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# ON CATEGORIES OF INDECOMPOSABLE MODULES II

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We have studied the Krull-Remak-Schmidt-Azumaya's theorem from point of view of categories in [4], §3. In this note, we shall study further properties of those categories.

We have defined an additive category $\mathfrak{A}$ induced from a family of completely indecomposable modules $\{M_\alpha\}$, namely whose objects consist of directsums of $M_\alpha$ and studied the quotient category $\mathfrak{A}/\mathfrak{J}$ with respect to the ideal $\mathfrak{J}$ in $\mathfrak{A}$ in [4].

In the first section, we characterize submodules $M_0$ in an object $M$ in $\mathfrak{A}$, which is in $\mathfrak{A}$ and $M_0 \equiv M$ (mod $\mathfrak{J}$), and show that every such $M_0$ coincides with $M$ if and only if $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{A}$, (see [7] for the radical).

In the second section, we consider Conditions II and III defined in [4], which are related with exchange property defined in [1]. We change slightly the definition of exchange property in this note and show that every direct summand of objects $M$ in $\mathfrak{A}$ has the exchange property in $M$ if and only if $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{A}$.

In the final section, we restrict ourselves to a case where $M_\alpha$'s are projective. We shall show, in this case, that objects in $\mathfrak{A}$ are closely related to semi-perfect modules defined in [9]. Especially we show that an object $P = \sum_I \oplus P_\alpha$ in $\mathfrak{A}$ is perfect if and only if $\{P_\alpha\}$ is an elementwise $T$-nilpotent system defined in [4] and $P$ is semi-perfect if and only if $\{P_\alpha\}$ is an elementwise semi-$T$-nilpotent system.

Let $R$ be a ring with unit element and all modules in this note be unitary right $R$-modules. An $R$-module $M$ is called *completely indecomposable* if $\mathrm{End}_R(M) = S_M$ is a (non-commutative) local ring. We assume here that indecomposable modules mean completely indecomposable.

## 1. Dense submodules

Let $M_0$ be an R-module and assume $M_0 = \sum_I \oplus M_\alpha$, where $M_\alpha$'s are indecomposable modules. We have defined an additive category $\mathfrak{A}$ in [4] from the above decomposition as follows: The objects of $\mathfrak{A}$ consist of some directsums of $M_\alpha$'s and the morphisms of $\mathfrak{A}$ consist of all $R$-homomorphisms. We denote those morphisms by $[M, N']_R$ and we call $\mathfrak{A}$ is induced from a family $\{M_\alpha\}$. Furthermore, we have defined an ideal $\mathfrak{J}$ in $\mathfrak{A}$ in [4] as follows: Let $N=\sum\oplus M'_\beta$, $N'=\sum\oplus M'_{\beta'}$ be in $\mathfrak{A}$. $\mathfrak{J}$ consists of all morphisms $f$ in $[N, N']_R$ such that $p_{\beta'}fi_\beta\in[M_\beta, M'_{\beta'}]_R$ is not isomorphic for any $\beta$ and $\beta'$, where $i_\beta$, $p_{\beta'}$ are injection and projection, respectively and $N$, $N'$ run through all objects in $\mathfrak{A}$. We know from Theorem 7 in [4], that $\mathfrak{J}$ defines an ideal in $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{J}$ is a $C_3$-completely reducible abelian category, (see [2] for the definition of ideals).

When we consider object $N$ and morphism $f$ in $\mathfrak{A}/\mathfrak{J}$, we denote them by $\bar{N}$ and $\bar{f}$. Furthermore, if $N$ is in $\mathfrak{A}$ and $N=N_1\oplus N_2$ as $R$-modules, then $\bar{N}_i$ means $\operatorname{Im}\bar{e}_i$, where $e_i$ is a projection of $N$ to $N_i$. Let $S$ be a ring. By $J(S)$ we denote the Jacobson radical of $S$ and by $\mathfrak{J}_M$ we denote $\mathfrak{J}\cap\operatorname{End}_R(M)$ for an $R$-module $M$.

Let $M\supseteq N$ be objects in $\mathfrak{A}$ and $i$ the inclusion of $N$ to $M$. If $i$ is isomorphic modulo $\mathfrak{J}$, i.e. $\bar{M}=\bar{N}$, then we call $N$ a *dense submodule* in $M$.

