injective, then it is an E_*-local spectrum.*

For any abelian group G we denote by SG the Moore spectrum of type G. Given a CW-spectrum E the corresponding spectrum with coefficients in G is defined by $EG=E\wedge SG$. In the $G=Q$ case we can particularly show

Lemma 1.3. *Assume that $EQ\neq pt$. Then a CW-spectrum W is EQ_*-injective if and only if $W=WQ$.*

Proof. We may regard as $E=S$, the sphere spectrum. The "if" part is easily verified since any SQ_*-monic map $f: X\to Y$ induces an epimorphism $f^*: [Y, WQ]\to[X, WQ]$. On the other hand, the "only if" part is immediate from Corollary 1.2.

We mean by *a quasi ring spectrum E* a ring spectrum with unit which is not necessarily associative and by *a quasi E-module spectrum F* a (left) E-module spectrum which is not necessarily associative. Notice that any quasi E-module spectrum F is always E_*-local when E is a quasi ring spectrum (see [2, Lemma III.13.1] or [14, Proposition 1.17]).

Lemma 1.4. *Let E be a quasi ring spectrum and F be a quasi E-module spectrum. If a CW-spectrum W is F_*-injective, then it is a quasi E-module spectrum. In particular, any EG_*-injective spectrum W is always a quasi E-module spectrum.*

Proof. Since the unit $\iota: S\to E$ induces a monomorphism $(\iota\wedge 1)_*: F_*W\to F_*(E\wedge W)$, there exists a map $\mu: E\wedge W\to W$ satisfying $\mu(\iota\wedge 1)=1$, where 1 denotes the identity map.

1.2. Let E be a fixed CW-spectrum and $D=\{D_n\}$ be a graded divisible abelian group. By Representability theorem there exists a CW-spectrum $\nabla E(D)=\prod_n \Sigma^n \nabla E(D_n)$ which is related to E and D by a natural isomorphism

$$(1.2) \qquad \kappa_{E,D}: [X, \nabla E(D)] \to \mathrm{Hom}(E_*X, D) = \prod_n \mathrm{Hom}(E_nX, D_n)$$

for any CW-spectrum X. Setting $\lambda_{E,D}=\kappa_{E,D}(1)\in\mathrm{Hom}(E_*\nabla E(D), D)$, the natural isomorphism $\kappa_{E,D}$ assigns to each map $f: X\to\nabla E(D)$ the composite $\lambda_{E,D}f_*: E_*X\to E_*\nabla E(D)\to D$.

For any graded abelian group $A=\{A_n\}$ we choose an injective resolution $0\to A\to D\xrightarrow{d} D'\to 0$ and denote by $\nabla E(A)=\prod_n \Sigma^n\nabla E(A_n)$ the fiber of the induced map $d_*: \nabla E(D)\to\nabla E(D')$. Then we obtain a universal coefficient sequence

$$0 \to \mathrm{Ext}(E_{*-1}X, A) \to [X, \nabla E(A)] \xrightarrow{\kappa_{E,A}} \mathrm{Hom}(E_*X, A) \to 0$$

for any CW-spectrum X. As is easily seen [15, I and II], $\nabla E(A)$ is independent of the choice of an injective resolution of A and it is just the function spectrum $F(E, \nabla S(A)) = \prod_n \Sigma^n F(E, \nabla S(A_n))$. We call $\nabla E(A)$ *the Anderson dual spectrum of E with coefficients in* $A = \{A_n\}$ (cf. [4]).

Using the natural isomorphism $\kappa_{E,D}$ of (1.2) we see immediately that

(1.3) every Anderson dual spectrum $\nabla E(D) = \prod_n \Sigma^n \nabla E(D_n)$ is E_*-injective if $D = \{D_n\}$ is divisible.

By virtue of (1.3) we can show the enough E_*-injectivity of the stable homotopy category of CW-spectra (cf. [13, Proposition 4]).

Proposition 1.5. *For any CW-spectrum X there exists an Anderson dual spectrum* $\nabla E(D)$ *with* $D = \{D_n\}$ *divisible, which is* E_**-injective, and an* E_**-monic map* $f\colon X \to \nabla E(D)$.

Proof. Choose a graded divisible abelian group $D = \{D_n\}$ so that E_nX is embedded into D_n for each n. Pick up a map $f\colon X \to \nabla E(D)$ such that $\kappa_{E,D}(f)\colon E_*X \to D$ is just the embedding of E_*X into D. Since $\kappa_{E,D}(f)$ is decomposed into the composite $\lambda_{E,D} f_*$, the map $f\colon X \to \nabla E(D)$ is certainly E_*-monic.

More generally we will next deal with the CW-spectrum EG with coefficients in G. For any divisible abelian group D_n we set $B_n = \mathrm{Hom}(G, D_n)$. Take a free resolution $0 \to \bigoplus_\alpha Z \to \bigoplus_\beta Z \to G \to 0$ and consider the commutative diagram

$$\begin{array}{ccccc} \nabla E(B_n) & \to & \nabla E(\prod_\beta D_n) & \to & \nabla E(\prod_\alpha D_n) \\ & & \downarrow & & \downarrow \\ F(SG, \nabla E(D_n)) & \to & \prod_\beta \nabla E(D_n) & \to & \prod_\alpha \nabla E(D_n) \end{array}$$

with two cofiber sequences. The two vertical arrows are equivalences because each of them induces the canonical isomorphism $\mathrm{Hom}(\pi_*E, \prod_\gamma D_n) \to \prod_\gamma \mathrm{Hom}(\pi_*E, D_n)$ in the homotopy group. By applying Five lemma we get an equivalence $\nabla E(B_n) \to F(SG, \nabla E(D_n)) = \nabla EG(D_n)$. Thus

$$\nabla E(B) = \prod_n \Sigma^n \nabla E(B_n) = \prod_n \Sigma^n \nabla EG(D_n) = \nabla EG(D) \tag{1.4}$$

if $D = \{D_n\}$ is divisible and $B = \{B_n = \mathrm{Hom}(G, D_n)\}$.

Using Proposition 1.5 combined with (1.1), (1.3) and (1.4) we can easily show

Proposition 1.6. *A CW-spectrum W is EG_*-injective if and only if it is a retract of a certain Anderson dual spectrum $\nabla E(B)=\prod_n \Sigma^n \nabla E(B_n)$ in which $B_n=$ $\mathrm{Hom}(G, D_n)$ for some divisible D_n.*

1.3. For any graded abelian group $A=\{A_n\}$ we denote by $\mathrm{Tor}\, A=\{\mathrm{Tor}\, A_n\}$ its torsion subgroup. The torsion subgroup $\mathrm{Tor}\, A$ is said to be bounded if $m\, \mathrm{Tor}\, A=0$ for some positive integer m.

Proposition 1.7. *Let E be a CW-spectrum such that $\mathrm{Tor}\, \pi_* E$ is bounded. If a CW-spectrum W is E_*-injective, then it is decomposed into the wedge sum $WQ \vee \Sigma^{-1} WQ/Z$.*

Proof. According to Proposition 1.6 W is a retract of a certain Anderson dual spectrum $\nabla E(D)$ with $D=\{D_n\}$ divisible. Set $A=\pi_* E$ thus $A_n=\pi_n E$. Since $\mathrm{Hom}(A/\mathrm{Tor}\, A, D)$ is divisible, the short exact sequences $0\to \mathrm{Hom}(A/\mathrm{Tor}\, A, D)\to \mathrm{Hom}(A, D)\to \mathrm{Hom}(\mathrm{Tor}\, A, D)\to 0$ and $0\to \mathrm{Hom}(A/\mathrm{Tor}\, A, D)*Q/Z\to \mathrm{Hom}(A/\mathrm{Tor}\, A, D)\to \mathrm{Hom}(A/\mathrm{Tor}\, A, D)\otimes Q\to 0$ are both split. Under our assumption on $\mathrm{Tor}\, A$, we note that $\mathrm{Hom}(\mathrm{Tor}\, A, D)\otimes Q=0$ and hence $\mathrm{Hom}(\mathrm{Tor}\, A, D)\otimes Q/Z=0$. This implies that $\mathrm{Hom}(A, D)\otimes Q/Z=0$. So the rationalization l: $\mathrm{Hom}(A, D)\to \mathrm{Hom}(A, D)\otimes Q$ has a right inverse k because $\mathrm{Hom}(A/\mathrm{Tor}\, A, D)\otimes Q\to \mathrm{Hom}\, (A, D)\otimes Q$ is an isomorphism. Thus the cofiber sequence $\Sigma^{-1}\nabla E(D)Q/Z\to \nabla E(D)\xrightarrow{i} \nabla E(D)Q$ gives rise to a split short exact sequence

$$0\to \pi_{*+1}\nabla E(D)Q/Z\to \pi_*\nabla E(D)\xrightarrow{l} \pi_*\nabla E(D)Q\to 0$$

in the homotopy group.

