OPTIMUM STRATIFICATION AND ITS ROBUSTNESS WITH COMPUTATIONAL CONSIDERATION

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ABSTRACT

Optimum Stratification Points (OSP), Minimum Variances and some kinds of stratification efficiencies were computed (a) for typical four distributions, including normal and exponential, (b) under three sample allocation methods; Neyman, Equal and Proportional, (c) up to 10 strata, (d) in estimating the population mean μ and variance σ^2, by using nonlinear programming algorithms. The optimum stratification in estimating μ was found to be attained by Interval Optimum Stratification with 5 or less strata usually, whereas General Optimum Stratification was very effective in estimating σ^2, especially for a symmetric distribution.

Based on these results, some sampling procedure was proposed, which was effective in decreasing the standard error of the estimator for μ in some representative practical examples by about 30-60% compared with the traditional procedure.

The author pointed out the importance of evaluating the robustness of the optimum stratification method with respect to a small change of (i) the distribution, (ii) sample sizes in respective strata and (iii) stratification points, and gave some formulae for the evaluation. Numerical studies with practical examples showed that (1) each of the measures of the three kinds of robustness was so small as less than 10%, (2) the proposed procedure, therefore, might be useful in practical fields, (3) Equal Allocation is recommendable for its simplicity, robustness and similarity to the Neyman allocation, and (4) symmetric distributions were more robust than unsymmetric distributions.

CONTENTS

CHAPTER 1

INTRODUCTION

Stratified random sampling is frequently employed in various fields in order to reduce the variance of some estimator of a population parameter. Many research workers therefore have studied "Optimum Stratification Problem". Technical problems in this subject are as follows:

(a) the choice of sampling design within strata — Kitagawa[26],

(b) the choice of a population parameter to be estimated — Wakimoto[45] - [47] and Taga et al.[34],

(c) the choice of stratification variable — Ericson[14] and Taga[33],

(d) the allocation of sample size — Aoyama[1], [2] and Cochran[5],

(e) The choice of the number of strata — Murthy[29] and Taguri et al.[41],

(f) the location of the boundary points of stratification — Hayashi et al.[18], [19], Dalenius[7], Isii et al.[23], [24], Taga[33], Sethi[32], Taguri et al.[36], [41] and so on,

(g) the estimation of expected gains from stratification — Taguri et al.[41].

The best method of stratified random sampling may consist of determining an optimal choice of a solution to the problems stated above, since they are interdependent.

Earlier studies on the stratified random sampling were, however, mainly restricted to the area of the allocation of sample size and the determination of Optimum Stratification Points (OSP). Hayashi et al.[18], [19] and Dalenius[7] suggested the significance of the determination problem of OSP and considered the simultaneous nonlinear equations to be satisfied by OSP. Various practical procedures or rules for attaining the approximate OSP were proposed by Dalenius et al.[8] - [10], Durbin[11], Eckman[12] and Kpedekpo[28]. Actually Sethi[32] and Taguri et al.[36], [41] computed OSP for some typical distributions by using iterative approaches or mathematical programming techniques. Comparison of several iterative procedures were reported by Cochran[4] and Hess et al.[20]. On the other hand, Ghosh[15] considered the bi-variate cases and Isii et al.[24] has extended the results of Ghosh. Furthermore, another optimum stratification were proposed by Isii[23] and extended by Taga[33], which were more general stratification than the traditional interval stratification.

As for the allocation of sample size, Neyman suggested the problem and many research workers have discussed it. Cochran[4] discovered the fact that Neyman Allocation was nearly equivalent to Equal Allocation. Although this has been numerically ascertained by Sethi[32] and Taguri et al.[41], it is not yet proved theoretically. Ghosh[16], Aoyama[2] discussed the multivariate case using concomitant variable. On the other hand, Jagannathan[25] and Nordbotten[30] suggested the formulation of the allocation problem as a mathematical programming problem and Bracken et al.[3] actually solved some optimum sample allocation problem. However, it may be said that this approach has, in general, received little attention up to this time, partly because of the statisticians' unfamiliarity with mathematical programming theory.

In almost all studies stated above, the population parameter to be estimated was the population mean. Wakimoto[45] - [47] suggested the problem to estimate the population variance, covariance and correlation coefficient, and Taguri[36] numerically gave OSP in estimating the population variance.

As for the number of strata and the gain from stratification, Murthy[29] and Taguri et al.[41] discussed the optimum number of strata under some appropriate cost function.

These studies were done mainly from the theoretical point of view. On the other hand, in practical sample surveys, stratification has been usually used without any consideration of theoretical results on optimum stratification. So I have studied this problem in order to apply the theoretical results to practical problems.

At the beginning of sample design, the information needed to perform it is assumed to be given. Therefore population distribution F is definitely specified from past surveys or a pilot survey. Although F could be different from the actual distribution, we can only apply our design to such specified F. For this reason, we will assume that F is known. Then it will make the sampling design more efficient to offer Optimum Stratification Points and Minimum Variances of the estimator for some typical distributions numerically, so as to provide approximate optimum stratification points through specifying the population distribution F. In many practical problems, we may be able to guess the type of distribution approximately even if the true distribution itself is not obtainable.

(i) From this standpoint, we firstly make tables giving Optimum Stratification Points (OSP), Minimum Variances (Min.Var.) and some sorts of efficiencies for some distributions (Taguri[36], [41]).

(ii) Secondly we propose a method how to use these tables, and then apply our method to actual data sets; Current Survey on Petroleum Products Demand and Supply which was performed by the Ministry of International Trade and Industry (MITI) in 1981 (Taguri[37]).

(iii) Now in applying the tables to practical problems, the optimum values in stratification will not be always practicable because of various constraints in practical fields. So it may be important to analyse the problem how much the value of an objective function (the variance of an estimator) is influenced by small deviation of a distribution function (Taguri[38]), sample sizes in respective strata (Taguri[40]) and/or stratification points (Taguri[39]). Therefore we, finally, investigate these facets of robustness analytically and numerically.

Through these studies, the theoretical results on optimum stratification could be effectively useful in many practical jobs.

CHAPTER 2

NOTATIONS AND PRELIMINARIES

2.1. Quantities of Population

Throughout the present paper, the distribution function $F(x)$ corresponding to the population Π is assumed to be absolutely continuous and to have finite fourth order moment. Let Π be decomposed into ℓ strata Π_i $(i=1,2,\ldots,\ell)$, where ℓ is a preassigned positive integer. Corresponding to Π_i, $F(x)$ can be decomposed into $F_i(x)(i=1,2,\ldots,\ell)$ satisfying the following relation;

$$F(x)=\sum_{i=1}^{\ell}F_i(x) \qquad \text{for} \quad x\in R^1,$$

where $F_i(x)$ is non-negative and non-decreasing in x for $i=1,2,\ldots,\ell$. This is called "an ℓ-decomposition of F". Put $w_i=\lim_{x\to\infty} F_i(x)-\lim_{x\to-\infty} F_i(x)$, then w_i represents the weight of Π_i and $F_i(x)/w_i$ is a distribution function of Π_i. Let us denote the population mean and variance by μ, $\sigma^2(\sigma>0)$ and the mean and variance of the i-th stratum by μ_i, $\sigma_i^2(\sigma_i>0)$ respectively;

$$\begin{aligned} \mu_i &= \int_{-\infty}^{\infty} x dF_i(x)/w_i, \\ \sigma_i^2 &= \int_{-\infty}^{\infty} (x-\mu_i)^2 dF_i(x)/w_i, \end{aligned} \qquad (i=1,2,\ldots\ldots,\ell).$$

The assumption implies that there exist the moments $r_{\nu i}$ up to the 4-th order of the i-th stratum;

$$r_{\nu i} = \int_{-\infty}^{\infty} x^{\nu} dF_i(x) < \infty \qquad (\nu=0,1,2,3,4). \tag{2.1}$$

Since the population distribution is absolutely continuous by the assumption, there exists the probability density function (p.d.f.) $f(x)$.

2.2. Stratification Method

Now for any ℓ-decomposition $\{F_i\}$ there corresponds the following vector-valued function $\boldsymbol{\phi}=(\phi_1,\phi_2,\ldots\ldots,\phi_\ell)$ defined uniquely except for F-measure 0(see Taga[33]);

$$\phi_i(x)\geqq 0 \quad (i=1,2,\ldots,\ell); \qquad \sum_{i=1}^{\ell}\phi_i(x)=1 \quad \text{for a.e. } F,$$

where $\phi_i(x)$ is the Radon-Nikodym derivative dF_i/dF of the measure F_i with respect to the measure F. Therefore we may take such a vector-valued function $\boldsymbol{\phi}(x)$ for a stratification or "a General Stratification (GS)" and designate a stratification by $\boldsymbol{\phi}$ instead of $\{F_i(x)\}$ hereafter.

Let $\mathcal{U}$ be the set of all open, half open or closed intervals in $\mathbf{R}^1$. Let $\mathcal{E}$ be the empty set. If $\boldsymbol{\phi}(x)$ satisfies

$$\phi_i(x)=\begin{cases}1 \text{ on } I_i\varepsilon\mathcal{U}, \\ 0 \text{ on } \mathbf{R}^1-I_i,\end{cases} \quad (i=1,2,..,\ell); \; \cup_{i=1}^{\ell}I_i=\mathbf{R}^1; \quad I_i\cap I_j=\mathcal{E} \text{ if } i\neq j,$$

then we call it "Interval Stratification (IS)". If the F-measure for the set $\{x;0<\phi_i(x)<1\}$ is positive for some $i\varepsilon\{1,2,\ldots,\ell\}$, then we call it "Randomized Stratification (RS)".

2.3. Quantities Obtained from Sample

Let $(X_{i1},X_{i2},\ldots\ldots,X_{in_i})$ be a random sample with size n_i drawn from the i-th stratum Π_i for $i=1,2,\ldots,\ell$. Total sample size is fixed and

$$n = \sum_{i=1}^{\ell}n_i.$$

Throughout this paper, we consider the following estimator of the population mean μ based on the stratified random sample:

$$\overline{X}=\sum_{i=1}^{\ell}w_i\overline{X}_i; \qquad \overline{X}_i=\sum_{j=1}^{n_i}X_{ij}/n_i \quad (i=1,2,\ldots,\ell). \tag{2.2}$$

The estimator $\overline{X}$ is unbiased provided that the weights w_i are known(see Section 6.1). We also consider the following estimator of the population variance σ^2.

$$U_{st} = \sum_{i=1}^{\ell} w_i U_i + \sum_{i=1}^{\ell} w_i (\overline{X}_i - \overline{X})^2 - \sum_{i=1}^{\ell} w_i (1-w_i) U_i / n_i,$$
$$U_i = \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2/(n_i - 1) \quad (i=1,2,\ldots,\ell).$$

If a stratification method ϕ and an allocation method $\{n_i\}$ of sample sizes in respective strata are determined, then the variance $V(\overline{X}|\phi)$ of the estimator $\overline{X}$ is given by

$$\begin{aligned} V(\overline{X}|\phi) &= \sum_{i=1}^{\ell} w_i^2 \sigma_i^2 / n_i \\ &= \sum_{i=1}^{\ell} [\int_{-\infty}^{\infty} \phi_i(x) dF(x) \int_{-\infty}^{\infty} x^2 \phi_i(x) dF(x) - \{\int_{-\infty}^{\infty} x \phi_i(x) dF(x)\}^2]/n_i. \end{aligned} \qquad (2.3)$$

2.4. Sample Allocation Method

Let us consider the following three sample allocation methods:

Neyman Allocation (NA) : $n_i = n w_i \sigma_i / \sum_{j=1}^{\ell} w_j \sigma_j$,

Equal Allocation (EA) : $n_i = n/\ell$, $(i=1,2,\ldots\ldots,\ell)$.

Proportional Allocation (PA) : $n_i = n w_i$,

Then the variance of the estimator $\overline{X}$ is given by

$V_N(\overline{X}|\phi) = (\sum_{i=1}^{\ell} w_i \sigma_i)^2 / n$ for NA,

$V_E(\overline{X}|\phi) = \ell \sum_{i=1}^{\ell} w_i^2 \sigma_i^2 / n$ for EA,

$V_P(\overline{X}|\phi) = \sum_{i=1}^{\ell} w_i \sigma_i^2 / n$ for PA.

If we decide a number of strata ℓ and a total sample size n, the problem of optimum stratification for the population parameter μ is reduced to determining the stratification method ϕ under a given allocation method so as to minimize the variance of $\overline{X}$. Therefore we may adopt the following functions $\psi(\phi)$ as our objective functions to be minimized:

$$\Psi_N(\phi)=nV_N(\overline{X}|\phi)=(\sum_{i=1}^{\ell} w_i\sigma_i)^2 \quad \text{for NA,} \tag{2.4}$$

$$\Psi_E(\phi)=nV_E(\overline{X}|\phi)=\ell\sum_{i=1}^{\ell} w_i^2\sigma_i^2 \quad \text{for EA,} \tag{2.5}$$

$$\Psi_P(\phi)=nV_P(\overline{X}|\phi)=\sum_{i=1}^{\ell} w_i\sigma_i^2 \quad \text{for PA.} \tag{2.6}$$

In estimating the population variance σ^2, we consider only the case of PA for the simplicity of computation. Then the estimator U_{st} is unbiased and its variance is given by

$$V(U_{st}|\phi)=\frac{1}{n}[\sum_{i=1}^{\ell} w_i\{\int_{-\infty}^{\infty}(x-\mu_i)^4\phi_i(x)dF(x)-\sigma_i^4+4(\mu_i-\mu)\int_{-\infty}^{\infty}(x-\mu_i)^3\phi_i(x)dF(x)$$
$$+4(\mu_i-\mu)^2\sigma_i^2\}+\frac{2}{n}(\sum_{i=1}^{\ell} w_i\sigma_i^2)^2+\frac{2}{n^2}\sum_{i=1}^{\ell} w_i^2\sigma_i^4/(n_i-1)].$$

It is proved by Wakimoto [45] that

$$V(U)-V(U_{st}|\phi)=\frac{1}{n}\sum_{i<j}^{\ell} w_i w_j\{\sigma_i^2-\sigma_j^2+(\mu_i-\mu)^2-(\mu_j-\mu)^2\}^2$$
$$+\frac{2}{n^2}\{\sigma^4-(\sum_{i=1}^{\ell} w_i\sigma_i^2)^2\}+\frac{1}{n^3}\{\sum_{i=1}^{\ell} w_i(\sigma^4-\sigma_i^4)+\sum_{i=1}^{\ell}(\frac{\sigma^4}{n-1}-\frac{\sigma_i^4}{n_i-1})\}, \tag{2.7}$$

where U is an unbiased estimator of σ^2 based on a simple random sample $(X_1,X_2,\ldots\ldots,X_n)$ and is given by $U=\sum_{i=1}^{n}(X_i-\overline{\overline{X}})^2/(n-1)$ $(\overline{\overline{X}}=\sum_{i=1}^{n}X_i/n)$. Provided that a number of strata ℓ and total sample size n are given, minimizing $V(U_{st}|\phi)$ is equivalent to maximizing (2.7) because of $V(U)$ being constant. If n is so large that the last two terms of the right-hand of (2.7) may be neglected, the Asymptotic General Optimum Stratification (AGOS) ϕ^{**} is given by one which attains the supremum of $\sum_{i<j}^{\ell} w_i w_j\{\sigma_i^2-\sigma_j^2+(\mu_i-\mu)^2-(\mu_j-\mu)^2\}^2$. Therefore we will take up the following as the objective function to be maximized;

$$\Psi_S(\phi)=\sum_{i<j}^{\ell} w_i w_j\{\sigma_i^2-\sigma_j^2+(\mu_i-\mu)^2-(\mu_j-\mu)^2\}^2. \tag{2.8}$$

Let us briefly summarize main results obtained up to now as to the optimum stratification method $\phi^*=(\phi_1^*,\phi_2^*,\ldots,\phi_\ell^*)$ in estimating μ. We denote OSP by x_i^* for $i=0,1,\ldots,M$ ($x_{i-1}^*<x_i^*$ for $i=1,2,\ldots,M$), where x_0^* and

x_M^* are the both end points of the domain of a distribution including $\pm\infty$. w_i^*, μ_i^* and σ_i^{*2} are the weight, mean and variance of the i-th stratum corresponding to ϕ^*.

(a) In the case of NA (Isii[23])

There exists some ϕ^* attaining $\inf \Psi_N(\phi)$ such as

$$\phi_i^*(x)=\begin{cases}1 & \text{if } g_i(x)<g_j(x) \text{ for all } j\neq i,\\ 0 & \text{if } g_i(x)>g_j(x) \text{ for some } j\neq i,\end{cases} \qquad (i=1,2,\ldots,\ell), \tag{2.9}$$

where

$$g_i(x)=\frac{1}{\sigma_i^*}(x-\mu_i^*)^2+\sigma_i^* \quad (\sigma_i^*>0), \quad (i=1,2,\ldots,\ell). \tag{2.10}$$

Therefore in general, ϕ^* is GS. If we limit a stratification method to IS, then x_i^* should satisfy the following relations (Dalenius[7]);

$$\frac{1}{\sigma_i^*}(x_i^*-\mu_i^*)^2+\sigma_i^*=\frac{1}{\sigma_{i+1}^*}(x_i^*-\mu_{i+1}^*)^2+\sigma_{i+1}^* \quad (i=1,2,\ldots,\ell-1). \tag{2.11}$$

(b) In the case of EA

If we limit a stratification method to IS, it is easily shown by differentiating (2.5) by x_i that x_i^* should satisfy

$$w_i^*\{(x_i^*-\mu_i^*)^2+\sigma_i^{*2}\}= w_{i+1}^*\{(x_i^*-\mu_{i+1}^*)^2+\sigma_{i+1}^{*2}\} \quad (i=1,2,\ldots,\ell-1). \tag{2.12}$$

(c) In the case of PA (Taga[33])

There exists some ϕ^* attaining $\inf \Psi_P(\phi)$ such as

$$\phi_i^*(x)=\begin{cases}1 & \text{if } x_{i-1}^*\leq \mu_i^*< x_i^*,\\ 0 & \text{otherwise},\end{cases} \qquad (i=1,2,\ldots,\ell).$$

That is, ϕ^* is nothing but IS, and x_i^* satisfies the following (Hayashi et al.[18], Dalenius[7]);

$$x_i^*=\frac{1}{2}(\mu_i^*+\mu_{i+1}^*) \qquad (i=1,2,\ldots,\ell-1). \tag{2.13}$$

In the case of estimating σ^2, there exists some AGOS $\phi^{**}=(\phi_1^{**},\ldots\ldots,\phi_\ell^{**})$ attaining sup $\Psi_S(\phi)$ such as

$$\phi_i^{**}(x)=\begin{cases}1 & \text{if } h_i(x)>h_j(x) \text{ for all } j\neq i,\\ 0 & \text{if } h_i(x)<h_j(x) \text{ for some } j\neq i,\end{cases} \qquad (i=1,2,\ldots\ldots,\ell),$$

where

$$h_i(x)=a_i\{(x-\mu)^2-a_i/2\}, \tag{2.14}$$

$$a_i=\int_{-\infty}^{\infty}(t-\mu)^2\phi_i^{**}(t)dF(t)/\int_{-\infty}^{\infty}\phi_i^{**}(t)dF(t).$$

[**Remark 2.1**] In the optimum stratification problem, we need not consider Randomized Stratification (RS) as stated above. So hereafter, RS is excluded from our consideration.

2.5. Some Efficiencies

Let us define some efficiencies in estimating μ :

$$e_N(\ell)=\{V_N^*(1)-V_N^*(\ell)\}/V_N^*(1) \quad \text{for NA,}$$

$$e_E(\ell)=\{V_E^*(1)-V_E^*(\ell)\}/V_E^*(1) \quad \text{for EA,}$$

$$e_P(\ell)=\{V_P^*(1)-V_P^*(\ell)\}/V_P^*(1) \quad \text{for PA,}$$

where $V_\cdot^*(\ell)$ is the variance of $\overline{X}$ under the optimum stratification ϕ^* with ℓ strata. $e_\cdot(\ell)$ therefore shows the relative efficiency of stratified random sampling with ℓ strata to simple random sampling under each sample allocation method. Moreover we will define the following efficiency which represents the degree of relative improvement of NA to EA or PA:

$$c_E(\ell)=\{V_E^*(\ell)-V_N^*(\ell)\}/V_N^*(\ell) \quad \text{for EA,}$$

$$c_P(\ell)=\{V_P^*(\ell)-V_N^*(\ell)\}/V_N^*(\ell) \quad \text{for PA.}$$

In the case of estimating σ^2, the following two kinds of efficiencies are defined:

$$e_I(\ell)=\{V_I^{**}(1)-V_I^{**}(\ell)\}/V_I^{**}(1),$$

$$e_G(\ell)=\{V_G^{**}(1)-V_G^{**}(\ell)\}/V_G^{**}(1),$$

$$c_S(\ell)=\{V_I^{**}(\ell)-V_G^{**}(\ell)\}/V_I^{**}(\ell),$$

where $V_I^{**}(\ell)$ and $V_G^{**}(\ell)$ are the variances of U_{st} under Asymptotic Interval Optimum Stratification (AIOS) and AGOS in the case of ℓ strata, respectively. $e.(\ell)$ therefore shows the relative efficiency of stratified random sampling with ℓ strata to simple random sampling under each stratification method, and $c_S(\ell)$ represents the degree of relative improvement of AGOS to AIOS.

2.6. Conditions of Our Study

Our study was performed under the following conditions:

(A) Types of distributions

1° Equilateral triangular distribution : $f(x)=1-|x|$, $-1 \leq x \leq 1$.

2° Normal distribution : $f(x)=e^{-x^2/2}/\sqrt{2\pi}$, $-\infty < x < \infty$.

3° Rightangled triangular distribution : $f(x)=1-x/2$, $0 \leq x \leq 2$.

4° Exponential distribution : $f(x)=e^{-x}$, $0 \leq x < \infty$.

The reason why we have selected these four types of distributions are as follows: First we adopt the distribution 1° or 2° as an example of a symmetric one, while 3° or 4° as an unsymmetric one. Furthermore the distribution 1° or 3° is considered as an example of a straight line type distribution, and 2° or 4° as a curved line type one. In some papers of the author([36], [41]), truncated versions of 2°, 4° and a (truncated) gamma distribution were also taken into consideration in addition to these four distributions.