**Proposition 1.** *Every dense submodule of $M$ is $R$-isomorphic to $M$.*

Proof. Since $\bar{M}=\bar{N}$, $M$ and $N$ have isomorphic direct summands by [4], Corollary 1 to Theorem 7.

Let $\{M_\alpha\}$ be a family of completely indecomposable modules and $\{f_{\alpha i}\}_{i=1}^\infty$ any sequence of non isomorphic $R$-homomorphisms of $M_{\alpha i}$ to $M_{\alpha i+1}$ in $\{M_\alpha\}$ ($M_{\alpha i+1}$ may be equal to $M_{\alpha i}$). If there exists $n$, which depends on the above sequence and for any element $m$ in $M_{\alpha_1}$, such that $f_{\alpha n}f_{\alpha n-1}\cdots f_{\alpha_1}(m)=0$, then we call $\{M_\alpha\}$ a (*elementwise*) *T-nilpotent system* (cf. [4]). If the above condition is satistied for any sequence $\{f_{\alpha i}\}$ such that $M_{\alpha i}\neq M_{\alpha j}$ if $i\neq i$, then we call $\{M_\alpha\}$ a (*elementwise*) *semi-T-nilpotent system*. In general, a semi-$T$-nilpotent system is not $T$-nilpotent.

Let $M=\sum_I\oplus M_\alpha$ and $J$ a subset of $I$, then we denote a submodule $\sum_J\oplus M_{\alpha'}$ by $M_J$.

**Proposition 2.** *Let $M$ and $P$ be in $\mathfrak{A}$ and $\bar{M}\supseteq\bar{P}$. Then there exists a submodule $P_0$ in $M$ satisfying the following conditions.*

1 *$P_0$ is an object in $\mathfrak{A}$; $P_0=\sum_I\oplus M_\alpha$.*

2 *$P_{0J}$ is a direct summand of $M$ for any finite subset $J$ of $I$, (if $\{M_{\alpha'}\}_{J'}$ is $T$-nilpotent system, $J'$ need not be finite).*

3 *$\bar{P}_0=\bar{P}$.*

*Furthermore, if $\bar{P}=\operatorname{Im}\bar{e}$ and $e$ is an idempotent in $S_M=\operatorname{End}_R(M)$, then we can choose $P_0$ in* $\operatorname{Im} e$.

Proof. Let $P=\sum_I\oplus M_\lambda$. Since $\bar{M}\supseteq\bar{P}$ and $\mathfrak{A}/\mathfrak{J}$ is completely reducible, there exist $i\in[P, M]_R$ and $p\in[M, P]_R$ such that $pi\equiv 1_P \pmod{\mathfrak{J}}$. Let $J$ be a subset of $I$ and $i_J$, $p_J$ be inclusion and projection, respectively. Since $p_Jpii_J$ is isomorphic, $p_J p i i_J$ is $R$-isomorphic by [4], Lemma 8 and Theorem 8 if $J$ is finite or $\{M_\lambda\}_J$ is a $T$-nilpotent system. Hence, $\alpha_J = i i_J$ splits, namely $\mathrm{Im}\,\alpha_J$ is a direct summand of $M$. Therefore, $i$ is $R$-monomorphic. We put $P_0 = \mathrm{Im}\, i$ as $R$-module, then $P_0$ satisfies $1 \sim 3$. If $P = \mathrm{Im}\, e$, we have a relation $pei \equiv pi \equiv 1_P$ (mod $\mathfrak{J}$). Hence, if we take $\alpha_J = e i i_J$, we know that $\mathrm{Im}\, ei = P_0 \subseteq eM$.

The following theorem gives a special answer for Condition III in [4].

**Theorem 1.** *Let $M$ be in $\mathfrak{A}$ and $M = \sum_K \oplus N_\gamma$ as $R$-module. Then each $N_\gamma$ contains a submodule $P_\gamma$ such that $P_\gamma$ is in $\mathfrak{A}$ and $\sum \oplus P_\gamma$ is a dense submodule of $M$.*