Note that XQ is just the generalized Mooie spectrum $\bigvee_n \Sigma^n S(\pi_n X\otimes Q)$ for each CW-spectrum X. Consider the commutative diagram

$$\begin{array}{ccccccc} 0\to & \mathrm{Ext}(\pi_{*-1}XQ, \pi_*\nabla E(D)) & \to & [XQ, \nabla E(D)] & \xrightarrow{\kappa_S} & \mathrm{Hom}(\pi_*XQ, \pi_*\nabla E(D)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \mathrm{Ext}(\pi_{*-1}XQ, \pi_*\nabla E(D)Q) & \to & [XQ, \nabla E(D)Q] & \xrightarrow{\kappa_S} & \mathrm{Hom}(\pi_*XQ, \pi_*\nabla E(D)Q) & \to 0 \end{array}$$

involving the universal coefficient sequences [11]. Since the left Ext-terms are both vanishing, the two assignments κ_S are exactly isomorphisms. Taking $X=\nabla E(D)$, we can pick up a map f: $\nabla E(D)Q\to \nabla E(D)$ inducing the right inverse k of l in the homotopy group. This map f is certainly a right inverse of i: $\nabla E(D)\to \nabla E(D)Q$. Thus the cofiber sequence $\Sigma^{-1}\nabla E(D)Q/Z\to \nabla E(D)\to \nabla E(D)Q$ is split. Then it is easily verified that the cofiber sequence $\Sigma^{-1}WQ/Z\to W\to WQ$ is split, too.

By virtue of Proposition 1.7 we obtain

Theorem 1.8. *Let E be a CW-spectrum such that* $\operatorname{Tor}\pi_*E$ *is bounded. Then a CW-spectrum W is E_*-injective if and only if its p-local spectrum $WZ_{(p)}$ is $EZ_{(p)*}$-injective for each prime p.*

Proof. The "only if" part: Assume that a CW-spectrum W is E_*-injective. Then Proposition 1.7 implies that the p-local spectrum $WZ_{(p)}$ is decomposed into the wedge sum of E_*-injective spectra WQ and $\Sigma^{-1}WZ/p^\infty$, in which $WQ=pt$ whenever $EQ=pt$. The E_*-injective spectrum WQ is obviously $EZ_{(p)*}$-injective because of Lemma 1.3. On the other hand, the E_*-injective spectrum WZ/p^∞ is a retract of a certain Anderson dual spectrum $\nabla E(D_p)$. Since we can take $D_p=\{D_{p,n}\}$ to be divisible p-torsion, it is also $EZ_{(p)*}$-injective by means of Proposition 1.6. Therefore $WZ_{(p)}=WQ\vee\Sigma^{-1}WZ/p^\infty$ is $EZ_{(p)*}$-injective for each prime p.

The "if" part: Assume that the p-local spectrum $WZ_{(p)}$ is $EZ_{(p)*}$-injective for each prime p. Then Proposition 1.7 asserts that WQ and WZ/p^∞ are $EZ_{(p)*}$-injective and the canonical map $j_p\colon WQ\to WZ/p^\infty$ is trivial for each prime p, where $WQ=pt$ if $EQ=pt$. Set $\bar{W}=\prod_p WZ/p^\infty$, which is E_*-injective. Using Proposition 1.7 again we observe that the direct product $\bar{W}$ is decomposed into the wedge sum of $\bar{W}Q$ and $\Sigma^{-1}\bar{W}Q/Z$. Note that WZ/p^∞ is a retract of $\Sigma^{-1}\bar{W}Z/p^\infty$ because $\bar{W}Z/p^\infty=(WZ/p^\infty\wedge SZ/p^\infty)\vee(\prod_{q\neq p}WZ/q^\infty)\wedge SZ/p^\infty$. Therefore $WQ/Z=\bigvee_p WZ/p^\infty$ is a retract of the E_*-injective spectrum $\bar{W}=\prod_p WZ/p^\infty$. Thus the canonical map $l\colon WQ/Z\to\bar{W}$ has a left inverse. Now it is easy to check that the canonical map $j\colon WQ\to WQ/Z$ becomes trivial because $l_*\colon[WQ, WQ/Z]\to[WQ, \bar{W}]=\prod_p[WQ, WZ/p^\infty]$ is a monomorphism. Since the CW-spectrum W is written into the wedge sum of the E_*-injective spectra WQ and $\Sigma^{-1}WQ/Z$, it is E_*-injective as desired.

1.4. Assume that E is a ring spectrum. Then E_*X admits an E_*-module structure for each CW-spectrum X where π_*E is abbreviated as E_*. Given an injective E_*-module M there exists a CW-spectrum V_M so that

$$\kappa_{E,M}\colon [X, V_M]\to \operatorname{Hom}_{E_*}(E_*X, M) \tag{1.5}$$

is a natural isomorphism for any CW-spectrum X, by applying Representability theorem similarly to (1.2). The natural isomorphism $\kappa_{E,M}$ assings to each map $f\colon X\to V_M$ the composite $\lambda_{E,M}f_*\colon E_*X\to E_*V_M\to M$ where $\lambda_{E,M}=\kappa_{E,M}(1)\in \operatorname{Hom}_{E_*}(E_*V_M, M)$. As a similar result to (1.3) we have

(1.6) each represented spectrum V_M is E_*-injective if M is an injective E_*-module.

Every injective E_*-module can be realized by a certain E-module spectrum as follows.

Lemma 1.9. *Let E be a ring spectrum and M be an injective E_*-module. Then the represented spectrum V_M is an E-module spectrum such that the composite $\lambda_{E,M}(\iota\wedge 1)_*: \pi_*V_M \to E_*V_M \to M$ is an isomorphism of E_*-modules.*

Proof. Pick up a map $\mu: E\wedge V_M \to V_M$ such that $\kappa_{E,M}(\mu)=\lambda_{E,M}(m\wedge 1)_* \in \mathrm{Hom}_{E_*}(E_*(E\wedge V_M), M)$ where $m: E\wedge E\to E$ denotes the multiplication of E. Using the equality $\lambda_{E,M}(1\wedge\mu)_*=\lambda_{E,M}(m\wedge 1)_*$ we can easily check that $\mu(\iota\wedge 1)=1: V_M\to V_M$ and $\mu(1\wedge\mu)=\mu(m\wedge 1): E\wedge E\wedge V_M\to V_M$, thus V_M becomes an E-module spectrum.

Consider the following diagram

$$\begin{array}{ccc} [Y, E]\otimes[X, V_M] & \xrightarrow[1\otimes\kappa_{E,M}]{} & [Y, E]\otimes\mathrm{Hom}_{E_*}(E_*X, M) \\ \mu_\sharp\downarrow & & \downarrow m_\sharp \\ [Y\wedge X, V_M] & \xrightarrow[\kappa_{E,M}]{} & \mathrm{Hom}_{E_*}(E_*(Y\wedge X), M) \end{array}$$

where the vertical arrows $\mu_\sharp$ and $m_\sharp$ are respectively defined to be $\mu_\sharp(f\otimes g)=\mu(f\wedge g)$ and $m_\sharp(f\otimes a)=a(m\wedge 1)_*(1\wedge f\wedge 1)_*$. By a routine computation we can observe that the above square is commutative. Thus $\kappa_{E,M}: [X, V_M]\to \mathrm{Hom}_{E_*}(E_*X, M)$ is an isomorphism of E_*-modules. In particular, this implies that the composite $\lambda_{E,M}(\iota\wedge 1)_*: \pi_*V_M\to E_*V_M\to M$ is an isomorphism of E_*-modules.

By virtue of Lemma 1.9 each injective E_*-module M can be identified with π_*V_M. Then the natural isomorphism $\kappa_{E,M}: [X, V_M]\to\mathrm{Hom}_{E_*}(E_*X, M)$ may be regarded as the canonical morphism

(1.7) $\quad \kappa_{E,V_M}=\varphi\kappa_E: [X, V_M]\to\mathrm{Hom}_{E_*}(E_*X, E_*V_M)\to\mathrm{Hom}_{E_*}(E_*X, \pi_*V_M)$

where φ is induced by the E-module structure map $\mu: E\wedge V_M\to V_M$, because $\lambda_{E,M}(\iota\wedge 1)_*\mu_*=\lambda_{E,M}(1\wedge\mu)_*(\iota\wedge 1\wedge 1)_*=\lambda_{E,M}(m\wedge 1)_*(\iota\wedge 1\wedge 1)_*=\lambda_{E,M}$.