(B) Population parameters to be estimated

1° The population mean μ

2° The population variance σ^2

In Chapter 6 - 9, we consider only μ in discussing the problem of robustness.

(C) Stratification methods

1° Interval Optimum Stratification (IOS)

2° General Optimum Stratification (GOS)

In estimating σ^2, Asymptotic IOS or GOS is considered.

(D) Sample allocation methods

1° Neyman Allocation (NA)

2° Equal Allocation (EA)

3° Proportional Allocation (PA)

In estimating σ^2, only PA is taken into consideration.

(E) Number of strata ℓ

$\ell = 2, 3, \ldots\ldots, 10.$

Under GOS, we consider only the case of $\ell = 2, 3, 4$.

CHAPTER 3

OSP, Min.Var. AND SOME EFFICIENCIES IN ESTIMATING THE POPULATION MEAN

3.1. Formulation as a Nonlinear Programming Problem

As stated in Section 2.4, the optimum stratification problem can be formulated as a problem to minimize the appropriate objective function under some constraints:

$$\text{Minimize } \Psi(\mathbf{x}) \quad \text{subject to } x_0 \leq x_1 \leq \cdots\cdots \leq x_M, \tag{3.1}$$

where $\mathbf{x}=(x_1,x_2,\ldots\ldots,x_{M-1})$ and $\Psi(\mathbf{x})$ is given by (2.4), (2.5) or (2.6) according to each sample allocation method.

This formulation and its direct solution have received little attention as reviewed in Chapter 1. However, we are convinced that this formulation as a mathematical programming problem is superior to alternative approaches (for example, [8] - [12]), since it is simpler and easier in describing the whole environments of the problem.

Then we applied a general iterative nonlinear programming algorithm and successfully solved the optimum stratification problem for some typical distributions as shown later. The nonlinear programming algorithms could treat quite easily the problem with many strata, while approximate procedures are, in general, very cumbersome. It should be noted that superiority of the use of nonlinear programming techniques is clear as the number of strata grows larger.

3.2. Algorithm for Obtaining General Optimum Stratification

Let us examine feasible decompositions in General Stratification. The number of intersecting points of quadratic functions (2.10) is at most ${}_{\ell}C_2 \times 2=\ell(\ell-1)$, but from (2.9) the number of points being significant to GOS is easily shown by mathematical induction to be at most $2(\ell-1)$. Therefore $\mathbf{R}^1$ must be decomposed into at most $2\ell-1$ intervals. Since GOS having just ℓ intervals is nothing but IOS, the number M of intervals corresponding to GOS except for IOS is given by

$$\ell+1 \leq M \leq 2\ell-1. \tag{3.2}$$

Hereupon let us denote the coefficient of the term x^2 in $g_i(x)$ by c_i $(i=1,2,\ldots,\ell)$ and M-1 stratification points by $x_1, x_2, \ldots\ldots, x_{M-1}$ $(x_1<x_2<\ldots<x_{M-1})$. Moreover, without loss of generality we may assume that $c_1 \geq c_2 \geq \ldots \geq c_{\ell}$. If $c_1>c_2$ is satisfied, then

$$\begin{cases} \phi_1(x)=1, \\ \phi_i(x)=0 \ (i\neq 1), \end{cases} \quad \text{for } -\infty<x\leq x_1 \text{ or } x_{M-1}<x<\infty \tag{3.3}$$

hold. Hence it is obvious from (3.2) and (3.3) that the interval (x_1, x_{M-1}) should be decomposed into sub-intervals the number of which is from $\ell-1$ to $2\ell-3$. The case of $c_1=c_2$ is excluded from our examination formally but is taken into consideration in our computational process.

Now let us assign number α to the interval (x_i, x_{i+1}) when $\phi_{\alpha}(x)=1$, $x_i<x\leq x_{i+1}$ $(i=1,2,\ldots,M-2)$ for $\alpha=1,2,\ldots\ldots,\ell$. Then our problem of examining feasible decompositions reduces to finding all possible sequences of stratum numbers $1,2,\ldots\ldots,\ell$ assigned to M intervals corresponding to feasible GOS's. It is easily shown from our assumptions that any sequence corresponding to feasible GOS must satisfy the following five conditions:

(i) Each number $1,2,\ldots,\ell$ should be assigned to at least one interval.

(ii) The same number should not be assigned to adjacent intervals.

(iii) The number ℓ should appear only once: This is derived from the assumption that c_ℓ is minimum among $c_1, c_2, \ldots, c_\ell$.

(iv) For any positive integer β, the number β must be adjacent each other in a new sequence obtained by eliminating numbers greater than or equal to $\beta+1$ ($\beta=1,2,\ldots,\ell-1$) from the original one: This is derived from the assumption of $c_1 \geqq c_2 \geqq \cdots \geqq c_\ell$. For example, consider the sequence 1213421 in the case of $\ell=4$, $M=7$. If we eliminate the numbers greater than or equal to 3, then the obtained sequence is 12121. In this one the number 2 is not adjacent, therefore the original sequence is not feasible. On the contrary the sequence 1213431 is feasible, for example.

(v) Remove the sequence that is equivalent to another by interchanging stratum numbers.

Hereupon let us examine the case that the decomposed region is finite. For example, consider the sequence 1213141 in the case of $\ell=4$, $M=7$ for the distribution 1°. In this case the decomposed region being finite, the feasible sequence may be 121314, 213141 or 21314. However from Table 3.13, the leftmost and rightmost intervals of OSP corresponding to the sequence 1213141 are both degenerate. Therefore these cases are taken into consideration in our computational process. For the next example, consider another sequence 123241. In this case the feasible sequence may be 12324, 23241 or 3241. As the interval corresponding to the leftmost number 1 is degenerate from Table 3.13, the sequence 23241 has become the object of our computation. As for the sequence 12324, by interchanging the stratum number, it can be equivalent to 21314 which is already considered in the first example. The sequence 3241 may be excepted from our examination because this case is nothing but IS. In this manner we can show that all the case of the decomposed domain being finite is taken into consideration in our computational process.

Now let us give some examples in which x_0 and x_M (M=3,4,...,7) denote both end points of the domain for each distribution.

[**Example 3.1**]

When ℓ=2, (3.2) implies M=3. Therefore the sequence corresponding to (x_0,x_1), (x_1,x_2), (x_2,x_M) is 121.

[**Example 3.2**]

When ℓ=3, (3.2) implies M=4 or 5. Although the sequences satisfying the constraints (i) - (iv) are 1231 and 1321 in case of M=4, from (v) it is only 1231. On the other hand, in case of M=5, though the sequences satisfying the constraints (i) - (iv) are 12131, 12321 and 13121, from (v) the sequence 13121 is rejected. Therefore it is sufficient to consider only two sequences; 12131 and 12321.

[**Example 3.3**]

When ℓ=4, (3.2) implies M=5, 6 or 7. Examining feasible decompositions by the same procedure as in [Example 3.2], the resulting sequences are as follows;

12341 for M=5,

121341, 123141, 123241, 123421, 123431 for M=6

1213141, 1213431, 1232141, 1232421, 1234321 for M=7.

If we examine all of these decompositions, the GOS should exist among them. In our computation, this will be performed; to save space only a part of them will be represented though.

3.3. Computational Scheme

In order to solve the problem (3.1), we used the nonlinear programming algorithm proposed by Sakakibara[31] and Hooke-Jeeves[22]. Initial values in optimization process were determined by the modified Monte Carlo method given by Taguri[35]. This strategy gave quite satisfactory solutions within reasonable computing time for each distribution. In order to improve hopefully the solutions in the case of IS, we optimized the same objective function $\Psi(\mathbf{x})$ under the equality constraints (2.11), (2.12) and (2.13) in addition to the inequality constraints as $x_0 \leqq x_1 \leqq \cdots \leqq x_M$ by using the same solution strategy. This did give the slight improvement of Min.Var. and the difference between the two sides in each of (2.11), (2.12) or (2.13) was reduced by the factor about 10^{-2}.

Our computation was mostly performed on HITAC-8700/8800 of Tokyo University Computer Center and on M-170 of Chiba University Computer Center.

Since the values w_i and σ_i^2 for the standard normal distribution N(0,1) can not be calculated analytically, the following scheme given in Erdelyi[13] was employed in the optimization process:

$$\int_0^x e^{-t^2/2}dt=\frac{1}{\sqrt{2}}\int_0^{x^2/2} y^{1/2-1}e^{-y}dy$$
$$\doteqdot\frac{1}{\sqrt{2}}\sum_{j=0}^{25}(-1)^j(x^2/2)^{1/2+j}/\{j!(1/2+j)\} \quad \text{for } x\geqq 0. \qquad (3.4)$$

The error of approximation in (3.4) is less than 2×10^{-11}, as is easily seen from the fact that $\int_0^x t^{\alpha-1}e^{-t}dt=\sum_{j=0}^{\infty}(-1)^j x^{\alpha+j}/\{j!(\alpha+j)\}$ is a converging alternating series. Therefore after some calculation, it is seen that the truncation error in our objective function induced by this approximation is less than 2×10^{-6} for all cases. The value μ_i for N(0,1) and the values w_i, μ_i and σ_i^2 for any other distribution can be derived analytically.

As for the round-off error, we use double precision arithmetics during the optimization process and the error of each term in our objective function is less than 10^{-18}. Consequently our computational schemes are expected to give sufficient precision.

[**Remark 3.1**] In the computation of w_i for N(0,1), we also employed the polynomial approximation originally proposed by Hastings[17] and improved by Toda et al.[42]-[44]. To evaluate σ_i^2 for N(0,1), we examined several approaches such as the use of numerical integration, series expansion of the incomplete gamma function or application of the spline function. As the numerical integration is expected to require heavy computational works, it is not tested in practice. In this paper, we present the results by using series expansion for the incomplete gamma function, since it gives quite satisfactory values for each computation. Application of the spline function is also recommendable as it gives enough accuracy for practical use and requires less computation time than the series expansion.

3.4. Results

3.4.1. Interval Optimum Stratification

OSP x_i^* (i=0,1,...,10), the value of the objective function $\Psi_{\cdot}^*$ and the efficiency e.(ℓ) for ℓ=1,2,...,10 are listed in Table 3.1 - Table 3.12 for the four distributions(see [Remark 3.2]). The first column of the table shows the number indicating individual distribution stated in Section 2.6, (A). $\Psi_{\cdot}^*$ means the value of $nV(\overline{X}|\phi^*)$ for ℓ=1,2,...,10.

These tables show that the relative efficiency e.(ℓ) does not much depend on the type of distributions. The relative efficiency for the straight line type distribution 3° is greater than for the curved line type one 4°, and the same conclusion is also true in the comparison of 1° and 2°. This tendency is remarkable in the case of PA. Besides, the relative efficiency for the unsymmetric distribution 3° is greater than for the symmetric one 1°, and the same tendency is observed between 2° and 4°. This is remarkable in the case of NA or EA. Generally speaking, the degree of the improvement of the relative efficiency owing to stratification is considerably large, and it is most remarkable in the case of ℓ=2.

[**Remark 3.2.**] We have also computed OSP, $\Psi_{\cdot}^*$ and e.(ℓ) for some other distributions; truncated normal, truncated exponential and (truncated) gamma distribution. However to save space, they are not shown in this paper(see Section 2.6, (A)).

3.4.2. General Optimum Stratification

The results of our computation are tabulated in Table 3.13 and Table 3.14. The sequences in the second column of these tables express those considered in Section 3.2. In each sequence in Table 3.13, some decomposed intervals are degenerate. Changing the stratum number, this is equal to the case of IOS, besides the value of Min.Var. under GOS is equal to the one under IOS. For example, consider the sequence 12321 on the case of $\ell=3$, $M=5$. As shown in Table 3.13, the leftmost two intervals are both degenerate. Therefore the original sequence 12321 is reduced to the sequence 321, which is nothing but the case of IS. Moreover the value 0.02556 of Ψ^*_G in Table 3.13 is equal to the one of Ψ^*_N in Table 3.1. Consequently for the distribution 1°, GOS should coincide with IOS.

For all the other distributions and numbers of strata treated in this paper, the same discussion can be true as partly shown in Table 3.14. Therefore it is scarcely needed in practice to consider GOS and we may deal with only IOS. The theoretical investigation of this issue is an open problem.

3.4.3. Comparison of NA and PA

Let us study differences between the sample allocation methods. From Section 3.4.2, since GOS can be regarded to approximately coincide with IOS in the case of NA, we will consider only IOS hereafter. The values $c_E(\ell)$ and $c_P(\ell)$ are shown in Table 3.15. From this table, it may be concluded that NA and EA give quite similar results for all the distributions considered here, since the values $c_E(\ell)$ are very small. So let us investigate the difference between NA and PA in the following.

Firstly, it is shown in Table 3.15 that as expected theoretically $V_P^* \geq V_N^*$ for all ℓ and for all distributions and the equality sign holds if and only if $\ell=2$ for the symmetric distribution.

Next from Table 3.15, we can see that the degree of relative improvement $c_P(\ell)$ for the curved line type distribution 2° or 4° is a little greater than the one for the straight line type one 1° or 3°. And $c_P(\ell)$ for the unsymmetric distribution 3° or 4° is considerably larger than the one for the symmetric distribution 1° or 2°. Therefore we may expect that the effect depending upon the difference of allocation methods is large when the form of a distribution is curved and unsymmetric.

The values of $c_P(\ell)$ show that PA reveals quite different behaviors, since the differences are significant and grow larger as the number of strata increases. EA may, therefore, be the best allocation method for practical uses.

3.5. Optimal Relation between Sample Sizes and Number of Strata

In this section, we consider the determination problem of the sample size n and the number of strata ℓ under some simple cost model. Assume that the sampling cost is expressed as

$$C=c_0+c_1 n+c_2 \ell^{\alpha}, \quad (c_0, c_1, c_2>0\ ; \alpha \geq 1), \tag{3.5}$$

where C is the total cost, c_0 is the fixed cost for sampling and does not depend on n nor ℓ. c_1 and c_2 are unit cost relative to the total sample size and the number of strata, respectively. As EA is the useful allocation from the practical view-point, we consider EA in the following discussions. Same discussions as below are possible in case of NA or PA.

In the case of EA, (2.5) gives

$$V_E(\overline{X})=\frac{1}{n}\Psi_E(\mathbf{x}). \tag{3.6}$$

Let us assume that the total sampling cost C must not exceed the available total cost C*. That is,

$$c_0+c_1n+c_2\ell^{\alpha}\leq C^*.$$

Then the problem can be described as the following optimization problem:

$$\text{Minimize}\quad V_E(\overline{X})\quad \text{subject to } c_0+c_1n+c_2\ell^{\alpha}\leq C^*,$$

where n and ℓ are both positive integers.

It is known that the objective function $V_E(\overline{X})$ is monotone decreasing with respect to n and ℓ. Let C** be total cost for appropriate integer n, ℓ. Then we can derive the following relation by combining (3.6) and the equation for C**.

$$\underset{n,\ell,\mathbf{x}}{\text{Minimum}}\ V_E(\overline{X}) = \underset{C^{**},\ell,\mathbf{x}}{\text{Minimum}}\ (c_1/c_2)/\{(C^{**}-c_0)/c_2-\ell^{\alpha}\}\cdot\Psi_E(\mathbf{x}).$$

Namely, in order to solve the minimization problem (3.7), it is sufficient to consider the minimization of

$$\psi_{\alpha}(\ell) = \frac{1}{k-\ell^{\alpha}}\Psi_E(\mathbf{x})$$

with respect to the integral value of ℓ and the appropriate value of $\mathbf{x}$, where $k=(C^{**}-c_0)/c_2$.

When k is specified, we can compute the minimum value of $\psi_{\alpha}(\ell)$ for the typical distributions in Section 2.6, (A), since we have computed the minimum values of $\Psi_E(\mathbf{x})$. For some values of k and for $\alpha=1$ and $\alpha=2$, minimum values of $\psi_{\alpha}(\ell)$ with respect to ℓ are summarized in Table 3.16 - Table 3.23, where $\min_{\ell}\psi_{\alpha}(\ell)$ are marked by the underlines. Judging from these tables, if $\alpha=1$, it may be said that the number of strata should be selected as large as possible in so far as the value of k is not so small.

In other words, to increase the number of strata is beneficial as far as the stratification cost is not so expensive. On the other hand, if $\alpha=2$, it may be sufficient to take the number of strata less than or equal to 5 or 6 so far as the stratification cost is not very cheap.

CHAPTER 4

OSP, Min.Var. AND SOME EFFICIENCIES IN ESTIMATING THE POPULATION VARIANCE

4.1. Computation

The optimum stratification problem in estimating the population variance σ^2 can be also described as the following mathematical programming problem;

$$\text{Maximize } \Psi_S(\mathbf{x}) \quad \text{subject to } x_0 \leqq x_1 \leqq \cdots \cdots \leqq x_M, \tag{4.1}$$

where $\Psi_S(\mathbf{x})$ is given by (2.8) and $\mathbf{x}=(x_1,x_2,\ldots,x_{M-1})$.

Let us consider feasible AGOS by the same procedure as in Section 3.2. In this case, since the axes of ℓ quadratic functions $h_i(x)=a_i\{(x-\mu)^2-a_i/2\}$ in (2.14) are common(that is, $x=\mu$), feasible decompositions for $\ell=2,3,4$ are; 121 for $\ell=2(M=3)$, 12321 for $\ell=3(M=5)$ and 1234321 for $\ell=4(M=7)$. Therefore examining these decompositions, AGOS should exist among them. In Section 4.2, calculation for such cases is carried out.

The nonlinear programming algorithm used for solving (4.1) is Hooke-Jeeves'[22] and initial values are determined by the modified Monte Carlo method(Taguri[35]). The values w_i and σ_i^2 for the standard normal distribution were computed by using the approximation formula (3.4) with the same truncation and round-off errors as in Section 3.3.

4.2. Results

4.2.1. Asymptotic Interval Optimum Stratification and Asymptotic General Optimum Stratification

OSP $x_i^{**}(i=0,1,\ldots,5)$, the value of the variance of U_{st}, $V_I^{**}=n.\inf_{\phi} V(U_{st}|\phi^{**})$, and the efficiency $e_I(\ell)$ for $\ell=1,2,\ldots,5$ in the case of AIOS are shown in Table 4.1, where V_I^{**} for $\ell=1$ means an approximate value of $nV(U)$ in (2.7).

This table shows that the relative efficiency in the unsymmetric distribution 3° or 4° is much greater than the one in the symmetric distribution 1° or 2°. In case of the unsymmetric distributions, the effect of stratification in the curved line type distribution 4° is greater than the one in the straight line type distribution 3°. On the other hand, in case of the symmetric distributions the relative efficiency is fairly bad when $\ell=2$, because the stratification method is restricted to AIOS and AGOS is not taken into consideration.

In the case of AGOS, the results are summarized in Table 4.2, which shows that AGOS does not always coincide with AIOS in estimating σ^2.

4.2.2. Comparison of AIOS and AGOS

Let us compare the value V_I^{**} with V_G^{**} when the number of strata ℓ is fixed. The last column of Table 4.1 shows the value of $c_I(\ell)$.

Firstly for the symmetric distribution $1°$ or $2°$, the relative efficiency in AGOS is far greater than the one in the case of AIOS, and therefore it is much effective to consider the case of AGOS. Hereupon the remarkable point is that for these distributions, $V^{**}_{I3}=V^{**}_{G2}$ and $V^{**}_{I5}=V^{**}_{G3}$, where $V^{**}_{\cdot\ell}$ represents the value of V^{**} in the case of ℓ strata. This means that each decomposed interval under AGOS has the effect equivalent to each stratum under AIOS. The theoretical study of this point is an open problem. Furthermore it is corrobolated from Table 4.2 that the stratification given by symmetric pairs of intervals around μ is optimum, that is theoretically proved by Wakimoto[45].

Secondly, for the distribution $3°$, AGOS coincides with AIOS in case of $\ell=2$ but not in case of $\ell=3$ or 4. For example in case of $\ell=3$, V^{**}_{G3} is slightly greater than V^{**}_{I4} and much smaller than V^{**}_{I3}. This may be caused by the fact that AGOS has much information in estimating σ^2 compared with AIOS. Now the sequence 12321 in this case degenerates to 2321. Moreover $\frac{1}{2}(x^{**}_2+x^{**}_3)=0.667$ holds and this is nearly equal to the mean value 0.66667. This fact is consistent with above-mentioned theoretical result given in Wakimoto[45]. The same discussion holds also in case of $\ell=4$.

Thirdly, for the distribution $4°$, AGOS exactly coincides with AIOS. Therefore we may only consider the case of AIOS for this distribution.

CHAPTER 5

APPLICATIONAL SCHEME OF OUR RESULTS AND SOME REPRESENTATIVE EXAMPLE

5.1. Working Procedure

In the following, we will only consider the estimation of the population mean. In actual sample surveys, the procedure of application of the tables giving OSP and Min.Var. should be carried out by the following steps:

1° Guess the type of a distribution for a given practical problem, and decide the type of distribution among those shown in Section 2.6, (A) which should be fitted to the histogram made from the given data.

2° Determine the population parameters of the fitted distribution under some criterion of goodness of fit by using the information of past surveys or a pilot survey.

3° Compute optimum values of a number of strata and a total sample size with or without using a cost function such as (3.5).

4° From the tables giving OSP, compute approximate values of OSP for the given distribution, and construct strata.

5° Compute values of w_i and/or σ_i, and then determine sample sizes in respective strata.

6° Of cource, it is necessary to proceed random sampling within each stratum, and to estimate the population parameters and their estimated standard error by ordinary methods.

If strata are determined in advance, the steps 2°–4° are needless. If we should compute OSP by using some auxiliary information, it is required to get an approximate regression function from some information before the step 4°(see Taga[33]).

Since the four distributions given in Section 2.6, (A) are all represented in standardized form, we have to make some appropriate variable transformation in practical situations(see Section 5.2.2).

5.2. Representative Example of Application

In this section, we will apply our procedure stated in the previous section to the data of "Current Survey on Petroleum Product Demand and Supply", which was performed by the Ministry of International Trade and Industry (MITI) in Japan. Our working procedure is applicable to any data of a similar kind as this example(for example, to the data of "The Current Statistics of Commerce").