Proof. Let $\pi_\gamma$ be a projection of $M$ to $N_\gamma$. Then from Proposition 2 we have $P_\gamma$ in $\mathfrak{A}$ such that $\bar{P}_\gamma = \mathrm{Im}\,\bar{\pi}_\gamma = \bar{N}_\gamma$. We shall show $\bar{M} = \sum_K \oplus \bar{P}_\gamma$. Let $i_\gamma$, $i_\gamma'$ and $i_\gamma''$ be inculsions of $P_\gamma$ to $M$, of $P_\gamma$ to $N_\gamma$ and of $N_\gamma$ to $M$ such that $i_\gamma = i_\gamma'' i_\gamma'$, respectively. Since $\bar{P}_\gamma = \mathrm{Im}\,\bar{\pi}_\gamma$, there exists an $R$-homomorphism $p_\gamma$ of $M$ to $P_\gamma$ such that $i_\gamma p_\gamma = \pi_\gamma$ (mod $\mathfrak{J}$). Let $\{\bar{f}_\gamma\}$ be an element in $\prod_\gamma [P_\gamma, N]_{\mathfrak{A}/\mathfrak{J}}$, where $N$ is an object in $\mathfrak{A}$ and $f_\gamma \in [P_\gamma, N]_R$. We put $f_\gamma'' = f_\gamma p_\gamma i_\gamma'' \in [N_\gamma, N]_R$ and $f = \prod_K f_\gamma' \in [M, N]_R$. Then we have $f i_\gamma = f i_\gamma'' i_\gamma' = f_\gamma'' i_\gamma' = f_\gamma p_\gamma i_\gamma$. Hence, $f i_\gamma \equiv f_\gamma$ (mod $\mathfrak{J}$). We shall show that $\bar{f}$ does not depend on a choice of representative $f_\gamma$. It is sufficient to show that if $f \mid P_\gamma = f i_\gamma$ is in $\mathfrak{J}$ for all $\gamma$, then $f$ is in $\mathfrak{J}$ for any $f$ in $[M, N]_R$. Let $N = \sum_L \oplus M_\delta$; $M_\delta$'s are indecomposable. If $f$ is not in $\mathfrak{J}$, there exists an idecomposable direct summand $T$ of $M$ such that $p_\delta' f i_T$ is isomorphic, where $i_T\colon T \to M$, $p_\delta'\colon N \to N_\delta$ are inclusion and projection, respectively. Since $\{\pi_\gamma\}_K$ is summable, $1_M = \sum_K \pi_\gamma$ and $f = \sum_K f\pi_\gamma$. Furthermore, since $\{p_\gamma' f \pi_\gamma i_T\}_K$ is summable and $p_\delta' f i_T = \sum p_\delta' f \pi_\gamma i_T$, there exists a finite subset $K'$ in $K$ such that $\sum_{K-K'} p_\delta' f \pi_\gamma i_T$ is not isomorphic, and $\sum_{K'} p_\delta' f \pi_\gamma i_T$ is isomorphic. Therefore, there exists $\gamma$ in $K'$ such that $p_\delta' f \pi_\gamma i_T$ is isomorphic. On the other hand $p_\delta' f \pi_\gamma i_T \equiv p_\delta' f i_\gamma \pi_\gamma i_T \equiv 0$ (mod $\mathfrak{J}$), which is a contradiction. Conversely, we take a morphism $\bar{f} \in [M, N]_{\mathfrak{A}/\mathfrak{J}}$ and $f \in [M, N]_R$. Put $f_\gamma = f i_\gamma$, then $\bar{f}_\gamma$ does not depend on a choice of $f$ by Proposition 2 and [4], Lemma 5. Thus, we have shown that $[M, N]_{\mathfrak{A}/\mathfrak{J}} = \prod [P_\gamma, N]_{\mathfrak{A}/\mathfrak{J}}$.

We call such $P_\gamma$ a *dense submodule of* $N_\gamma$.

**Theorem 2.** *Let $M$ be in $\mathfrak{A}$ induced from a family $\{M_\alpha\}$ of completely indecomposable modules $M_\alpha$, and $N = \sum_I \oplus M_\rho'$ in $\mathfrak{A}$ be a submodule of $M$. Then the following statements are equivalent.*

1 *$N$ is a dense submodule of $M$.*

2 *There exists a finite subset $J$ of $I$, for any direct summand $P$ of $M$, such that either $P \cap N_J \neq 0$ or $P \oplus N_J$ is not a direct summand of $M$.*

3 *N contains Im* $(1-f)$ *for some element* $f$ *in* $\mathfrak{J}_M$ *and* $\bar{i}_N$ *is monomorphic.*

*In those cases* $N_{J'}$ *is a direct summand of* $M$ *for any finite subset* $J'$ *of* $I$. *Furthermore, Im* $(1-f)$ *is always a dense submodule of* $M$.