If $D=\{D_n\}$ is a graded divisible abelain group, then the E_*-module $\mathrm{Hom}(E_*, D)_*=\{\mathrm{Hom}(E_{*-n}, D)\}$ becomes injective. Setting $M=\mathrm{Hom}(E_*, D)_*$, we note that V_M coincides with the Anderson dual spectrum $\nabla E(D)$ since η: $\mathrm{Hom}_{E_*}(E_*X, \mathrm{Hom}(E_*, D)_*)\to\mathrm{Hom}_{E_*}(E_*X, D)$ is an isomorphism. Hence Lemma 1.9 implies

Corollary 1.10. *Let E be a ring spectrum and $D=\{D_n\}$ be a graded divisible abelian group. Then the Anderson dual spectrum $\nabla E(D)$ is an E-module spectrum such that $\kappa_{E,D}: \pi_*\nabla E(D)\to\mathrm{Hom}(E_*, D)_*$ is an isomorphism of E_*-modules.*

Taking $M=\mathrm{Hom}(E_*, D)_*$ with $D=\{D_n\}$ divisible, we restate (1.7) as the natural isomorphism $\kappa_{E,D}\colon [X, \nabla E(D)] \to \mathrm{Hom}(E_*X, D)$ can be decomposed into the composite

$$(1.8)\quad \eta\kappa_{E,\nabla E(D)} = \eta\varphi\kappa_E\colon [X, \nabla E(D)] \to \mathrm{Hom}_{E_*}(E_*X, E_*\nabla E(D)) \to \mathrm{Hom}_{E_*}(E_*X, \pi_*\nabla E(D)) \xrightarrow{\cong} \mathrm{Hom}(E_*X, D)\,.$$

1.5. Assume that a ring spectrum E satisfies the following property:

(F) E_*E is flat as an E_*-module.

Then E_*X is regarded to be an E_*E-comodule as well as an E_*-module for each CW-spectrum X. Given an injective E_*E-comodule I there exists a CW-spectrum W_I so that

$$(1.9)\qquad \kappa_{E,I}\colon [X, W_I] \to \mathrm{Hom}_{E_*E}(E_*X, I)$$

is a natural isomorphism for any CW-spectrum X, by means of Representability theorem. Setting $\lambda_{E,I}=\kappa_{E,I}(1)\in \mathrm{Hom}_{E_*E}(E_*W_I, I)$, the natural isomorphism $\kappa_{E,I}$ is given by $\kappa_{E,I}(f)=\lambda_{E,I}f_*$ for each $f\colon X\to W_I$. For any family $\{I_\lambda\}$ of injective E_*E-comodules it is obvious that

(1.10) the direct product $\prod_\lambda W_{I_\lambda}$ coincides with the represented spectrum W_I where $I=\prod_\lambda I_\lambda$.

Corresponding to (1.3) or (1.6) we have

(1.11) each represented spectrum W_I is E_*-injective if I is an injective E_*E-comodule.

Let M be an E_*-module such that the extended comodule $E_*E\otimes_{E_*}M$ is injective. Put $I=E_*E\otimes_{E_*}M$, and consider the composite

$$(1.12)\qquad \theta\kappa_{E,I}\colon [X, W_I]\to \mathrm{Hom}_{E_*E}(E_*X, E_*E\otimes_{E_*}M)\to \mathrm{Hom}_{E_*}(E_*X, M)$$

in which θ is defined to be $\theta(a)=(m_*\otimes 1)a$. Since θ is an isomorphism, the above composite $\theta\kappa_{E,I}$ becomes a natural isomorphism for any CW-spectrum X. If an E_*-module M is injective, then the extended comodule $I=E_*E\otimes_{E_*}M$ is exactly injective. In this case we notice that $W_I=V_M$ and $\theta\kappa_{E,I}=\kappa_{E,M}$. By the quite same argument as in the proof of Lemma 1.9 we can show

Lemma 1.11. *Let E be a ring spectrum satisfying the property* (F), *and M be an E_*-module such that the extended comodule $I=E_*E\otimes_{E_*}M$ is injective. Then the represented spectrum W_I is an E-module spectrum such that the composite $\lambda_{E,M}(\iota\wedge 1)_*\colon \pi_*W_I\to E_*W_I\to M$ is an isomorphism of E_*-modules, where $\lambda_{E,M}=$*

$(m_*\otimes 1)\kappa_{E,I}(1)\in \mathrm{Hom}_{E_*}(E_*W_I, M)$.

A similar discussion to (1.7) shows that the above natural isomorphism $\theta\kappa_{E,I}$: $[X, W_I]\to \mathrm{Hom}_{E_*}(E_*X, M)$ is rewritten into the canonical morphism

$$(1.13)\quad \kappa_{E,W_I}=\varphi\kappa_E\colon [X, W_I]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W_I)\to \mathrm{Hom}_{E_*}(E_*X, \pi_*W_I)$$

where M is identified with π_*W_I by the aid of Lemma 1.11.

For each E_*E-comodule I we denote by $\mathrm{injdim}_{E_*E}I$ the injective dimension of I as an E_*E-comodule.

Proposition 1.12. *Let E be a ring spectrum satisfying the property* (F), *and L be an E_*-module with* $\mathrm{injdim}_{E_*E}E_*E\underset{E_*}{\otimes}L\leqq 1$. *Then there exists a quasi E-module spectrum Y such that π_*Y has an E_*-module structure and it is isomorphic to L as E_*-modules.*

Proof. Choose a short exact sequence $0\to L\overset{i}{\to}M\overset{j}{\to}N\to 0$ of E_*-modules with M injective. Set $I=E_*E\underset{E_*}{\otimes}M$ and $J=E_*E\underset{E_*}{\otimes}N$, both of which are injective E_*E-comodules. According to Lemma 1.11 both W_I and W_J are E-module spectra such that the composites $\lambda_{E,M}(\iota\wedge 1)_*\colon \pi_*W_I\to M$ and $\lambda_{E,N}(\iota\wedge 1)_*\colon \pi_*W_J\to N$ are isomorphisms of E_*-modules. Pick up a map $g\colon W_I\to W_J$ such that $\theta\kappa_{E,J}(g)=j\lambda_{E,M}\in \mathrm{Hom}_{E_*}(E_*W_I, N)$, and then consider the cofiber sequence $Y\overset{f}{\to}W_I\overset{g}{\to}W_J$. Since $\lambda_{E,N}(1\wedge g)_*=j\lambda_{E,M}$, there exists a homomorphism $\lambda_{E,L}\colon E_*Y\to L$ of E_*-modules such that $\lambda_{E,M}(1\wedge f)_*=i\lambda_{E,L}\in \mathrm{Hom}_{E_*}(E_*Y, M)$. Denote by $\mu_I\colon E\wedge W_I\to W_I$ and $\mu_J\colon E\wedge W_J\to W_J$ the E-module structure maps of W_I and W_J respectively, which satisfy that $\lambda_{E,M}(1\wedge \mu_I)_*=\lambda_{E,M}(m\wedge 1)_*$ and $\lambda_{E,N}(1\wedge\mu_J)_*=\lambda_{E,N}(m\wedge 1)_*$. As is easily checked, there holds the equality $g\mu_I=\mu_J(1\wedge g)\colon E\wedge W_I\to W_J$. So we get a map $\mu_Y\colon E\wedge Y\to Y$ such that $f\mu_Y=\mu_I(1\wedge f)$ and $\mu_Y(\iota\wedge 1)=1$. Thus Y is a quasi E-module spectrum which gives π_*Y an associative E_*-module structure. Because the cofiber sequence $Y\to W_I\to W_J$ induces a short exact sequence $0\to\pi_*Y\to\pi_*W_I\to\pi_*W_J\to 0$ of E_*-modules. By applying Five lemma we can moreover see that the composite $\lambda_{E,L}(\iota\wedge 1)_*\colon \pi_*Y\to E_*Y\to L$ is an isomorphism of E_*-modules.