Now in order to estimate the sale of some kinds of petroleum, MITI had a plan to do a stratified random sampling in 1981. Let us consider the estimation of the sales of LPG and benzine.

5.2.1. Sampling Procedure

In the case of estimating the sale of LPG, we omit establishments whose sale are 0, and the stratification variable is the sale of LPG. For the establishments with the LPG sale being 0, the stratification is done by the sale of benzine, where we omit establishments whose sale of benzine are 0. In the traditional procedure which has been used up to this time, the strata are constructed as in Table 5.1 for LPG and benzine, respectively. The sample allocation method is the Neyman allocation by utilizing the values of the within-strata weight w_i and variance σ_i^2 in the sample survey performed in March 1980, which was a pilot survey for 1981.

Now the strata shown in Table 5.1 have been used without any theoretical consideration. Let us stratify the population over again by our procedure proposed in Section 5.1 and compare this with the traditional one, where the number of strata ℓ and the total sample size n are the same as before.

5.2.2. Fitting of a Distribution

MITI performed pilot surveys on some kinds of petroleum in March 1978 and March 1980. Table 5.2 and Table 5.3 show the data obtained, which are classified into three types; TYPE I, II and III. The data of TYPE III are the sale of LPG for the establishments dealing only LPG, and the data of TYPE II are for the establishments dealing some kinds of oils including LPG. The data of TYPE I are the total of TYPE II and TYPE III. Some histograms of these data are shown in Figure 5.1 and Figure 5.2. The K-th(right-most) class is constructed by $[z_{K-1}, z_K)$, where $z_K=z_{K-1}+2(m_K-z_{K-1})$. m_K is the mean of the K-th class and was given from the results of sample survey. The value of m_K is shown in Table 5.2 and Table 5.3. From these figures and/or tables, it can be seen that the distribution is skew, with its mode at the lower part, and is monotone decreasing, roughly speaking. We, therefore, may fit the exponential or gamma (or their truncated) distribution to the data(see [Remark 3.2] and [Remark 5.1]). In this section, let us adopt the untruncated exponential distribution among these, because the degree of goodness of fit is slightly better than in other cases. The variable transformation $x=\beta z$ $(\beta>0)$ is done, since OSP were given for the standardized probability density function (p.d.f.) $f(x)$ in Section 3.4.1(see Section 5.1). Then the p.d.f. of z is $f_z(z)=\beta f(\beta z)$ $(0<z<\infty)$. We will use the following T as the criterion of goodness of fit between two distributions(see [Remark 5.2]);

$$T=\sum_{j=1}^{K+1}\int_{z_{j-1}}^{z_j}[\{f_z(z)-h(z)\}/f_z(z)]^2 f_z(z)dz,$$

where $h(z)$ represents the p.d.f. of the histogram and K is the number of class. z_{j-1} or z_j $(z_{j-1}<z_j)$ is the lower or upper end point of the j-th class respectively, and $h(z)=q_j$ on $[z_{j-1}, z_j)$ for $j=1,2,\ldots,K$ $(z_0=0)$. In order to let the domains of $h(z)$ and $f_z(z)$ coincide, let us consider the K+1-th class $[z_K, z_{K+1})$, on which $h(z)=q_{K+1}=0$. Then T is given by

$$T=\sum_{j=1}^{K+1}\int_{z_{j-1}}^{z_j}\{f_z(z)-h(z)\}^2/f_z(z)dz$$

$$=\sum_{j=1}^{K}q_j^2(e^{\beta z_j}-e^{\beta z_{j-1}})/\beta^2-1. \tag{5.1}$$

The most-fitted distribution $f_z^*(z)$ to $h(z)$ can be determined under this criterion if the value of β minimizing (5.1) is obtained. T is unimodal on β since $\lim_{\beta\to+0}\partial T/\partial\beta=-\infty$, $\lim_{\beta\to\infty}\partial T/\partial\beta=\infty$ and $\partial^2T/\partial\beta^2>0$. The optimum value β^* is, therefore, obtainable by using the linear search(for example, Golden section method). The values of β^* in the eight cases are given in the bottom row of Table 5.2 and Table 5.3. We may, then, fit the p.d.f. $f_z^*(z)=\beta^* e^{-\beta^* z}$ to $h(z)$, which is shown in Figure 5.1 and Figure 5.2.

[**Remark 5.1**] From the standpoint of fitting a distribution, we should fit some other one; for example, a beta distribution in the Pearson system. Our objective is, however, to stratify the population by using OSP computed in advance. In the preceding section, we gave OSP for some distributions, only which can be now utilized for us. We, therefore, should fit the exponential or gamma distribution among them.

[**Remark 5.2**] As the criterion of goodness of fit, we may adopt the traditional χ^2. However, by our preliminary computation, obtained result is more preferable in the case of using T than in the case of using χ^2; that is, the variance of the estimator is smaller. We, therefore, decide to use the criterion T instead of χ^2.

5.2.3. Construction of Strata and Allocation of Sample

Let x_i^* and z_i^* be OSP for $f(x)$ and $f_z^*(z)$ respectively, then $z_i^*=x_i^*/\beta^*$. The i-th stratum Π_i may be constructed by $\Pi_i=[z_{i-1}^*, z_i^*)$ $(i=1,2,\ldots,\ell;z_0^*=0, z_\ell^*=\infty)$. Computational results are shown in Table 5.4, which shows that the stratification points $\{z_i^*\}$ considerably differ from the traditional ones(cf. Table 5.1).

Next we will determine the sample size n_i^* in the i-th stratum by Neyman allocation. Let n be the total sample size, then n_i^* is given by the following;

$$n_i^* = nw_i^*\sigma_i^*/\sum_{i=1}^{\ell} w_i^*\sigma_i^*, \tag{5.2}$$

where w_i^* and σ_i^* are the weight and standard deviation of $f_z^*(z)$ in the i-th stratum. The values of n_i^* are also summarized in Table 5.4.

5.2.4. Comparison of the Proposed and Traditional Method

Let us compare the proposed method with the traditional one. The comparison of the two methods is impossible in the strict sense, since the true distribution g(z) in the practical field is unknown for us. We, therefore, assume that the histogram h(z) is satisfactorily near to the true distribution g(z), and investigate this problem. Let w_i and σ_i be the weight and standard deviation of h(z) in the i-th stratum Π_i respectively. The standard error of the estimator $\overline{X}$ given by (2.2) is $S(\overline{X})=\sqrt{\Sigma_{i=1}^{\ell} w_i^2 \sigma_i^2 / n_i}$. In the case of our proposed method, the standard error of $\overline{X}$ is given by (2.3) as

$$S_1=\sqrt{\frac{1}{n}\sum_{i=1}^{\ell} w_i^* \sigma_i^* \sum_{i=1}^{\ell} (w_i^2 \sigma_i^2 / w_i^* \sigma_i^*)}.$$

In the case of the traditional method, the standard error of $\overline{X}$ is

$$S_2=\sum_{i=1}^{\ell} w_i \sigma_i / \sqrt{n},$$

since $n_i=nw_i\sigma_i/\Sigma_{i=1}^{\ell} w_i\sigma_i$. Computational results of the values S_1 and S_2 are given in Table 5.4, which shows that our method decreases the standard error of $\overline{X}$ by about 30 – 60 % compared with the traditional method, in this example.

5.2.5. The Influence on $S(\overline{X})$ by Small Change of z_i^* and n_i^*

OSP $\{z_i^*\}$ obtained in Section 5.2.3 were computed from the theoretical point of view. On the contrary, the list of establishments is classified in the classes given in Table 5.2 or Table 5.3. Therefore we have to utilize the end point of some class near to z_i^* in place of the exact z_i^*.

Moreover n_i^* given by (5.2) is not generally integer or we usually round off n_i^* appropriately for the sake of the computational convenience.

Thus, how is the standard error of $\overline{X}$ influenced by small change of z_i^* and n_i^*? Let z_i^{**} and n_i^{**} be the stratification point and sample size in the i-th stratum after small change, and S_3 be the value of $S(\overline{X})$ in this case. Computational results are given in Table 5.5. The influence on $S(\overline{X})$ is about less than 10% and is considered not to be serious.

CHAPTER 6

PROBLEM OF ROBUSTNESS IN OPTIMUM STRATIFICATION

We found in the previous section that the proposed method might be useful in practical sample surveys. However there have been some important unsolved problems for further study. For example, as shown in the preceding example, OSP are usually impracticable from the constraint of sampling frame and the optimum sample sizes in respective strata may be changed for the sake of convenience in the analysis. As for the weights $\{w_i\}$, they are generally unknown for us. Moreover the distribution in a given practical problem is different from the distribution fitted to it. Thus, strictly speaking, the optimum stratification is almost always impracticable. We should therefore make a study of so-called "problem of robustness in optimum stratification". Through this study, theoretical results on optimum stratification are effectively useful in many practical jobs.

Now situations in which the optimum stratification can not be executed in practice are classified as follows:

(a) The type of a fitted distribution is different from that of an actual distribution.

(b) The parameters of an actual distribution must be estimated.

(c) For convenience' sake of the ensuing analysis, sample sizes in respective strata may be changed from their optimum values.

(d) From the constraint of sampling frame or for the sake of computational convenience, stratification points may be changed from the computed OSP.

6.1. Bias of the Estimator $\overline{X}$

The estimator of the population mean μ is usually given by (2.2), in which w_i has been assumed to be known in the preceding section. However in practical stratified random sampling, w_i is usually not available in advance. For instance, as the value of w_i is out of date in the preceding example, it differs from the true value w_i^0. It is generally impossible to obtain a complete list for sampling, so that there often exist cases in which w_i is not precisely equal to w_i^0.

Let us denote the difference between w_i and w_i^0 by v_i, that is,

$$w_i = w_i^0 + \nu_i, \qquad (i = 1,2,\ldots,\ell).$$

In this case, $\overline{X}$ is not an unbiased estimator of μ in general. Therefore we should adopt the Mean Square Error MSE($\overline{X}$) of $\overline{X}$ as our objective function to be minimized and should not adopt the variance V($\overline{X}$) as in Section 2.4. But if the bias term is small, the optimum stratification may be approximately attained by the stratification method minimizing V($\overline{X}$). In the following we will examine a condition under which the bias term is small compared to V($\overline{X}$).

Practically in almost all cases, we might have some information as to an upper bound of $|v_i/w_i^0|$. Therefore suppose that a value of λ satisfying the following is known;

$$|\nu_i/w_i^0| \leqq \lambda \ (\lambda > 0), \quad (i = 1,2,\ldots,\ell).$$

MSE($\overline{X}$) is given by

$$\mathrm{MSE}(\overline{X}) = V(\overline{X}) + (\sum_{i=1}^{\ell} \nu_i \mu_i)^2,$$

where μ_i is the mean of the i-th stratum.

Now let us examine the condition that

$$\left| MSE(\overline{X}) - V(\overline{X}) \right| / V(\overline{X}) = \left(\sum_{i=1}^{\ell} \nu_i \mu_i \right)^2 / V(\overline{X}) < \delta^2 \tag{6.1}$$

for a preassigned value δ $(0<\delta<1)$. For this purpose we will find an upper bound of $\left| \sum_{i=1}^{\ell} \nu_i \mu_i \right|$. This problem reduces to that of linear programming with linear constraints;

$$\text{Maximize or minimize} \qquad \phi(\nu_1, \nu_2, \ldots, \nu_\ell) = \sum_{i=1}^{\ell} \mu_i \nu_i, \tag{6.2}$$

$$\text{subject to} \quad \begin{cases} -\lambda w_i^0 \leqq \nu_i \leqq \lambda w_i^0, \ (i = 1, 2, \ldots, \ell), \\ \sum_{i=1}^{\ell} \nu_i = 0. \end{cases}$$

In order to solve this problem, we prepare the following lemma.

[**Lemma 6.1**]

The solution of the problem;

$$\text{Maximize} \qquad \Phi(\xi_1, \xi_2, \ldots, \xi_\ell) = \sum_{i=1}^{\ell} a_i \xi_i,$$

$$\text{subject to} \quad \begin{cases} k_i \leqq \xi_i \leqq K_i, \ (i = 1, 2, \ldots, \ell), \\ \sum_{i=1}^{\ell} \xi_i = \xi_0 \end{cases} \tag{6.3}$$

is given by the following:

(Solution) Rearrange $a_1, a_2, \ldots, a_\ell$ in a descending order of magnitude and put $a_{(1)} \geqq a_{(2)} \geqq \cdots \geqq a_{(\ell)}$. Let $\xi_{(i)}$, $k_{(i)}$, $K_{(i)}$ be the value of ξ_i, k_i, K_i corresponding to $a_{(i)}$ respectively, then the optimal solution $\xi^*_{(i)}$ is given by

$$\begin{cases} \xi^*_{(i)} = K_{(i)}, & (i = 1, 2, \ldots, \ell_1 - 1), \\ \xi^*_{(\ell_1)} = \xi_0 - \sum_{i=1}^{\ell_1 - 1} K_{(i)} - \sum_{i=\ell_1+1}^{\ell} k_{(i)}, & \\ \xi^*_{(i)} = k_{(i)}, & (i = \ell_1 + 1, \ldots, \ell), \end{cases}$$

where ℓ_1 is the integer satisfying the followings simultaneously;

$$\begin{cases} \sum_{i=1}^{\ell_1 - 1} K_{(i)} + \sum_{i=\ell_1}^{\ell} k_{(i)} \leqq \xi_0, \\ \sum_{i=1}^{\ell_1} K_{(i)} + \sum_{i=\ell_1+1}^{\ell} k_{(i)} > \xi_0. \end{cases} \tag{6.4}$$

The probem; minimize $\Phi(\xi_1,\xi_2,\ldots,\xi_\ell)=\sum_{i=1}^{\ell} a_i\xi_i$, subject to (6.3), can be transformed to the maximizing problem in the above lemma by replacing the objective function Φ with $-\Phi$.

Now let us solve the problem (6.2) by using this lemma. The continuous function $\phi(v_1,v_2,\ldots,v_\ell)$, defined on the hyperplane in the ℓ-dimensional closed interval, must have a maximum value ϕ_U and a minimum value ϕ_L. Let us calculate ϕ_U:

In [Lemma 6.1], put

$$\begin{cases} \Phi=\phi, & k_i=-\lambda w_i^0, \\ a_i=\mu_i, & K_i=\lambda w_i^0, \qquad (i=1,2,\ldots,\ell), \\ \xi_i=\nu_i, & \xi_0=0. \end{cases}$$

Without loss of generality, we may assume that the relation $\mu_1<\mu_2<\ldots\ldots<\mu_\ell$ is satisfied in this problem. Now let us obtain the integer ℓ_1 satisfying (6.4), then

$$\begin{cases} \sum_{i=\ell-\ell_1+2}^{\ell} w_i^0-\sum_{i=1}^{\ell-\ell_1+1} w_i^0<0, \\ \sum_{i=\ell-\ell_1+1}^{\ell} w_i^0-\sum_{i=1}^{\ell-\ell_1} w_i^0\geqq 0 \end{cases}$$

must hold simultaneously. Therefore ℓ_1 is the integer satisfying

$$\sum_{i=1}^{\ell-\ell_1} w_i^0\leqq 1/2<\sum_{i=1}^{\ell-\ell_1+1} w_i^0, \tag{6.5}$$

and is uniquely determined.

[**Rmark 6.1**] In the case of symmetric distributions, $\ell_1=[(\ell+1)/2]$ holds, where $[x]$ expresses the maximum integer smaller than or equal to x.

Then the optimal solution is given by

$$\begin{cases} \nu_i^*=-\lambda w_i^0, & (i=1,2,\ldots,\ell-\ell_1), \\ \nu_{\ell-\ell_1+1}^*=\lambda\left(\sum_{i=1}^{\ell-\ell_1} w_i^0-\sum_{i=\ell-\ell_1+2}^{\ell} w_i^0\right), & \\ \nu_i^*=\lambda w_i^0, & (i=\ell-\ell_1+2,\ldots,\ell), \end{cases}$$

and ϕ_U is

$$\phi_U = \sum_{i=1}^{\ell} \mu_i \nu_i^* = \sum_{i=1}^{\ell-\ell_1} (-\lambda w_i^0)\mu_i + \lambda(\sum_{i=1}^{\ell-\ell_1} w_i^0 - \sum_{i=\ell-\ell_1+2}^{\ell} w_i^0)\mu_{\ell-\ell_1+1} + \sum_{i=\ell-\ell_1+2}^{\ell} \lambda w_i^0 \mu_i$$

$$= \lambda\left\{\sum_{i=\ell-\ell_1+2}^{\ell} w_i^0(\mu_i - \mu_{\ell-\ell_1+1}) - \sum_{i=1}^{\ell-\ell_1} w_i^0(\mu_i - \mu_{\ell-\ell_1+1})\right\} \equiv \lambda M_U,$$

where ℓ_1 is the integer satisfying (6.5).

Next by the same procedure as mentioned above, ϕ_L is given by

$$\phi_L = \lambda\left\{\sum_{i=1}^{\ell_1-1} w_i^0(\mu_i - \mu_{\ell_1}) - \sum_{i=\ell_1+1}^{\ell} w_i^0(\mu_i - \mu_{\ell_1})\right\} \equiv \lambda M_L,$$

where ℓ_1 is the integer satisfying $\sum_{i=1}^{\ell_1-1} w_i^0 < 1/2 \leq \sum_{i=1}^{\ell_1} w_i^0$. Therefore in both cases the following is satisfied;

$$\left|\sum_{i=1}^{\ell} \nu_i \mu_i\right| \leqq \lambda M,$$

where $M=\max(M_U, -M_L)$. After all these calculations, a sufficient condition satisfying (6.1) is given by

$$\lambda < \delta\sqrt{V(\overline{X})}\,/M,$$

where $V(\overline{X})$ is the variance of $\overline{X}$ in the case of ℓ strata.

Let λ_u be an upper bound of λ, where $\lambda_u=\delta\sqrt{V(\overline{X})}/M$. We compute the values of λ_u for the three sample allocation methods and for the four distributions; the equilateral triangular, the normal, the rightangled triangular and the exponential distribution. Table 6.1 gives the values of λ_u in case of the rightangled triangular distribution under NA, for which λ_u is smallest among all cases. On the other hand Table 6.2 gives the values of λ_u in case of the exponential distribution under PA, when λ_u is largest among all cases.

6.2. Formulation of the Problem of Robustness

In general, under a fixed total sample size and a p.d.f. $h(x)$, our objective function Ψ to be minimized is a function of a stratification method ϕ_h and an allocation method A_h. Let us express the objective function by $\Psi(\phi_h, A_h; h)$. The true p.d.f. in a particular problem is denoted by $g(x)$ and the p.d.f. fitted to it by $f(x)$. Ideally, we should execute the optimum stratification method ϕ_g^* and the optimum allocation method A_g^* for $g(x)$. However, it is impossible in practice because $g(x)$ is unknown. Then we perform the stratification by using the optimum stratification method ϕ_f^* and the optimum allocation method A_f^* for some $f(x)$ which is available and approximates $g(x)$. But in many practical works, even this stratification is often not practical enough as stated above. In such situations, we would have to be content with a stratification method ϕ_f and an allocation method A_f which approximate ϕ_f^* and A_f^*, respectively. Therefore, in general, we have to evaluate the quantity R_0 given by

$$R_0 = \Psi(\phi_f, A_f; g) - \Psi(\phi_g^*, A_g^*; g)$$

(see Figure 6.1). If we put

$$\begin{cases} R_1 = \Psi(\phi_f^*, A_f^*; g) - \Psi(\phi_g^*, A_g^*; g), \\ R_2 = \Psi(\phi_f^*, A_f; g) - \Psi(\phi_f^*, A_f^*; g), \\ R_3 = \Psi(\phi_f, A_f; g) - \Psi(\phi_f^*, A_f; g), \end{cases} \tag{6.6}$$

then

$$R_0 = R_1 + R_2 + R_3.$$

From the practical point of view, it is convenient to give the efficiency of R_j $(j=0,1,2,3)$ against the optimum value $\Psi(\phi_g^*, A_g^*; g)$. However the latter can be represented by using the values $\Psi(\phi_f^*, A_f^*; f)$ and q defined by (7.1) and (7.3) in Chapter 7 as follows;

$$\Psi(\phi_g^*, A_g^*; g) = (1-q)\,\Psi(\phi_f^*, A_f^*; f). \tag{6.7}$$

Therefore we may calculate the efficiency Q_j which is defined by

$$Q_j = R_j / \Psi(\phi_f^*, A_f^*; f) \qquad (j=0,1,2,3) \tag{6.8}$$

instead of $Q_j^* = R_j / \Psi(\phi_g^*, A_g^*; g)$. The reason of this is as follows: (1) We are interested in the value of q itself. (2) The value of $\Psi(\phi_f^*, A_f^*; f)$ is independent on a small change of distributions and the value of q is not so large on the whole. We, therefore, consider that it is convenient to give the value of Q_j instead of Q_j^* in order to roughly estimate the degree of robustness.

Now Q_1 means the degree of the effect caused by changing a p.d.f. from g(x) to f(x), that is to say, Q_1 represents the degree of robustness on distributions. In order to evaluate the influence deriving from the cause (a) or (b), described in the first part of Chapter 6, we may calculate the value Q_1. Next Q_2 means the effect caused by changing an allocation method from A_f^* to A_f, that is, it represents the degree of robustness on sample sizes in respective strata. Calculation of Q_2 is needed for evaluating the influence deriving from the cause (c). Similarly, Q_3 represents the degree of robustness on stratification points, and calculation of Q_3 is needed for evaluating the influence deriving from the cause (d).

