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# $E_{*}$-INJECTIVE SPECTRA AND INJECTIVE $E_{*}$ E-COMODULES 

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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## 0. Introduction

In [13] Ohkawa introduced the notion of the injective hull of spaces and spectra with respect to homology and proved the existence theorem [13, Theorem 1]. Following [13, Definition 1 i)] we call a CW-spectrum $W E_{*}$-injective if any map $f: X \rightarrow Y$ induces an epimorphism $f^{*}:[Y, W] \rightarrow[X, W]$ whenever $f_{*}: E_{*} X \rightarrow E_{*} Y$ is a monomorphism, for a fixed CW-spectrum $E$. A CWspectrum $W$ is $E_{*}$-injective if and only if the homomorphism $\kappa_{E}:[X, W] \rightarrow$ $\operatorname{Hom}\left(E_{*} X, E_{*} W\right)$ defined by $\kappa_{E}(f)=f_{*}$ is a monomorphism for any CW spectrum $X$ (see [13, Proposition 7]). In this note we will be concerned about $E_{*}$-injective spectra.

For each CW-spectrum $X, E_{*} X$ is regarded as a module over the algebra $E^{*} E$ of cohomology operations. Under the restriction that $E$ is finite, Ohkawa [13, Theorem 3 i) and iii)] gave the following characterization.

Theorem 0. Assume that a $C W$-spectrum $E$ is finite. Then the following conditions are equivalent:
i) $W$ is an $E_{*}$-injective spectrum.
ii) $W$ is an $E_{*}$-local spectrum such that $E_{*} W$ is injective as an $E^{*} E$-module.
iii) $\left.\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E_{E}} E^{( } E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.

According to [2, Proposition III.13.4] (or see [1]), the well known ring spectra $E=S, H Z \mid p, M O, M U, M S p, K U$ and $K O$ satisfy some of nice properties as stated in the beginning of $\S 2$. For example, $E_{*} E$ becomes flat as an $E_{*}$-module, and then $E_{*} X$ may be regarded as a comodule over the coalgebra $E_{*} E$. In $\S 2$ we will prove the following result (Theorem 2.1 ) for such a nice ring spectrum $E$, corresponding to Theorem 0 for a finite spectrum $E$.

Theorem 1. Let $E$ be a ring spectrum such that $E_{*} E$ is flat as an $E_{*}-m o-$ dule. Assume that $E$ satisfies the property $\left(\mathrm{K}^{\prime \prime}\right)$ stated in the beginning of $\S 2$. Then the following conditions are equivalent:
i) $W$ is an $E_{*}$-injective spectrum.
ii) $W$ is an $E_{*}$-local spectrum such that $E_{*} W$ is injective as an $E_{*} E$-comodule.
iii) $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E_{* E}}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.

In $\S 3$ we will next study the $E G_{*}$-injectivity where $E G$ denotes the $C W$ spectrum $E_{\wedge} S G$ with coefficients in $G$. There exists a partial order among $C W$-spectra by writing $\langle E\rangle_{I} \leqq\langle F\rangle_{I}$ when each $E_{*}$-injective spectrum is $F_{*^{-}}$ injective. In order to decide $\langle E G\rangle_{I} \leqq\left\langle E G^{\prime}\right\rangle_{I}$ we will find a certain relation between the abelian groups $G$ and $G^{\prime}$ (Theorem 3.5). Moreover we will prove the following complete result (Theorem 3.9), especially when $E=H$ "the Eilenberg-MacLane spectrum''.

Theorem 2. A $C W$-spectrum $W$ is $H G_{*}$-injective if and only if it is a generalized Eilenberg-MacLane spectrum $\vee \Sigma^{n} H A_{n}$ in which $A_{n}$ is a direct summand of $\operatorname{Hom}\left(G, D_{n}\right)$ with $D_{n}$ divisible.

When $E=K U$ "the complex K-spectrum", we will finally show a partial result (Theorem 3.11) corresponding to Theorem 2.

In this note we will work in the stable homotopy category of $C W$-spectra. We mean by a ring spectrum $E$ an associative ring spectrum with unit, and by an $E$-module spectrum $F$ an associative (left) $E$-module spectrum. If $E$ or $F$ is not necessarily assumed to be associative, then it is called a quasi ring spectrum or a quasi E-module spectrum.

## 1. The Anderson dual spectra $\nabla \boldsymbol{E}(\boldsymbol{D})$

1.1. Let us fix a $C W$-spectrum $E$. Given $C W$-spectra $X$ and $Y$ a map $f: X \rightarrow Y$ is said to be $E_{*}$-monic if it induces a monomorphism $f_{*}: E_{*} X \rightarrow E_{*} Y$. Following [13, Definition 1 i)] (cf. [8, §9]) we call a $C W$-spectrum $W$ an $E_{*}$ injective spectrum if any $E_{*}$-monic map $f: X \rightarrow Y$ induces always an epimorphism $f^{*}:[Y, W] \rightarrow[X, W]$. For any family $\left\{W_{\lambda}\right\}$ of $C W$-spectra it is obvious by definition that
(1.1) each $C W$-spectrum $W_{\lambda}$ is $E_{*}$-injective if and only if the direct product $\prod_{\lambda} W_{\lambda}$ is $E_{*}$-injective.

Consider the homomorphism $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}\left(E_{*} X, E_{*} W\right)$ assigning to each map $f: X \rightarrow W$ its induced homomorphism $f_{*}: E_{*} X \rightarrow E_{*} W$. Then the following result involving $\kappa_{E}$ can be easily verified.

Proposition 1.1. ([13, Proposition 7]). A $C W$-spectrum $W$ is $E_{* \text {-injec- }}$ tive if and only if $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}\left(E_{*} X, E_{*} W\right)$ is a monomorphism for any $C W$ spectrum $X$.

A $C W$-spectrum $W$ is said to be $E_{*}$-local (see [6], [7] or [14]) if [ $X, W$ ]= 0 for all $C W$-spectra $X$ with $E_{*} X=0$. As an immediate result we have

Corollary 1.2. If a $C W$-spectrum $W$ is $E_{*}$-injective, then it is an $E_{*}$-local spectrum.

For any abelian group $G$ we denote by $S G$ the Moore spectrum of type $G$. Given a $C W$-spectrum $E$ the corresponding spectrum with coefficients in $G$ is defined by $E G=E_{\wedge} S G$. In the $G=Q$ case we can particularly show

Lemma 1.3. Assume that $E Q \neq p t$. Then a $C W$-spectrum $W$ is $E Q_{*}$-injective if and only if $W=W Q$.

Proof. We may regard as $E=S$, the sphere spectrum. The "if" part is easily verified since any $S Q_{*}$-monic map $f: X \rightarrow Y$ induces an epimorphism $f^{*}$ : $[Y, W Q] \rightarrow[X, W Q]$. On the other hand, the "only if" part is immediate from Corollary 1.2.

We mean by a quasi ring spectrum $E$ a ring spectrum with unit which is not necessarily associative and by a quasi $E$-module spectrum $F$ a (left) $E$-module spectrum which is not necessarily associative. Notice that any quasi $E$-module spectrum $F$ is always $E_{*}$-local when $E$ is a quasi ring spectrum (see [2, Lemma III.13.1] or [14, Proposition 1.17]).

Lemma 1.4. Let $E$ be a quasi ring spectrum and $F$ be a quasi E-module spectrum. If a $C W$-spectrum $W$ is $F_{*}$-injective, then it is a quasi $E$-module spectrum. In particular, any $E G_{*}$-injective spectrum $W$ is always a quasi $E$-module spectrum.

Proof. Since the unit $\iota: S \rightarrow E$ induces a monomorphism $(\iota \wedge 1)_{*}: F_{*} W \rightarrow$ $F_{*}\left(E_{\wedge} W\right)$, there exists a map $\mu: E_{\wedge} W \rightarrow W$ satisfying $\mu(\iota \wedge 1)=1$, where 1 denotes the identity map.
1.2. Let $E$ be a fixed $C W$-spectrum and $D=\left\{D_{n}\right\}$ be a graded divisible abelian group. By Representability theorem there exists a $C W$-spectrum $\nabla E(D)=\prod_{n} \Sigma^{n} \nabla E\left(D_{n}\right)$ which is related to $E$ and $D$ by a natural isomorphism

$$
\begin{equation*}
\kappa_{E, D}:[X, \nabla E(D)] \rightarrow \operatorname{Hom}\left(E_{*} X, D\right)=\prod_{n} \operatorname{Hom}\left(E_{n} X, D_{n}\right) \tag{1.2}
\end{equation*}
$$

for any $C W$-spectrum $X$. Setting $\lambda_{E, D}=\kappa_{E, D}(1) \in \operatorname{Hom}\left(E_{*} \nabla E(D), D\right)$, the natural isomorphism $\kappa_{E, D}$ assigns to each map $f: X \rightarrow \nabla E(D)$ the composite $\lambda_{E, D} f_{*}$ : $E_{*} X \rightarrow E_{*} \nabla E(D) \rightarrow D$.

For any graded abelian group $A=\left\{A_{n}\right\}$ we choose an injective resolution $0 \rightarrow A \rightarrow D \xrightarrow{d} D^{\prime} \rightarrow 0$ and denote by $\nabla E(A)=\prod_{n} \Sigma^{n} \nabla E\left(A_{n}\right)$ the fiber of the induced map $d_{*}: \nabla E(D) \rightarrow \nabla E\left(D^{\prime}\right)$. Then we obtain a universal coefficient se-
quence

$$
0 \rightarrow \operatorname{Ext}\left(E_{*-1} X, A\right) \rightarrow[X, \nabla E(A)] \xrightarrow{\kappa_{E, A}} \operatorname{Hom}\left(E_{*} X, A\right) \rightarrow 0
$$

for any $C W$-spectrum $X$. As, is easily seen [15, I and II], $\nabla E(A)$ is independent of the choice of an injective resolution of $A$ and it is just the function spectrum $F(E, \nabla S(A))=\prod_{n} \Sigma^{n} F\left(E, \nabla S\left(A_{n}\right)\right)$. We call $\nabla E(A)$ the Anderson dual spectrum of $E$ with coefficients in $A=\left\{A_{n}\right\}$ (cf. [4]).

Using the natural isomorphism $\kappa_{E, D}$ of (1.2) we see immediately that
(1.3) every Anderson dual spectrum $\nabla E(D)=\prod_{n} \Sigma^{n} \nabla E\left(D_{n}\right)$ is $E_{*}$-injective if $D=\left\{D_{n}\right\}$ is divisible.

By virtue of (1.3) we can show the enough $E_{*}$-injectivity of the stable homotopy category of $C W$-spectra (cf. [13, Proposition 4]).

Proposition 1.5. For any $C W$-spectrum $X$ there exists an Anderson dual spectrum $\nabla E(D)$ with $D=\left\{D_{n}\right\}$ divisible, which is $E_{*}$-injective, and an $E_{*}$-monic map $f: X \rightarrow \nabla E(D)$.

Proof. Choose a graded divisible abelian gıoup $D=\left\{D_{n}\right\}$ so that $E_{n} X$ is embedded into $D_{n}$ for each $n$. Pick up a map $f: X \rightarrow \nabla E(D)$ such that $\kappa_{E, D}(f)$ : $E_{*} X \rightarrow D$ is just the embedding of $E_{*} X$ into $D$. Since $\kappa_{E, D}(f)$ is decomposed into the composite $\lambda_{E, D} f_{*}$, the map $f: X \rightarrow \nabla E(D)$ is certainly $E_{*}$-monic.