Finally we show

Lemma 1.13. *Let E be a ring spectrum satisfying the property* (F) *and W be an E-module spectrum such that the extended comodule $I=E_*E\underset{E_*}{\otimes}W_*$ is injective. Then W coincides with the represented spectrum W_I.*

Proof. Use the natural isomorphism $\theta\kappa_{E,I}$: $[X, W_I]\to \mathrm{Hom}_{E_*E}(E_*X, I)\to \mathrm{Hom}_{E_*}(E_*X, \pi_*W)$ of (1.12). We then get a map $f\colon W\to W_I$ such that $\lambda_{E,W_*}f_*$

$=\mu_*: E_*W \to \pi_*W$, where $\lambda_{E,W_*}=\theta\kappa_{E,I}(1)$ and $\mu: E\wedge W\to W$ denotes the E-module structure map of W. Consider the composite

$$(\theta\kappa_{E,I})^{-1}\kappa_{E,W}: [X, W] \to \mathrm{Hom}_{E_*}(E_*X, \pi_*W) \overset{\cong}{\leftarrow} [X, W_I]$$

which is certainly induced by the map f, where $\kappa_{E,W}=\varphi\kappa_E$ and it is defined to be $\kappa_{E,W}(g)=\mu_*(1\wedge g)_*$. Taking $X=\Sigma^k$ for every k, $\kappa_{E,W}: [\Sigma^k, W]\to \mathrm{Hom}_{E_*}(E_*\Sigma^k, \pi_*W)$ is evidently an isomorphism. Therefore we can easily observe that $f: W \to W_I$ is an equivalence.

2. Injective E_*E-comodules

2.1. Let E be a ring spectrum and F be an E-module spectrum. The E-module structure map $\mu: E\wedge F\to F$ gives rise to homomorphisms $\nu_{E,F}: X_*E\underset{E_*}{\otimes}F_*\to X_*F$ and $\kappa_{E,F}: [X, F]\to \mathrm{Hom}_{E_*}(E_*X, F_*)$ defined in the canonical ways. We are interested in ring spectra E which satisfy some of the following nice properties (see [1] or [2]):

(F) E_*E is flat as an E_*-module.
(K) $\nu_{E,F}: X_*E\underset{E_*}{\otimes}F_*\to X_*F$ is an isomorphism for any E-module spectrum F if E_*X is a flat E_*-module.
(U) $\kappa_{E,F}: [X, F]\to \mathrm{Hom}_{E_*}(E_*X, F_*)$ is an isomorphism for any E-module spectrum F if E_*X is a projective E_*-module.
(P) For every CW-spectrum X there exists a CW-spectrum Y and a map $g: Y\to X$ such that E_*Y is a projective E_*-module and $g_*: E_*Y\to E_*X$ is an epimorphism.

If a ring spectrum E satisfies the property (F), then the condition (K) implies that

(K′) $\nu_{E,F}: E_*E\underset{E_*}{\otimes}F_*\to E_*F$ is an isomorphism for any E-module spectrum F,

and in particular that

(K″) $\nu_{E,\nabla E(D)}: E_*E\underset{E_*}{\otimes}\nabla E(D)_*\to E_*\nabla E(D)$ is an isomorphism for each graded divisible abelian group $D=\{D_n\}$,

because the Anderson dual spectrum $\nabla E(D)$ is an E-module spectrum by Corollary 1.10.

As typical examples of ring spectra satisfying all of the properties (F), (K) ,(U) and (P) the following spectra are well known (see [1] or [2, Proposition III.13.4]):

(2.1) S, HZ/p, MO, MU, MSp, KU and KO.

In this section we will give some characterizations of E_*-injective spectra for such a nice ring spectrum E as in (2.1). Putting their results together we can summarize as follows (cf. [13, Theorem 3 i) and iii)] or [8, §9]).

Theorem 2.1. *Assume that a ring spectrum E satisfies the properties* (F) *and* (K''). *Then the following five conditions are all equivalent:*
i) *W is an E_*-injective spectrum.*
ii) *W is a quasi E-module spectrum such that E_*W is injective as an E_*E-module.*
iii) *W is an E_*-local spectrum such that E_*W is injective as an E_*E-comodule.*
iv) *$\kappa_E: [X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*
v) *$\kappa_E: [X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is a monomorphism for any CW-spectrum X.*

2.2. Let E be a ring spectrum satisfying the property (F). Then the homomorphism $\kappa_E: [X, W] \to \mathrm{Hom}(E_*X, E_*W)$ defined by $\kappa_E(f)=f_*$ is evidently factorized through $\mathrm{Hom}_{E_*E}(E_*X, E_*W)$. Moreover we note that $\nu_{E,E\wedge W}$: $E_*E \underset{E_*}{\otimes} E_*W \to E_*(E \wedge W)$ defined in the canonical way is an isomorphism, even if E is not assumed to satisfy the property (K').

Proposition 2.2. *Let E be a ring spectrum satisfying the property* (F). *If W is a quasi E-module spectrum such that E_*W is injective as an E_*E-comodule, then it is an E_*-injective spectrum and $\kappa_E: [X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*

Proof. Set $I=E_*W$, which is an injective E_*E-comodule. By means of (1.9) and (1.11) there exists an E_*-injective spectrum W_I so that $\kappa_{E,I}: [X, W_I] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is a natural isomorphism for any CW-spectrum X. Taking $X=W$, we get a map $f: W \to W_I$ with $\kappa_{E,I}(f)=1$. Notice the composite

$$\kappa_{E,I}^{-1}\kappa_E: [X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W) \overset{\cong}{\leftarrow} [X, W_I]$$

is exactly induced by the map f. Consider the composite $\varphi\kappa_E: [X, E \wedge W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*(E \wedge W)) \to \mathrm{Hom}_{E_*}(E_*X, E_*W)$ in which φ is defined to be $\varphi(a)=(m \wedge 1)_*a$ and it is an isomorphism because $\nu_{E,E\wedge W}: E_*E \underset{E_*}{\otimes} E_*W \to E_*(E \wedge W)$ is an isomorphism. Taking $X=\Sigma^k$ for every k, it is easily checked that $\kappa_E: [\Sigma^k, E \wedge W] \to \mathrm{Hom}_{E_*E}(E_*\Sigma^k, E_*(E \wedge W))$ is an isomorphism. Since W is a retract of $E \wedge W$, $\kappa_E: [\Sigma^k, W] \to \mathrm{Hom}_{E_*E}(E_*\Sigma^k, E_*W)$ becomes an isomorphism, too. This implies that $f: W \to W_I$ is an equivalence because $\kappa_{E,I}^{-1}\kappa_E = f_*: [X, W] \to [X, W_I]$. Hence we observe that W is an E_*-injective spectrum and moreover $\kappa_E: [X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism.

Adding the condition (K'') on E we can show the following result, which contains the converse of Proposition 2.2.

Proposition 2.3. *Let E be a ring spectrum satisfying the properties* (K$''$) *as well as* (F). *If a CW-spectrum W is E_*-injective, then it is a quasi E-module spectrum such that E_*W is injective as an E_*E-comodule and $\kappa_E\colon [X, W]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*

Proof. From Lemma 1.4 it follows that W is a quasi E-module spectrum. On the other hand, Proposition 1.6 asserts that W is a retract of a certain Anderson dual spectrum $\nabla E(D)=\prod_n \Sigma^n\nabla E(D_n)$ with $D=\{D_n\}$ divisible. So it is sufficient to show our result for $W=\nabla E(D)$. According to Corollary 1.10, $\nabla E(D)$ is an E-module spectrum and $\kappa_{E,D}\colon \pi_*\nabla E(D)\to \mathrm{Hom}(E_*, D)_*$ is an isomorphism of E_*-modules. Under the condition (K$''$) on E the E_*E-comodule $E_*\nabla E(D)$ is isomorphic to the extended comodule $E_*E\otimes_{E_*}\mathrm{Hom}(E_*, D)_*$, which is certainly injective as an E_*E-comodule. Moreover we can easily observe by use of (1.8) that $\kappa_E\colon [X, \nabla E(D)]\to \mathrm{Hom}_{E_*E}(E_*X, E_*\nabla E(D))$ is an isomorphism for any CW-spectrum X.

By making use of Proposition 2.3 we obtain the following result, which resembles Proposition 2.2.

Proposition 2.4. *Let E be a ring spectrum satisfying the properties* (K$''$) *as well as* (F). *If W is an E_*-local spectrum such that E_*W is injective as an E_*E-comodule, then it is an E_*-injective spectrum and $\kappa_E\colon [X, W]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*

Proof. Set $I=E_*W$, then there exists an E_*-injective spectrum W_I so that $\kappa_{E,I}\colon [X, W_I]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is a natural isomorphism for any CW-spectrum X. On the other hand, Proposition 2.3 asserts that $\kappa_E\colon [X, W_I]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W_I)$ is also an isomorphism for any CW-spectrum X. Taking $X=W$ in the former and $X=W_I$ in the latter, we get maps $f\colon W\to W_I$ and $g\colon W_I\to W_I$ such that $\lambda_{E,I}f_*=1\colon E_*W\to E_*W$ and $g_*=f_*\lambda_{E,I}\colon E_*W_I\to E_*W_I$ where $\lambda_{E,I}=\kappa_{E,I}(1)\in \mathrm{Hom}_{E_*E}(E_*W_I, E_*W)$. As is easily checked, the map g becomes just the identity. This implies that $f_*\colon E_*W\to E_*W_I$ is an isomorphism. Since both W and W_I are E_*-local spectra, we then observe that $f\colon W\to W_I$ becomes an equivalence. Thus W is an E_*-injective spectrum and moreover $\kappa_E\colon [X, W]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism.