Thus the problem of evaluating Q_0 is decomposed into three sub-problems on robustness, which will be examined in the following chapters:

$$Q_0 = Q_1 + Q_2 + Q_3.$$

6.3. Expression of g(x) by Using f(x)

In the preceding section, we considered some typical distributions f(x) defined on some interval (x_0, x_M). In many applied fields, however, we often need to encounter p.d.f.'s which are slightly different from f(x). It is then natural to express g(x) in a series of orthonormal polynomials;

$$g(x)=f(x)\{1+a_1P_1(x)+a_2P_2(x)+.....\}, \tag{6.9}$$

where the a_i's are constant coefficients, whereas $P_i(x)$ is the orthonormal polynomial of degree i with respect to the weight function f(x) over the interval (x_0, x_M) and has the following form:

$$P_i(x)=b_{i0}+b_{i1}x+.....+b_{ii}x^i.$$

Let us take the terms up to $P_2(x)$ in (6.9) into consideration for the sake of computational convenience. Assume that the mean and variance of g(x) are different from the mean μ_f and the variance σ_f^2 of f(x) by $100\alpha\%$ and $100\beta\%$ respectively, then it is easily shown that a_1 and a_2 are represented as follows:

$$\begin{cases} a_1=\int_{x_0}^{x_M}P_1(x)g(x)dx=b_{10}+b_{11}(1+\alpha)\mu_f, \\ a_2=\int_{x_0}^{x_M}P_2(x)g(x)dx=b_{20}+(1+\alpha)\mu_f\{b_{21}+b_{22}(1+\alpha)\mu_f\}+b_{22}(1+\beta)\sigma_f^2. \end{cases} \tag{6.10}$$

Note that the p.d.f. f(x) and the values of α, β are given, and then the p.d.f. g(x) is uniquely determined. This p.d.f. g(x) is slightly different from the p.d.f. f(x). In the following discussion, only such g(x)'s are taken into consideration. Now for the convenience' sake of the analysis in the later chapters, let us rewrite g(x) as follows;

$$g(x)=f(x)\cdot(C_0+C_1x+C_2x^2), \tag{6.11}$$

where

$$C_0=1+a_1b_{10}+a_2b_{20}, \quad C_1=a_1b_{11}+a_2b_{21}, \quad C_2=a_2b_{22}. \tag{6.12}$$

6.4. Some Lemma

We will give some lemma used effectively in the following chapters.

[**Lemma 6.2**] (Mean value theorem for the function of many variables)

Let $f(x_1,\ldots,x_p)$ be a real-valued function of p variables. If $f(x_1,\ldots,x_p)$ has partial derivatives $f_{x_i}(x_1,\ldots,x_p)=\partial f(x_1,\ldots,x_p)/\partial x_i$ for $i=1,2,\ldots,p$ in a neighbourhood $U(x_1^0,\ldots,x_p^0)$ of $(x_1^0,\ldots,x_p^0)$, then for any $(x_1,\ldots,x_p)\varepsilon U(x_1^0,\ldots,x_p^0)$ there exists some real value θ such that

$$f(x_1,\ldots,x_p)=f(x_1^0,\ldots,x_p^0)+\sum_{i=1}^{p} h_i f_{x_i}(x_1^0,\ldots,x_{i-1}^0,x_i^0+h_i\theta,x_{i+1},\ldots,x_p);0<\theta<1,$$

where

$$x_i=x_i^0+h_i \qquad (i=1,2,\ldots,p).$$

For a proof of this lemma, see e.g. Coffman[6] or Hitotsumatsu[21].

CHAPTER 7

ROBUSTNESS ON DISTRIBUTIONS

In this section, we will evaluate the quantity R_1 defined by (6.6). Let us decompose R_1 as follows:

$$R_1 = R_{11} + R_{12} \tag{7.1}$$

$$\begin{cases} R_{11} = \Psi(\phi_f^*, A_f^*; f) - \Psi(\phi_g^*, A_g^*; g), \\ R_{12} = \Psi(\phi_f^*, A_f^*; g) - \Psi(\phi_f^*, A_f^*; f). \end{cases}$$

These are evaluated in the followings.

7.1. Computational Scheme for R_{11}

Let us describe a method of evaluating numerically the quantity R_{11} defined by (7.1), since the analytical evaluation of R_{11} is difficult and is an open problem. Now the values of $\Psi(\phi_f^*, A_f^*; f)$ in (7.1) have already been computed in Section 3.4.1. Therefore we should compute the values of $\Psi(\phi_g^*, A_g^*; g)$ under the given values of α, β in (6.10). This problem can be formulated as a nonlinear programming problem as shown in Section 3.1, and could be successfully solved for the four distributions by the nonlinear programming algorithm utilizing augmented Lagrangian function(see Konno et al.[27]).

In order to compute the ν-th moments r_{ν_i} in respective strata for $\nu=0,1,2,3,4$, the same approximation formula as (3.4) is employed in the optimization processes for the standard normal distribution. For the other distributions, analytical expressions of r_{ν_i} are easily derived. These computational schemes are expected to give sufficient precision if we use double precision arithmetics during the optimization process (see Section 3.3).

7.2. Analytical Examination for R_{12}

From the definition (7.1) of R_{12}, we consider the case where the stratification method is ϕ^*_f and the allocation method of sample sizes is A^*_f. Let ξ_i be the i-th optimum stratification point for f(x), then R_{12} is represented as follows;

$$R_{12} = \sum_{i=1}^{\ell} \frac{1}{n_i}\left[\int_{\xi_{i-1}}^{\xi_i} g(t)\,dt \int_{\xi_{i-1}}^{\xi_i} t^2 g(t)\,dt - \left\{\int_{\xi_{i-1}}^{\xi_i} tg(t)\,dt\right\}^2\right] - \sum_{i=1}^{\ell} \frac{1}{n_i}\left[\int_{\xi_{i-1}}^{\xi_i} f(t)dt \int_{\xi_{i-1}}^{\xi_i} t^2 f(t)dt - \left\{\int_{\xi_{i-1}}^{\xi_i} tf(t)\,dt\right\}^2\right].$$

From (2.1) and (6.11),

$$\int_{\xi_{i-1}}^{\xi_i} t^\nu g(t)\,dt = \int_{\xi_{i-1}}^{\xi_i} t^\nu (C_0 + C_1 t + C_2 t^2) f(t)\,dt = C_0 \gamma_{\nu i} + C_1 \gamma_{\nu+1 i} + C_2 \gamma_{\nu+2 i}$$

holds for ν=0,1,2. Then we get

$$R_{12} = \sum_{i=1}^{\ell} \frac{1}{n_i}\Big\{ (C_0\gamma_{0i} + C_1\gamma_{1i} + C_2\gamma_{2i})\ (C_0\gamma_{2i} + C_1\gamma_{3i} + C_2\gamma_{4i}) - (C_0\gamma_{1i} + C_1\gamma_{2i} + C_2\gamma_{3i})^2 - (\gamma_{0i}\gamma_{2i} - \gamma_{1i}^2)\Big\}, \tag{7.2}$$

where $r_{\nu i}$ and C_1, C_2 are given by (2.1) and (6.10), (6.12). Since n_i and $r_{\nu i}$ are obtainable, the value of R_{12} can be computed for given values of α and β in (6.10).

7.3. Results for q

As stated in Section 6.2, we need to represent $\Psi(\phi^*_g, A^*_g; g)$ by using $\Psi(\phi^*_f, A^*_f; f)$. For this purpose, the quantity R_{11} is numerically examined in this section. The efficiency of R_{11} is now denoted by q and is defined as follows;

$$q=R_{11}/\Psi(\phi_f^*,A_f^*;f). \tag{7.3}$$

This quantity q is computed for some values of α and β (see [Remark 7.1]), and is shown in Table 7.1 - Table 7.4. From these tables, the followings are ascertained:

(i) The differences of the values of q under NA and EA are less than 0.1%.

(ii) On the whole, the value of q under PA is slightly smaller than the one under NA in the range of α, β treated here.

(iii) q is fairly robust with respect to the variation of α, β. If $|\alpha|\leq 0.2$ and $|\beta|\leq 0.3$, then $|q|$ are less than 0.25 except for a few cases. Therefore we may conclude that the difference of the optimum stratification for g(x) and that for f(x) is not so remarkable.

(iv) For the distribution 3°, we need to pay much attention because of the singular behaviours of q. For example, when $\alpha=-0.1$ and $\beta=-0.01$, q are large and nearly equal to 0.20 for all $\ell=2-10$.

[Remark 7.1] The values of α and β are determined as follows: As for the value of α, we take up the cases $\alpha=\pm 0.1$ and $\beta=\pm 0.2$. The value of β is determined so that the ratio of the variance against the mean of g(x) is equal to ± 0.1. Our computation is intended to perform for all combination of these values. However for some values of (α, β), the p.d.f. g(x) defined by (6.11) and (6.12) is negative on some interval in (x_0, x_M). We, therefore, omit such values out of our computation.

7.4. Results for Q_1

In the preceding section, we have computed the value of R_{11} and the quantity R_{12} has been evaluated by (7.2). We can now compute the value of

Q_1 by using the relation (7.1). These computational results are summarized in Table 7.5 - Table 7.8, where the efficiency Q_1 defined by (6.8) is given for some values of α and β (see [Remark 7.1]). From these tables, the followings are ascertained:

(i) As expected theoretically, $Q_1 \geq 0$ is satisfied under NA for all cases, where the equality sign holds if and only if $\ell=2$ for the symmetric distributions.

(ii) The differences of the values of Q_1 under NA and EA are quite small and less than 0.4%.

(iii) On the whole, the value of Q_1 under NA is more robust than the one under PA with respect to the variation of α, β. The value of Q_1 under PA for the case of $\alpha>0$, $\beta>0$, is fairly large compared with the one under NA.

(iv) The value of Q_1 under NA is fairly robust with respect to the variation of α, β. If $|\alpha| \leqq 0.2$ and $|\beta| \leqq 0.3$, then they are less than 0.2 except for the case of $\alpha=0.2$, $\beta=0.32$ for the distribution 3°.

(v) For the unsymmetric distribution under PA, even if the values of $|\alpha|$, $|\beta|$ are small ($\alpha=-0.1$, $\beta=-0.01$), the value of Q_1 may be fairly large ($Q_1=0.213$ for the distribution 4°). On the contrary, even if the values of $|\alpha|$, $|\beta|$ are large ($\alpha=-0.2$, $\beta=-0.28$), there exists a case when $|Q_1|$ is fairly small ($Q_1=-0.087$ for the distribution 4°).

(vi) In general, Q_1 is more robust for symmetric distributions than for unsymmetric ones with respect to the variation of α, β.

As a conclusion, it may be seen that the method proposed in Section 5.1 is practicable. In the analysis of a given practical problem we may fit f(x) to g(x) and then apply the optimum stratification for f(x), where NA or EA should be used as the allocation method of sample sizes.

7.5. Some Practical Example

Consider the frequency distribution of the sale of benzine shown in Table 5.3, Case 8. In Section 5.2.2, we fitted the exponential distribution $f_z^*(z)=\beta^* e^{-\beta^* z}$ ($\beta^*=0.00309$) to the histogram $g_z(z)$, where the number of strata ℓ was 6 and the sample allocation method was Neyman Allocation. Let us examine the values of q, Q_1 in this case.

The distribution g(x) standardizing the histogram $g_z(z)$ is given by $g(x)=g_z(x/\beta^*)/\beta^*$. The mean μ_g, variance σ_g^2 for g(x) and the mean μ_f, variance σ_f^2 for f(x) are

$$\mu_g = 0.879,\ \sigma_g^2 = 1.010;\ \mu_f = 1.000,\ \sigma_f^2 = 1.000$$

respectively, so

$$\alpha = \mu_g/\mu_f - 1 = -0.121, \beta = \sigma_g^2 / \sigma_f^2 - 1 = 0.010.$$

We will consider that $\alpha \fallingdotseq -0.1$, $\beta \fallingdotseq -0.01$, then from Table 7.4 and Table 7.8

$$q=0.039, \qquad Q_1=0.035.$$

The efficiency Q_1^*, which is the ratio of R_1 against the optimum value $\Psi(\phi_g^*, A_g^*; g)$ is given as follows from (6.7) and (6.8);

$$Q_1^*=Q_1/(1-q)=0.036.$$

Therefore we may conclude that the loss owing to fitting f(x) to g(x) is very small.

CHAPTER 8

ROBUSTNESS ON SAMPLE SIZES IN RESPECTIVE STRATA

8.1. Analytical Examination

In this section, we analytically evaluate the quantity R_2 defined by (6.6). This problem is formulated as follows:

[**Problem**] Calculate the degree of the influence on our objective functions (2.4) - (2.6) caused by changing a sample size $n_i(\geq 1)$ in the i-th stratum to $n_i+m_i(\geq 1)$, where total sample size n is fixed.

From the definition of R_2, we may discuss the case where a p.d.f. is g(x) and a stratification method is ϕ_f^*. Then our objective function being a function of only sample sizes in respective strata, we will denote it by $\Psi(n_1,\ldots\ldots,n_\ell)$. In (2.4) - (2.6), note that w_i and σ_i^2 are the weight and variance of the i-th stratum for g(x), and a stratification method is the optimum method ϕ_f^* for f(x). Suppose that sample sizes in respective strata change from n_i to n_i+m_i. Since $\sum_{i=1}^{\ell}n_i=\sum_{i=1}^{\ell}(n_i+m_i)=n$ by the assumption,

$$\sum_{i=1}^{\ell}m_i=0 \tag{8.1}$$

must be satisfied. Then the given problem is to evaluate

$$R_2=\Psi(n_1+m_1,\ldots\ldots,n_\ell+m_\ell)-\Psi(n_1,\ldots,n_\ell) \tag{8.2}$$

under the condition (8.1), where n_i is determined by the allocation method for f(x). For this quantity, the following lemma holds:

[**Lemma 8.1**]

$$R_2=n\sum_{i=1}^{\ell-1} m_i\left\{w_\ell^2\sigma_\ell^2/n_\ell^2(1-\beta_i\theta-\beta_{i+1}-\cdots-\beta_{\ell-1})^2-w_i^2\sigma_i^2/n_i^2(1+\alpha_i\theta)^2\right\},\ \text{for some } \theta\ (0<\theta<1) \tag{8.3}$$

and

$$\alpha_i=m_i/n_i,\quad \beta_i=m_i/n_\ell,\quad (i=1,2,\ldots\ldots,\ell\text{-}1).$$

(Proof) Substituting the relation $\sum_{i=1}^{\ell} n_i=n$ into (2.3), we obtain

$$\Psi(n_1,\ldots\ldots,n_\ell)=n\left\{\sum_{i=1}^{\ell-1} w_i^2\sigma_i^2/n_i+w_\ell^2\sigma_\ell^2/(n-\sum_{j=1}^{\ell-1} n_j)\right\}.$$

Treating n_i's as continuous arguments, we have

$$\frac{\partial}{\partial n_i}\Psi(n_1,\ldots\ldots,n_\ell)=n\left\{w_\ell^2\sigma_\ell^2/(n-\sum_{j=1}^{\ell-1} n_j)^2-w_i^2\sigma_i^2/n_i^2\right\},\ (i=1,2,\ldots\ldots,\ell\text{-}1).$$

Since it is possible to expand $\Psi(n_1+m_1,\ldots\ldots,n_\ell+m_\ell)$ in a neighbourhood of the point $(n_1,\ldots\ldots,n_\ell)$, by using [Lemma 6.2] we obtain

$$\begin{aligned}R_2&=n\sum_{i=1}^{\ell-1} m_i\left\{w_\ell^2\sigma_\ell^2/(n-\sum_{j=1}^{\ell-1} n_j-m_i\theta-m_{i+1}-\cdots-m_{\ell-1})^2-w_i^2\sigma_i^2/(n_i+m_i\theta)^2\right\}\\&=n\sum_{i=1}^{\ell-1} m_i\left\{w_\ell^2\sigma_\ell^2/n_\ell^2(1-\beta_i\theta-\beta_{i+1}-\cdots-\beta_{\ell-1})^2-w_i^2\sigma_i^2/n_i^2(1+\alpha_i\theta)^2\right\}.\end{aligned}$$

The proof is completed.

To evaluate R_2, we use the inequality $0<\theta<1$ and obtain the following theorem:

[**Theorem 8.1**]

If

$$\begin{cases}1-\sum_{i=k}^{\ell-1}\beta_i>0, & (k=1,2,\ldots,\ell\text{-}1),\\ 1+\alpha_i>0, & (i=1,2,\ldots,\ell\text{-}1)\end{cases} \tag{8.4}$$

are satisfied simultaneously, then the following relation holds;

$$t<R_2<T, \tag{8.5}$$

where

$$\begin{cases} t = \sum_{i=1}^{\ell-1} \frac{m_i}{n}\{w_\ell^2\sigma_\ell^2/(\frac{n_\ell}{n})^2 (1-\beta_{i+1}-\dots-\beta_{\ell-1})^2 - w_i^2\sigma_i^2/(\frac{n_i}{n})^2\}, \\ T = \sum_{i=1}^{\ell-1} \frac{m_i}{n}\{w_\ell^2\sigma_\ell^2/(\frac{n_\ell}{n})^2(1-\beta_i-\dots-\beta_{\ell-1})^2 - w_i^2\sigma_i^2/(\frac{n_i}{n})^2(1+\alpha_i)^2\}. \end{cases} \tag{8.6}$$

(Proof) Let us consider R_2 in (8.3) as a function of θ, and write it as $R_2(\theta)$. Then

$$\frac{\partial R_2(\theta)}{\partial\theta} = 2n\sum_{i=1}^{\ell-1} m_i^2\{w_\ell^2\sigma_\ell^2/n_\ell^3(1-\beta_i\theta-\beta_{i+1}-\dots-\beta_{\ell-1})^3 + w_i^2\sigma_i^2/n_i^3(1+\alpha_i\theta)^3\}.$$

Therefore from the given conditions (8.4), $\partial R_2(\theta)/\partial\theta>0$ is obtained in the limit of $0\leqq\theta\leqq1$, and $R_2(\theta)$ is a strictly monotone increasing function with respect to θ. From the definition (8.6), we see that $t=R_2(0)$ and $T=R_2(1)$. Therefore

$$t<R_2(\theta)<T, \qquad \text{for } 0<\theta<1,$$

and the proof is complete.

In (8.6), $w_i^2\sigma_i^2$ and n_i/n are known for us as follows: Firstly $w_i^2\sigma_i^2$ is expressed by

$$w_i^2\sigma_i^2 = \int_{-\infty}^{\infty}\phi_i(x)g(x)dx\int_{-\infty}^{\infty}x^2\phi_i(x)g(x)dx - \{\int_{-\infty}^{\infty}x\phi_i(x)g(x)dx\}^2. \tag{8.7}$$

From (6.11),

$$\int_{-\infty}^{\infty}x^\nu\phi_i(x)g(x)dx = \int_{-\infty}^{\infty}x^\nu(c_0+c_1x+c_2x^2)\phi_i(x)f(x)dx = c_0\gamma_{\nu i}+c_1\gamma_{\nu+1 i}+c_2\gamma_{\nu+2 i}$$

holds for $\nu=0,1,2$, where $r_{\nu i}$ is defined by (2.1). If we substitute this into (8.7), the following is obtained;

$$w_i^2\sigma_i^2 = (c_0\gamma_{0i}+c_1\gamma_{1i}+c_2\gamma_{2i})(c_0\gamma_{2i}+c_1\gamma_{3i}+c_2\gamma_{4i}) - (c_0\gamma_{1i}+c_1\gamma_{2i}+c_2\gamma_{3i})^2.$$

Secondly, n_i/n is given as follows according to the three allocation methods;

$n_i/n = \sqrt{\gamma_{0i}\gamma_{2i}-\gamma_{1i}^2} \Big/ \sum_{j=1}^{\ell}\sqrt{\gamma_{0j}\gamma_{2j}-\gamma_{1j}^2}$ under NA,

$n_i/n = 1/\ell$ under EA,

$n_i/n = \gamma_{0i}$ under PA.

Therefore the values of t and T are computable if we determine the values of m_i and α, β given in (6.10).

8.2. Results

In order to estimate the efficiency Q_2 roughly, we will compute it in some special case, that is, in the case of $g(x)=f(x)$. The computational results are summarized in Table 8.1 – Table 8.4.

In the first row of the tables, we show the value of $|\alpha_i|$. The value of β_i is determined by the relation $\beta_i=\alpha_i n_i/n$. As for α_i, (8.1) implies $\sum_{i=1}^{\ell}\alpha_i=0$ under EA. Moreover by our empirical results the value of $w_i\sigma_i$ are approximately equal to each other under NA (see Chapter 1). Then $\sum_{i=1}^{\ell}\alpha_i \doteqdot 0$. Therefore we suppose $\alpha_i>0$ for $i=1,3,..,\ell-1$ and $\alpha_i<0$ for $i=2,4,...,\ell$. For the sake of computational convenience, we assumed that $|\alpha_i|$ were constant for any stratum. Our computation has been performed for the case of $|\alpha_i|=0.01$, 0.05, 0.07, 0.10, 0.15 and 0.20. We give the value of $u=t/\psi(\phi_f^*,A_f^*;f)$ in the upper row and the value of $U=T/\psi(\phi_f^*,A_f^*;f)$ in the lower row. From these computational results for the four distributions, we obtained the followings:

(i) The difference of the values of u and U is not so remarkable under NA or EA; If $|\alpha_i|\leqq 0.10$, then $U-u<0.04$ and if $|\alpha_i|\leqq 0.20$, then $U-u\leqq 0.17$.

(ii) The value of Q_2 under NA is nearly independent of distributions (see [Remark 8.1]).

(iii) For all cases, the value of Q_2 under EA is nearly equal to that under NA. In the case of PA, the value of Q_2 is largest.

(iv) If $|\alpha_i|\leqq 0.10$ holds under NA or EA, then the value of Q_2 is less than about 0.03 and if $|\alpha_i|\leqq 0.20$, then $Q_2\leqq 0.13$. Therefore in such cases there may be no riskiness from the practical point of view.

(v) For the unsymmetric distributions under PA, even if the values of $|\alpha_i|$ and ℓ are small ($|\alpha_i|=0.07$, $\ell=4$), the value of Q_2 may be fairly large ($U=5.029$ for the distribution 4°).

(vi) In general, the value of Q_2 for the symmetric distribution is more robust than the one for the unsymmetric distribution with respect to α_i under PA.

In the preceding Section 7.4, we described that NA or EA should be used in practical fields, because the efficiency $Q_1=R_1/\Psi(\phi_f^*, A_f^*; f)$ is small. From the above-mentioned results, the efficiency Q_2 is also small in the case of NA or EA.