More generally we will next deal with the $C W$-spectrum $E G$ with coefficients in $G$. For any divisible abelian group $D_{n}$ we set $B_{n}=\operatorname{Hom}\left(G, D_{n}\right)$. Take a free resolution $0 \rightarrow \underset{\alpha}{\oplus} Z \rightarrow \underset{\beta}{\oplus} Z \rightarrow G \rightarrow 0$ and consider the commutative diagram

$$
\begin{array}{ccccc}
\nabla E\left(B_{n}\right) & \rightarrow & \nabla E\left(\prod_{\beta} D_{n}\right) & \rightarrow & \nabla E\left(\prod_{\alpha} D_{n}\right) \\
& & \downarrow & & \downarrow \\
F\left(S G, \nabla E\left(D_{n}\right)\right) & \rightarrow & \prod_{\beta} \nabla E\left(D_{n}\right) & \rightarrow & \prod_{\alpha} \nabla E\left(D_{n}\right)
\end{array}
$$

with two cofiber sequences. The two vertical arrows are equivalences because each of them induces the canonical isomorphism $\operatorname{Hom}\left(\pi_{*} E, \prod_{\gamma} D_{n}\right) \rightarrow \prod_{\gamma} \operatorname{Hom}\left(\pi_{*} E, D_{n}\right)$ in the homotopy group. By applying Five lemma we get an equivalence $\nabla E\left(B_{n}\right)$ $\rightarrow F\left(S G, \nabla E\left(D_{n}\right)\right)=\nabla E G\left(D_{n}\right)$. Thus

$$
\begin{equation*}
\nabla E(B)=\prod_{n} \Sigma^{n} \nabla E\left(B_{n}\right)=\prod_{n} \Sigma^{n} \nabla E G\left(D_{n}\right)=\nabla E G(D) \tag{1.4}
\end{equation*}
$$

if $D=\left\{D_{n}\right\}$ is divisible and $B=\left\{B_{n}=\operatorname{Hom}\left(G, D_{n}\right)\right\}$.
Using Proposition 1.5 combined with (1.1), (1.3) and (1.4) we can easily show

Proposition 1.6. A CW-spectrum $W$ is $E G_{* \text {-injective if and only if it is a }}$ retract of a certain Anderson dual spectrum $\nabla E(B)=\prod_{n} \Sigma^{n} \nabla E\left(B_{n}\right)$ in which $B_{n}=$ $\operatorname{Hom}\left(G, D_{n}\right)$ for some divisible $D_{n}$.
1.3. For any graded abelian group $A=\left\{A_{n}\right\}$ we denote by Tor $A=$ $\left\{\right.$ Tor $\left.A_{n}\right\}$ its torsion subgroup. The torsion subgroup Tor $A$ is said to be bounded if $m$ Tor $A=0$ for some positive integer $m$.

Proposition 1.7. Let $E$ be a $C W$-spectrum such that Tor $\pi_{*} E$ is bounded. If a $C W$-spectrum $W$ is $E_{*}$-injective, then it is decomposed into the wedge sum $W Q \vee \Sigma^{-1} W Q / Z$.

Proof. According to Proposition 1.6 $W$ is a retract of a certain Anderson dual spectrum $\nabla E(D)$ with $D=\left\{D_{n}\right\}$ divisible. Set $A=\pi_{*} E$ thus $A_{n}=\pi_{n} E$. Since $\operatorname{Hom}(A / \operatorname{Tor} A, D)$ is divisible, the short exact sequences $0 \rightarrow$ $\operatorname{Hom}(A / \operatorname{Tor} A, D) \rightarrow \operatorname{Hom}(A, D) \rightarrow \operatorname{Hom}(\operatorname{Tor} A, D) \rightarrow 0$ and $0 \rightarrow \operatorname{Hom}(A /$ Tor $A$, $D) * Q / Z \rightarrow \operatorname{Hom}(A / \operatorname{Tor} A, D) \rightarrow \operatorname{Hom}(A / \operatorname{Tor} A, D) \otimes Q \rightarrow 0$ are both split. Under our assumption on $\operatorname{Tor} A$, we note that $\operatorname{Hom}(\operatorname{Tor} A, D) \otimes Q=0$ and hence $\operatorname{Hom}(\operatorname{Tor} A, D) \otimes Q / Z=0$. This implies that $\operatorname{Hom}(A, D) \otimes Q / Z=0$. So the rationalization $l: \operatorname{Hom}(A, D) \rightarrow \operatorname{Hom}(A, D) \otimes Q$ has a right inverse $k$ because $\operatorname{Hom}(A / \operatorname{Tor} A, D) \otimes Q \rightarrow \operatorname{Hom}(A, D) \otimes Q$ is an isomorphism. Thus the cofiber sequence $\Sigma^{-1} \nabla E(D) Q / Z \rightarrow \nabla E(D) \xrightarrow{i} \nabla E(D) Q$ gives rise to a split short exact sequence

$$
0 \rightarrow \pi_{*+1} \nabla E(D) Q / Z \rightarrow \pi_{*} \nabla E(D) \xrightarrow{l} \pi_{*} \nabla E(D) Q \rightarrow 0
$$

in the homotopy group.
Note that $X Q$ is just the generalized Moore spectrum ${\underset{n}{ }}_{\vee} \Sigma^{n} S\left(\pi_{n} X \otimes Q\right)$ for each $C W$-spectrum $X$. Consider the commutative diagram

$$
\begin{gathered}
0 \rightarrow \operatorname{Ext}\left(\pi_{*-1} X Q, \pi_{*} \nabla E(D)\right) \rightarrow[X Q, \nabla E(D)] \xrightarrow{\kappa_{S}} \operatorname{Hom}\left(\pi_{*} X Q, \pi_{*} \nabla E(D)\right) \rightarrow 0 \\
\downarrow \\
0 \rightarrow \operatorname{Ext}\left(\pi_{*-1} X Q, \pi_{*} \nabla E(D) Q\right) \rightarrow[X Q, \nabla E(D) Q] \rightarrow{ }_{\downarrow}^{\kappa_{S}} \operatorname{Hom}\left(\pi_{*} X Q, \pi_{*} \nabla E(D) Q\right) \rightarrow 0
\end{gathered}
$$

involving the universal coefficient sequences [11]. Since the left Ext-terms are both vanishing, the two assignments $\kappa_{s}$ are exactly isomorphisms. Taking $X=\nabla E(D)$, we can pick up a map $f: \nabla E(D) Q \rightarrow \nabla E(D)$ inducing the right inverse $k$ of $l$ in the homotopy group. This map $f$ is certainly a right inverse of $i: \nabla E(D) \rightarrow \nabla E(D) Q$. Thus the cofiber sequence $\Sigma^{-1} \nabla E(D) Q / Z \rightarrow \nabla E(D)$ $\rightarrow \nabla E(D) Q$ is split. Then it is easily verified that the cofiber sequence $\Sigma^{-1} W Q / Z \rightarrow W \rightarrow W Q$ is split, too.

By virtue of Proposition 1.7 we obtain

Theorem 1.8. Let E be a CW-spectrum such that Tor $\pi_{*} E$ is bounded. Then a $C W$-spectrum $W$ is $E_{*}$-injective if and only if its p-local spectrum $W Z_{(p)}$ is $E Z_{(p) *-i n j e c t i v e ~ f o r ~ e a c h ~ p r i m e ~} p$.

Proof. The "only if" part: Assume that a $C W$-spectrum $W$ is $E_{*}$-injective. Then Proposition 1.7 implies that the $p$-local spectrum $W Z_{(p)}$ is decomposed into the wedge sum of $E_{*}$-injective spectra $W Q$ and $\Sigma^{-1} W Z / p^{\infty}$, in which $W Q=p t$ whenever $E Q=p t$. The $E_{*}$-injective spectrum $W Q$ is obviously $E Z_{(p)^{*-}}$ injective because of Lemma 1.3. On the other hand, the $E_{*}$-injective spectrum $W Z \mid p^{\infty}$ is a retract of a certain Anderson dual spectrum $\nabla E\left(D_{p}\right)$. Since we can take $D_{p}=\left\{D_{p, n}\right\}$ to be divisible $p$-torsion, it is also $E Z_{(p) * \text {-injective by means of }}$ Proposition 1.6. Therefore $W Z_{(p)}=W Q \vee \Sigma^{-1} W Z \mid p^{\infty}$ is $E Z_{(p)^{*} \text {-injective for each }}$ prime $p$.

The "if" part: Assume that the $p$-local spectrum $W Z_{(p)}$ is $E Z_{(p) * * \text {-injective }}$ for each prime $p$. Then Proposition 1.7 asserts that $W Q$ and $W Z / p^{\infty}$ are $E Z_{(p)^{*-}}$ injective and the canonical map $j_{p}: W Q \rightarrow W Z / p^{\infty}$ is trivial for each prime $p$, where $W Q=p t$ if $E Q=p t$. Set $\bar{W}=\prod_{p} W Z / p^{\infty}$, which is $E_{*}$-injective. Using Proposition 1.7 again we observe that the direct product $\bar{W}$ is decomposed into the wedge sum of $\bar{W} Q$ and $\Sigma^{-1} \bar{W} Q / Z$. Note that $W Z / p^{\infty}$ is a retract of $\Sigma^{-1} \bar{W} Z / p^{\infty}$ because $\bar{W} Z\left|p^{\infty}=\left(W Z\left|p^{\infty} \wedge S Z\right| p^{\infty}\right) \vee\left(\prod_{q \neq p} W Z / q^{\infty}\right) \wedge S Z\right| p^{\infty}$. Therefore $W Q / Z=$ $\bigvee_{p} W Z / p^{\infty}$ is a retract of the $E_{*}$-injective spectrum $\bar{W}=\prod_{p} W Z \mid p^{\infty}$. Thus the canonical map $l: W Q / Z \rightarrow \bar{W}$ has a left inverse. Now it is easy to check that the canonical map $j: W Q \rightarrow W Q / Z$ becomes trivial because $l_{*}:[W Q, W Q / Z] \rightarrow$ $[W Q, \bar{W}]=\prod_{p}\left[W Q, W Z / p^{\infty}\right]$ is a monomorphism. Since the $C W$-spectrum $W$ is written into the wedge sum of the $E_{*}$-injective spectra $W Q$ and $\Sigma^{-1} W Q / Z$, it is $E_{*}$-injective as desired.
1.4. Assume that $E$ is a ring spectrum. Then $E_{*} X$ admits an $E_{*}$-module structure for each $C W$-spectrum $X$ where $\pi_{*} E$ is abbreviated as $E_{*}$. Given an injective $E_{*}$-module $M$ there exists a $C W$-spectrum $V_{M}$ so that

$$
\begin{equation*}
\kappa_{E, M}:\left[X, V_{M}\right] \rightarrow \operatorname{Hom}_{E *}\left(E_{*} X, M\right) \tag{1.5}
\end{equation*}
$$

is a natural isomorphism for any $C W$-spectrum $X$, by applying Representability theorem similarly to (1.2). The natural isomorphism $\kappa_{E, M}$ assings to each map $f: X \rightarrow V_{M}$ the composite $\lambda_{E, M} f_{*}: E_{*} X \rightarrow E_{*} V_{M} \rightarrow M$ where $\lambda_{E, M}=\kappa_{E, M}(1) \in$ $\operatorname{Hom}_{E_{*}}\left(E_{*} V_{M}, M\right)$. As a similar result to (1.3) we have
(1.6) each represented spectrum $V_{M}$ is $E_{*}$-injective if $M$ is an injective $E_{*^{-}}$ module.