Proof of Theorem 2.1. The implication i)→ii)→iii)→i) is immediately shown by use of Propositions 2.3 and 2.4. On the other hand, the implication i)→iv)→v)→i) is done by use of Propositions 2.3 and 1.1.

2.3. Making use of Proposition 2.3 again we can realize every injective E_*E-comodule by a certain E_*-injective spectrum when E satisfies the properties (F) and (K$''$) (cf. Lemma 1.11).

Proposition 2.5. *Let E be a ring spectrum satisfying the properties* (F) *and* (K″) *and I be an injective E_*E-comodule. Then the represented spectrum W_I is E_*-injective such that $\lambda_{E,I}: E_*W_I \to I$ is an isomorphism of E_*E-comodules.*

Proof. Choose a graded divisible abelian group D so that I is embedded into $M = \mathrm{Hom}(E_*, D)_*$ as E_*-modules. Consider the monomorphism $i: I \to E_*E \underset{E_*}{\otimes} I \to E_*E \underset{E_*}{\otimes} M$ of E_*E-comodules, which has a left inverse $j: E_*E \underset{E_*}{\otimes} M \to I$. Identify the injective E_*-module M with $\pi_*\nabla E(D)$ by use of Corollary 1.10, and the extended comodule $E_*E \underset{E_*}{\otimes} \nabla E(D)_*$ with $E_*\nabla E(D)$ under our assumption on (K″). Then we get maps $f: W_I \to \nabla E(D)$ and $g: \nabla E(D) \to W_I$ making the following diagram commutative

$$\begin{array}{ccccc} [X, W_I] & \xrightarrow{f_*} & [X, \nabla E(D)] & \xrightarrow{g_*} & [X, W_I] \\ \downarrow \kappa_{E,I} & & \downarrow \kappa_E & & \downarrow \kappa_{E,I} \\ \mathrm{Hom}_{E_*E}(E_*X, I) & \xrightarrow[i_*]{} & \mathrm{Hom}_{E_*E}(E_*X, E_*\nabla E(D)) & \xrightarrow[j_*]{} & \mathrm{Hom}_{E_*E}(E_*X, I) \end{array}$$

where the vertical arrows $\kappa_{E,I}$ and κ_E are all isomorphisms by (1.9) and Proposition 2.3. Note that the composite $gf: W_I \to W_I$ is exactly the identity. Since $f_* = i\lambda_{E,I}: E_*W_I \to E_*\nabla E(D)$ and $j = \lambda_{E,I}g_*: E_*\nabla E(D) \to I$, it is easily checked that $\lambda_{E,I}: E_*W_I \to I$ is an isomorphism of E_*E-comodules where $\lambda_{E,I} = \kappa_{E,I}(1)$.

By a quite similar discussion to [8, Theorem 10.1] using Proposition 2.5 with (1.9) we can show

Corollary 2.6. *Let E be a ring spectrum satisfying the properties* (F) *and* (K″) *and C be an E_*E-comodule with* $\mathrm{injdim}_{E_*E} C \leqq 2$. *Then there exists an E_*-local spectrum Z such that E_*Z is isomorphic to C as E_*E-comodules.*

Combining Propositions 2.4 and 2.5 with (1.10) we obtain

Proposition 2.7. *Let E be a ring spectrum satisfying the properties* (F) *and* (K″) *and $\{I_\lambda\}$ be a family of injective E_*E-comodules. Then W is an E_*-local spectrum such that E_*W is isomorphic to the direct product $\prod_\lambda I_\lambda$ as E_*E-comodules if and only if it coincides with the direct product $\prod_\lambda W_{I_\lambda}$.*

Proof. The "if" part follows immediately from (1.10) and Proposition 2.5. On the other hand, the "only if" part is easily shown by observing (1.10) and the proof of Proposition 2.4.

2.4. Let E be a ring spectrum satisfying the properties (F) and (K′) and W be an E-module spectrum such that the extended comodule $E_*E \underset{E_*}{\otimes} W_*$ is injective. Then the E_*E-comodule E_*W is injective since $\nu_{E,W}: E_*E \underset{E_*}{\otimes} W_* \to$

E_*W is an isomorphism under the condition (K'). So Proposition 2.2 asserts that W is an E_*-injective spectrum and κ_E: $[X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X. Conversely we obtain

Lemma 2.8. *Let E be a ring spectrum satisfying the property* (F) *and W be an E-module spectrum such that the extended comodule $E_*E \otimes_{E_*} W_*$ is injective. Assume that κ_E: $[X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X. Then W is an E_*-injective spectrum such that $\nu_{E,W}$: $E_*E \otimes_{E_*} W_* \to E_*W$ is an isomorphism of E_*E-comodules.*

Proof. According to Lemma 1.13 W may be actually regarded as the represented spectrum W_I with $I = E_*E \otimes_{E_*} W_*$, which is E_*-injective. Under our assumption on κ_E (1.13) implies that κ_E: $[X, W] \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is decomposed into the composite $\nu\kappa_{E,I}$: $[X, W] \to \mathrm{Hom}_{E_*E}(E_*X, I) \to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ where ν is induced by $\nu_{E,W}$: $I \to E_*W$. Then it is obvious that $\nu_{E,W}$: $I \to E_*W$ becomes a left inverse of $\lambda_{E,I} = \kappa_{E,I}(1)$. In order to show that $\nu_{E,W}$ is a right inverse of $\lambda_{E,I}$ we here use the canonical isomorphism θ: $\mathrm{Hom}_{E_*E}(C, I) \to \mathrm{Hom}_{E_*}(C, \pi_*W)$ for any E_*E-comodule C. By using (1.13) again we note that $\theta(\lambda_{E,I}) = \mu_*$: $E_*W \to \pi_*W$ under our assumption on κ_E where μ: $E \wedge W \to W$ denotes the E-module structure map of W. Taking $C = I$, it is easily computed that $\theta(\lambda_{E,I}\nu_{E,W}) = \mu_*\nu_{E,W} = \theta(1)$. Consequently $\nu_{E,W}$: $I \to E_*W$ is in fact an isomorphism of E_*E-comodules.

Lemma 2.8 combined with Lemma 1.13 and Proposition 2.3 shows

Corollary 2.9. *Let E be a ring spectrum satisfying the properties* (F) *and* (K'') *and W be an E-module spectrum such that the extended comodule $E_*E \otimes_{E_*} W_*$ is injective. Then W is an E_*-injective spectrum such that $\nu_{E,W}$: $E_*E \otimes_{E_*} W_* \to E_*W$ is an isomorphism of E_*E-comodules.*

We will here deal with a ring spectrum E satisfying the properties (F), (U) and (P) in place of (F) and (K'').

Lemma 2.10. *Let E be a ring spectrum satisfying the properties* (F) *and* (U). *If E_*Y is projective as an E_*-module, then κ_E: $[Y, W] \to \mathrm{Hom}_{E_*E}(E_*Y, E_*W)$ is an isomorphism for any quasi E-module spectrum W.*

Proof. Consider the composite $\varphi\kappa_E$: $[Y, E \wedge W] \to \mathrm{Hom}_{E_*E}(E_*Y, E_*(E \wedge W)) \to \mathrm{Hom}_{E_*}(E_*Y, E_*W)$. Here φ is an isomorphism under the condition (F), and $\varphi\kappa_E$ is an isomorphism under the condition (U). Consequently κ_E: $[Y, W] \to \mathrm{Hom}_{E_*E}(E_*Y, E_*W)$ becomes an isomorphism when W is a quasi E-module spectrum.