[**Remark 8.1**] In the case of NA, $n_i=nw_i\sigma_i/\Sigma_{j=1}^{\ell} w_j\sigma_j$ holds. On the other hand, the values of $w_i\sigma_i$ are approximately equal to each other under NA by our empirical results. Therefore $n_i=n/\ell$ and $\beta_i=\alpha_i$ are approximately satisfied. Substituting these relation into (8.6), the following is obtained provided that (8.4) holds;

$$u_{NA}<Q_2<U_{NA},$$

where

$$u_{NA}=\sum_{i=1}^{\ell-1}\alpha_i\{1/(1-\alpha_{i+1}-\ldots-\alpha_{\ell-1})^2-1\}/\ell, \qquad U_{NA}=\sum_{i=1}^{\ell-1}\alpha_i\{1/(1-\alpha_i-\ldots-\alpha_{\ell-1})^2-1/(1+\alpha_i)^2\}/\ell.$$

Therefore the values of u_{NA} and U_{NA} are approximately independent of distributions.

8.3. Some Practical Example

We will take up once more the frequency distribution of the sale of benzine in Table 5.3, Case 8. The values of n_i's under Neyman Allocation are given in the second column of Table 8.5. In Section 5.2.5, we rounded off n_i for the sake of computational convenience and numerically examined this influence. Let us compute the value of Q_2 evaluated by (8.5) in this example.

[**Example 8.1**]

In Section 5.2.5, we changed the values of n_i such as shown in the third column of Table 8.5, where the values of m_i are given in the fourth column of this table. The distribution g(x) standardizing the histogram $g_z(z)$ is given by $g(x)=g_z(x/\beta^*)/\beta^*$. The mean μ_g, variance σ_g^2 for g(x) and the mean μ_f, variance σ_f^2 for f(x) are

$$\mu_g = 0.879,\quad \sigma_g^2 = 1.010\ ;\quad \mu_f = 1.000,\quad \sigma^2 f = 1.000$$

respectively, so

$$\alpha = \mu_g/\mu_f - 1 = -0.121,\ \beta = \sigma_g^2\ /\sigma^2 f - 1 = 0.010.$$

If these values are substituted into (8.6), then we obtain

$$t=0.0^4 695,\qquad T=0.0^4 762.$$

Let us denote the variances of $\overline{X}$ in the cases of $\{n_i\}$ and $\{n_i+m_i\}$ by V_{NA} and V'_{NA} respectively, then

$$(1/\beta^*)^2 \cdot t/n < V'_{NA} - V_{NA} < (1/\beta^*)^2 \cdot T/n$$

from (2.3) and (8.2). These values are computed as follows;

$$0.00170 < V'_{NA} - V_{NA} < 0.00187.$$

Since $V_{NA}=0.618$, the value of $Q'_2=(V'_{NA}-V_{NA})/V_{NA}$ is given by

$$0.00276 < Q'_2 < 0.00302.$$

The value of Q_2 defined by (6.8) is evaluated as follows:

$$0.00198<Q_2<0.00217.$$

Therefore we may conclude that the loss owing to changing $\{n_i\}$ to $\{n_i+m_i\}$ is very small.

[**Example 8.2**] (Equal Allocation)

Let us examine the influence owing to changing allocation method from NA to EA. The values of n_i+m_i and m_i are given in the fifth and sixth column of Table 8.5, respectively. By a similar computation as in [Example 8.1], we obtain

$$t=0.0^3460, \qquad T=0.0^3538.$$

If we denote the variance of $\overline{X}$ under EA by V_{EA}, then the value of $Q_2''=(V_{EA}-V_{NA})/V_{NA}$ is given by

$$0.0183<Q_2''<0.0213.$$

The value of Q_2 in this case is

$$0.0132<Q_2<0.0153.$$

Therefore we may use Equal Allocation in practical sample survey.

CHAPTER 9

ROBUSTNESS ON STRATIFICATION POINTS

9.1. Analytical Examination

In this section we analytically evaluate the quantity R_3 defined by (6.6). This problem is formulated as follows:

[Problem] Calculate the degree of the influence on the objective functions (2.4) - (2.6) caused by changing the stratification points ξ_i to $\xi_i+\eta_i$ in the i-th stratum.

From the definition of R_3, we consider the case where a p.d.f. and an allocation method of sample sizes under our consideration are g(x) and A_f, respectively. Suppose that $x_1, x_2, \ldots\ldots, x_{\ell-1}$ are the x-coordinates of stratification points and satisfy the relation $x_0 < x_1 < \ldots\ldots < x_\ell$, where the domain of g(x) is (x_0, x_ℓ). Since our objective function is a function of only stratification points in this problem, we denote it by $\Psi(x_1, \ldots, x_{\ell-1})$ and obtain

$$\Psi(x_1,\ldots,x_{\ell-1}) = n\sum_{i=1}^{\ell}\left[\int_{x_{i-1}}^{x_i} g(t)dt \int_{x_{i-1}}^{x_i} t^2 g(t)dt - \left\{\int_{x_{i-1}}^{x_i} tg(t)dt\right\}^2\right]/n_i.$$

Let $\Psi_{x_i}(x_1, \ldots, x_{\ell-1})$ be a partial derivative of $\Psi(x_1, \ldots, x_{\ell-1})$ with respect to the variable x_i, then it is given by

$$\Psi_{x_i}(x_1,\ldots,x_{\ell-1}) = ng(x_i)\left\{\int_{x_{i-1}}^{x_i}(t-x_i)^2 g(t)dt/n_i - \int_{x_i}^{x_{i+1}}(t-x_i)^2 g(t)dt/n_{i+1}\right\}.$$

From [Lemma 6.2], we can write for some θ $(0<\theta<1)$,

$$\begin{aligned} R_3 &= \Psi(\xi_1+\eta_1,\ldots,\xi_{\ell-1}+\eta_{\ell-1}) - \Psi(\xi_1,\ldots,\xi_{\ell-1}) \\ &= \sum_{i=1}^{\ell-1}\eta_i\Psi_{x_i}(\xi_1,\ldots,\xi_{i-1},\xi_i+\eta_i\theta,\xi_{i+1}+\eta_{i+1},\ldots,\xi_{\ell-1}+\eta_{\ell-1}), \end{aligned}$$

where

$$\Psi_{x_i}(\xi_1,\ldots,\xi_{i-1},\xi_i+\eta_i\theta,\xi_{i+1}+\eta_{i+1},\ldots,\xi_{\ell-1}+\eta_{\ell-1})$$
$$=ng(\xi_i+\eta_i\theta)\left\{\int_{\xi_{i-1}}^{\xi_i+\eta_i\theta}(t-\xi_i-\eta_i\theta)^2g(t)dt/\eta_i-\int_{\xi_i+\eta_i\theta}^{\xi_{i+1}+\eta_{i+1}}(t-\xi_i-\eta_i\theta)^2g(t)dt/\eta_{i+1}\right\} \quad (9.1)$$

with $\eta_\ell=0$. Evaluating each term in (9.1), we can obtain the following lemma:

[**Lemma 9.1**]

If we neglect the terms of order η_i^3, then R_3 is given by

$$R_3=n\sum_{i=1}^{\ell-1}\eta_i g(\xi_i+\eta_i\theta_0)[\{C_2S_{4i}+(C_1-2\xi_iC_2)S_{3i}-\xi_i(2C_1-\xi_iC_2)S_{2i}+\xi_i^2C_1S_{1i}\}$$
$$-2\eta_i\theta_0\{C_2S_{3i}+(C_1-\xi_iC_2)S_{2i}+(C_0-\xi_iC_1)S_{1i}-\xi_iC_0S_{0i}\}$$
$$-\eta_{i+1}d_{i+1}^2g(\xi_{i+1}+\eta_{i+1}\theta_1)/n_{i+1}],\qquad S_{\nu i}=\gamma_{\nu i}/n_i-\gamma_{\nu.i+1}/n_{i+1}\ ,\ (\nu=0,1,2,3,4)\ ;$$
$$0<\theta_k<1\ ,\quad (k=0,1)\ . \quad (9.2)$$

(Proof) The first term in the right-hand side of (9.1) can be decomposed so that

$$\int_{\xi_{i-1}}^{\xi_i+\eta_i\theta}(t-\xi_i-\eta_i\theta)^2g(t)dt=\int_{\xi_{i-1}}^{\xi_i}(t-\xi_i)^2g(t)dt-2\eta_i\theta\int_{\xi_{i-1}}^{\xi_i}(t-\xi_i)g(t)dt$$
$$+(\eta_i\theta)^2\int_{\xi_{i-1}}^{\xi_i}g(t)dt+\int_{\xi_i}^{\xi_i+\eta_i\theta}(t-\xi_i-\eta_i\theta)^2g(t)dt. \quad (9.3)$$

The first two terms in (9.3) is calculated as follows by using (6.11):

$$\int_{\xi_{i-1}}^{\xi_i}(t-\xi_i)^2g(t)dt=\int_{\xi_{i-1}}^{\xi_i}(t-\xi_i)^2(C_0+C_1t+C_2t^2)f(t)dt$$
$$=C_2\gamma_{4i}+(C_1-2\xi_iC_2)\gamma_{3i}+(C_0-2\xi_iC_1+\xi_i^2C_2)\gamma_{2i}$$
$$+(\xi_i^2C_1-2\xi_iC_0)\gamma_{1i}+\xi_i^2C_0\gamma_{0i}. \quad (9.4)$$

$$\int_{\xi_{i-1}}^{\xi_i}(t-\xi_i)g(t)dt=\int_{\xi_{i-1}}^{\xi_i}(t-\xi_i)(C_0+C_1t+C_2t^2)f(t)dt$$
$$=C_2\gamma_{3i}+(C_1-\xi_iC_2)\gamma_{2i}+(C_0-\xi_iC_1)\gamma_{1i}-\xi_iC_0\gamma_{0i}. \quad (9.5)$$

The third term in the right-hand side of (9.3) is clearly of order η_i^2. As for the last term in (9.3), transform the variable t to $t-\xi_i$ and apply the mean value theorem, then it is seen that the term is of order η_i^2. Therefore by substituting (9.4) and (9.5) into (9.3), the following can be obtained;

$$\int_{\xi_{i-1}}^{\xi_i+\eta_i\theta}(t-\xi_i-\eta_i\theta)^2 g(t)\,dt = C_2\gamma_{4i}+(C_1-2\xi_i C_2)\gamma_{3i}+(C_0-2\xi_i C_1+\xi_i^2 C_2)\gamma_{2i}$$
$$+(\xi_i^2 C_1-2\xi_i C_0)\gamma_{1i}+\xi_i^2 C_0\gamma_{0i}-2\eta_i\theta\{C_2\gamma_{3i}+(C_1-\xi_i C_2)\gamma_{2i}$$
$$+(C_0-\xi_i C_1)\gamma_{1i}-\xi_i C_0\gamma_{0i}\}+O(\eta_i^2). \tag{9.6}$$

Secondly, we will calculate the second term in the right-hand side of (9.1) by similar way. Now the following holds;

$$\int_{\xi_i+\eta_i\theta}^{\xi_{i+1}+\eta_{i+1}}(t-\xi_i-\eta_i\theta)^2 g(t)\,dt = \int_{\xi_i}^{\xi_{i+1}}(t-\xi_i)^2 g(t)\,dt-2\eta_i\theta\int_{\xi_i}^{\xi_{i+1}}(t-\xi_i)g(t)\,dt$$
$$+(\eta_i\theta)^2\int_{\xi_i}^{\xi_{i+1}}g(t)\,dt$$
$$-\int_{\xi_i}^{\xi_i+\eta_i\theta}(t-\xi_i-\eta_i\theta)^2 g(t)\,dt$$
$$+\int_{\xi_{i+1}}^{\xi_{i+1}+\eta_{i+1}}(t-\xi_i-\eta_i\theta)^2 g(t)\,dt. \tag{9.7}$$

The last term in (9.7) is calculated as follows by transforming the variable t to $u=t-\xi_{i+1}$ and applying the mean value theorem;

$$\int_{\xi_{i+1}}^{\xi_{i+1}+\eta_{i+1}}(t-\xi_i-\eta_i\theta)^2 g(t)\,dt = \int_0^{\eta_{i+1}}(d_{i+1}+u-\eta_i\theta)^2 g(u+\xi_{i+1})\,du$$
$$=\eta_{i+1}d^2_{i+1}g(\xi_{i+1}+\eta_{i+1}\theta') + O(\eta_i^2),$$
$$d_{i+1}=\xi_{i+1}-\xi_i\ , \qquad 0<\theta'<1\ , \tag{9.8}$$

where we should define $\eta_\ell=0$ since the value of (9.8) must be 0 for $i=\ell-1$. Therefore if we neglect the terms of order η_i^2 in (9.7), then we can obtain

$$\int_{\xi_i+\eta_i\theta}^{\xi_{i+1}+\eta_{i+1}}(t-\xi_i-\eta_i\theta)^2 g(t)\,dt = C_2\gamma_{4i+1}+(C_1-2\xi_i C_2)\gamma_{3i+1}+(C_0-2\xi_i C_1$$
$$+\xi_i^2 C_2)\gamma_{2i+1}+(\xi_i^2 C_1-2\xi_i C_0)\gamma_{1\cdot i+1}+\xi_i^2 C_0\gamma_{0\cdot i+1}$$
$$-2\eta_i\theta\{C_2\gamma_{3i+1}+(C_1-\xi_i C_2)\gamma_{2\cdot i+1}$$
$$+(C_0-\xi_i C_1)\gamma_{1i+1}-\xi_i C_0\gamma_{0.i+1}\}$$
$$+\eta_{i+1}d_{i+1}^2\,g(\xi_{i+1}+\eta_{i+1}\theta')+O(\eta_i^2),$$
$$0<\theta<1\ , \qquad 0<\theta'<1 \tag{9.9}$$

If we substitute (9.6), (9.9) into (9.1) and use the relation (2.12), then (9.2) can be obtained by using [Lemma 6.2]. The proof is completed.

Since unknown constants θ_0, θ_1 are included in (9.2), we can not compute the value of R_3 in practice. If we assume that η_i is small for $i=1,2,\ldots,\ell-1$ and $g(x)$ has not any sudden changes in the function value, then we may make the following approximations;

$$g(\xi_i+\eta_i\theta_k) \doteqdot g(\xi_i+\eta_i/2), \qquad (k=0,1 \ ; \ i=1,2,\cdots,\ell-1). \tag{9.10}$$

Furthermore if $\eta_i\theta_k$ is small, then

$$\eta_i\theta_k \doteqdot \eta_i/2, \qquad (k=0,1 \ ; \ i=1,2,\ldots,\ell-1) \tag{9.11}$$

may be satisfied approximately. Substituting (9.10) and (9.11) into (9.2), we can obtain the following theorem:

[**Theorem 9.1**]

R_3 can be approximately evaluated as follows;

$$\begin{aligned} R_3 \doteqdot n\sum_{i=1}^{\ell-1}\eta_i g(\xi_i+\eta_i/2)\big[&\{C_2S_{4i}+(C_1-2\xi_iC_2)S_{3i}-\xi_i(2C_1-\xi_iC_2)S_{2i}+\xi_i^2C_1S_{1i}\} \\ &-\eta_i\{C_2S_{3i}+(C_1-\xi_iC_2)S_{2i}+(C_0-\xi_iC_1)S_{1i}-\xi_iC_0S_{0i}\} \\ &-\eta_{i+1}d_{i+1}^2 g(\xi_{i+1}+\eta_{i+1}/2)/n_{i+1}\big] \ ; \ \eta_\ell=0. \end{aligned} \tag{9.12}$$

Since various quantities in (9.12) except for $\eta_1,\eta_2,\ldots,\eta_{\ell-1}$ are obtainable if we determine the values of α and β given in (6.10), an approximate value of R_3 can be computed by (9.12) if we give values of $\eta_1,\eta_2,\ldots\ldots,\eta_{\ell-1}$.

9.2. Results

In this section we will make tables giving approximate values of Q_3 defined by (6.8), which may be useful in practical fields. As the value of Q_3 depends upon the values of α and β in (6.10), there are many cases according to the combination of the values of α and β. So in order to find the value of Q_3 roughly, we will make the tables giving Q_3 for the case of $g(x)=f(x)$ under A^*_f. In the case that $g(x)$ differs from $f(x)$, an approximate value of Q_3 can be computed by using (6.8) and (9.12).

Examples of computational results are partly shown in Table 9.1 - Table 9.4. In these tables, the notation PLUS, MINUS and ALTERNATING have the following meanings, respectively;

PLUS : $\eta_i>0$ $(i=1,2,\ldots,\ell-1)$.

MINUS : $\eta_i<0$ $(i=1,2,\ldots,\ell-1)$.

ALTERNATING : $\eta_i>0$ $(i=1,3,\ldots,\ell-1)$; $\eta_i<0$ $(i=2,4,\ldots,\ell-2)$.

In the second row, we show the values of $|\eta_i|$ which is equal to 0.01σ, 0.05σ or 0.10σ, where σ represents the standard deviation of $f(x)$. For the sake of computational convenience we assumed that $|\eta_i|$ was constant for any stratum and a number of strata ℓ was even. From these computational results for the four distributions, we obtain the followings:

(i) For all cases, the value of Q_3 under EA is nearly equal to that under NA.

(ii) In the case of ALTERNATING, the value of Q_3 is fairly large.

(iii) In the case of ALTERNATING, the value of Q_3 under PA is smallest and in other cases the differences of the values of Q_3 under three allocation methods are not so remarkable as in the case of ALTERNATING.

(iv) In the case of ALTERNATING, if $\ell=10$ and $|\eta_i|=0.10\sigma$, then the value of Q_3 is nearly equal to 1 for some distributions.

(v) In the case of PLUS or MINUS if $|\eta_i|\leqq 0.05\sigma$, then the value of Q_3 is less than 0.13. Therefore in such cases there may be no riskiness from the practical point of view.

(vi) In general Q_3 is more robust for symmetric distributions than for unsymmetric ones with respect to η_i.

(vii) The value of Q_3 for the normal distribution is smallest among those for the four distributions.

9.3. Some Practical Example

As in Section 7.5 and Section 8.3, let us take up the frequency distribution of the sale of benzine shown in Table 5.3, Case 8. In this case, computed optimum stratification points z_i^*'s are as follows;

$$z_1^*=114.5, \quad z_2^*=253.8, \quad z_3^*=431.7, \quad z_4^*=678.6, \quad z_5^*=1086.3.$$

Let us change z_i^* as follows owing to the constraint of sampling frame (see Table 5.5), and evaluate the influence of this change by using (9.12);

$$z_1=120, \quad z_2=260, \quad z_3=500, \quad z_4=700, \quad z_5=1100,$$

The distribution $g(x)$ standardizing the histogram $g_z(z)$ is given by $g(x)=g_z(x/\beta^*)/\beta^*$. The mean μ_g, variance σ_g^2 for $g(x)$ and the mean μ_f, variance σ_f^2 for $f(x)$ are

$$\mu_g = 0.8787, \quad \sigma_g^2 = 1.0100 \;; \; \mu_f = 1.0000, \quad \sigma_f^2 = 1.0000$$

respectively, so

$$\alpha = \mu_g/\mu_f - 1 = -0.1213, \quad \beta = \sigma^2 g/\sigma_f^2 - 1 = 0.0100 .$$

The values of ξ_i in (9.12) are given by the followings (see Table 3.10);

$$\xi_1 = 0.3543, \quad \xi_2 = 0.7853, \quad \xi_3 = 1.3360,$$
$$\xi_4 = 2.0999, \quad \xi_5 = 3.3618.$$

The values of η_i are obtained as follows by the relation $\eta_i=\beta_i^*(z_i-z_i^*)$;

$$\eta_1 = 0.01706, \quad \eta_2 = 0.01928, \quad \eta_3 = 0.21108,$$
$$\eta_4 = 0.06627, \quad \eta_5 = 0.04228.$$

Since $\Psi(\phi_f^*, A_f^*; f)=0.03507$ from Table 3.10, we have

$$Q_3=0.1607.$$

If we consider that $\alpha \fallingdotseq -0.1$ and $\beta \fallingdotseq -0.01$, then the value of q in (6.7) is $q=0.039$ from Table 7.4. The efficiency Q_3^*, which is the ratio of R_3 against the optimum value $\Psi(\phi_g^*, A_g^*; g)$, is given as follows from (6.7) and (6.8);

$$Q_3^*=Q_3/(1-q)=0.174.$$

Therefore we may conclude that the loss owing to changing z_i^* to z_i is small and there is no riskiness from the practical point of view.

CHAPTER 10

CONCLUSION

We will now summarize the main results obtained in this paper.

In Chapter 3, we considered the estimation of the population mean μ, and obtained the followings:

(3-1) Optimum Stratification Points (OSP) and Minimum Variances (Min.Var.) for typical four distributions under three sample allocations were computed by using nonlinear programming algorithms up to 10 strata.

(3-2) Gains of the increase of the number of strata and efficiencies of each allocation were investigated for the above cases.

(3-3) Neyman Allocation (NA) and Equal Allocation (EA) turned out to give quite similar results.

(3-4) General Optimum Stratification (GOS) could be regarded to coincide with Interval Optimum Stratification (IOS).

(3-5) A method for determining the sample size and the number of strata were given under the assumption that the total sampling cost was constant.

In chapter 4, we considered the estimation of the population variance σ^2, and obtained the followings:

(4-1) OSP and Min.Var. for typical four distributions under Proportional Allocation (PA) were computed by using general nonlinear programming algorithm up to 4 or 5 strata.

(4-2) Gains of the increase of the number of strata and efficiencies of each stratification method were investigated for the above four distributions.

(4-3) Asymptotic GOS (AGOS) did not always coincide with Asymptotic IOS (AIOS), especially AGOS was much better than AIOS in case of the symmetric distributions.

Applicational scheme of the above-mentioned results to practical sample surveys was proposed in Chapter 5, and we applied it to the data of "Current Survey on Petroleum Products Demand and Supply" as representative examples. By these examples, it could be shown that our proposed procedure decreased the standard error of the estimator $\overline{X}$ for μ by about 30 - 60 % compared with the traditional procedure, and the influence on the standard error of $\overline{X}$ owing to small change of sample sizes in respective strata and stratification points was about less than 10 % and might not be serious.

In Chapters 7, 8 and 9, we discussed three sub-problems of robustness in optimum stratification based on the formulation given in Chapter 6.

As for the robustness on distributions, we obtained the followings in Chapter 7:

(7-1) The value of Q_1 under NA or EA was more robust than that under PA with respect to some kind of changes of distributions.

(7-2) The value of Q_1 under NA or EA was fairly robust and it was less than 20% in usual cases.