Every injective $E_{*}$-module can be realized by a certain $E$-module spectrum
as follows.
Lemma 1.9. Let $E$ be a ring spectrum and $M$ be an injective $E_{*}$-module. Then the represented spectrum $V_{M}$ is an $E$-module spectrum such that the composite $\lambda_{E, M}(\iota \wedge 1)_{*}: \pi_{*} V_{M} \rightarrow E_{*} V_{M} \rightarrow M$ is an isomorphism of $E_{*-m o d u l e s . ~}^{\text {. }}$

Proof. Pick up a map $\mu: E_{\wedge} V_{M} \rightarrow V_{M}$ such that $\kappa_{E, M}(\mu)=\lambda_{E, M}(m \wedge 1)_{*} \in$ $\operatorname{Hom}_{E_{*}}\left(E_{*}\left(E_{\wedge} V_{M}\right), M\right)$ where $m: E_{\wedge} E \rightarrow E$ denotes the multiplication of $E$. Using the equality $\lambda_{E, M}(1 \wedge \mu)_{*}=\lambda_{E, M}(m \wedge 1)_{*}$ we can easily check that $\mu(\iota \wedge 1)$ $=1: V_{M} \rightarrow V_{M}$ and $\mu(1 \wedge \mu)=\mu(m \wedge 1): E \wedge E \wedge V_{M} \rightarrow V_{M}$, thus $V_{M}$ becomes an $E$-module spectrum.

Consider the following diagram

where the vertical arrows $\mu_{*}$ and $m_{*}$ are respectively defined to be $\mu_{\xi}(f \otimes g)=$ $\mu(f \wedge g)$ and $m_{*}(f \otimes a)=a\left(m_{\wedge} 1\right)_{*}(1 \wedge f \wedge 1)_{*}$. By a routine computation we can observe that the above square is commutative. Thus $\kappa_{E, M}:\left[X, V_{M}\right] \rightarrow$ $\operatorname{Hom}_{E *}\left(E_{*} X, M\right)$ is an isomorphism of $E_{*}$-modules. In particular, this implies that the composite $\lambda_{E, M}(\iota \wedge 1)_{*}: \pi_{*} V_{M} \rightarrow E_{*} V_{M} \rightarrow M$ is an isomorphism of $E_{*}{ }^{-}$ modules.

By virtue of Lemma 1.9 each injective $E_{*}$-module $M$ can be identified with $\pi_{*} V_{M}$. Then the natural isomorphism $\kappa_{E, M}:\left[X, V_{M}\right] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, M\right)$ may be regarded as the canonical morphism

$$
\begin{equation*}
\kappa_{E, V_{M}}=\varphi \kappa_{E}:\left[X, V_{M}\right] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, E_{*} V_{M}\right) \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, \pi_{*} V_{M}\right) \tag{1.7}
\end{equation*}
$$

where $\varphi$ is induced by the $E$-module structure map $\mu: E \wedge V_{M} \rightarrow V_{M}$, because $\lambda_{E, M}(\iota \wedge 1)_{*} \mu_{*}=\lambda_{E, M}(1 \wedge \mu)_{*}(\iota \wedge 1 \wedge 1)_{*}=\lambda_{E, M}(m \wedge 1)_{*}(\iota \wedge 1 \wedge 1)_{*}=\lambda_{E, M}$.

If $D=\left\{D_{n}\right\}$ is a graded divisible abelain group, then the $E_{*}$-module $\operatorname{Hom}\left(E_{*}, D\right)_{*}=\left\{\operatorname{Hom}\left(E_{*-n}, D\right)\right\}$ becomes injective. Setting $M=\operatorname{Hom}\left(E_{*}, D\right)_{*}$, we note that $V_{M}$ coincides with the Anderson dual spectrum $\nabla E(D)$ since $\eta$ : $\operatorname{Hom}_{E_{*}}\left(E_{*} X, \operatorname{Hom}\left(E_{*}, D\right)_{*}\right) \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, D\right)$ is an isomorphism. Hence Lemma 1.9 implies

Corollary 1.10. Let $E$ be a ring spectrum and $D=\left\{D_{n}\right\}$ be a graded divisible abelian group. Then the Anderson dual spectrum $\nabla E(D)$ is an $E$-module spectrum such that $\kappa_{E, D}: \pi_{*} \nabla E(D) \rightarrow \operatorname{Hom}\left(E_{*}, D\right)_{*}$ is an isomorphism of $E_{*}$-modules.

Taking $M=\operatorname{Hom}\left(E_{*}, D\right)_{*}$ with $D=\left\{D_{n}\right\}$ divisible, we restate (1.7) as the
natural isomorphism $\kappa_{E, D}:[X, \nabla E(D)] \rightarrow \operatorname{Hom}\left(E_{*} X, D\right)$ can be decomposed into the composite

$$
\begin{align*}
\eta \kappa_{E, \nabla E(D)}=\eta \varphi \kappa_{E}:[X, & \nabla E(D)] \rightarrow \operatorname{Hom}_{E *}\left(E_{*} X, E_{*} \nabla E(D)\right)  \tag{1.8}\\
& \rightarrow \operatorname{Hom}_{E *}\left(E_{*} X, \pi_{*} \nabla E(D)\right) \xrightarrow{\cong} \operatorname{Hom}\left(E_{*} X, D\right) .
\end{align*}
$$

1.5. Assume that a ring spectrum $E$ satisfies the following property:
(F) $E_{*} E$ is flat as an $E_{*}$-module.

Then $E_{*} X$ is regarded to be an $E_{*} E$-comodule as well as an $E_{*}$-module for each $C W$-spectrum $X$. Given an injective $E_{*} E$-comodule $I$ there exists a $C W$ spectrum $W_{I}$ so that

$$
\begin{equation*}
\kappa_{E, I}:\left[X, W_{I}\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, I\right) \tag{1.9}
\end{equation*}
$$

is a natural isomorphism for any $C W$-spectrum $X$, by means of Representability theorem. Setting $\lambda_{E, I}=\kappa_{E, I}(1) \in \operatorname{Hom}_{E * E}\left(E_{*} W_{I}, I\right)$, the natural isomorphism $\kappa_{E, I}$ is given by $\kappa_{E, I}(f)=\lambda_{E, I} f_{*}$ for each $f: X \rightarrow W_{I}$. For any family $\left\{I_{\lambda}\right\}$ of injective $E_{*} E$-comodules it is obvious that
(1.10) the direct product $\prod_{\lambda} W_{I_{\lambda}}$ coincides with the represented spectrum $W_{I}$ where $I=\prod_{\lambda} I_{\lambda}$.

Corresponding to (1.3) or (1.6) we have
(1.11) each represented spectrum $W_{I}$ is $E_{*}$-injective if $I$ is an injective $E_{*} E$ comodule.

Let $M$ be an $E_{*}$-module such that the extended comodule $E_{*} E \otimes M$ is injective. Put $I=E_{*} E \otimes_{B_{*}} M$, and consider the composite

$$
\begin{equation*}
\theta \kappa_{E, I}:\left[X, W_{I}\right] \rightarrow \operatorname{Hom}_{E_{*} E}\left(E_{*} X, E_{*} E \underset{B_{*}}{\otimes} M\right) \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, M\right) \tag{1.12}
\end{equation*}
$$

in which $\theta$ is defined to be $\theta(a)=\left(m_{*} \otimes 1\right) a$. Since $\theta$ is an isomorphism, the above composite $\theta \kappa_{E, I}$ becomes a natural isomorphism for any $C W$-spectrum $X$. If an $E_{*}$-module $M$ is injective, then the extended comodule $I=E_{*} E \bigotimes_{B_{*}}^{\otimes} M$ is exactly injective. In this case we notice that $W_{I}=V_{M}$ and $\theta \kappa_{E, I}=\kappa_{E, M}$. By the quite same argument as in the proof of Lemma 1.9 we can show

Lemma 1.11. Let $E$ be a ring spectrum satisfying the property (F), and $M$ be an $E_{*}$-module such that the extended comodule $I=E_{*} E{\underset{E *}{*}}_{\otimes} M$ is injective. Then the represented spectrum $W_{I}$ is an E-module spectrum such that the composite $\lambda_{E, M}(\iota \wedge 1)_{*}: \pi_{*} W_{I} \rightarrow E_{*} W_{I} \rightarrow M$ is an isomorphism of $E_{*}$-modules, where $\lambda_{E, M}=$
$\left(m_{*} \otimes 1\right) \kappa_{E, I}(1) \in \operatorname{Hom}_{E *}\left(E_{*} W_{I}, M\right)$.
A similar discussion to (1.7) shows that the above natural isomorphism $\theta \kappa_{E, I}:\left[X, W_{I}\right] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, M\right)$ is rewritten into the canonical morphism

$$
\begin{equation*}
\kappa_{E, W_{I}}=\varphi \kappa_{E}:\left[X, W_{I}\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W_{I}\right) \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, \pi_{*} W_{I}\right) \tag{1.13}
\end{equation*}
$$

where $M$ is identified with $\pi_{*} W_{I}$ by the aid of Lemma 1.11.
For each $E_{*} E$-comodule $I$ we denote by $\operatorname{injdim}_{E * E} I$ the injective dimension of $I$ as an $E_{*} E$-comodule.

Proposition 1.12. Let $E$ be a ring spectrum satisfying the property ( F ), and $L$ be an $E_{*-m o d u l e}$ with $\operatorname{injdim}_{E_{*}} E_{*} E \underset{B_{*}}{\otimes} L \leqq 1$. Then there exists a quasi $E$ module spectrum $Y$ such that $\pi_{*} Y$ has an $E_{*}$-module structure and it is isomorphic to $L$ as $E_{*-m o d u l e s . ~}^{\text {. }}$