Using Lemmas 1.4 and 2.10 we obtain

Proposition 2.11. *Let E be a ring spectrum satisfying the properties* (F), (U) *and* (P). *If a CW-spectrum W is E_*-injective, then κ_E: $[X, W]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ is an isomorphism for any CW-spectrum X.*

Proof. Under the condition (P) we can choose for any CW-spectrum X a cofiber sequence $Z\to Y\to X$ such that $0\to E_*Z\to E_*Y\to E_*X\to 0$ is a short exact sequence and E_*Y is projective as an E_*-module. Use the commutative diagram

$$\begin{array}{ccccccc} & & [X, W] & \to & [Y, W] & \to & [Z, W] \\ & & \downarrow \kappa_E & & \downarrow \kappa_E & & \downarrow \kappa_E \\ 0 & \to & \mathrm{Hom}_{E_*E}(E_*X, E_*W) & \to & \mathrm{Hom}_{E_*E}(E_*Y, E_*W) & \to & \mathrm{Hom}_{E_*E}(E_*Z, E_*W) \end{array}$$

with two exact rows. The vertical arrows κ_E are all monomorphisms by Proposition 1.1, and in particular the central one is an isomorphism by means of Lemmas 1.4 and 2.10. Applying Four lemma we see that the left arrow κ_E: $[X, W]\to \mathrm{Hom}_{E_*E}(E_*X, E_*W)$ becomes an epimorphism, and hence an isomorphism as desired.

Putting Corollary 1.10, Lemma 2.8 and Proposition 2.11 together we can finally remark that

(2.2) a ring spectrum E satisfies the condition (K″) if it satisfies (F), (U) and (P).

3. EG_*-injective spectra

3.1. Among abelian groups we introduce a partial order as follows. Given abelian groups G and G' we write

$$\langle G\rangle \leqq \langle G'\rangle \tag{3.1}$$

if for each divisible abelian group D there exists a divisible one D' so that Hom (G,D) is a direct summand of Hom (G', D').

Let G and D be abelian groups. If D is divisible, then the short exact sequence

$$0 \to \mathrm{Hom}(G/\mathrm{Tor}\, G, D) \to \mathrm{Hom}(G, D) \to \mathrm{Hom}(\mathrm{Tor}\, G, D) \to 0 \tag{3.2}$$

is split. Here $\mathrm{Hom}(G/\mathrm{Tor}\, G, D)$ is a maximal divisible subgroup of $\mathrm{Hom}(G, D)$ and $\mathrm{Hom}(\mathrm{Tor} G, D)$ is a reduced algebraically compact group (see [9, Theorem 46.1]). Recall that every divisible abelian group D is written into the direct sum of indecomposable ones which are either isomorphic to Q or Z/p^∞ for various primes p. Using these facts we can show

Lemma 3.1. *$\langle G\rangle \leqq \langle G'\rangle$ if and only if for every prime p there exists a divisible abelian group D_p so that* $\mathrm{Hom}(G, Z/p^\infty)$ *is a direct summand of* $\mathrm{Hom}(G', D_p)$ *and in addition $G\otimes Q=0$ whenever $G'\otimes Q=0$.*

Proof. The "if" part: Obviously $\mathrm{Hom}(G, Q)$ is a direct summand of $\mathrm{Hom}(G', D_0)$ with $D_0=\mathrm{Hom}(G, Q)$. Each divisible abelain group D is written into the form of $(\bigoplus_{\alpha_0} Q) \oplus \bigoplus_p (\bigoplus_{\alpha_p} Z/p^\infty)$ and it is a direct summand of the direct product $(\prod_{\alpha_0} Q) \oplus \prod_p (\prod_{\alpha_p} Z/p^\infty)$. So the result is immediately obtained.

The "only if" part: Assume that $G'\otimes Q=0$. By definition $\mathrm{Hom}(G, Q)$ is a direct summand of $\mathrm{Hom}(G', D_0)$ for some divisible D_0. However $\mathrm{Hom}(G', D_0)$ is reduced, in other words $\mathrm{Hom}(Q, \mathrm{Hom}(G', D_0))=0$. This implies that $\mathrm{Hom}(G, Q)=0$, and hence $G\otimes Q=0$.

For each abelian group G we consider the subset

$$J(G) = \{p; (G/\mathrm{Tor}\, G)\otimes Z/p \neq 0\} \tag{3.3}$$

of primes p and set

$$\tilde{J}(G) = J(G) \cup \{0\} \text{ or } \phi \tag{3.4}$$

according as either $G\otimes Q\neq 0$ or $G\otimes Q=0$. For any divisible abelian group D, $\mathrm{Hom}(G/\mathrm{Tor}\, G, D)*Z/p$ is isomorphic to $\mathrm{Hom}(G/\mathrm{Tor}\, G\otimes Z/p, D*Z/p)$. Hence it is easily seen that

$$\mathrm{Hom}(G/\mathrm{Tor}\, G, D) \cong (\bigoplus_{\alpha_0} Q) \oplus \bigoplus_{p\in J(G)} (\bigoplus_{\alpha_p} Z/p^\infty) \tag{3.5}$$

where $\alpha_p\neq 0$ if $D*Z/p\neq 0$. In particular, taking $D=Z/p^\infty$ we have

$$\mathrm{Hom}(G/\mathrm{Tor}\, G, Z/p^\infty) \cong \begin{cases} (\bigoplus_{\alpha_0} Q)\oplus(\bigoplus_{\alpha_p} Z/p^\infty) & \text{if } p\in J(G) \\ \bigoplus_{\alpha_0} Q & \text{if } p\notin J(G) \end{cases} \tag{3.6}$$

where $\alpha_p\neq 0$.

For every prime p we can choose a p-basic subgroup B_p of $\mathrm{Tor}\, G$ which is unique up to isomorphism [9, Theorems 32.3 and 35.2]. The p-basic subgroup B_p is written into the form of a direct sum of cyclic p-groups Z/p^k and it is p-pure in $\mathrm{Tor}\, G$, and moreover $(\mathrm{Tor}\, G/B_p)\otimes Z_{(p)}$ is expressed as the direct sum of copies of Z/p^∞. Consider the subset

$$J_p(G) = \{k; Z/p^k \text{ is a direct sum component of } B_p\} \tag{3.7}$$

of positive integers k.

When $J_p(G)$ is an infinite set, the p-adic integers $Z_p^\wedge$ is embedded into the direct product $\prod_{k\in J_p(G)} Z/p^k$ as a direct summand (cf. [9, Corollary 38.2]). We here put

$$(3.8)\quad \tilde{J}_p(G)=\begin{cases} J_p(G) & \text{if } J_p(G) \text{ is finite and } (\mathrm{Tor}\, G/B_p)\otimes Z_{(p)}=0 \\ J_p(G)\cup\{\omega_p\} & \text{if otherwise.}\end{cases}$$

For any p-local divisible abelian group $D_{(p)}$ the short exact sequence

$$(3.9)\quad 0\to \mathrm{Hom}(\mathrm{Tor}\, G/B_p, D_{(p)})\to \mathrm{Hom}(\mathrm{Tor}\, G, D_{(p)})\to \mathrm{Hom}(B_p, D_{(p)})\to 0$$

is split because it is pure owing to [9, Proposition 44.7] and Hom(Tor G/B_p, $D_{(p)}$) is algebraically compact by [9, Theorem 44.7]. As is easily seen, we have

$$(3.10)\qquad \mathrm{Hom}(B_p, D_{(p)})\cong \prod_{k\in J_p(G)}\left(\prod_{\beta_k} D_{(p)}*Z/p^k\right)$$

with $\beta_k\neq 0$. On the other hand, Hom(Tor G/B_p, $D_{(p)}$) is the p-adic completion of the direct sum of copies of $Z_p^\wedge$(use [9, Exercise 47.7]). In particular, taking $D_{(p)}=Z/p^\infty$ we obtain

$$(3.11)\quad \mathrm{Hom}(\mathrm{Tor}\, G, Z/p^\infty)\cong\begin{cases} \prod_{k\in J_p(G)}\left(\prod_{\beta_k} Z/p^k\right)\oplus\prod_{\beta_\omega} Z_p^\wedge & \text{if } (\mathrm{Tor}\, G/B_p)\otimes Z_{(p)}\neq 0 \\ \prod_{k\in J_p(G)}\left(\prod_{\beta_k} Z/p^k\right) & \text{if } (\mathrm{Tor}\, G/B_p)\otimes Z_{(p)}=0\end{cases}$$

where $\beta_k\neq 0$ and $\beta_\omega\neq 0$ (cf. [9, Theorem 47.1]).

3.2. By applying Lemma 3.1 with the aid of (3.2), (3.6), (3.9), (3.10) and (3.11) we will prove the following criterion (cf. [5]).