(7-3) In general, the value of Q_1 was more robust for a symmetric distribution than for an unsymmetric one with respect to the change of a distribution. Especially for an unsymmetric distribution, PA is not recommendable, because the value of Q_1 for such a case may be large.

As for the robustness on sample sizes in respective strata, we obtained the followings in Chapter 8:

(8-1) The value of Q_2 under NA was nearly independent on a distribution.

(8-2) The value of Q_2 under NA or EA was more robust than that under PA.

(8-3) The value of Q_2 under NA or EA was very robust and was less than 3% in usual cases.

(8-4) In general, the value of Q_2 under PA for a symmetric distribution was more robust than that for an unsymmetric distribution, and so PA is not recommendable for an unsymmetric distribution.

As for the robustness on stratification points, we obtained the followings in Chapter 9:

(9-1) Although the value of Q_3 under PA was the smallest in the case of ALTERNATING, that under NA or EA was not so large. In other cases (PLUS or MINUS), the differences of the values of Q_3 under three allocations were not remarkable.

(9-2) We should pay much attention to the cases of ALTERNATING because the value of Q_3 was too large for some distributions. On the contrary, in the cases of PLUS or MINUS the value of Q_3 was usually less than 13%, so there may be no riskiness from the practical point of view.

(9-3) The value of Q_3 was more robust for a symmetric distribution than that for an unsymmetric distribution. Especially in the case of the normal distribution, the value of Q_3 was most robust.

Through all these discussions, EA gave quite similar results with NA as reported in Chapter 1.

In the chapters dealing with three kinds of robustness, we gave some practical examples. Now let us summarize these examples:

[**Example 10.1**]

Consider the data shown in Table 5.3, Case 8. In Section 5.2 we applied our proposed working procedure to this data, and then changed sample sizes in respective strata and stratification points as shown in the last two rows of Table 5.5. In Chapters 7, 8 and 9 we have evaluated the influence of such change, and have obtained

$$Q_1=0.035, \qquad Q_2=0.002, \qquad Q_3=0.161.$$

So the theoretical consideration yields

$$Q_0=0.198.$$

On the other hand as shown in Chapter 5, we have obtained

$$S_1=0.786, \qquad S_3=0.871.$$

Since S_1^2 and S_3^2 may be nearly equal to $\Psi(\phi_f^*, A_f^*; g)/\{n(\beta^*)^2\}$ and $\Psi(\phi_f, A_f; g)/\{n(\beta^*)^2\}$ respectively, the value of Q_0 can be given as follows by using $n=4272$, $\beta^*=0.00309$, $\Psi(\phi_f^*, A_f^*; f)=0.035068$ and $Q_1=0.035$;

$$Q_0=Q_1+Q_2+Q_3=Q_1+ n(\beta^*)^2(S_3^2-S_1^2)/\Psi(\phi_f^*, A_f^*; f)=0.199.$$

This value is almost the same as that obtained above. It may, therefore, be concluded that the evaluation methods for Q_1, Q_2 and Q_3 given in Chapters 7, 8 and 9 are useful in practice.

As a conclusion, we may apply our working procedure proposed in Section 5.1 in designing stratified random sampling to estimate the population mean, if the population distribution can be approximated by either of the distributions shown in Section 2.6, (A). As a stratification method, we could use Interval Optimum Stratification (IOS) because General Optimum Stratification (GOS) coincided with IOS in the range of the cases treated in this paper (cf. GOS was very effective in estimating the population variance, especially for symmetric distributions). The robustness of a sample design should be evaluated by using the tables given in Chapters 7, 8 and 9 or by the formulae (7.2), (8.5) and (9.12). The value of Q_0, which indicates the degree of robustness, is usually not so large. As for a sample allocation method, Equal Allocation is recommendable since it is simpler, more robust than Proportional Allocation and gives quite similar effects with the optimal allocation, Neyman Allocation. The number of strata may be sufficient to be less than or equal to 5 so far as the stratification cost is not so cheap, and then we can get satisfactory effect of stratification. The author hopes that the results given in this paper are effectively utilized in practical sample surveys.

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Table 3.1. OSP, ψ_N^* and e_N for the equilateral triangular distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_N^*	e_N
1						0.1666667	0.000
2	0.00000					0.0555551	0.667
3	−0.23132					0.0255607	0.847
4	−0.35425	0.00000				0.0150372	0.910
5	−0.44226	−0.13629				0.0097277	0.942
6	−0.50263	−0.22978	0.00000			0.0068784	0.959
7	−0.55072	−0.30425	−0.09669			0.0050797	0.970
8	−0.58745	−0.36112	−0.17052	0.00000		0.0039271	0.976
9	−0.61836	−0.40900	−0.23268	−0.07497		0.0031123	0.981
10	−0.64342	−0.44780	−0.28306	−0.13567	0.00000	0.0025363	0.985

Table 3.2. OSP, ψ_E^* and e_E for the equilateral triangular distribution

ℓ	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_E^*	e_E
1						0.1666667	0.000
2	0.00000					0.0555551	0.667
3	-0.23207					0.0255610	0.847
4	-0.35915	0.00000				0.0150434	0.910
5	-0.44668	-0.13656				0.0097309	0.942
6	-0.50789	-0.23207	0.00000			0.0068811	0.959
7	-0.55551	-0.30641	-0.09681			0.0050814	0.970
8	-0.59243	-0.36401	-0.17183	0.00000		0.0039285	0.976
9	-0.62304	-0.41173	-0.23393	-0.07499		0.0031133	0.981
10	-0.64812	-0.45087	-0.28490	-0.13652	0.00000	0.0025371	0.985

Table 3.3. OSP, ψ_P^* and e_P for the equilateral triangular distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_P^*	e_P
1						0.1666667	0.000
2	0.00000					0.0555552	0.667
3	−0.25000					0.0260417	0.844
4	−0.38197	0.00000				0.0154800	0.907
5	−0.47444	−0.14963				0.0100775	0.940
6	−0.53734	−0.25139	0.00000			0.0071608	0.957
7	−0.58679	−0.33141	−0.10688			0.0053049	0.968
8	−0.62427	−0.39205	−0.18789	0.00000		0.0041128	0.975
9	−0.65551	−0.44260	−0.25541	−0.08315		0.0032656	0.980
10	−0.68066	−0.48330	−0.30979	−0.15010	0.00000	0.0026661	0.984

Table 3.4. OSP, Ψ_N^* and e_N for the normal distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_N^*	e_N
1						1.000000	0.000
2	0.00000					0.363380	0.637
3	−0.54981					0.182473	0.818
4	−0.87569	0.00000				0.109128	0.891
5	−1.10410	−0.33585				0.072461	0.928
6	−1.27825	−0.57560	0.00000			0.051569	0.948
7	−1.41805	−0.76055	−0.24280			0.038556	0.961
8	−1.53427	−0.91021	−0.43182	0.00000		0.029909	0.970
9	−1.63339	−1.03532	−0.58579	−0.19034		0.023873	0.976
10	−1.71955	−1.14244	−0.71516	−0.34622	0.00000	0.019495	0.981

Table 3.5. OSP, Ψ_E^* and e_E for the normal distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_E^*	e_E
1						1.000000	0.000
2	0.00000					0.363380	0.637
3	−0.56888					0.182704	0.817
4	−0.90091	0.00000				0.109294	0.891
5	−1.13189	−0.34326				0.072570	0.927
6	−1.30729	−0.58671	0.00000			0.051643	0.948
7	−1.44770	−0.77377	−0.24674			0.038607	0.961
8	−1.56422	−0.92473	−0.43813	0.00000		0.029946	0.970
9	−1.66343	−1.05069	−0.59364	−0.19278		0.023901	0.976
10	−1.74960	−1.15838	−0.72407	−0.35031	0.00000	0.019516	0.980

Table 3.6. OSP, Ψ_P^* and e_P for the normal distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_P^*	e_P
1						1.000000	0.000
2	0.00000					0.363380	0.637
3	−0.61201					0.190175	0.810
4	−0.98158	0.00000				0.117483	0.883
5	−1.24435	−0.38228				0.079943	0.920
6	−1.44684	−0.65891	0.00000			0.057979	0.942
7	−1.61075	−0.87436	−0.28029			0.044001	0.956
8	−1.74792	−1.04995	−0.50055	0.00000		0.034549	0.965
9	−1.86552	−1.19759	−0.68122	−0.22182		0.027854	0.972
10	−1.96821	−1.32457	−0.83384	−0.40474	0.00000	0.022938	0.977

Table 3.7. OSP, Ψ_N^* and e_N for the rightangled triangular distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	Ψ_N^*	e_N
1										0.2222222	0.000
2	0.75850									0.0601488	0.729
3	0.45956	1.00526								0.0275137	0.876
4	0.34105	0.72225	1.17489							0.0157086	0.929
5	0.27133	0.56611	0.89559	1.28683						0.0101453	0.954
6	0.22534	0.46610	0.72767	1.02003	1.36718					0.0070887	0.968
7	0.19271	0.39634	0.61390	0.85026	1.11445	1.42816				0.0052311	0.977
8	0.16834	0.34483	0.53132	0.73057	0.94704	1.18899	1.47629			0.0040184	0.982
9	0.14945	0.30522	0.46852	0.64107	0.82543	1.02573	1.24960	1.51543		0.0031833	0.986
10	0.13438	0.27379	0.41909	0.57142	0.73238	0.90436	1.09120	1.30002	1.54799	0.0025839	0.988

Table 3.8. OSP, Ψ_E^* and e_E for the rightangled triangular distribution

ℓ	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	Ψ_E^*	e_E
1										0.2222222	0.000
2	0.71831									0.0601734	0.729
3	0.46412	1.01574								0.0275245	0.876
4	0.34366	0.72806	1.18489							0.0157140	0.929
5	0.27302	0.56980	0.90174	1.29624						0.0101484	0.954
6	0.22656	0.46870	0.73187	1.02624	1.37604					0.0070906	0.968
7	0.19357	0.39817	0.61685	0.85454	1.12037	1.43631				0.0052324	0.976
8	0.16902	0.34627	0.53362	0.73387	0.95150	1.19485	1.48403			0.0040193	0.982
9	0.14997	0.30631	0.47025	0.64356	0.82880	1.03012	1.25525	1.52279		0.0031840	0.986
10	0.13486	0.27479	0.42065	0.57360	0.73527	0.90803	1.09575	1.30564	1.55508	0.0025844	0.988

Table 3.9. OSP, Ψ_P^* and e_P for the rightangled triangular distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	Ψ_P^*	e_P
1										0.2222222	0.000
2	0.76397									0.0619201	0.721
3	0.50279	1.07467								0.0286434	0.871
4	0.37577	0.78409	1.24853							0.0164512	0.926
5	0.30021	0.61958	0.96661	1.36133						0.0106643	0.952
6	0.25002	0.51271	0.79215	1.09580	1.44117					0.0074701	0.966
7	0.21423	0.43747	0.67201	0.92152	1.19264	1.50103				0.0055226	0.975
8	0.18742	0.38158	0.58390	0.79646	1.02259	1.26830	1.54779			0.0042482	0.981
9	0.16659	0.33840	0.51639	0.70185	0.89671	1.10400	1.32925	1.58545		0.0033690	0.985
10	0.14992	0.30402	0.46295	0.62760	0.79916	0.97941	1.17116	1.37953	1.61653	0.0027370	0.988

Table 3.10. OSP, Ψ_N^* and e_N for the exponential distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	Ψ_N^*	e_N
1										1.000000	0.000
2	1.26191									0.285434	0.715
3	0.76396	2.02587								0.133225	0.867
4	0.55065	1.31461	2.57652							0.076868	0.923
5	0.43103	0.98167	1.74563	3.00754						0.049969	0.950
6	0.35428	0.78531	1.33595	2.09992	3.36182					0.035068	0.965
7	0.30081	0.65509	1.08612	1.63676	2.40073	3.66263				0.025961	0.974
8	0.26139	0.56220	0.91648	1.34751	1.89815	2.66212	3.92403			0.019991	0.980
9	0.23112	0.49252	0.79333	1.14761	1.57864	2.12928	2.89325	4.15515		0.015867	0.984
10	0.20715	0.43827	0.69967	1.00048	1.35476	1.78579	2.33643	3.10040	4.36230	0.012898	0.987

Table 3.11. OSP, Ψ_E^* and e_E for the exponential distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	Ψ_E^*	e_E
1										1.000000	0.000
2	1.30008									0.285810	0.714
3	0.77906	2.07914								0.133378	0.867
4	0.55875	1.33781	2.63789							0.076946	0.923
5	0.43609	0.99485	1.77391	3.07399						0.050013	0.950
6	0.35774	0.79383	1.35258	2.13164	3.43172					0.035096	0.965
7	0.30332	0.66107	1.09716	1.65591	2.43496	3.73504				0.025980	0.974
8	0.26331	0.56664	0.92439	1.36048	1.91923	2.69830	3.99837			0.020004	0.980
9	0.23264	0.49595	0.79928	1.15703	1.59313	2.15188	2.93095	4.23102		0.015879	0.984
10	0.20836	0.44098	0.70429	1.00761	1.36536	1.80145	2.36020	3.13926	4.43933	0.012905	0.987

Table 3.12. OSP, Ψ_P^* and e_P for the exponential distribution

l	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	Ψ_P^*	e_P
1										1.000000	0.000
2	1.59359									0.352390	0.658
3	1.01758	2.61120								0.179737	0.820
4	0.75403	1.77161	3.36523							0.108952	0.891
5	0.60043	1.35447	2.37205	3.96567						0.073090	0.927
6	0.49932	1.09976	1.85379	2.87137	4.46500					0.052427	0.948
7	0.42757	0.92689	1.52733	2.28137	3.29895	4.89257				0.039439	0.961
8	0.37394	0.80152	1.30085	1.90128	2.65532	3.67290	5.26652			0.030745	0.969
9	0.33233	0.70629	1.13387	1.63321	2.23365	2.98769	4.00527	5.59890		0.024640	0.975
10	0.29906	0.63137	1.00531	1.43289	1.93219	2.53262	3.28660	4.30423	5.89786	0.020189	0.980

Table 3.13. OSP, Ψ_G^* for the equilateral triangular distribution

ℓ		x_0^*	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	Ψ_G^*
2	1 2 1	−1.000	0.000	1.000	1.000					0.05556
3	1 2 3 1	−1.000	−1.000	−0.231	0.231	1.000				0.02556
	1 2 1 3 1	−1.000	−1.000	−0.231	0.231	1.000	1.000			0.02556
	1 2 3 2 1	−1.000	−1.000	−1.000	−0.231	0.231	1.000			0.02556
4	1 2 3 4 1	−1.000	−0.354	0.000	0.354	1.000	1.000			0.01504
	1 2 1 3 4 1	−1.000	−1.000	−0.354	0.000	0.354	1.000	1.000		0.01504
	1 2 3 1 4 1	−1.000	−0.354	0.000	0.354	0.354	1.000	1.000		0.01504
	1 2 3 2 4 1	−1.000	−1.000	−1.000	−0.354	0.000	0.354	1.000		0.01504
	1 2 3 4 2 1	−1.000	−1.000	−1.000	−0.354	0.000	0.354	1.000		0.01504
	1 2 3 4 3 1	−1.000	−0.354	0.000	0.354	1.000	1.000	1.000		0.01504
	1 2 1 3 1 4 1	−1.000	−1.000	−0.354	−0.354	0.000	0.354	1.000	1.000	0.01504
	1 2 1 3 4 3 1	−1.000	−1.000	−0.354	0.000	0.354	1.000	1.000	1.000	0.01504
	1 2 3 2 1 4 1	−1.000	−1.000	−1.000	−0.354	0.000	0.354	1.000	1.000	0.01504
	1 2 3 2 4 2 1	−1.000	−0.354	0.000	0.354	0.354	1.000	1.000	1.000	0.01504
	1 2 3 4 3 2 1	−1.000	−0.354	0.000	0.354	1.000	1.000	1.000	1.000	0.01504

Table 3.14. OSP, Ψ_G^* for the normal, rightangled triangular and exponential distribution

	ℓ		x_0^*	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	Ψ_G^*
2°	2	1 2 1	−3.000	−3.000	0.000	3.000			0.34741
	3	1 2 3 1	−3.000	−3.000	−0.545	0.545	3.000		0.17256
	4	1 2 3 4 1	−3.000	−3.000	−0.866	0.000	0.866	3.000	0.10236
3°	2	1 2 1	0.000	0.709	2.000	2.000			0.06015
	3	1 2 3 1	0.000	0.460	1.005	2.000	2.000		0.02751
	4	1 2 3 4 1	0.000	0.341	0.722	1.175	2.000	2.000	0.01571
4°	2	1 2 1	0.000	1.225	6.000	6.000			0.25323
	3	1 2 3 1	0.000	0.741	1.938	6.000	6.000		0.11639
	4	1 2 3 4 1	0.000	0.533	1.262	2.430	6.000	6.000	0.06647

Table 3.15. c_E and c_P for the four distributions

Name of distribution	$c \backslash \ell$	2	3	4	5	6	7	8	9	10
Equilateral triangular distribution	c_E	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	c_P	0.000	0.019	0.029	0.036	0.041	0.043	0.047	0.049	0.051
Normal distribution	c_E	0.000	0.001	0.002	0.002	0.001	0.001	0.001	0.001	0.001
	c_P	0.000	0.042	0.058	0.103	0.124	0.141	0.155	0.167	0.176
Rightangled triangular distribution	c_E	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	c_P	0.029	0.041	0.047	0.051	0.054	0.056	0.057	0.058	0.059
Exponential distribution	c_E	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	c_P	0.235	0.349	0.417	0.463	0.495	0.519	0.538	0.553	0.565

Table 3.16. $\psi_1(\ell)$ for the equilateral triangular distribution

($\times 10^{-7}$)

$1/k$	k \ l	1	2	3	4	5	6	7	8	9	10
0.010	100	16835	5669	2635	1567	1024	732	546	427	342	282
0.020	50	34014	11574	5439	3270	2162	1563	1181	935	759	634
0.033	30	57471	19841	9467	5786	3891	2866	2209	1785	1482	1268
0.040	25	69444	24154	11619	7164	4864	3620	2822	2310	1945	1691
0.050	20	87719	30864	15036	9402	6485	4913	3907	3273	2829	2536
0.067	15	119048	42735	21301	13676	9728	7643	6350	5610	5187	5073
0.100	10	185185	69444	36516	25072	19455	17196	16932	19636	31123	
0.111	9	208333	79364	42602	30087	24319	22928	25399	39271		
0.125	8	238095	92592	51122	37609	32426	34392	50797			
0.143	7	277778	111110	63903	50145	48639	68784				
0.167	6	333333	138888	85203	75217	97277					
0.200	5	416667	185184	127805	150434						
0.250	4	555555	277776	255610							
0.333	3	833334	555551								
0.500	2	1666667									

Table 3.17. $\psi_2(\ell)$ for the equilateral triangular distribution

($\times 10^{-7}$)

1/k	k \ ℓ	1	2	3	4	5	6	7	8	9
0.010	100	16835	5787	2809	1791	1297	1075	996	1091	1639
0.020	50	34014	12077	6234	4425	3892	4915	50814		
0.033	30	57471	21367	12172	10745	19462				
0.040	25	69444	26455	15976	16715					
0.050	20	87719	34722	23237	37609					
0.067	15	119048	50505	42602						
0.100	10	185186	92592	255610						
0.111	9	208333	111110							
0.125	8	238095	138888							
0.143	7	277778	185184							
0.167	6	333333	277776							
0.200	5	416667	555551							
0.250	4	555556								
0.333	3	833334								
0.500	2	1666667								

Table 3.18. $\psi_1(\ell)$ for the normal distribution

($\times 10^{-6}$)

$1/k$	k \ l	1	2	3	4	5	6	7	8	9	10
0.010	100	10101	3708	1884	1138	764	549	415	326	263	217
0.020	50	20408	7570	3887	2376	1613	1174	898	713	583	488
0.033	30	34482	12978	6767	4204	2903	2152	1679	1361	1138	976
0.040	25	41667	15799	8305	5204	3629	2718	2145	1762	1494	1301
0.050	20	52632	20188	10747	6831	4838	3689	2970	2496	2173	1952
0.067	15	71429	27952	15225	9936	7257	5738	4826	4278	3984	3903
0.100	10	111111	45423	26101	18216	14514	12911	12869	14973	23901	
0.111	9	125000	51911	30451	21859	18143	17214	19304	29946		
0.125	8	142857	60563	36541	27324	24190	25822	38607			
0.143	7	166667	72676	45674	36431	36285	51643				
0.167	6	200000	90845	60901	54647	72570					
0.200	5	250000	121127	91352	109294						
0.250	4	333333	181690	182704							
0.333	3	500000	363380								
0.500	2	1000000									

Table 3.19. $\psi_2(\ell)$ for the normal distribution

($\times 10^{-6}$)

1/k	k \ ℓ	1	2	3	4	5	6	7	8	9
0.010	100	10101	3785	2008	1301	968	807	757	832	2173
0.020	50	20408	7900	4456	3215	2903	3689	38607		
0.033	30	34483	13976	8700	7807	14514				
0.040	25	41667	17304	11419	12144					
0.050	20	52632	22711	16609	27324					
0.067	15	71429	33035	30451						
0.100	10	111111	60563	182704						
0.111	9	125000	72676							
0.125	8	142857	90845							
0.143	7	166667	121127							
0.167	6	200000	181690							
0.200	5	250000	363380							
0.250	4	333333								
0.333	3	500000								
0.500	2	1000000								

Table 3.20. $\psi_1(\ell)$ for the rightangled triangular distribution

($\times 10^{-7}$)

$1/k$	k \ l	1	2	3	4	5	6	7	8	9	10
0.010	100	22447	6138	2836	1636	1068	754	562	437	350	287
0.020	50	45351	12531	5854	3415	2255	1611	1217	957	776	646
0.033	30	76628	21482	10190	6042	4058	2594	2274	1827	1516	1292
0.040	25	92593	26152	12506	7480	5073	3731	2906	2364	1990	1723
0.050	20	116959	33416	16185	9818	6764	5063	4024	3349	2894	2584
0.067	15	158730	46268	22928	14281	10145	7876	6539	5741	5306	5168
0.100	10	246914	75186	39305	26181	20291	17722	17437	20092	31833	
0.111	9	277778	85927	45856	31417	25363	23629	26156	40184		
0.125	8	317460	100248	55027	39272	33818	35444	52311			
0.143	7	370370	120298	68784	52362	50727	70887				
0.167	6	444444	150372	91712	78543	101453					
0.200	5	555555	200496	137569	157086						
0.250	4	740741	300744	275137							
0.333	3	1111111	601488								
0.500	2	2222222									