Proof. Choose a short exact sequence $0 \rightarrow L \stackrel{i}{\rightarrow} M \xrightarrow{j} N \rightarrow 0$ of $E_{*}$-modules with $M$ injective. Set $I=E_{*} E \underset{B_{*}}{\otimes} M$ and $J=E_{*} E \underset{B_{*}}{B_{*}} N$, both of which are injective $E_{*} E$-comodules. According to Lemma 1.11 both $W_{I}$ and $W_{J}$ are $E$-module spectra such that the composites $\lambda_{E, M}(\iota \wedge 1)_{*}: \pi_{*} W_{I} \rightarrow M$ and $\lambda_{E, N}(\iota \wedge 1)_{*}: \pi_{*} W_{J} \rightarrow N$ are isomorphisms of $E_{*}$-modules. Pick up a map $g: W_{I} \rightarrow W_{J}$ such that $\theta \kappa_{E, J}(g)=$ $j \lambda_{E, M} \in \operatorname{Hom}_{E_{*}}\left(E_{*} W_{I}, N\right)$, and then consider the cofiber sequence $Y \xrightarrow{f} W_{I} \xrightarrow{g} W_{J}$. Since $\lambda_{E, N}(1 \wedge g)_{*}=j \lambda_{E, M}$, there exists a homomorphism $\lambda_{E, L}: E_{*} Y \rightarrow L$ of $E_{*-}$ modules such that $\lambda_{E, M}(1 \wedge f)_{*}=i \lambda_{E, L} \in \operatorname{Hom}_{E *}\left(E_{*} Y, M\right)$. Denote by $\mu_{I}$ : $E \wedge W_{I} \rightarrow W_{I}$ and $\mu_{J}: E \wedge W_{J} \rightarrow W_{J}$ the $E$-module structure maps of $W_{I}$ and $W_{J}$ respectively, which satisfy that $\lambda_{E, M}\left(1 \wedge \mu_{I}\right)_{*}=\lambda_{E, M}(m \wedge 1)_{*}$ and $\lambda_{E, N}\left(1 \wedge \mu_{J}\right)_{*}=$ $\lambda_{E, N}(m \wedge 1)_{*} . \quad$ As is easily checked, there holds the equality $g \mu_{I}=\mu_{J}(1 \wedge g)$ : $E \wedge W_{I} \rightarrow W_{J}$. So we get a map $\mu_{Y}: E \wedge Y \rightarrow Y$ such that $f \mu_{Y}=\mu_{I}(1 \wedge f)$ and $\mu_{Y}(\iota \wedge 1)=1$. Thus $Y$ is a quasi $E$-module spectrum which gives $\pi_{*} Y$ an associative $E_{*}$-module structure. Because the cofiber sequence $Y \rightarrow W_{I} \rightarrow W_{J}$ induces a short exact sequence $0 \rightarrow \pi_{*} Y \rightarrow \pi_{*} W_{I} \rightarrow \pi_{*} W_{J} \rightarrow 0$ of $E_{*}$-modules. By applying Five lemma we can moreover see that the composite $\lambda_{E, L}(\iota \wedge 1)_{*}: \pi_{*} Y$ $\rightarrow E_{*} Y \rightarrow L$ is an isomorphism of $E_{*}$-modules.

Finally we show
Lemma 1.13. Let $E$ be a ring spectrum satisfying the property $(\mathrm{F})$ and $W$ be an $E$-module spectrum such that the extended comodule $I=E_{*} E \otimes W_{*}$ is injective. Then $W$ coincides with the represented spectrum $W_{I}$.

Proof. Use the natural isomorphism $\theta \kappa_{E, I}:\left[X, W_{I}\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, I\right) \rightarrow$ $\operatorname{Hom}_{E *}\left(E_{*} X, \pi_{*} W\right)$ of (1.12). We then get a map $f: W \rightarrow W_{I}$ such that $\lambda_{E, W *} f_{*}$
$=\mu_{*}: E_{*} W \rightarrow \pi_{*} W$, where $\lambda_{E, W *}=\theta \kappa_{E, I}(1)$ and $\mu: E \wedge W \rightarrow W$ denotes the $E$ module structure map of $W$. Consider the composite

$$
\left(\theta \kappa_{E, I}\right)^{-1} \kappa_{E, W}:[X, W] \rightarrow \operatorname{Hom}_{E *}\left(E_{*} X, \pi_{*} W\right) \cong\left[X, W_{I}\right]
$$

which is certainly induced by the map $f$, where $\kappa_{E, W}=\varphi \kappa_{E}$ and it is defined to be $\kappa_{E, W}(g)=\mu_{*}(1 \wedge g)_{*} . \quad$ Taking $X=\Sigma^{k}$ for every $k, \kappa_{E, W}:\left[\Sigma^{k}, W\right] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} \Sigma^{k}\right.$, $\left.\pi_{*} W\right)$ is evidently an isomorphism. Therefore we can easily observe that $f: W$ $\rightarrow W_{I}$ is an equivalence.

## 2. Injective $\boldsymbol{E}_{*} \boldsymbol{E}$-comodules

2.1. Let $E$ be a ring spectrum and $F$ be an $E$-module spectrum. The $E$-module structure map $\mu: E_{\wedge} F \rightarrow F$ gives rise to homomorphisms $\nu_{E, F}$ : $X_{*} E \underset{E_{*}}{\otimes} F_{*} \rightarrow X_{*} F$ and $\kappa_{E, F}:[X, F] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, F_{*}\right)$ defined in the canonical ways. We are interested in ring spectra $E$ which satisfy some of the following nice properties (see [1] or [2]):
(F) $E_{*} E$ is flat as an $E_{*}$-module.
(K) $\nu_{E, F}: X_{*} E \otimes F_{*} \rightarrow X_{*} F$ is an isomorphism for any $E$-module spectrum $F$ if $E_{*} X$ is a flat $E_{*}$-module.
(U) $\kappa_{E, F}:[X, F] \rightarrow \operatorname{Hom}_{E *}\left(E_{*} X, F_{*}\right)$ is an isomorphism for any $E$-module spectrum $F$ if $E_{*} X$ is a projective $E_{*}$-module.
(P) For every $C W$-spectrum $X$ there exists a $C W$-spectrum $Y$ and a map $g: Y \rightarrow X$ such that $E_{*} Y$ is a projective $E_{*}$-module and $g_{*}: E_{*} Y \rightarrow E_{*} X$ is an epimorphism.

If a ring spectrum $E$ satisfies the property $(\mathrm{F})$, then the condition $(\mathrm{K})$ implies that
( $\left.\mathrm{K}^{\prime}\right) \nu_{E, F}: E_{*} E \underset{B_{*}}{\otimes} F_{*} \rightarrow E_{*} F$ is an isomorphism for any $E$-module spectrum $F$, and in particular that
$\left(\mathrm{K}^{\prime \prime}\right) \nu_{E, \nabla E(D)}: E_{*} E \underset{B_{*}}{\bigotimes} \nabla E(D)_{*} \rightarrow E_{*} \nabla E(D)$ is an isomorphism for each graded divisible abelian group $D=\left\{D_{n}\right\}$,
because the Anderson dual spectrum $\nabla E(D)$ is an $E$-module spectrum by Corollary 1.10 .

As typical examples of ring spectra satisfying all of the properties (F), $(\mathrm{K}),(\mathrm{U})$ and (P) the following spectra are well known (see [1] or [2, Proposition III.13.4]):
$S, H Z \mid p, M O, M U, M S p, K U$ and $K O$.

In this section we will give some characterizations of $E_{*}$-injective spectra for such a nice ring spectrum $E$ as in (2.1). Putting their results together we can summarize as follows (cf. [13, Theorem 3 i) and iii)] or [8, §9]).

Theorem 2.1. Assume that a ring spectrum $E$ satisfies the properties (F) and $\left(\mathrm{K}^{\prime \prime}\right)$. Then the following five conditions are all equivalent :
i) $W$ is an $E_{*}$-injective spectrum.
ii) $W$ is a quasi $E$-module specirum such that $E_{*} W$ is injective as an $E_{*} E$-module.
iii) $W$ is an $E_{*}$-local spectrum such that $E_{*} W$ is injective as an $E_{*} E$-comodule.
iv) $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.
v) $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, E_{*} W\right)$ is a monomorphism for any $C W$-spectrum $X$.
2.2. Let $E$ be a ring spectrum satisfying the property $(\mathrm{F})$. Then the homomorphism $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}\left(E_{*} X, E_{*} W\right)$ defined by $\kappa_{E}(f)=f_{*}$ is evidently factorized through $\operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$. Moreover we note that $\nu_{E, E_{\wedge} W}$ : $E_{*} E \underset{E_{*}}{\otimes} E_{*} W \rightarrow E_{*}\left(E_{\wedge} W\right)$ defined in the canonical way is an isomorphism, even if $E$ is not assumed to satisfy the property $\left(\mathrm{K}^{\prime}\right)$.

Proposition 2.2. Let $E$ be a ring spectrum satisfying the property (F). If $W$ is a quasi $E$-module spectrum such that $E_{*} W$ is injective as an $E_{*} E$-comodule, then it is an $E_{*}$-injective spectrum and $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.

Proof. Set $I=E_{*} W$, which is an injective $E_{*} E$-comodule. By means of (1.9) and (1.11) there exists an $E_{*}$-injective spectrum $W_{I}$ so that $\kappa_{E, I}:\left[X, W_{I}\right] \rightarrow$ $\operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is a natural isomorphism for any $C W$-spectrum $X$. Taking $X=W$, we get a map $f: W \rightarrow W_{I}$ with $\kappa_{E, I}(f)=1$. Notice the composite

$$
\kappa_{E, I}^{-1} \kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right) \stackrel{\cong}{\leftrightarrows}\left[X, W_{I}\right]
$$

is exactly induced by the map $f$. Consider the composite $\varphi \kappa_{E}:[X, E \wedge W] \rightarrow$ $\operatorname{Hom}_{E * E}\left(E_{*} X, E_{*}\left(E_{\wedge} W\right)\right) \rightarrow \operatorname{Hom}_{E *}\left(E_{*} X, E_{*} W\right)$ in which $\varphi$ is defined to be $\varphi(a)=(m \wedge 1)_{*} a$ and it is an isomorphism because $\nu_{E, E_{\wedge} W}: E_{*} E \otimes_{B_{*}} E_{*} W \rightarrow$ $E_{*}\left(E_{\wedge} W\right)$ is an isomorphism. Taking $X=\Sigma^{k}$ for every $k$, it is easily checked that $\kappa_{E}:\left[\Sigma^{k}, E \wedge W\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} \Sigma^{k}, E_{*}\left(E_{\wedge} W\right)\right)$ is an isomorphism. Since $W$ is a retıact of $E \wedge W, \kappa_{E}:\left[\Sigma^{k}, W\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} \Sigma^{k}, E_{*} W\right)$ becomes an isomorphism, too. This implies that $f: W \rightarrow W_{I}$ is an equivalence because $\kappa_{E, I}^{-1} \kappa_{E}=$ $f_{*}:[X, W] \rightarrow\left[X, W_{I}\right]$. Hence we observe that $W$ is an $E_{*}$-injective spectrum and moreover $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is an isomorphism.

Adding the condition ( $\mathrm{K}^{\prime \prime}$ ) on $E$ we can show the following result, which contains the converse of Proposition 2.2.

Proposition 2.3. Let $E$ be a ring spectrum satisfying the properties ( $\mathrm{K}^{\prime \prime}$ ) as well as (F). If a $C W$-spectrum $W$ is $E_{*}$-injective, then it is a quasi E-module spectrum such that $E_{*} W$ is injective as an $E_{*} E$-comodule and $\kappa_{E}:[X, W] \rightarrow$ $\operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.

Proof. From Lemma 1.4 it follows that $W$ is a quasi $E$-module spectrum. On the other hand, Proposition 1.6 asserts that $W$ is a retract of a certain Anderson dual spectrum $\nabla E(D)=\prod_{n} \Sigma^{n} \nabla E\left(D_{n}\right)$ with $D=\left\{D_{n}\right\}$ divisible. So it is sufficient to show our result for $W=\nabla E(D)$. According to Corollary 1.10, $\nabla E(D)$ is an $E$-module spectrum and $\kappa_{E, D}: \pi_{*} \nabla E(D) \rightarrow \operatorname{Hom}\left(E_{*}, D\right)_{*}$ is an isomorphism of $E_{*}$-modules. Under the condition ( $\mathrm{K}^{\prime \prime}$ ) on $E$ the $E_{*} E$-comodule $E_{*} \nabla E(D)$ is isomorphic to the extended comodule $E_{*} E \otimes_{B_{*}}^{\otimes} \operatorname{Hom}\left(E_{*}, D\right)_{*}$, which is certainly injective as an $E_{*} E$-comodule. Moreover we can easily observe by use of (1.8) that $\kappa_{E}:[X, \nabla E(D)] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} \nabla E(D)\right)$ is an isomorphism for any $C W$-spectrum $X$.