Proposition 3.2. *$\langle G\rangle\leqq\langle G'\rangle$ if and only if $\tilde{J}(G)\subset\tilde{J}(G')$ and $\tilde{J}_p(G)\subset \tilde{J}_p(G')$ for every prime p.*

Proof. The "if" part: If $G'\otimes Q=0$, then $\tilde{J}(G')=\phi$ by definition. This implies immediately that $G\otimes Q=0$. We will next show that for each prime p there exist divisible abelian groups D_p' and D_p'' so that Hom(G/Tor G, Z/p^∞) and Hom(Tor G, Z/p^∞) are respectively direct summands of Hom(G'/Tor G', D_p') and Hom(TorG', D_p''). In order to find a suitable divisible D_p' we may assume that $G'\otimes Q\neq 0$. Evidently Q is a direct summand of Hom(G'/Tor G', Q). If $p\in J(G)$, then (3.6) shows that Z/p^∞ is contained in Hom(G'/Tor G', Z/p^∞) as a direct summand because $J(G)\subset J(G')$. Taking $D_p'=(\prod_{\alpha_0} Q)\oplus(\prod_{\alpha_p} Z/p^\infty)$ or $\prod_{\alpha_0} Q$ according as $p\in J(G)$ or $p\notin J(G)$, it is easily seen that Hom(G/Tor G, Z/p^∞) expressed as in (3.6) becomes a direct summand of Hom(G'/Tor G', D_p').

Using (3.11) in place of (3.6) we will similarly find a suitable divisible D_p''. If $k\in J_p(G)$, then (3.11) shows that Z/p^k is contained in Hom(Tor G', Z/p^∞) as a direct summand because $J_p(G)\subset J_p(G')$. Assume that (Tor $G/B_p)\otimes Z_{(p)}\neq 0$. In this situation $J_p(G')$ is an infinite set or (Tor $G'/B_p')\otimes Z_{(p)}\neq 0$ because the element ω_p is belonging to $\tilde{J}_p(G)\subset\tilde{J}_p(G')$. Set $D_\omega=\prod_{k\in J_p(G')} Z/p^\infty$ or Z/p^∞ according as $J_p(G')$ is infinite or (Tor $G'/B_p')\otimes Z_{(p)}\neq 0$. By use of (3.11) we see easily that Hom(Tor G', D_ω) contains $Z_p^\wedge$ as a direct summand. Taking $D_p''=\prod_{k\in J_p(G)}(\prod_{\beta_k} Z/p^\infty)\oplus\prod_{\beta_\omega} D_\omega$, it follows immediately that Hom(Tor G, Z/p^∞) expressed as in (3.11) is a direct summand of Hom(Tor G', D_p'') as desired. By the aid of (3.2) we may now apply Lemma 3.1 to obtain $\langle G\rangle\leqq\langle G'\rangle$.

The "only if" part: For every prime p we choose a divisible abelian group D_p so that Hom(G, Z/p^∞) is a direct summand of Hom(G', D_p). As is easily verified, Hom(G/Tor G, Z/p^∞) and Hom(Tor G, Z/p^∞) are respectively direct summands of Hom(G'/Tor G', D_p) and Hom(Tor G', D_p). When $J(G)=\phi$, it follows from Lemma 3.1 that $\tilde{J}(G)\subset\tilde{J}(G')$. Assume that $J(G)\neq\phi$. For each prime $p\in J(G)$, (3.6) asserts that as a direct summand Z/p^∞ is contained in Hom(G/Tor G,Z/p^∞) and also in Hom(G'/Tor G', D_p). Therefore we see that Hom(G'/Tor $G'\otimes Z/p$, $D_p)\neq 0$, and hence $p\in J(G')$. Thus $J(G)\subset J(G')$, which implies that $\tilde{J}(G)\subset\tilde{J}(G')$.

Pick up a positive integer $k\in J_p(G)$ for a fixed prime p. By virtue of (3.11) Z/p^k is contained in Hom(Tor G, Z/p^∞) as a direct summand and also in Hom(Tor G', $D_p\otimes Z_{(p)}$). However Z/p^k is actually embedded into Hom(B_p', $D_p\otimes Z_{(p)}$) because Hom(Tor G'/B_p', $D_p\otimes Z_{(p)}$) is torsion free. Using (3.10) we can easily see that $k\in J_p(G')$. Thus $J_p(G)\subset J_p(G')$. In order to show $\tilde{J}_p(G)\subset\tilde{J}_p(G')$ we here assume that $\tilde{J}_p(G)$ includes the element ω_p, thus $J_p(G)$ is an infinite set or (Tor $G/B_p)\otimes Z_{(p)}\neq 0$. For our purpose we may consider only the case when $J_p(G')$ is finite and (Tor $G/B_p)\otimes Z_{(p)}\neq 0$. In this situation $Z_p^\wedge$ is contained in Hom(Tor G, Z/p^∞) as a direct summand by use of (3.11), and hence in Hom(Tor G', $D_p\otimes Z_{(p)}$). If Hom(Tor G'/B_p', $D_p\otimes Z_{(p)})=0$, then Hom(TorG', $D_p\otimes Z_{(p)})\cong$Hom(B_p', $D_p\otimes Z_{(p)}$) is bounded because of (3.10). This is a contradiction. Thus Hom(Tor G'/B_p', $D_p\otimes Z_{(p)})\neq 0$ which implies that (Tor $G'/B_p')\otimes Z_{(p)}\neq 0$. Therefore the element ω_p is belonging to $\tilde{J}_p(G')$ in our case. Consequently $\tilde{J}_p(G)\subset\tilde{J}_p(G')$.

3.3. For any CW-spectra E and F we write

$$\langle E\rangle_I\leqq\langle F\rangle_I \tag{3.12}$$

if every E_*-injective spectrum W becomes always F_*-injective (cf. [6], [7] or [14]).

Lemma 3.3. *$\langle E\rangle_I\leqq\langle F\rangle_I$ if and only if every F_*-monic map $f\colon X\to Y$*

is always E_*-monic.

Proof. The "if" part is immediate by definition.

The "only if" part: Given an F_*-monic map $f: X \to Y$ we may choose an E_*-injective spectrum W and an E_*-monic map $g: X \to W$ by virtue of Proposition 1.5. Since W becomes F_*-injective, there exists a map $h: Y \to W$ with $hf=g$. This equality implies that $f: X \to Y$ is E_*-monic.

As a relation between the partial orders (3.1) and (3.12) we have

Lemma 3.4. i) *If $\langle G\rangle \leqq \langle G'\rangle$, then $\langle EG\rangle_I \leqq \langle EG'\rangle_I$ for any CW-spectrum E.*

ii) *Assume that π_*E is torsion free and $\pi_*E \otimes Z/p \neq 0$ for each prime p. Then the converse of* i) *is valid.*

Proof. i) is immediate from Proposition 1.6 and Definitions (3.1) and (3.12).

ii) We may assume that $\pi_0 E \otimes Z/p \neq 0$ for a fixed prime p. Set $A_p = \mathrm{Hom}(G, Z/p^\infty)$. By means of (1.3) and (1.4) the Anderson dual spectrum $\nabla E(A_p)$ becomes EG_*-injective, and hence it is EG'_*-injective. According to Proposition 1.6 $\nabla E(A_p)$ is a retract of a certain Anderson dual spectrum $\nabla E(B) = \prod_n \Sigma^n \nabla E(B_n)$ where $B_n = \mathrm{Hom}(G', D_n)$ for some divisible D_n. So $\pi_0 \nabla E(A_p)$ becomes a direct summand of $\pi_0 \nabla E(B) = \prod_n \pi_{-n} \nabla E(B_n)$. As is easily computed, $\pi_0 \nabla E(A_p) \cong \mathrm{Hom}(\pi_0 E, A_p) \cong \mathrm{Hom}(G, \mathrm{Hom}(\pi_0 E, Z/p^\infty))$ and similarly $\pi_{-n} \nabla E(B_n) \cong \mathrm{Hom}(G', \mathrm{Hom}(\pi_n E, D_n))$. By (3.5) the divisible group $\mathrm{Hom}(\pi_0 E, Z/p^\infty)$ contains Z/p^∞ as a direct summand under the assumption that $\pi_0 E \otimes Z/p \neq 0$. Therefore $A_p = \mathrm{Hom}(G, Z/p^\infty)$ is contained in $\pi_0 \nabla E(A_p)$ as a direct summand, and hence in $\pi_0 \nabla E(B) \cong \mathrm{Hom}(G', \tilde{D}_p)$ where $\tilde{D}_p = \prod_n \mathrm{Hom}(\pi_n E, D_n)$ and it is divisible.

Similarly we can choose a divisible abelian group $\tilde{D}_0$ so that $A_0 = \mathrm{Hom}(G, Q)$ is a direct summand of $\mathrm{Hom}(G', \tilde{D}_0)$. This implies that $G \otimes Q = 0$ whenever $G' \otimes Q = 0$, as was shown in the proof of Lemma 3.1. Consequently it follows from Lemma 3.1 that $\langle G\rangle \leqq \langle G'\rangle$.