Table 3.21. $\psi_2(\ell)$ for the rightangled triangular distribution

($\times 10^{-7}$)

$1/k$	k \ ℓ	1	2	3	4	5	6	7	8	9
0.010	100	22447	6268	3025	1871	1353	1108	1026	1116	1676
0.020	50	45351	13081	6713	4622	4059	5065	52324		
0.033	30	76628	23144	13107	11224	20297				
0.040	25	92593	28654	17203	17460					
0.050	20	116959	37608	25022	39285					
0.067	15	158730	54703	45874						
0.100	10	246914	100289	275245						
0.111	9	277778	120347							
0.125	8	317460	150434							
0.143	7	370370	200578							
0.167	6	444444	300867							
0.200	5	555556	601734							
0.250	4	740741								
0.333	3	1111111								
0.500	2	2222222								

Table 3.22. $\psi_1(\ell)$ for the exponential distribution

$(\times 10^{-6})$

1/k	k \ l	1	2	3	4	5	6	7	8	9	10
0.010	100	10101	2916	1375	802	526	373	279	217	174	143
0.020	50	20408	5954	2838	1673	1111	798	604	476	387	323
0.033	30	34482	10208	4940	2959	2001	1462	1130	909	756	645
0.040	25	41667	12427	6063	3664	2501	1847	1443	1177	992	860
0.050	20	52632	15878	7846	4809	3334	2507	1998	1667	1443	1291
0.067	15	71429	21985	11115	6995	5001	3900	3248	2858	2646	2581
0.100	10	111111	35726	19054	12824	10003	8774	8660	10002	15876	
0.111	9	125000	40830	22230	15389	12503	11699	12990	20004		
0.125	8	142857	47635	26676	19237	16671	17548	25980			
0.143	7	166667	57162	33345	25649	25007	35096				
0.167	6	200000	71455	44459	38473	50013					
0.200	5	250000	95270	66689	76946						
0.250	4	333333	142905	133378							
0.333	3	500000	285810								
0.500	2	1000000									

Table 3.23. $\psi_2(\ell)$ for the exponential distribution

$(\times 10^{-6})$

1/k	k \ ℓ	1	2	3	4	5	6	7	8	9
0.010	100	10101	2977	1466	916	667	548	509	556	1443
0.020	50	20408	6213	3253	2263	2001	2507	25980		
0.033	30	34483	10993	6351	5496	10003				
0.040	25	41667	13610	8336	8550					
0.050	20	52632	17863	12125	19237					
0.067	15	71429	25983	22230						
0.100	10	111111	47635	133378						
0.111	9	125000	57162							
0.125	8	142857	71453							
0.143	7	166667	95270							
0.167	6	200000	142905							
0.200	5	250000	285810							
0.250	4	333333								
0.333	3	500000								
0.500	2	1000000								

Table 4.1. OSP, V_I^{**}, e_I and c_S for the four distributions

	ℓ	x_0^{**}	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	V_I^{**}	e_I	c_S
	1	−1.000	1.000					0.03889	0.000	0.000
	2	−1.000	0.570	1.000				0.02624	0.325	0.604
1°	3	−1.000	−0.529	0.529	1.000			0.01038	0.733	0.547
	4	−1.000	−0.518	0.423	0.677	1.000		0.00765	0.803	0.651
	5	−1.000	−0.668	−0.405	0.405	0.668	1.000	0.00470	0.879	
	1	−3.000	3.000					1.73269	0.000	0.000
	2	−3.000	−1.515	3.000				1.18517	0.316	0.537
2°	3	−3.000	−1.423	1.423	3.000			0.54928	0.683	0.527
	4	−3.000	−1.394	1.130	1.893	3.000		0.40950	0.764	0.635
	5	−3.000	−1.867	−1.084	1.084	1.867	3.000	0.25997	0.850	
	1	0.000	2.000					0.06914	0.000	0.000
	2	0.000	1.382	2.000				0.02484	0.641	0.000
3°	3	0.000	0.205	1.351	2.000			0.01751	0.747	0.429
	4	0.000	0.226	1.198	1.554	2.000		0.00894	0.871	0.319
	5	0.000	0.235	1.121	1.399	1.653	2.000	0.00605	0.912	
	1	0.000	6.000					4.41936	0.000	0.000
	2	0.000	3.309	6.000				1.35384	0.694	0.000
4°	3	0.000	2.674	4.156	6.000			0.65524	0.852	0.000
	4	0.000	2.375	3.449	4.598	6.000		0.40864	0.908	0.000
	5	0.000	2.201	3.058	3.920	4.870	6.000	0.29656	0.933	

Table 4.2. OSP, V_G^{**} and e_G for the four distributions

	ℓ		x_0^{**}	x_1^{**}	x_2^{**}	x_3^{**}	x_4^{**}	x_5^{**}	x_6^{**}	x_7^{**}	V_G^{**}	e_G
1°	2	1 2 1	−1.000	−0.529	0.529	1.000					0.01038	0.733
	3	1 2 3 2 1	−1.000	−0.668	−0.405	0.405	0.668	1.000			0.00470	0.879
	4	1 2 3 4 3 2 1	−1.000	−0.739	−0.547	−0.337	0.337	0.547	0.739	1.000	0.00267	0.931
2°	2	1 2 1	−3.000	−1.423	1.423	3.000					0.54928	0.683
	3	1 2 3 2 1	−3.000	−1.867	−1.084	1.084	1.867	3.000			0.25997	0.850
	4	1 2 3 4 3 2 1	−3.000	−2.110	−1.493	−0.900	0.900	1.493	2.110	3.000	0.14956	0.914
3°	2	1 2 1	0.000	0.000	1.382	2.000					0.02484	0.641
	3	1 2 3 2 1	0.000	0.000	0.205	1.128	1.490	2.000			0.00999	0.856
	4	1 2 3 4 3 2 1	0.000	0.000	0.000	0.232	1.102	1.378	1.642	2.000	0.00609	0.912
4°	2	1 2 1	0.000	3.309	6.000	6.000					1.35384	0.694
	3	1 2 3 2 1	0.000	2.674	4.156	6.000	6.000	6.000			0.65524	0.852
	4	1 2 3 4 3 2 1	0.000	2.375	3.449	4.598	6.000	6.000	6.000	6.000	0.40864	0.908

Table 5.1. The traditional strata

Stratum No.	Sale of LPG (ton)	Sale of benzine (Kl)
1	1 ~ 49	1 ~ 49
2	50 ~ 99	50 ~ 99
3	100 ~ 199	100 ~ 199
4	200 ~ 249	200 ~ 259
5	250 ~	260 ~ 499
6	———	500 ~

Table 5.2. The sale of LPG and the optimum value β^*

Class \ Case No.	1	2	3	4	5	6
1 ~ 4	240	215	230	205	10	7
5 ~ 9	260	246	240	226	20	19
10 ~ 19	438	426	324	310	114	109
20 ~ 29	237	228	159	146	78	77
30 ~ 49	233	222	165	150	68	66
50 ~ 69	117	114	80	77	37	35
70 ~ 99	114	111	75	73	39	37
100 ~ 119	54	51	37	31	17	15
120 ~ 139	41	39	30	29	11	10
140 ~ 159	50	48	35	31	15	15
160 ~ 179	44	44	29	29	15	13
180 ~ 199	31	30	16	15	15	14
200 ~ 249	101	94	51	48	50	43
250 ~	491	471	268	247	223	209
m'_K	785	765	785	765	740	750
β^*	0.00432	0.00429	0.00490	0.00491	0.00329	0.00332

[Remark] Case 1 : LPG (Type I), 1978
Case 2 : LPG (Type I), 1980
Case 3 : LPG (Type II), 1978
Case 4 : LPG (Type II), 1980
Case 5 : LPG (Type III), 1978
Case 6 : LPG (Type III), 1980

Table 5.3. The sale of benzine and the optimum value β^*

Class \ Case No.	7	8
1 ~ 9	132	106
10 ~ 19	89	85
20 ~ 29	162	159
30 ~ 39	239	234
40 ~ 49	268	263
50 ~ 59	245	239
60 ~ 79	369	360
80 ~ 99	256	253
100 ~ 119	221	218
120 ~ 139	165	163
140 ~ 159	129	126
160 ~ 179	121	117
180 ~ 199	127	124
200 ~ 219	123	120
220 ~ 239	130	129
240 ~ 259	134	134
260 ~ 279	109	108
280 ~ 299	91	91
300 ~ 349	197	196
350 ~ 399	175	171
400 ~ 499	246	243
500 ~	643	633
m_K	1045	1075
β^*	0.00311	0.00309

[Remark] Case 7 : Benzine, 1978
Case 8 : Benzine, 1980

Table 5.4. OSP$\{z_i^*\}$, sample sizes$\{n_i^*\}$, the standard errors S_1, S_2 and some efficiency

Case No.	Stratum / z_i^*, n_i^*	1	2	3	4	5	6		n	S_1	S_2	$\frac{S_2 - S_1}{S_2}$
		$Z_0^* = 0$	Z_1^*	Z_2^*	Z_3^*	Z_4^*	Z_5^*	$Z_6^* = \infty$				
		n_1^*	n_2^*	n_3^*	n_4^*	n_5^*	n_6^*					
1	z_i^*	0	99.8	227.3	404.1	696.3	∞			1.189	1.953	0.391
	n_i^*	475.5	476.0	477.0	480.7	541.8			2451			
2	z_i^*	0	100.4	228.6	406.5	700.4	∞			1.241	1.995	0.392
	n_i^*	453.8	454.2	455.2	458.7	517.0			2339			
3	z_i^*	0	88.0	200.4	356.4	614.1	∞			1.331	1.829	0.273
	n_i^*	337.4	337.7	338.5	341.1	384.4			1739			
4	z_i^*	0	87.7	199.8	355.3	612.2	∞			1.378	1.888	0.270
	n_i^*	313.7	314.0	314.7	317.1	357.4			1617			
5	z_i^*	0	131.0	298.5	530.7	914.4	∞			2.172	5.468	0.603
	n_i^*	138.1	138.3	138.6	139.6	157.4			712			
6	z_i^*	0	129.9	295.9	526.2	906.7	∞			2.236	5.516	0.595
	n_i^*	129.8	129.9	130.2	131.2	147.9			669			
7	z_i^*	0	114.1	252.9	430.2	676.2	1082.6	∞		0.777	1.474	0.473
	n_i^*	709.9	710.3	710.9	712.6	718.0	809.2		4371			
8	z_i^*	0	114.5	253.8	431.7	678.6	1086.3	∞		0.786	1.501	0.476
	n_i^*	693.8	694.2	694.8	696.4	701.8	791.0		4272			

Table 5.5. Changed OSP$\{z_i^{**}\}$, sample sizes$\{n_i^{**}\}$, the standard error S_3 and some efficiencies

Case No.	Stratum / z_i^{**}, n_i^{**}	1	2	3	4	5	6		n	S_3	$\frac{S_3 - S_1}{S_1}$	$\frac{S_2 - S_3}{S_2}$
		$Z_0^{**} = 0$	Z_1^{**}	Z_2^{**}	Z_3^{**}	Z_4^{**}	Z_5^{**}	$Z_6^{**} = \infty$				
		n_1^{**}	n_2^{**}	n_3^{**}	n_4^{**}	n_5^{**}	n_6^{**}					
1	z_i^{**}	0	100	250	400	700	∞			1.197	0.007	0.387
	n_i^{**}	480	480	480	480	531			2451			
2	z_i^{**}	0	100	250	400	700	∞			1.223	0.008	0.387
	n_i^{**}	455	455	455	455	519			2339			
3	z_i^{**}	0	100	200	350	600	∞			1.436	0.079	0.215
	n_i^{**}	340	340	340	340	379			1739			
4	z_i^{**}	0	100	200	350	600	∞			1.484	0.077	0.214
	n_i^{**}	315	315	315	315	357			1617			
5	z_i^{**}	0	140	300	550	900	∞			2.206	0.016	0.597
	n_i^{**}	140	140	140	140	152			712			
6	z_i^{**}	0	140	300	550	900	∞			2.261	0.011	0.590
	n_i^{**}	130	130	130	130	149			669			
7	z_i^{**}	0	120	260	500	700	1100	∞		0.863	0.111	0.415
	n_i^{**}	710	710	710	710	720	811		4371			
8	z_i^{**}	0	120	260	500	700	1100	∞		0.871	0.108	0.420
	n_i^{**}	700	700	700	700	700	772		4272			

Table 6.1. λ_u for the rightangled triangular distribution under NA

ℓ \ δ	0.05	0.10	0.15	0.20
2	0.174	0.246	0.301	0.347
4	0.076	0.108	0.132	0.153
6	0.050	0.070	0.086	0.099
8	0.037	0.052	0.064	0.074
10	0.029	0.041	0.051	0.058

Table 6.2. λ_u for the exponential distribution under PA

ℓ \ δ	0.05	0.10	0.15	0.20
2	0.327	0.462	0.566	0.653
4	0.110	0.156	0.192	0.220
6	0.078	0.110	0.135	0.156
8	0.058	0.082	0.101	0.116
10	0.046	0.066	0.080	0.093

Table 7.1. q for the equilateral triangular distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	−0.2	−0.12	0.095	0.084	0.080	0.077	0.076	0.075	0.074	0.074	0.073
	−0.1	−0.19	0.159	0.143	0.138	0.133	0.131	0.129	0.128	0.127	0.126
	−0.1	−0.01	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.005	0.005
	0.1	−0.01	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.005	0.005
	0.1	0.21	−0.121	−0.102	−0.099	−0.096	−0.096	−0.095	−0.094	−0.094	−0.094
	0.2	0.08	−0.053	−0.045	−0.043	−0.042	−0.041	−0.041	−0.041	−0.040	−0.040
	0.2	0.32	−0.162	−0.139	−0.136	−0.132	−0.132	−0.130	−0.130	−0.129	−0.130
PA	−0.2	−0.12	0.095	0.079	0.073	0.067	0.065	0.063	0.061	0.060	0.059
	−0.1	−0.19	0.159	0.137	0.127	0.119	0.115	0.111	0.109	0.106	0.105
	−0.1	−0.01	0.007	0.006	0.005	0.005	0.005	0.004	0.004	0.004	0.004
	0.1	−0.01	0.007	0.006	0.005	0.005	0.005	0.004	0.004	0.004	0.004
	0.1	0.21	−0.121	−0.092	−0.084	−0.078	−0.076	−0.073	−0.072	−0.071	−0.070
	0.2	0.08	−0.053	−0.041	−0.037	−0.034	−0.033	−0.032	−0.032	−0.031	−0.031
	0.2	0.32	−0.162	−0.124	−0.114	−0.106	−0.103	−0.099	−0.098	−0.096	−0.096

Table 7.2. q for the normal distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	0.1	0.21	−0.191	−0.188	−0.186	−0.187	−0.186	−0.186	−0.186	−0.186	−0.186
	0.2	0.08	−0.077	−0.077	−0.076	−0.076	−0.076	−0.076	−0.076	−0.076	−0.076
	0.2	0.32	−0.275	−0.271	−0.270	−0.270	−0.270	−0.270	−0.271	−0.271	−0.271
EA	0.1	0.21	−0.191	−0.188	−0.187	−0.187	−0.187	−0.186	−0.186	−0.186	−0.186
	0.2	0.08	−0.077	−0.077	−0.076	−0.076	−0.076	−0.076	−0.076	−0.076	−0.076
	0.2	0.32	−0.275	−0.271	−0.270	−0.270	−0.270	−0.270	−0.271	−0.271	−0.271
PA	0.1	0.21	−0.191	−0.184	−0.180	−0.178	−0.177	−0.176	−0.175	−0.174	−0.174
	0.2	0.08	−0.077	−0.076	−0.075	−0.075	−0.074	−0.074	−0.074	−0.074	−0.074
	0.2	0.32	−0.275	−0.262	−0.256	−0.253	−0.250	−0.249	−0.247	−0.247	−0.246

Table 7.3. q for the rightangled triangular distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	-0.2	-0.12	0.106	0.131	0.139	0.142	0.144	0.145	0.146	0.146	0.147
	-0.1	-0.01	0.213	0.202	0.197	0.195	0.193	0.192	0.191	0.190	0.190
	0.1	-0.01	0.026	0.043	0.048	0.050	0.052	0.052	0.053	0.053	0.053
	0.1	0.21	-0.125	-0.113	-0.107	-0.105	-0.104	-0.108	-0.102	-0.102	-0.102
	0.2	0.08	-0.107	-0.113	-0.115	-0.116	-0.116	-0.117	-0.117	-0.117	-0.117
	0.2	0.32	-0.163	-0.156	-0.150	-0.147	-0.145	-0.144	-0.144	-0.143	-0.143
PA	-0.2	-0.12	0.064	0.083	0.089	0.091	0.093	0.094	0.095	0.096	0.096
	-0.1	-0.01	0.214	0.199	0.192	0.187	0.184	0.182	0.181	0.180	0.179
	0.1	-0.01	0.005	0.019	0.024	0.027	0.028	0.029	0.030	0.031	0.031
	0.1	0.21	-0.115	-0.097	-0.088	-0.084	-0.082	-0.080	-0.079	-0.078	-0.078
	0.2	0.08	-0.083	-0.084	-0.083	-0.083	-0.083	-0.083	-0.083	-0.083	-0.083
	0.2	0.32	-0.145	-0.126	-0.119	-0.114	-0.111	-0.109	-0.108	-0.107	-0.106

Table 7.4. q for the exponential distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	-0.2	-0.28	0.259	0.253	0.253	0.255	0.257	0.259	0.261	0.262	0.263
	-0.2	-0.12	0.109	0.135	0.154	0.164	0.169	0.171	0.172	0.172	0.172
	-0.1	-0.19	0.189	0.189	0.188	0.188	0.188	0.188	0.188	0.188	0.188
	-0.1	-0.01	0.013	0.024	0.032	0.036	0.039	0.041	0.042	0.043	0.043
	0.1	0.21	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209
EA	-0.2	-0.28	0.258	0.253	0.254	0.256	0.258	0.260	0.261	0.263	0.264
	-0.2	-0.12	0.110	0.135	0.155	0.165	0.169	0.171	0.172	0.172	0.172
	-0.1	-0.19	0.189	0.189	0.188	0.188	0.188	0.188	0.188	0.188	0.188
	-0.1	-0.01	0.014	0.025	0.033	0.037	0.040	0.041	0.042	0.043	0.043
	0.1	0.21	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209	-0.209
PA	-0.2	-0.28	0.210	0.172	0.158	0.155	0.155	0.156	0.156	0.156	0.156
	-0.2	-0.12	-0.025	-0.017	-0.010	-0.009	-0.008	-0.007	-0.006	-0.005	-0.005
	-0.1	-0.19	0.189	0.187	0.186	0.186	0.185	0.184	0.184	0.183	0.183
	-0.1	-0.01	-0.067	-0.079	-0.079	-0.079	-0.078	-0.077	-0.077	-0.076	-0.076
	0.1	0.21	-0.209	-0.208	-0.208	-0.207	-0.207	-0.207	-0.207	-0.206	-0.206

Table 7.5. Q_1 for the equilateral triangular distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	−0.2	−0.12	0.000	0.010	0.015	0.017	0.019	0.021	0.021	0.022	0.022
	−0.1	−0.19	0.000	0.026	0.039	0.046	0.050	0.054	0.055	0.057	0.058
	−0.1	−0.01	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
	0.1	−0.01	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
	0.1	0.21	0.000	0.025	0.038	0.047	0.052	0.055	0.059	0.060	0.062
	0.2	0.08	0.000	0.004	0.006	0.007	0.008	0.008	0.008	0.009	0.009
	0.2	0.32	0.000	0.054	0.085	0.105	0.118	0.127	0.133	0.138	0.141
PA	−0.2	−0.12	0.000	−0.004	−0.006	−0.009	−0.011	−0.011	−0.013	−0.014	−0.015
	−0.1	−0.19	0.000	0.006	0.007	0.007	0.006	0.006	0.005	0.004	0.004
	−0.1	−0.01	0.000	−0.001	−0.002	−0.002	−0.002	−0.003	−0.003	−0.003	−0.003
	0.1	−0.01	0.000	−0.001	−0.002	−0.002	−0.002	−0.003	−0.003	−0.003	−0.003
	0.1	0.21	0.000	0.054	0.083	0.102	0.114	0.123	0.129	0.133	0.137
	0.2	0.08	0.000	0.014	0.022	0.028	0.031	0.033	0.034	0.036	0.036
	0.2	0.32	0.000	0.099	0.155	0.192	0.216	0.233	0.245	0.254	0.260

Table 7.6. Q_1 for the normal distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	0.1	0.21	0.000	0.022	0.039	0.050	0.060	0.067	0.072	0.076	0.081
	0.2	0.08	0.000	0.003	0.006	0.008	0.009	0.010	0.011	0.012	0.012
	0.2	0.32	0.000	0.049	0.085	0.112	0.133	0.149	0.161	0.171	0.180
EA	0.1	0.21	0.000	0.023	0.039	0.051	0.060	0.068	0.073	0.078	0.082
	0.2	0.08	0.000	0.003	0.006	0.008	0.009	0.010	0.011	0.012	0.012
	0.2	0.32	0.000	0.050	0.088	0.115	0.136	0.152	0.164	0.174	0.183
PA	0.1	0.21	0.000	0.058	0.107	0.145	0.175	0.201	0.222	0.241	0.256
	0.2	0.08	0.000	0.016	0.029	0.039	0.047	0.054	0.059	0.063	0.067
	0.2	0.32	0.000	0.108	0.199	0.271	0.331	0.379	0.421	0.455	0.485