By making use of Proposition 2.3 we obtain the following result, which resembles Proposition 2.2.

Proposition 2.4. Let E be a ring spectrum satisfying the properties ( $\mathrm{K}^{\prime \prime}$ ) as well as ( F ). If $W$ is an $E_{*}$-local spectrum such that $E_{*} W$ is injective as an $E_{*} E$ comodule, then it is an $E_{*}$-injective spectrum and $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E_{*} E}\left(E_{*} X\right.$, $\left.E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.

Proof. Set $I=E_{*} W$, then there exists an $E_{*}$-injective spectrum $W_{I}$ so that $\kappa_{E, I}:\left[X, W_{I}\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is a natural isomorphism for any $C W$ spectrum $X$. On the other hand, Proposition 2.3 asserts that $\kappa_{E}:\left[X, W_{I}\right] \rightarrow$ $\operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W_{I}\right)$ is also an isomorphism for any $C W$-spectrum $X$. Taking $X=W$ in the former and $X=W_{I}$ in the latter, we get maps $f: W \rightarrow W_{I}$ and $g: W_{I} \rightarrow W_{I}$ such that $\lambda_{E, I} f_{*}=1: E_{*} W \rightarrow E_{*} W$ and $g_{*}=f_{*} \lambda_{E, I}: E_{*} W_{I} \rightarrow E_{*} W_{I}$ where $\lambda_{E, I}=\kappa_{E, I}(1) \in \operatorname{Hom}_{E * E}\left(E_{*} W_{I}, E_{*} W\right)$. As is easily checked, the map $g$ becomes just the identity. This implies that $f_{*}: E_{*} W \rightarrow E_{*} W_{I}$ is an isomorphism. Since both $W$ and $W_{I}$ are $E_{*}$-local spectra, we then observe that $f: W \rightarrow W_{I}$ becomes an equivalence. Thus $W$ is an $E_{*}$-injective spectrum and moreover $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} X, E_{*} W\right)$ is an isomorphism.

Proof of Theorem 2.1. The implication i) $\rightarrow$ ii) $\rightarrow \mathrm{iii}) \rightarrow \mathrm{i}$ ) is immediately shown by use of Propositions 2.3 and 2.4. On the other hand, the implication i) $\rightarrow \mathrm{iv}) \rightarrow \mathrm{v}) \rightarrow \mathrm{i}$ ) is done by use of Propositions 2.3 and 1.1.
2.3. Making use of Proposition 2.3 again we can realize every injective $E_{*} E$-comodule by a certain $E_{*}$-injective spectrum when $E$ satisfies the properties $(\mathrm{F})$ and ( $\mathrm{K}^{\prime \prime}$ ) (cf. Lemma 1.11).

Proposition 2.5. Let $E$ be a ring spectrum satisfying the properties ( F ) and $\left(\mathrm{K}^{\prime \prime}\right)$ and I be an injective $E_{*} E$-comodule. Then the represented spectrum $W_{I}$ is $E_{*}$-injective such that $\lambda_{E, I}: E_{*} W_{I} \rightarrow I$ is an isomorphism of $E_{*} E$-comodules.

Proof. Choose a graded divisible abelian group $D$ so that $I$ is embedded into $M=\operatorname{Hom}\left(E_{*}, D\right)_{*}$ as $E_{*}$-modules. Consider the monomorphism $i: I \rightarrow$ $E_{*} E \otimes_{Z_{*}} I \rightarrow E_{*} E \otimes_{\nabla_{*}} M$ of $E_{*} E$-comodules, which has a left inverse $j: E_{*} E \otimes_{H_{*}} M \rightarrow$ I. Identify the injective $E_{*}$-module $M$ with $\pi_{*} \nabla E(D)$ by use of Corollary ${ }^{B_{*}}$ 1.10, and the extended comodule $E_{*} E \otimes \nabla E(D)_{*}$ with $E_{*} \nabla E(D)$ under our assumption on $\left(\mathrm{K}^{\prime \prime}\right)$. Then we get maps $f: \stackrel{A_{*}}{W_{I}} \rightarrow \nabla E(D)$ and $g: \nabla E(D) \rightarrow W_{I}$ making the following diagram commutative

$$
\begin{array}{cccccc}
{\left[X, W_{I}\right]} & \xrightarrow{f_{*}} & {[X, \nabla E(D)]} & \xrightarrow[\rightarrow]{g_{*}} & {\left[X, W_{I}\right]} \\
\downarrow \kappa_{E, I} & & \downarrow \kappa_{E} & & \downarrow \kappa_{E, I} \\
\operatorname{Hom}_{E * E}\left(E_{*} X, I\right) & \overrightarrow{i_{*}} & \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} \nabla E(D)\right) & \overrightarrow{j_{*}} & \operatorname{Hom}_{E * E}\left(E_{*} X, I\right)
\end{array}
$$

where the vertical arrows $\kappa_{E, I}$ and $\kappa_{E}$ are all isomorphisms by (1.9) and Proposition 2.3. Note that the composite $g f: W_{I} \rightarrow W_{I}$ is exactly the identity. Since $f_{*}=i \lambda_{E, I}: E_{*} W_{I} \rightarrow E_{*} \nabla E(D)$ and $j=\lambda_{E, I} g_{*}: E_{*} \nabla E(D) \rightarrow I$, it is easily checked that $\lambda_{E, I}: E_{*} W_{I} \rightarrow I$ is an isomorphism of $E_{*} E$-comodules where $\lambda_{E, I}=\kappa_{E, I}(1)$.

By a quite similar discussion to [8, Theorem 10.1] using Proposition 2.5 with (1.9) we can show

Corollary 2.6. Let $E$ be a ring spectrum satisfying the properties $(\mathrm{F})$ and $\left(\mathrm{K}^{\prime \prime}\right)$ and $C$ be an $E_{*} E$-comodule with injdim ${ }_{E * E} C \leqq 2$. Then there exists an $E_{*}$ local spectrum $Z$ such that $E_{*} Z$ is isomorphic to $C$ as $E_{*} E$-comodules.

Combining Propositions 2.4 and 2.5 with (1.10) we obtain
Proposition 2.7. Let $E$ be a ring spectrum satisfying the properties ( F ) and $\left(\mathrm{K}^{\prime \prime}\right)$ and $\left\{I_{\lambda}\right\}$ be a family of injective $E_{*}$ E-comodules. Then $W$ is an $E_{*}$-local spectrum such that $E_{*} W$ is isomorphic to the direct product $\prod_{\lambda} I_{\lambda}$ as $E_{*} E$-comodules if and only if it coincides with the direct product $\prod_{\lambda} W_{I_{\lambda}}$.

Proof. The "if" part follows immediately from (1.10) and Proposition 2.5. On the other hand, the "only if" part is easily shown by observing (1.10) and the proof of Proposition 2.4.
2.4. Let $E$ be a ring spectrum satisfying the properties $(\mathrm{F})$ and $\left(\mathrm{K}^{\prime}\right)$ and $W$ be an $E$-module spectrum such that the extended comodule $E_{*} E \otimes_{E_{*}} W_{*}$ is injective. Then the $E_{*} E$-comodule $E_{*} W$ is injective since $\nu_{E, W}: E_{*} E \otimes_{B_{*}} W_{*} \rightarrow$
$E_{*} W$ is an isomorphism under the condition $\left(\mathrm{K}^{\prime}\right)$. So Proposition 2.2 asserts that $W$ is an $E_{*}$-injective spectrum and $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$. Conversely we obtain

Lemma 2.8. Let $E$ be a ring spectrum satisfying the property $(\mathrm{F})$ and $W$ be an E-module spectrum such that the extended comodule $E_{*} E \otimes_{F_{*}} W_{*}$ is injective. Assume that $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E_{* E}}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$ spectrum $X$. Then $W$ is an $E_{*}$-injective spectrum such that $\nu_{E, W}: E_{*} E{\underset{B_{*}}{ } W_{*} \rightarrow}^{\text {and }}$ $E_{*} W$ is an isomorphism of $E_{*} E$-comodules.

Proof. According to Lemma $1.13 W$ may be actually regarded as the represented spectrum $W_{I}$ with $I=E_{*} E \otimes_{B_{*}} W_{*}$, which is $E_{*}$-injective. Under our assumption on $\kappa_{E}(1.13)$ implies that $\kappa_{E}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is decomposed into the composite $\nu \kappa_{E, I}:[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, I\right) \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X\right.$, $E_{*} W$ ) where $\nu$ is induced by $\nu_{E, W}: I \rightarrow E_{*} W$. Then it is obvious that $\nu_{E, W}: I \rightarrow$ $E_{*} W$ becomes a left inverse of $\lambda_{E, I}=\kappa_{E, I}(1)$. In order to show that $\nu_{E, W}$ is a right inverse of $\lambda_{E, I}$ we here use the canonical isomorphism $\theta: \operatorname{Hom}_{E * E}(C, I) \rightarrow$ $\operatorname{Hom}_{E_{*}}\left(C, \pi_{*} W\right)$ for any $E_{*} E$-comodule $C$. By using (1.13) again we note that $\theta\left(\lambda_{E, I}\right)=\mu_{*}: E_{*} W \rightarrow \pi_{*} W$ under our assumption on $\kappa_{E}$ where $\mu: E_{\wedge} W \rightarrow W$ denotes the $E$-module structure map of $W$. Taking $C=I$, it is easily computed that $\theta\left(\lambda_{E, I} \nu_{E, W}\right)=\mu_{*} \nu_{E, W}=\theta(1)$. Consequently $\nu_{E, W}: I \rightarrow E_{*} W$ is in fact an isomorphism of $E_{*} E$-comodules.

## Lemma 2.8 combined with Lemma 1.13 and Proposition 2.3 shows

Corollary 2.9. Let $E$ be a ring spectrum satisfying the properties ( F ) and $\left(\mathrm{K}^{\prime \prime}\right)$ and $W$ be an $E$-module spectrum such that the extended comodule $E_{*} E \otimes_{B_{*}} W_{*}$ is injective. Then $W$ is an $E_{*}$-injective spectrum such that $\nu_{E, W}: E_{*} E{\underset{B}{*}}^{\otimes} W_{*} \rightarrow$ $E_{*} W$ is an isomorphism of $E_{*} E$-comodules.

We will here deal with a ring spectrum $E$ satisfying the properties ( F ), $(\mathrm{U})$ and $(\mathrm{P})$ in place of $(\mathrm{F})$ and ( $\left.\mathrm{K}^{\prime \prime}\right)$.

Lemma 2.10. Let $E$ be a ring spectrum satisfying the properties ( F ) and (U). If $E_{*} Y$ is projective as an $E_{*}$-module, then $\kappa_{E}:[Y, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} Y\right.$, $E_{*} W$ ) is an isomorphism for any quasi $E$-module spectrum $W$.