Combining Proposition 3.2 with Lemma 3.4 we obtain

Theorem 3.5. i) *If $\tilde{J}(G) \subset \tilde{J}(G')$ and $\tilde{J}_p(G) \subset \tilde{J}_p(G')$ for every prime p, then $\langle EG\rangle_I \leqq \langle EG'\rangle_I$ for any CW-spectrum E.*

ii) *Assume that π_*E is torsion free and $\pi_*E \otimes Z/p \neq 0$ for each prime p. Then the converse of* i) *is valid.*

As an immediate result we have the following criterion (cf. [7, Proposition 2.7]).

Corollary 3.6. *Assume that π_*E is torsion free and $\pi_*E \otimes Z/p \neq 0$ for* *each prime p. Then* $\langle EG\rangle_I=\langle EG'\rangle_I$ *if and only if* $\tilde{J}(G)=\tilde{J}(G')$ *and* $\tilde{J}_p(G)=\tilde{J}_p(G')$ *for every prime p.*

3.4. Let us denote by H the Eilenberg-MacLane spectrum and by KO, KU and KT the real, the complex and the self-conjugate K-spectrum respectively. Recall that $\nabla H(G)=HG$, $\nabla KO(G)=\Sigma^4 KOG$, $\nabla KU(G)=KUG$ and $\nabla KT(G)=\Sigma^1 KTG$ for any abelian group G (see [4] or [15, I]). Note that the canonical map $\omega: \bigvee_n \Sigma^n HA_n \to \prod_n \Sigma^n HA_n$ is an equivalence for any graded abelian group $A=\{A_n\}$. On the other hand, it is well known that the K-spectra $K=KO$, KU and KT possess the period $p(K)=8$, 2 and 4 respectively.

Taking $E=H$, KO, KU or KT in Proposition 1.6 we can immediately show

Proposition 3.7. i) *A CW-spectrum W is* HG_**-injective if and only if it is a retract of a generalized Eilenberg-MacLane spectrum* $\bigvee_n \Sigma^n HB_n$ *in which* $B_n=Hom(G, D_n)$ *for some divisible* D_n.

ii) *Let K denote the periodic K-spectrum KO, KU or KT. Then a CW-spectrum W is* KG_**-injective if and only if it is a retract of a certain finite wedge sum* $\bigvee_{0\leq n<p(K)} \Sigma^n KB_n$ *in which* $B_n=\mathrm{Hom}(G, D_n)$ *for some divisible* D_n.

The following easy result is useful in studying the EG_*-injectivity when $E=H$ or KU.

Lemma 3.8. i) *If W is a quasi H-module spectrum, then it is a generalized Eilenberg-MacLane spectrum written into the form* $\bigvee_n \Sigma^n H(\pi_n W)$.

ii) *If W is a quasi KU-module spectrum such that the composite* $\mu_*\beta^n(\iota\wedge 1)_*: \pi_i W\to KU_i W\cong KU_{2n+i}W\to \pi_{2n+i}W$ $(i=0, 1)$ *is an isomorphism for any n, then it is written into the wedge sum* $KU(\pi_0 W)\vee\Sigma^1 KU(\pi_1 W)$.

Proof. i) Our proof is due to [2, Lemma III.6.1]). Set $A_n=\pi_n W$ and choose a map $f_n: \Sigma^n SA_n\to W$ inducing the identity isomorphism between the n-th homotopy groups. Construct a map $f: \bigvee_n \Sigma^n HA_n\to W$ whose n-th component is the composite $\mu(1\wedge f_n): \Sigma^n HA_n\to H\wedge W\to W$ where $\mu: H\wedge W\to W$ denotes the H-module structure map of W. As is easily checked, the map f is an equivalence.

ii) is similarly shown to i).

Conserning the HG_*-injectivity we obtain the following characterization.

Thoerem 3.9. *The following three conditions are equivalent:*

i) *W is an* HG_**-injective spectrum.*

ii) *W is a quasi H-module spectrum such that for each n* $\pi_n W$ *is a direct summand of* $\mathrm{Hom}(G, D_n)$ *with* D_n *divisible.*

iii) *W is a generalized Eilenberg-MacLane spectrum* $\bigvee_n \Sigma^n HA_n$ *in which* A_n *is a direct summand of* $\mathrm{Hom}(G, D_n)$ *with* D_n *divisible.*

Proof. The implication iii)→i)→ii) follows immediately from Proposition 3.7 i), and the implication ii)→iii) is immediate from Lemma 3.8 i).

Let G be an abelian group whose torsion subgroup Tor G is bounded. Since the bounded torsion group Tor G is a direct sum of cyclic torsion groups, the sets $J_p(G)$ are finite but they are empty except only finite numbers of primes p. By means of (3.2) and (3.5) we then observe

$$\text{(3.13)} \qquad \mathrm{Hom}(G, D) \cong (\bigoplus_{\alpha_0} Q) \oplus \bigoplus_{p \in J(G)} (\bigoplus_{\alpha_p} Z/p^\infty) \oplus \bigoplus_p \bigoplus_{k \in J_p(G)} (\bigoplus_{\beta_{p,k}} Z/p^k)$$

for each divisible abelian group D, where $\alpha_p \neq 0$ and $\beta_{p,k} \neq 0$ if $D * Z/p \neq 0$.

Corollary 3.10. *Assume that the torsion subgroup* Tor G *is bounded. Then a CW-spectrum W is HG_*-injective if and only if it is a generalized Eilenberg-MacLane spectrum $\bigvee_n \Sigma^n HA_n$ in which each A_n is a direct sum of divisible groups and cyclic torsion groups as given in the right side of* (3.13).

3.5. Concerning the KUG_*-injectivity we obtain the following characterization under some restriction to G.

Theorem 3.11. *Assume that the torsion subgroup* Tor G *is bounded. The following three conditions are equivalent:*
i) *W is a KUG_*-injective spectrum.*
ii) *W is a quasi KU-module spectrum such that $KU \wedge W$ is KUG_*-injective.*
iii) *W is a quasi KU-module spectrum such that $KU_i W$ is a direct sum of divisible groups and cyclic torsion groups as given in the right side of* (3.13).

Proof. By virtue of Corollary 3.6 we may regard as Tor G is finitely generated.

The implication i)→ii): According to Proposition 3.7 ii), W is a retract of the wedge sum $KUB_0 \vee \Sigma^1 KUB_1$ for some $B_i = \mathrm{Hom}(G, D_i)$ with D_i divisible ($i=0, 1$). Recall that $KU_0 KU$ is a countable free abelian group and $KU_1 KU = 0$ (see [3, Theorem 2.1]). Then Lemma 3.8 ii) shows that the smash product $KU \wedge KU$ is just the wedge sum $\vee KU$ of countable copies of KU. Therefore $KU \wedge W$ is a retract of the wedge sum $KU(\oplus B_0) \vee \Sigma^1 KU(\oplus B_1)$. Consider the short exact sequence $0 \to \bigoplus_\lambda \mathrm{Hom}(G, D_\lambda) \to \mathrm{Hom}(G, \bigoplus_\lambda D_\lambda) \to C \to 0$ with D_λ divisible. Since Tor G may be regarded to be finitely generated, the canonical homomorphism $\bigoplus_\lambda \mathrm{Hom}(\mathrm{Tor}\, G, D_\lambda) \to \mathrm{Hom}(\mathrm{Tor}\, G, \bigoplus_\lambda D_\lambda)$ is an isomorphism. Hence we get a short exact sequence $0 \to \bigoplus_\lambda \mathrm{Hom}(G/\mathrm{Tor}\, G, D_\lambda) \to \mathrm{Hom}(G/\mathrm{Tor}\, G, \bigoplus_\lambda D_\lambda)$ $\to C \to 0$, which is clearly split. This implies that the previous short exact sequence is split, too. Consequently we see that $\oplus B_i = \oplus \mathrm{Hom}(G, D_i)$ is a direct summand of $\tilde{B}_i = \mathrm{Hom}(G, \oplus D_i)$ for each $i=0, 1$. So $KU \wedge W$ is KUG_*-

injective by use of Proposition 3.7 ii) again.

The implication ii)→iii) follows immediately from Proposition 3.7 ii) and (3.13).

The implication iii)→i): By Lemma 3.8 ii) $KU\wedge W$ is just the wedge sum $KUA_0\vee\Sigma^1 KUA_1$ with $A_i=KU_iW$ ($i=0, 1$). From Proposition 3.7 ii) and (3.13) it is immediate that $KU\wedge W$ is KUG_*-injective. Then W becomes also KUG_*-injective since W is a retract of $KU\wedge W$.

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