Table 7.7. Q_1 for the rightangled triangular distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	-0.2	-0.12	0.076	0.106	0.124	0.132	0.138	0.141	0.144	0.145	0.147
	-0.1	-0.01	0.207	0.191	0.189	0.189	0.188	0.188	0.188	0.188	0.188
	0.1	-0.01	0.034	0.091	0.107	0.114	0.120	0.122	0.124	0.125	0.126
	0.1	0.21	0.017	0.086	0.123	0.144	0.157	0.161	0.173	0.177	0.180
	0.2	0.08	0.073	0.126	0.135	0.139	0.142	0.143	0.144	0.145	0.145
	0.2	0.32	0.073	0.252	0.356	0.417	0.455	0.480	0.497	0.510	0.520
PA	-0.2	-0.12	-0.022	0.014	0.032	0.041	0.048	0.052	0.055	0.058	0.060
	-0.1	-0.01	0.211	0.170	0.169	0.168	0.167	0.167	0.168	0.168	0.168
	0.1	-0.01	0.045	0.084	0.099	0.106	0.108	0.112	0.115	0.116	0.117
	0.1	0.21	0.073	0.166	0.223	0.256	0.276	0.291	0.302	0.309	0.314
	0.2	0.08	0.175	0.217	0.236	0.245	0.249	0.253	0.256	0.257	0.258
	0.2	0.32	0.176	0.422	0.562	0.647	0.699	0.737	0.762	0.781	0.795

Table 7.8. Q_1 for the exponential distribution

Allocation method	α	β	Number of strata ℓ								
			2	3	4	5	6	7	8	9	10
NA	-0.2	-0.28	0.062	0.066	0.069	0.072	0.074	0.076	0.079	0.080	0.082
	-0.2	-0.12	0.023	0.040	0.058	0.074	0.088	0.100	0.112	0.123	0.134
	-0.1	-0.19	0.025	0.033	0.036	0.038	0.039	0.040	0.041	0.041	0.041
	-0.1	-0.01	0.003	0.010	0.019	0.027	0.035	0.042	0.049	0.055	0.060
	0.1	0.21	0.028	0.041	0.048	0.053	0.056	0.059	0.061	0.063	0.064
EA	-0.2	-0.28	0.061	0.067	0.070	0.072	0.075	0.077	0.079	0.081	0.083
	-0.2	-0.12	0.022	0.038	0.059	0.076	0.090	0.103	0.115	0.127	0.138
	-0.1	-0.19	0.025	0.033	0.036	0.038	0.039	0.040	0.041	0.041	0.042
	-0.1	-0.01	0.003	0.010	0.020	0.029	0.037	0.044	0.050	0.057	0.062
	0.1	0.21	0.029	0.041	0.049	0.053	0.057	0.060	0.062	0.063	0.065
PA	-0.2	-0.28	-0.076	-0.120	-0.134	-0.133	-0.126	-0.117	-0.107	-0.097	-0.087
	-0.2	-0.12	-0.186	-0.172	-0.118	-0.050	0.027	0.107	0.188	0.267	0.343
	-0.1	-0.19	-0.035	-0.050	-0.057	-0.060	-0.064	-0.066	-0.067	-0.069	-0.070
	-0.1	-0.01	-0.102	-0.093	-0.053	-0.008	0.040	0.087	0.131	0.174	0.213
	0.1	0.21	0.120	0.177	0.230	0.262	0.286	0.304	0.319	0.333	0.344

Table 8.1. Q_2 for the equilateral triangular distribution

Allocation method	ℓ	$\lvert\alpha_i\rvert$ 0.01	0.05	0.07	0.10	0.15	0.20
NA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.005	0.010	0.020	0.047	0.087
	4	0.000	− 0.001	− 0.002	− 0.005	− 0.013	− 0.025
		0.000	0.006	0.012	0.026	0.060	0.111
	6	0.000	− 0.002	− 0.003	− 0.007	− 0.017	− 0.033
		0.000	0.007	0.013	0.027	0.064	0.120
	8	0.000	− 0.002	− 0.004	− 0.008	− 0.019	− 0.037
		0.000	0.007	0.014	0.028	0.066	0.124
	10	0.000	− 0.002	− 0.004	− 0.008	− 0.021	− 0.040
		0.000	0.007	0.014	0.029	0.068	0.127
EA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.005	0.010	0.020	0.047	0.087
	4	0.000	− 0.001	− 0.003	− 0.006	− 0.015	− 0.029
		0.000	0.006	0.013	0.026	0.062	0.116
	6	0.000	− 0.002	− 0.004	− 0.008	− 0.020	− 0.039
		0.000	0.007	0.014	0.028	0.067	0.126
	8	0.000	− 0.002	− 0.004	− 0.009	− 0.022	− 0.044
		0.000	0.007	0.014	0.030	0.070	0.131
	10	0.000	− 0.002	− 0.005	− 0.010	− 0.024	− 0.047
		0.000	0.007	0.014	0.030	0.071	0.134
PA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.005	0.010	0.020	0.047	0.087
	4	0.000	− 0.004	− 0.008	− 0.018	− 0.047	− 0.100
		0.000	0.009	0.018	0.038	0.094	0.187
	6	0.000	− 0.007	− 0.014	− 0.031	− 0.079	− 0.166
		0.000	0.012	0.024	0.051	0.126	0.253
	8	0.000	− 0.010	− 0.020	− 0.044	− 0.115	− 0.248
		0.001	0.015	0.030	0.064	0.163	0.335
	10	0.000	− 0.012	− 0.025	− 0.056	− 0.147	− 0.317
		0.001	0.017	0.035	0.076	0.194	0.403

Table 8.2. Q_2 for the normal distribution

Allocation method	ℓ	$\|\alpha_i\|$ 0.01	0.05	0.07	0.10	0.15	0.20
NA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.005	0.010	0.020	0.047	0.087
	4	0.000	− 0.001	− 0.002	− 0.005	− 0.011	− 0.022
		0.000	0.006	0.012	0.025	0.058	0.109
	6	0.000	− 0.001	− 0.003	− 0.006	− 0.016	− 0.030
		0.000	0.007	0.013	0.027	0.063	0.117
	8	0.000	− 0.002	− 0.003	− 0.007	− 0.018	− 0.035
		0.000	0.007	0.013	0.028	0.065	0.122
	10	0.000	− 0.002	− 0.004	− 0.008	− 0.020	− 0.039
		0.000	0.007	0.014	0.029	0.067	0.126
EA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.005	0.010	0.020	0.047	0.087
	4	0.000	− 0.001	− 0.003	− 0.006	− 0.015	− 0.029
		0.000	0.006	0.013	0.027	0.062	0.116
	6	0.000	− 0.002	− 0.004	− 0.008	− 0.020	− 0.040
		0.000	0.007	0.014	0.029	0.067	0.126
	8	0.000	− 0.002	− 0.004	− 0.009	− 0.022	− 0.044
		0.000	0.007	0.014	0.030	0.070	0.131
	10	0.000	− 0.002	− 0.004	− 0.009	− 0.023	− 0.045
		0.000	0.007	0.014	0.030	0.070	0.131
PA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.005	0.010	0.020	0.047	0.087
	4	0.000	− 0.007	− 0.015	− 0.034	− 0.095	− 0.219
		0.000	0.012	0.025	0.054	0.142	0.306
	6	− 0.001	− 0.019	− 0.041	− 0.094	− 0.273	− 0.671
		0.001	0.024	0.051	0.115	0.320	0.758
	8	− 0.001	− 0.039	− 0.084	− 0.200	− 0.640	− 1.991
		0.002	0.044	0.094	0.220	0.687	2.078
	10	− 0.002	− 0.065	− 0.142	− 0.357	− 1.390	− 7.780
		0.002	0.070	0.151	0.377	1.437	7.867

Table 8.3. Q_2 for the rightangled triangular distribution

Allocation method	ℓ	$\lvert\alpha_i\rvert$					
		0.01	0.05	0.07	0.10	0.15	0.20
NA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.006	0.012	0.025	0.059	0.111
	4	0.000	− 0.001	− 0.002	− 0.005	− 0.013	− 0.025
		0.000	0.006	0.012	0.025	0.058	0.108
	6	0.000	− 0.002	− 0.003	− 0.007	− 0.017	− 0.033
		0.000	0.007	0.013	0.027	0.063	0.117
	8	0.000	− 0.002	− 0.004	− 0.008	− 0.019	− 0.038
		0.000	0.007	0.013	0.028	0.065	0.122
	10	0.000	− 0.002	− 0.004	− 0.009	− 0.021	− 0.040
		0.000	0.007	0.014	0.029	0.067	0.125
EA	2	0.000	0.001	0.002	0.002	0.003	0.005
		0.000	0.006	0.012	0.023	0.051	0.092
	4	0.000	− 0.001	− 0.002	− 0.005	− 0.013	− 0.027
		0.000	0.007	0.014	0.028	0.064	0.119
	6	0.000	− 0.001	− 0.003	− 0.007	− 0.019	− 0.037
		0.000	0.007	0.014	0.029	0.068	0.128
	8	0.000	− 0.002	− 0.004	− 0.009	− 0.022	− 0.043
		0.000	0.007	0.015	0.030	0.071	0.132
	10	0.000	− 0.002	− 0.004	− 0.009	− 0.023	− 0.046
		0.000	0.008	0.015	0.031	0.072	0.135
PA	2	0.004	0.019	0.026	0.037	0.056	0.074
		0.004	0.029	0.046	0.081	0.168	0.304
	4	0.003	0.008	0.007	0.000	− 0.030	− 0.094
		0.003	0.031	0.053	0.102	0.233	0.460
	6	0.002	0.000	− 0.007	− 0.030	− 0.104	− 0.251
		0.003	0.034	0.062	0.123	0.299	0.625
	8	0.002	− 0.006	− 0.020	− 0.057	− 0.176	− 0.417
		0.003	0.038	0.071	0.145	0.368	0.808
	10	0.001	− 0.013	− 0.033	− 0.084	− 0.252	− 0.604
		0.003	0.042	0.080	0.170	0.444	1.022

Table 8.4. Q_2 for the exponential distribution

Allocation method	ℓ	$\lvert\alpha_i\rvert$					
		0.01	0.05	0.07	0.10	0.15	0.20
NA	2	0.000	0.000	0.000	0.000	0.000	0.000
		0.000	0.004	0.009	0.018	0.041	0.074
	4	0.000	– 0.001	– 0.002	– 0.005	– 0.012	– 0.023
		0.000	0.006	0.012	0.024	0.056	0.103
	6	0.000	– 0.002	– 0.003	– 0.007	– 0.016	– 0.031
		0.000	0.006	0.013	0.026	0.061	0.113
	8	0.000	– 0.002	– 0.003	– 0.007	– 0.018	– 0.035
		0.000	0.007	0.013	0.027	0.063	0.118
	10	0.000	– 0.002	– 0.004	– 0.008	– 0.019	– 0.037
		0.000	0.007	0.013	0.028	0.065	0.121
EA	2	0.000	0.002	0.003	0.004	0.006	0.008
		0.001	0.007	0.013	0.024	0.053	0.096
	4	0.000	0.000	– 0.001	– 0.004	– 0.012	– 0.026
		0.000	0.007	0.014	0.029	0.066	0.121
	6	0.000	– 0.001	– 0.003	– 0.007	– 0.019	– 0.037
		0.000	0.008	0.015	0.030	0.070	0.130
	8	0.000	– 0.002	– 0.004	– 0.008	– 0.022	– 0.043
		0.000	0.008	0.015	0.031	0.072	0.135
	10	0.000	– 0.002	– 0.004	– 0.009	– 0.024	– 0.047
		0.000	0.008	0.015	0.031	0.073	0.137
PA	2	0.018	0.092	0.129	0.184	0.276	0.368
		0.020	0.156	0.275	0.577	1.952	9.659
	4	0.025	– 0.004	– 0.105	– *	–	–
		0.047	1.087	5.029			
	6	0.010	–	–	–	–	–
		0.110					
	8	– 0.052	–	–	–	–	–
		0.276					
	10	– 0.223	–	–	–	–	–
		0.743					

(*) – means that the condition (8.4) is not satisfied.

Table 8.5. Change of sample sizes $\{n_i\}$

Stratum No.	n_i	[Example 8.1]		[Example 8.2]	
		$n_i + m_i$	mi	$n_i + m_i$	m_i
1	693.8	700	6.2	712	18.2
2	694.2	700	5.8	712	17.8
3	694.8	700	5.2	712	17.2
4	696.4	700	3.6	712	15.6
5	701.8	700	– 1.8	712	10.2
6	791.0	772	– 19.0	712	– 79.0

Table 9.1. Q_3 for the equilateral triangular distribution

Allocation method	ℓ \ $\lvert\eta_i\rvert$	PLUS			MINUS			ALTERNATING		
		0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
NA	2	0.000	0.005	0.020	0.000	0.005	0.020	0.000	0.005	0.020
	4	0.001	0.016	0.064	0.001	0.016	0.064	0.001	0.034	0.137
	6	0.001	0.034	0.135	0.001	0.034	0.135	0.003	0.085	0.339
	8	0.002	0.058	0.233	0.002	0.058	0.233	0.006	0.156	0.623
	10	0.004	0.089	0.357	0.004	0.089	0.357	0.010	0.248	0.989
EA	2	0.000	0.005	0.020	0.000	0.005	0.020	0.000	0.005	0.020
	4	0.001	0.016	0.064	0.001	0.016	0.064	0.001	0.034	0.135
	6	0.001	0.034	0.134	0.001	0.034	0.134	0.003	0.084	0.335
	8	0.002	0.058	0.231	0.002	0.058	0.231	0.006	0.155	0.617
	10	0.004	0.089	0.355	0.004	0.089	0.355	0.010	0.246	0.981
PA	2	0.000	0.005	0.020	0.000	0.005	0.020	0.000	0.005	0.020
	4	0.001	0.015	0.061	0.001	0.015	0.061	0.001	0.031	0.123
	6	0.001	0.031	0.125	0.001	0.031	0.125	0.003	0.075	0.300
	8	0.002	0.053	0.213	0.002	0.053	0.213	0.006	0.138	0.549
	10	0.003	0.081	0.324	0.003	0.081	0.324	0.009	0.218	0.870

Table 9.2. Q_3 for the normal distribution

Allocation method	ℓ \ $\lvert\eta_i\rvert$	PLUS			MINUS			ALTERNATING		
		0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
NA	2	0.000	0.004	0.017	0.000	0.004	0.017	0.000	0.004	0.017
	4	0.001	0.014	0.058	0.001	0.014	0.058	0.001	0.031	0.123
	6	0.001	0.029	0.117	0.001	0.029	0.117	0.003	0.075	0.298
	8	0.002	0.050	0.198	0.002	0.050	0.198	0.005	0.135	0.539
	10	0.003	0.075	0.301	0.003	0.075	0.301	0.008	0.211	0.844
EA	2	0.000	0.004	0.017	0.000	0.004	0.017	0.000	0.004	0.017
	4	0.001	0.014	0.056	0.001	0.014	0.056	0.001	0.030	0.119
	6	0.001	0.029	0.115	0.001	0.029	0.115	0.003	0.073	0.291
	8	0.002	0.049	0.195	0.002	0.049	0.195	0.005	0.132	0.529
	10	0.003	0.074	0.297	0.003	0.074	0.297	0.008	0.208	0.833
PA	2	0.000	0.004	0.017	0.000	0.004	0.017	0.000	0.004	0.017
	4	0.001	0.013	0.051	0.001	0.013	0.051	0.001	0.025	0.099
	6	0.001	0.024	0.097	0.001	0.024	0.097	0.002	0.057	0.229
	8	0.002	0.040	0.158	0.002	0.040	0.158	0.004	0.102	0.407
	10	0.002	0.059	0.236	0.002	0.059	0.236	0.006	0.160	0.638

Table 9.3. Q_3 for the rightangled triangular distribution

Allocation method	$\lvert\eta_i\rvert$ / ϱ	PLUS 0.01	PLUS 0.05	PLUS 0.10	MINUS 0.01	MINUS 0.05	MINUS 0.10	ALTERNATING 0.01	ALTERNATING 0.05	ALTERNATING 0.10
NA	2	0.000	0.005	0.018	0.000	0.005	0.019	0.000	0.005	0.018
	4	0.001	0.019	0.075	0.001	0.019	0.076	0.002	0.038	0.150
	6	0.002	0.042	0.167	0.002	0.042	0.169	0.004	0.098	0.391
	8	0.003	0.074	0.294	0.003	0.074	0.297	0.007	0.086	0.741
	10	0.005	0.115	0.458	0.005	0.116	0.461	0.012	0.301	1.199
EA	2	0.000	0.005	0.019	0.000	0.005	0.019	0.000	0.005	0.019
	4	0.001	0.019	0.075	0.001	0.019	0.076	0.001	0.037	0.149
	6	0.002	0.042	0.166	0.002	0.042	0.168	0.004	0.098	0.389
	8	0.003	0.074	0.293	0.003	0.074	0.295	0.007	0.185	0.736
	10	0.005	0.115	0.456	0.005	0.115	0.459	0.012	0.300	1.193
PA	2	0.000	0.005	0.018	0.000	0.005	0.019	0.000	0.005	0.018
	4	0.001	0.017	0.069	0.001	0.017	0.070	0.001	0.034	0.136
	6	0.002	0.0 6	0.150	0.002	0.038	0.152	0.003	0.088	0.350
	8	0.003	0.066	0.263	0.003	0.067	0.266	0.007	0.166	0.660
	10	0.004	0.103	0.408	0.004	0.103	0.412	0.011	0.268	1.066

Table 9.4. Q_3 for the exponential distribution

Allocation method	$\lvert\eta_i\rvert$ / ϱ	PLUS 0.01	PLUS 0.05	PLUS 0.10	MINUS 0.01	MINUS 0.05	MINUS 0.10	ALTERNATING 0.01	ALTERNATING 0.05	ALTERNATING 0.10
NA	2	0.000	0.004	0.016	0.000	0.004	0.018	0.000	0.004	0.016
	4	0.001	0.020	0.078	0.001	0.020	0.083	0.001	0.036	0.144
	6	0.002	0.045	0.180	0.002	0.046	0.187	0.004	0.099	0.393
	8	0.003	0.081	0.321	0.003	0.082	0.332	0.008	0.191	0.761
	10	0.005	0.127	0.503	0.005	0.128	0.561	0.013	0.313	1.251
EA	2	0.000	0.004	0.015	0.000	0.004	0.017	0.000	0.004	0.015
	4	0.001	0.019	0.076	0.001	0.020	0.081	0.001	0.035	0.140
	6	0.002	0.045	0.177	0.002	0.046	0.184	0.004	0.097	0.385
	8	0.003	0.080	0.317	0.003	0.081	0.324	0.008	0.188	0.750
	10	0.005	0.125	0.497	0.005	0.127	0.511	0.012	0.310	1.236
PA	2	0.000	0.003	0.011	0.000	0.003	0.012	0.000	0.003	0.011
	4	0.000	0.011	0.042	0.000	0.011	0.045	0.001	0.020	0.078
	6	0.001	0.023	0.091	0.001	0.024	0.095	0.002	0.050	0.199
	8	0.002	0.040	0.158	0.002	0.040	0.163	0.004	0.094	0.373
	10	0.002	0.061	0.242	0.002	0.062	0.249	0.006	0.151	0.602

Figure 5.1. The histogram of Case 6 [LPG(Type III), 1980]

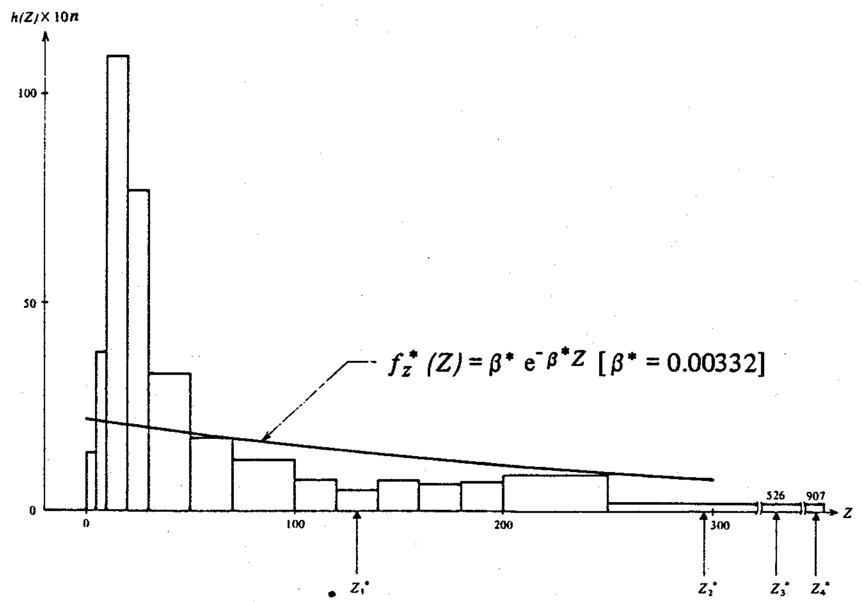

Figure 5.2. The histogram of Case 8 [Benzine, 1980]

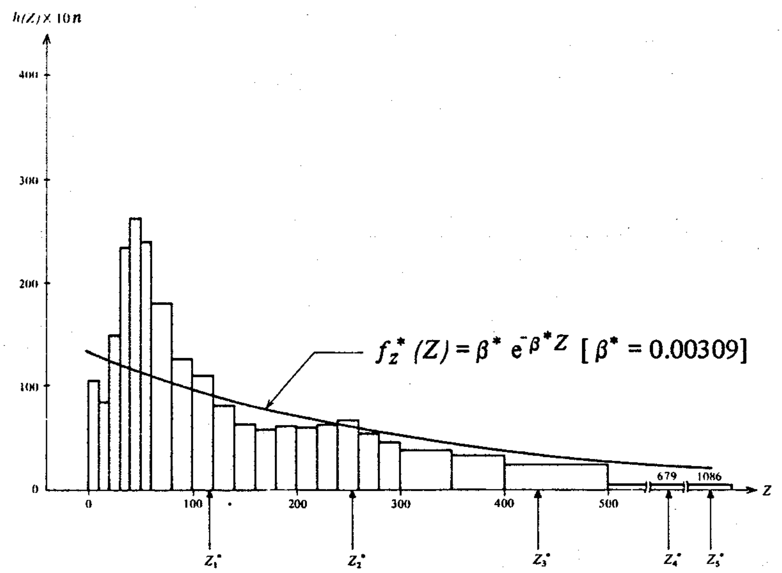

Figure 6.1. Decomposition of robustness