Proof. Consider the composite $\varphi \kappa_{E}:\left[Y, E_{\wedge} W\right] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} Y, E_{*}\left(E_{\wedge} W\right)\right)$ $\rightarrow \operatorname{Hom}_{E_{*}}\left(E_{*} Y, E_{*} W\right)$. Here $\varphi$ is an isomorphism under the condition (F), and $\phi \kappa_{E}$ is an isomorphism under the condition (U). Consequently $\kappa_{E}:[Y, W] \rightarrow$ $\operatorname{Hom}_{E * E}\left(E_{*} Y, E_{*} W\right)$ becomes an isomorphism when $W$ is a quasi $E$-module spectrum.

Using Lemmas 1.4 and 2.10 we obtain
Proposition 2.11. Let $E$ be a ring spectrum satisfying the properties (F), (U) and (P). If a CW-spectrum $W$ is $E_{*}$-injective, then $\kappa_{E}:[X, W] \rightarrow$ $\operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ is an isomorphism for any $C W$-spectrum $X$.

Proof. Under the condition ( P ) we can choose for any $C W$-spectrum $X$ a cofiber sequence $Z \rightarrow Y \rightarrow X$ such that $0 \rightarrow E_{*} Z \rightarrow E_{*} Y \rightarrow E_{*} X \rightarrow 0$ is a short exact sequence and $E_{*} Y$ is projective as an $E_{*}$-module. Use the commutative diagram

$$
\begin{array}{ccccc}
{[X, W]} & \rightarrow & {[Y, W]} & \rightarrow & {[Z, W]} \\
\downarrow \kappa_{E} & & \downarrow \kappa_{E} & & \downarrow \kappa_{E} \\
0 \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right) & \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} Y, E_{*} W\right) & \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} Z, E_{*} W\right)
\end{array}
$$

with two exact rows. The vertical arrows $\kappa_{E}$ are all monomorphisms by Proposition 1.1, and in particular the central one is an isomorphism by means of Lemmas 1.4 and 2.10. Applying Four lemma we see that the left arrow $\kappa_{E}$ : $[X, W] \rightarrow \operatorname{Hom}_{E * E}\left(E_{*} X, E_{*} W\right)$ becomes an epimorphism, and hence an isomorphismi as desired.

Putting Corollary 1.10, Lemma 2.8 and Proposition 2.11 together we can finally remark that
(2.2) a ring spectrum $E$ satisfies the condition ( $\mathrm{K}^{\prime \prime}$ ) if it satisfies (F), (U) and (P).

## 3. $E G_{*}$-injective spectra

3.1. Among abelian groups we introduce a partial order as follows. Given abelian groups $G$ and $G^{\prime}$ we write

$$
\begin{equation*}
\langle G\rangle \leqq\left\langle G^{\prime}\right\rangle \tag{3.1}
\end{equation*}
$$

if for each divisible abelian group $D$ there exists a divisible one $D^{\prime}$ so that Hom $(G, D)$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}, D^{\prime}\right)$.

Let $G$ and $D$ be abelian groups. If $D$ is divisible, then the short exact sequence

$$
\begin{equation*}
0 \rightarrow \operatorname{Hom}(G / \operatorname{Tor} G, D) \rightarrow \operatorname{Hom}(G, D) \rightarrow \operatorname{Hom}(\operatorname{Tor} G, D) \rightarrow 0 \tag{3.2}
\end{equation*}
$$

is split. Here $\operatorname{Hom}(G / \operatorname{Tor} G, D)$ is a maximal divisible subgroup of $\operatorname{Hom}(G, D)$ and $\operatorname{Hom}(\operatorname{Tor} G, D)$ is a reduced algebraically compact group (see [9, Theorem 46.1]). Recall that every divisible abelian group $D$ is written into the direct sum of indecomposable ones which are either isomorphic to $Q$ or $Z / p^{\infty}$ for vari-
ous primes $p$. Using these facts we can show
Lemma 3.1. $\langle G\rangle \leqq\left\langle G^{\prime}\right\rangle$ if and only if for every prime $p$ there exists a divisible abelian group $D_{p}$ so that $\operatorname{Hom}\left(G, Z \mid p^{\infty}\right)$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}, D_{p}\right)$ and in addition $G \otimes Q=0$ whenever $G^{\prime} \otimes Q=0$.

Proof. The "if" part: Obviously $\operatorname{Hom}(G, Q)$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}, D_{0}\right)$ with $D_{0}=\operatorname{Hom}(G, Q)$. Each divisible abelain group $D$ is written into the form of $\left(\underset{\alpha_{0}}{\oplus} Q\right) \oplus \underset{p}{\oplus}\left(\underset{\alpha_{p}}{\oplus} Z / p^{\infty}\right)$ and it is a direct summand of the direct product $\left(\prod_{a_{0}} Q\right) \oplus \prod_{p}\left(\prod_{\alpha_{p}} Z / p^{\infty}\right)$. So the result is immediately obtained.

The "only if" part: Assume that $G^{\prime} \otimes Q=0$. By definition $\operatorname{Hom}(G, Q)$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}, D_{0}\right)$ for some divisible $D_{0}$. However $\operatorname{Hom}\left(G^{\prime}, D_{0}\right)$ is reduced, in other words $\operatorname{Hom}\left(Q, \operatorname{Hom}\left(G^{\prime}, D_{0}\right)\right)=0$. This implies that $\operatorname{Hom}(G, Q)=0$, and hence $G \otimes Q=0$.

For each abelian group $G$ we consider the subset

$$
\begin{equation*}
J(G)=\{p ;(G / \operatorname{Tor} G) \otimes Z \mid p \neq 0\} \tag{3.3}
\end{equation*}
$$

of primes $p$ and set

$$
\begin{equation*}
\tilde{J}(G)=J(G) \cup\{0\} \text { or } \phi \tag{3.4}
\end{equation*}
$$

according as either $G \otimes Q \neq 0$ or $G \otimes Q=0$. For any divisible abelian group $D$, $\operatorname{Hom}(G / \operatorname{Tor} G, D) * Z \mid p$ is isomorphic to $\operatorname{Hom}(G / \operatorname{Tor} G \otimes Z|p, D * Z| p)$. Hence it is easily seen that

$$
\begin{equation*}
\operatorname{Hom}(G / \operatorname{Tor} G, D) \cong\left(\underset{\alpha_{0}}{\oplus} Q\right) \oplus \underset{p \in J(G)}{\oplus}\left(\oplus_{\alpha_{p}} Z \mid p^{\infty}\right) \tag{3.5}
\end{equation*}
$$

where $\alpha_{p} \neq 0$ if $D * Z \mid p \neq 0$. In particular, taking $D=Z / p^{\infty}$ we have

$$
\operatorname{Hom}\left(G / \operatorname{Tor} G, Z / p^{\infty}\right) \cong \begin{cases}\left(\underset{\alpha_{0}}{\oplus} Q\right) \oplus\left(\underset{\alpha_{p}}{\oplus} Z \mid p^{\infty}\right) & \text { if } p \in J(G)  \tag{3.6}\\ \underset{\alpha_{0}}{\oplus} Q & \text { if } p \notin J(G)\end{cases}
$$

where $\alpha_{p} \neq 0$.
For every prime $p$ we can choose a $p$-basic subgroup $B_{p}$ of Tor $G$ which is unique up to isomorphism [9, Theorems 32.3 and 35.2]. The $p$-basic subgroup $B_{p}$ is written into the form of a direct sum of cyclic $p$-groups $Z \mid p^{k}$ and it is $p$-pure in Tor $G$, and moreover (Tor $\left.G / B_{p}\right) \otimes Z_{(p)}$ is expressed as the direct sum of copies of $Z / p^{\infty}$. Consider the subset

$$
\begin{equation*}
J_{p}(G)=\left\{k ; Z / p^{k} \text { is a direct sum component of } B_{p}\right\} \tag{3.7}
\end{equation*}
$$

of positive integers $k$.

When $J_{p}(G)$ is an infinite set, the $p$-adic integers $Z_{p}^{\wedge}$ is embedded into the direct product $\prod_{k \in J_{p}(\epsilon)} Z / p^{k}$ as a direct summand (cf. [9, Corollary 38.2]). We here put

$$
\tilde{J}_{p}(G)= \begin{cases}J_{p}(G) & \text { if } J_{p}(G) \text { is finite and }\left(\operatorname{Tor} G / B_{p}\right) \otimes Z_{(p)}=0  \tag{3.8}\\ J_{p}(G) \cup\left\{\omega_{p}\right\} & \text { if otherwise. }\end{cases}
$$

For any $p$-local divisible abelian group $D_{(p)}$ the short exact sequence

$$
\begin{equation*}
0 \rightarrow \operatorname{Hom}\left(\operatorname{Tor} G / B_{p}, D_{(p)}\right) \rightarrow \operatorname{Hom}\left(\operatorname{Tor} G, D_{(p)}\right) \rightarrow \operatorname{Hom}\left(B_{p}, D_{(p)}\right) \rightarrow 0 \tag{3.9}
\end{equation*}
$$

is split because it is pure owing to [9, Proposition 44.7] and $\operatorname{Hom}\left(\operatorname{Tor} G / B_{p}, D_{(p)}\right)$ is algebraically compact by [9, Theorem 44.7]. As is easily seen, we have

$$
\begin{equation*}
\operatorname{Hom}\left(B_{p}, D_{(p)}\right) \cong \prod_{k \in J_{p}(\theta)}\left(\prod_{\beta_{k}} D_{(p)} * Z / p^{k}\right) \tag{3.10}
\end{equation*}
$$

with $\beta_{k} \neq 0$. On the other hand, $\operatorname{Hom}\left(\operatorname{Tor} G / B_{p}, D_{(p)}\right)$ is the $p$-adic completion of the direct sum of copies of $Z_{p}^{\wedge}$ (use [9, Exercise 47.7]). In particular, taking $D_{(p)}=Z / p^{\infty}$ we obtain

$$
\operatorname{Hom}\left(\operatorname{Tor} G, Z \mid p^{\infty}\right) \cong \begin{cases}\prod_{k \in J_{p}(G)}\left(\Pi_{\beta_{k}} Z \mid p^{k}\right) \oplus \Pi_{\beta_{\omega}} Z_{p}^{\hat{p}} & \text { if }\left(\operatorname{Tor} G \mid B_{p}\right) \otimes Z_{(p)} \neq 0  \tag{3.11}\\ \prod_{k \in J_{p}(G)}\left(\Pi_{\beta_{k}} Z \mid p^{k}\right) & \text { if }\left(\operatorname{Tor} G / B_{p}\right) \otimes Z_{(p)}=0\end{cases}
$$

where $\beta_{k} \neq 0$ and $\beta_{\omega} \neq 0$ (cf. [9, Theorem 47.1]).
3.2. By applying Lemma 3.1 with the aid of (3.2), (3.6), (3.9), (3.10) and (3.11) we will prove the following criterion (cf. [5]).

Proposition 3.2. $\langle G\rangle \leqq\left\langle G^{\prime}\right\rangle$ if and only if $\widetilde{J}(G) \subset \tilde{J}\left(G^{\prime}\right)$ and $\widetilde{J}_{p}(G) \subset$ $\tilde{J}_{p}\left(G^{\prime}\right)$ for every prime $p$.