Proof. $1\to 2$ Since every direct summand of $M$ contains an indecomposable module by [4], Corollary 1 to Theorem 7, we may assume so is $P$. Since $\bar{P}$ is a minimal object, $\bar{P}$ is small (cf. [5], Theorem 1.4). Hence, $\bar{P}\subseteq \sum_J \oplus \bar{M}_{\alpha_i}$ for some finite set $J$. We assume that $P\cap N_J=0$ and $P\oplus N_J$ is a direct summand of $M$. Let $e$, $f$ and $E$ be projections of $M$ to $P$, $N_J$ and $P\oplus N_J$, respectively. We may assume $E=e+f$ and $ef=fe=0$. We denote the inclusion of submodules to $M$ by $i$. Since $\bar{P}\subseteq \bar{N}_J$, there exists $\alpha$ in $[P, N_J]_R$ such that $i_J\alpha\equiv i_P \pmod{\mathfrak{J}}$. Since $\bar{1}_P=\bar{e}\bar{i}_P$ and $\bar{f}\bar{e}=0$, $\bar{1}_P=\bar{e}\bar{i}_J\bar{\alpha}=\bar{0}$. Hence, $P=0$, which is a contradiction.

$2\to 1$ If $\bar{M}\neq\bar{N}$, there exists an indecomposable module $P$ such that $\bar{P}\oplus\bar{N}_J$ is a direct summand of $\bar{M}$ for any finite subset $J$ of $I$. Since $\bar{P}\oplus\bar{N}_J$ is a directsum of finite many of minimal objects, there exists a direct summand $P_0$ of $M$ such that $P_0\cap N_J=0$ and $P_0\oplus N_J$ is a direct summand of $M$ (see the proof of Proposition 2).

$1\to 3$ Let $i$ be an inclusion of $N$ to $M$. Since $\bar{i}$ is isomorphic, there exists $j$ in $[M, N]_R$ such that $\bar{i}\bar{j}=\bar{1}_M$. Put $f=1-ij$, then $f\in\mathfrak{J}_M$ and $1-f=ij$. Since $i$ is monomorphic, $N\supseteq \mathrm{Im}\,(1-f)$.

$3\to 1$ First we shall show that $N'=\mathrm{Im}\,(1-f)$ is a dense submodule of $M$. We know from the proof of Proposition 10 in [4] that $1-f$ is monomorphic and hence, $N'$ is in $\mathfrak{A}$. Let $i$ be an inclusion of $N'$ to $M$ and $1-f=i\,(1-f)'$; $(1-f)'\in[M, N']_R$. Since $1\equiv 1-f\equiv i(1-f)'$ and $(1-f)'$ is isomorphic, so is $\bar{i}$. Therefore, $N'$ is a dense submodule. Hence, $M\supseteq N\supseteq \mathrm{Im}\,(1-f)$ implies that $\bar{i}_N$ is epimorphic. In order to get the last part we put $\bar{M}=\bar{N}=\bar{P}$ in Proposition 2, then $N_{J'}$, is a direct summand of $M$.

**Corollary 1.** *Let* $M$ *be an object in* $\mathfrak{A}$ *and* $P$ *a dense submodule of* $M$. *If for a direct summand* $N=\sum_J\oplus M_\alpha'$ *of* $M$ *in* $\mathfrak{A}$, $J$ *is finite or* $\{M_\alpha'\}$ *is a T-nilpotent system, there exists an automorphism* $\sigma$ *of* $M$ *such that* $\sigma(N)$ *is a direct summand of* $P$.

Proof. $P$ contains a submodule $N_1'$ which is isomorphic to $N_1$ and is a direct summand of $M$ by Proposition 2; say $M=N_1'\oplus N_2'=N_1\oplus N_2$. Since $N_2'\approx N_2$, we obtain the corollary.

**Corollary 2.** *Let* $\{M_\alpha\}$ *be a family of completely indecomposable modules and* $\mathfrak{A}$ *the induced additive category from* $\{M_\alpha\}$. *Then the following conditions are equivalent.*

1 $\{M_\alpha\}$ *is an elementwise T-nilpotent system.*

2 $\mathfrak{J}$ *is the Jacobson radical of* $\mathfrak{A}$.