Proof. The "if" part: If $G^{\prime} \otimes Q=0$, then $\widetilde{J}\left(G^{\prime}\right)=\phi$ by definition. This implies immediately that $G \otimes Q=0$. We will next show that for each prime $p$ there exist divisible abelian groups $D_{p}^{\prime}$ and $D_{p}^{\prime \prime}$ so that $\operatorname{Hom}(G / \operatorname{Tor} G$, $\left.Z \mid p^{\infty}\right)$ and $\operatorname{Hom}\left(\operatorname{Tor} G, Z \mid p^{\infty}\right)$ are respectively direct summands of $\operatorname{Hom}\left(G^{\prime} \mid\right.$ Tor $G^{\prime}, D_{p}^{\prime}$ ) and $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, D_{p}^{\prime \prime}\right)$. In order to find a suitable divisible $D_{p}^{\prime}$ we may assume that $G^{\prime} \otimes Q \neq 0$. Evidently $Q$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}\right)$ Tor $\left.G^{\prime}, Q\right)$. If $p \in J(G)$, then (3.6) shows that $Z \mid p^{\infty}$ is contained in $\operatorname{Hom}\left(G^{\prime} \mid\right.$ Tor $\left.G^{\prime}, Z / p^{\infty}\right)$ as a direct summand because $J(G) \subset J\left(G^{\prime}\right)$. Taking $D_{p}^{\prime}=\left(\prod_{\alpha_{0}} Q\right) \oplus$ ( $\prod_{\alpha_{p}} Z \mid p^{\infty}$ ) or $\prod_{\alpha_{0}} Q$ according as $p \in J(G)$ or $p \notin J(G)$, it is easily seen that $\operatorname{Hom}\left(G / \operatorname{Tor} G, Z / p^{\infty}\right)$ expressed as in (3.6) becomes a direct summand of $\operatorname{Hom}\left(G^{\prime} / \operatorname{Tor} G^{\prime}, D_{p}^{\prime}\right)$.

Using (3.11) in place of (3.6) we will similarly find a suitable divisible $D_{p}^{\prime \prime}$. If $k \in J_{p}(G)$, then (3.11) shows that $Z / p^{k}$ is contained in $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, Z / p^{\infty}\right)$ as a direct summand because $J_{p}(G) \subset J_{p}\left(G^{\prime}\right)$. Assume that ( $\left.\operatorname{Tor} G / B_{p}\right) \otimes Z_{(p)} \neq 0$. In this situation $J_{p}\left(G^{\prime}\right)$ is an infinite set or $\left(\operatorname{Tor} G^{\prime} \mid B_{p}^{\prime}\right) \otimes Z_{(p)} \neq 0$ because the element $\omega_{p}$ is belonging to $\tilde{J}_{p}(G) \subset \tilde{J}_{p}\left(G^{\prime}\right)$. Set $D_{\omega}=\prod_{k \in J_{p}\left(G^{\prime}\right)} Z / p^{\infty}$ or $Z / p^{\infty}$ according as $J_{p}\left(G^{\prime}\right)$ is infinite or $\left(\operatorname{Tor} G^{\prime} \mid B_{p}^{\prime}\right) \otimes Z_{(p)} \neq 0$. By use of (3.11) we see easily that $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, D_{\omega}\right)$ contains $Z_{p}^{\wedge}$ as a direct summand. Taking $D_{p}^{\prime \prime}=$ $\prod_{k \in J_{p}(\mathcal{A})}\left(\prod_{\beta_{k}} Z / p^{\infty}\right) \oplus \prod_{\beta_{\omega}} D_{\omega}$, it follows immediately that $\operatorname{Hom}\left(\operatorname{Tor} G, Z / p^{\infty}\right)$ expressed as in (3.11) is a direct summand of $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, D_{p}^{\prime \prime}\right)$ as desired. By the aid of (3.2) we may now apply Lemma 3.1 to obtain $\langle G\rangle \leqq\left\langle G^{\prime}\right\rangle$.

The "only if" part: For every prime $p$ we choose a divisible abelian group $D_{p}$ so that $\operatorname{Hom}\left(G, Z \mid p^{\infty}\right)$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}, D_{p}\right)$. As is easily verified, $\operatorname{Hom}\left(G / \operatorname{Tor} G, Z / p^{\infty}\right)$ and $\operatorname{Hom}\left(\operatorname{Tor} G, Z / p^{\infty}\right)$ are respectively direct summands of $\operatorname{Hom}\left(G^{\prime} / \operatorname{Tor} G^{\prime}, D_{p}\right)$ and $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, D_{p}\right)$. When $J(G)=\phi$, it follows from Lemma 3.1 that $\widetilde{J}(G) \subset \widetilde{J}\left(G^{\prime}\right)$. Assume that $J(G) \neq \phi$. For each prime $p \in J(G)$, (3.6) asserts that as a direct summand $Z / p^{\infty}$ is contained in $\operatorname{Hom}\left(G / \operatorname{Tor} G, Z / p^{\infty}\right)$ and also in $\operatorname{Hom}\left(G^{\prime} / \operatorname{Tor} G^{\prime}, D_{p}\right)$. Therefore we see that $\operatorname{Hom}\left(G^{\prime} / \operatorname{Tor} G^{\prime} \otimes Z \mid p, D_{p}\right) \neq 0$, and hence $p \in J\left(G^{\prime}\right)$. Thus $J(G) \subset J\left(G^{\prime}\right)$, which implies that $\widetilde{J}(G) \subset \widetilde{J}\left(G^{\prime}\right)$.

Pick up a positive integer $k \in J_{p}(G)$ for a fixed prime $p$. By virtue of (3.11) $Z \mid p^{k}$ is contained in $\operatorname{Hom}\left(\operatorname{Tor} G, Z \mid p^{\infty}\right)$ as a direct summand and also in $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, D_{p} \otimes Z_{(p)}\right)$. However $Z / p^{k}$ is actually embedded into $\operatorname{Hom}\left(B_{p}^{\prime}, D_{p} \otimes Z_{(p)}\right)$ because $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime} \mid B_{p}^{\prime}, D_{p} \otimes Z_{(p)}\right)$ is torsion free. Using (3.10) we can easily see that $k \in J_{p}\left(G^{\prime}\right)$. Thus $J_{p}(G) \subset J_{p}\left(G^{\prime}\right)$. In order to show $\tilde{J}_{p}(G) \subset \widetilde{J}_{p}\left(G^{\prime}\right)$ we here assume that $\widetilde{J}_{p}(G)$ includes the element $\omega_{p}$, thus $J_{p}(G)$ is an infinite set or $\left(\operatorname{Tor} G / B_{p}\right) \otimes Z_{(p)} \neq 0$. For our purpose we may consider only the case when $J_{p}\left(G^{\prime}\right)$ is finite and ( $\left.\operatorname{Tor} G / B_{p}\right) \otimes Z_{(p)} \neq 0$. In this situation $Z_{p}^{\wedge}$ is contained in $\operatorname{Hom}\left(\operatorname{Tor} G, Z \mid p^{\infty}\right)$ as a direct summand by use of (3.11), and hence in $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}, D_{p} \otimes Z_{(p)}\right)$. If $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime} \mid B_{p}^{\prime}, D_{p} \otimes Z_{(p)}\right)=0$, then $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime}\right.$, $\left.D_{p} \otimes Z_{(p)}\right) \cong \operatorname{Hom}\left(B_{p}^{\prime}, D_{p} \otimes Z_{(p)}\right)$ is bounded because of (3.10). This is a contradiction. Thus $\operatorname{Hom}\left(\operatorname{Tor} G^{\prime} \mid B_{p}^{\prime}, D_{p} \otimes Z_{(p)}\right) \neq 0$ which implies that $\left(\operatorname{Tor} G^{\prime} \mid B_{p}^{\prime}\right) \otimes$ $Z_{(p)} \neq 0$. Therefore the element $\omega_{p}$ is belonging to $\widetilde{J}_{p}\left(G^{\prime}\right)$ in our case. Consequently $\tilde{J}_{p}(G) \subset \widetilde{J}_{p}\left(G^{\prime}\right)$.

### 3.3. For any $C W$-spectra $E$ and $F$ we write

$$
\begin{equation*}
\langle E\rangle_{I} \leqq\langle F\rangle_{I} \tag{3.12}
\end{equation*}
$$

if every $E_{*}$-injective spectrum $W$ becomes always $F_{*}$-injective (cf. [6], [7] or [14]).

Lemma 3.3. $\langle E\rangle_{I} \leqq\langle F\rangle_{I}$ if and only if every $F_{*}$-monic map $f: X \rightarrow Y$
is always $E_{*-m o n i c . ~}^{\text {. }}$
Proof. The "if" part is immediate by definition.
The "only if" part: Given an $F_{*}$-monic map $f: X \rightarrow Y$ we may choose an $E_{*}$-injective spectrum $W$ and an $E_{*}$-monic map $g: X \rightarrow W$ by virtue of Proposition 1.5. Since $W$ becomes $F_{*}$-injective, there exists a map $h: Y \rightarrow W$ with $h f=g$. This equality implies that $f: X \rightarrow Y$ is $E_{*}$-monic.

As a relation between the partial orders (3.1) and (3.12) we have
Lemma 3.4. i) If $\langle G\rangle \leqq\left\langle G^{\prime}\right\rangle$, then $\langle E G\rangle_{I} \leqq\left\langle E G^{\prime}\right\rangle_{I}$ for any $C W$-spectrum $E$.
ii) Assume that $\pi_{*} E$ is torsion free and $\pi_{*} E \otimes Z \mid p \neq 0$ for each prime $p$. Then the converse of i) is valid.

Proof. i) is immediate from Proposition 1.6 and Definitions (3.1) and (3.12).
ii) We may assume that $\pi_{0} E \otimes Z \mid p \neq 0$ for a fixed prime $p$. Set $A_{p}=$ $\operatorname{Hom}\left(G, Z \mid p^{\infty}\right)$. By means of (1.3) and (1.4) the Anderson dual spectrum $\nabla E\left(A_{p}\right)$ becomes $E G_{*}$-injective, and hence it is $E G_{*}^{\prime}$-injective. According to Proposition $1.6 \nabla E\left(A_{p}\right)$ is a retract of a certain Anderson dual spectrum $\nabla E(B)=\prod_{n} \Sigma^{n} \nabla E\left(B_{n}\right)$ where $B_{n}=\operatorname{Hom}\left(G^{\prime}, D_{n}\right)$ for some divisible $D_{n}$. So $\pi_{0} \nabla E\left(A_{p}\right)$ becomes a direct summand of $\pi_{0} \nabla E(B)=\prod_{n} \pi_{-n} \nabla E\left(B_{n}\right)$. As is easily computed, $\pi_{0} \nabla E\left(A_{p}\right) \cong$ $\operatorname{Hom}\left(\pi_{0} E, A_{p}\right) \cong \operatorname{Hom}\left(G, \operatorname{Hom}\left(\pi_{0} E, Z \mid p^{\infty}\right)\right)$ and similarly $\pi_{-n} \nabla E\left(B_{n}\right) \cong \operatorname{Hom}\left(G^{\prime}\right.$, $\left.\operatorname{Hom}\left(\pi_{n} E, D_{n}\right)\right) . \quad B y(3.5)$ the divisible group $\operatorname{Hom}\left(\pi_{0} E, Z \mid p^{\infty}\right)$ contains $Z \mid p^{\infty}$ as a direct summand under the assumption that $\pi_{0} E \otimes Z \mid p \neq 0$. Therefore $A_{p}=$ $\operatorname{Hom}\left(G, Z \mid p^{\infty}\right)$ is contained in $\pi_{0} \nabla E\left(A_{p}\right)$ as a direct summand, and hence in $\pi_{0} \nabla E(B) \cong \operatorname{Hom}\left(G^{\prime}, \widetilde{D}_{p}\right)$ where $\tilde{D}_{p}=\prod_{n} \operatorname{Hom}\left(\pi_{n} E, D_{n}\right)$ and it is divisible.

Similarly we can choose a divisible abelian group $\widetilde{D}_{0}$ so that $A_{0}=\operatorname{Hom}(G, Q)$ is a direct summand of $\operatorname{Hom}\left(G^{\prime}, \tilde{D}_{0}\right)$. This implies that $G \otimes Q=0$ whenever $G^{\prime} \otimes Q=0$, as was shown in the proof of Lemma 3.1. Consequently it follows from Lemma 3.1 that $\langle G\rangle \leqq\left\langle G^{\prime}\right\rangle$.

Combining Proposition 3.2 with Lemma 3.4 we obtain
Theorem 3.5. i) If $\tilde{J}(G) \subset \widetilde{J}\left(G^{\prime}\right)$ and $\widetilde{J}_{p}(G) \subset \widetilde{J}_{p}\left(G^{\prime}\right)$ for every prime $p$, then $\langle E G\rangle_{I} \leqq\left\langle E G^{\prime}\right\rangle_{I}$ for any $C W$-spectrum $E$.
ii) Assume that $\pi_{*} E$ is torsion free and $\pi_{*} E \otimes Z \mid p \neq 0$ for each prime $p$. Then the converse of i) is valid.

As an immediate result we have the following criterion (cf. [7, Proposition 2.7]).

Corollary 3.6. Assume that $\pi_{*} E$ is torsion free and $\pi_{*} E \otimes Z \mid p \neq 0$ for
each prime $p$. Then $\langle E G\rangle_{I}=\left\langle E G^{\prime}\right\rangle_{I}$ if and only if $\tilde{J}(G)=\tilde{J}\left(G^{\prime}\right)$ and $\tilde{J}_{p}(G)=$ $\breve{J}_{p}\left(G^{\prime}\right)$ for every prime $p$.
3.4. Let us denote by $H$ the Eilenberg-MacLane spectrum and by $K O$, $K U$ and $K T$ the real, the complex and the self-conjugate $K$-spectrum respectively. Recall that $\nabla H(G)=H G, \nabla K O(G)=\Sigma^{4} K O G, \nabla K U(G)=K U G$ and $\nabla K T(G)=\Sigma^{1} K T G$ for any abelian group $G$ (see [4] or [15, I]). Note that the canonical map $\omega: \bigvee_{n} \Sigma^{n} H A_{n} \rightarrow \prod_{n} \Sigma^{n} H A_{n}$ is an equivalence for any graded abelian group $A=\left\{A_{n}\right\}$. On the other hand, it is well known that the $K$-spectra $K=$ $K O, K U$ and $K T$ possess the period $p(K)=8,2$ and 4 respectively.

Taking $E=H, K O, K U$ or $K T$ in Proposition 1.6 we can immediately show
Proposition 3.7. i) $A C W$-spectrum $W$ is $H G_{*}$-injective if and only if it is a retract of a generalized Eilenberg-MacLane spectrum $\bigvee_{n} \Sigma^{n} H B_{n}$ in which $B_{n}=\operatorname{Hom}\left(G, D_{n}\right)$ for some divisible $D_{n}$.
ii) Let $K$ denote the periodic $K$-spectrum $K O, K U$ or $K T$. Then a $C W$ spectrum $W$ is $K G_{*}$-injective if and only if it is a retract of a certain finive wedge sum $\vee_{0 \leqq n<p(K)} \Sigma^{n} K B_{n}$ in which $B_{n}=\operatorname{Hom}\left(G, D_{n}\right)$ for some divisible $D_{n}$.

The following easy result is useful in studying the $E G_{*}$-injectivity when $E=H$ or $K U$.

Lemma 3.9. i) If $W$ is a quasi $H$-module spectrum, then it is a generalized Eilenberg-MacLane spectrum written into the form ${\underset{n}{n}} \Sigma^{n} H\left(\pi_{n} W\right)$.
ii) If $W$ is a quasi $K U$-module spectrum such that the composite $\mu_{*} \beta^{n}(\iota \wedge 1)_{*}$ : $\pi_{i} W \rightarrow K U_{i} W \cong K U_{2 n+i} W \rightarrow \pi_{2 n+i} W(i=0,1)$ is an isomorphism for any $n$, then it is written into the wedge sum $K U\left(\pi_{0} W\right) \vee \Sigma^{1} K U\left(\pi_{1} W\right)$.

Proof. i) Our proof is due to [2, Lemma III.6.1]). Set $A_{n}=\pi_{n} W$ and choose a map $f_{n}: \Sigma^{n} S A_{n} \rightarrow W$ inducing the identity isomorphism between the $n$-th homotopy groups. Construct a map $f: \bigvee_{n} \Sigma^{n} H A_{n} \rightarrow W$ whose $n$-th component is the composite $\mu\left(1 \wedge f_{n}\right): \Sigma^{n} H A_{n} \rightarrow H_{\wedge} W \rightarrow W$ where $\mu: H \wedge W \rightarrow W$ denotes the $H$-module structure map of $W$. As is easily checked, the map $f$ is an equivalence.
ii) is similarly shown to i).

Conserning the $H G_{*}$-injectivity we obtain the following characterization.
Thoerem 3.9. The following three condıtions are equivalent:
i) $W$ is an $H G_{*}$-injectıve spectrum.
ii) $W$ is a quasi $H$-module spectrum such that for each $n \pi_{n} W$ is a direct summand of $\operatorname{Hom}\left(G, D_{n}\right)$ wuth $D_{n}$ divisible.
iii) $W$ is a generalized Eilenberg-MacLane spectrum $\bigvee_{n} \Sigma^{n} H A_{n}$ in which $A_{n}$ is a direct summand of $\operatorname{Hom}\left(G, D_{n}\right)$ with $D_{n}$ divisible.

Proof. The implication iii) $\rightarrow$ i) $\rightarrow$ ii) follows immediately from Proposition 3.7 i), and the implication ii) $\rightarrow$ iii) is immediate from Lemma 3.8 i ).

Let $G$ be an abelian group whose torsion subgroup Tor $G$ is bounded. Since the bounded torsion group Tor $G$ is a direct sum of cyclic torsion groups, the sets $J_{p}(G)$ are finite but they are empty except only finite numbers of primes $p$. By means of (3.2) and (3.5) we then observe

$$
\begin{equation*}
\operatorname{Hom}(G, D) \cong\left(\underset{\alpha_{0}}{\oplus} Q\right) \oplus \underset{p \in J(G)}{\oplus}\left(\oplus_{\alpha_{p}} Z \mid p^{\infty}\right) \oplus \underset{p}{\oplus} \underset{k \in J_{p}(G)}{\oplus}\left(\underset{\beta_{p, k}}{\oplus} Z \mid p^{k}\right) \tag{3.13}
\end{equation*}
$$

for each divisible abelian group $D$, where $\alpha_{p} \neq 0$ and $\beta_{p, k} \neq 0$ if $D * Z \mid p \neq 0$.
Corollary 3.10. Assume that the torsion subgroup Tor $G$ is bounded. Then a $C W$-spectrum $W$ is $H G_{*}$-injective if and only if it is a generalized EilenbergMacLane spectrum $\bigvee_{n}^{\vee} \Sigma^{n} H A_{n}$ in which each $A_{n}$ is a direct sum of divisible groups and cyclic torsion groups as given in the right side of (3.13).
3.5. Concerning the $K U G_{*}$-injectivity we obtain the following characterization under some restriction to $G$.

Theorem 3.11. Assume that the torsion subgroup Tor $G$ is bounded. The following three conditions are equivalent:
i) $W$ is a $K U G_{*}$-injeciive spectrum.
ii) $W$ is a quasi $K U$-module spectrum such that $K U \wedge W$ is $K U G_{*}$-injective.
iii) $W$ is a quasi $K U$-module spectrum such that $K U_{i} W$ is a direct sum of divisible groups and cyclic torsion groups as given in the right side of (3.13).

Proof. By virtue of Corollary 3.6 we may regard as Tor $G$ is finitely generated.

The implication i) $\rightarrow$ ii): According to Proposition 3.7 ii), $W$ is a retract of the wedge sum $K U B_{0} \vee \Sigma^{1} K U B_{1}$ for some $B_{i}=\operatorname{Hom}\left(G, D_{i}\right)$ with $D_{i}$ divisible ( $i=0,1$ ). Recall that $K U_{0} K U$ is a countable free abelian group and $K U_{1} K U=0$ (see [3, Theorem 2.1]). Then Lemma 3.8 ii) shows that the smash product $K U \wedge K U$ is just the wedge sum $\vee K U$ of countable copies of $K U$. Therefore $K U \wedge W$ is a retract of the wedge sum $K U\left(\oplus B_{0}\right) \vee \Sigma^{1} K U\left(\oplus B_{1}\right)$. Consider the short exact sequence $0 \rightarrow \underset{\lambda}{\oplus} \operatorname{Hom}\left(G, D_{\lambda}\right) \rightarrow \operatorname{Hom}\left(G, \underset{\lambda}{\oplus} D_{\lambda}\right) \rightarrow C \rightarrow 0$ with $D_{\lambda}$ divisible. Since Tor $G$ may be regarded to be finitely generated, the canonical homomorphism $\underset{\lambda}{\oplus} \operatorname{Hom}\left(\operatorname{Tor} G, D_{\lambda}\right) \rightarrow \operatorname{Hom}\left(\operatorname{Tor} G, \underset{\lambda}{\oplus} D_{\lambda}\right)$ is an isomorphism. Hence we get a short exact sequence $0 \rightarrow \underset{\lambda}{\oplus} \operatorname{Hom}\left(G / \operatorname{Tor} G, D_{\lambda}\right) \rightarrow \operatorname{Hom}\left(G / \operatorname{Tor} G, \underset{\lambda}{\oplus} D_{\lambda}\right)$ $\rightarrow C \rightarrow 0$, which is clearly split. This implies that the previous short exact sequence is split, too. Consequently we see that $\oplus B_{i}=\oplus \operatorname{Hom}\left(G, D_{i}\right)$ is a direct summand of $\tilde{B}_{i}=\operatorname{Hom}\left(G, \oplus D_{i}\right)$ for each $i=0,1$. So $K U \wedge W$ is $K U G_{*^{-}}$
injective by use of Proposition 3.7 ii) again.
The implication ii) $\rightarrow$ iii) follows immediately from Proposition 3.7 ii) and (3.13).

The implication iii) $\rightarrow$ i): By Lemma 3.8 ii) $K U \wedge W$ is just the wedge sum $K U A_{0} \vee \Sigma^{1} K U A_{1}$ with $A_{i}=K U_{i} W(i=0,1)$. From Proposition 3.7 ii) and (3.13) it is immediate that $K U \wedge W$ is $K U G_{*}$-injective. Then $W$ becomes also $K U G_{*}$-injective since $W$ is a retract of $K U \wedge W$.

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