3 *Every dense submodule of any* $M$ *in* $\mathfrak{A}$ *coincides with* $M$.

Proof. $1 \leftrightarrow 2$ is obtained in [4], Theorem 8.

$1 \rightarrow 3$ Let $N$ be a dense submodule of $M$. We know from 1 and Proposition 2 that $N$ is a direct summand of $M$. Hence, $N=M$ by Theorem 2.

$3 \rightarrow 2$ Let $f$ be in $\mathfrak{J}_M$ and $N=\mathrm{Im}\,(1-f)$. Since $N$ is a dense submodule by Theorem 2, $N=M$. Therefore, $1-f$ is isomorphic, which implies $\mathfrak{J}_M$ is equal to $J(S_M)$.

REMARK. Let $M=\sum_{i=1}^{\infty} \oplus M_i$ as in Theorem 2. We assume that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of monomorphisms but not epimorphisms $f_i$ of $M_i$ to $M_{i+1}$. Then for any finite set $J$ of $I$ there exists a dense submodule $N$ in $M$ such that $N \cap M_J=(0)$. Because, we make use of matrix representation of $[M, M]_R$ and by $\{e_{im}\}$ we denote a system of matrix unites. Put $f=\sum_i \sum_{k=1}^{J} f_{i+k-1} f_{i+k-2} \cdots f_i e_{i+k\,i}$, then $f$ is in $\mathfrak{J}$. Hence, $P=\mathrm{Im}\,(1-f)$ is a dense submodule and $P \cap M_J=(0)$.

If we use the same argument for any set $I$, we can give an example in which for some subset $J$ with $|J| \leqslant |I|$ there exists a dense submodule $P$ in $M$ such that $P \cap M_J=(0)$. Furthermore, we can give an example in which there exists a dense submodule $P$ in $M=\sum_{i=1}^{\infty} \oplus M_i$ such that $P \cap M_i \neq (0)$ for all $i$ and $P \neq M$.

In the above corollary, we have a situation $\mathfrak{J}_M=J(S_M)$. In this case we obtain a further result.

**Lemma 1.** *Let $M$ be in $\mathfrak{A}$ and $\mathfrak{J}_M=J(S_M)$. Then for every direct summand $N$ of $M$ we have $\mathfrak{J}_N=J(S_N)$.*

Proof. Since $\mathfrak{J}_M=J(S_M)$, $N$ is in $\mathfrak{A}$ by [4], Corollary 2 to Theorem 7. Put $N=eM$ for some idempotent $e$ in $S_M$. Then it is clear that $e\mathfrak{J}_M e=\mathfrak{J}_N$, since $\mathfrak{J}$ does not depend on decompositions of $M$ by [4], Lemma 5. Furthermore, $J(S_N)=eJ(S_M)e$. Hence, $J(S_N)=\mathfrak{J}_M$.

**Theorem 3.** *Let $P$ be in $\mathfrak{A}$ and $\mathfrak{J}_P=J(S_P)$. Then every idempotent $a$ in $S/J(S)$ is lifted to $S$.*

Proof. Let $a$ be idempotent modulo $\mathfrak{J}_P$. Then there exist a module $P_0$ in $\mathfrak{A}$ and $a' \in [P, P_0]_R$, $b' \in [P_0, P]_R$ such that $b'a' \equiv a$ and $a'b' \equiv 1_P \pmod{\mathfrak{J}}$. Since $P_0$ is isomorphic to a direct summand of $P$ by [4], Theorem 7, $\mathfrak{J}_P=J(S_P)$ by Lemma 1. Hence $b'$ is $R$-monomorphic and $\varepsilon'=a'b'$ is $R$-isomorphic on $P_0$. We may regard $P_0$ as a direct summand of $P$ via $b'$; $P=\mathrm{Im}\, b' \oplus Q$. We put $\varepsilon=b'\varepsilon'^{-1}b'^{-1}+1_Q$, then $\varepsilon \equiv 1 \pmod{\mathfrak{J}}$. Put $e=\varepsilon b'a'$, then $e\,|\,P_0=1_P$ and $\mathrm{Im}\, e=P_0$. Hence, $e$ is idempotent in $S_P$ and $e=\varepsilon b'a' \equiv b'a' \equiv a \pmod{\mathfrak{J}}$.

**Corollary.** *Let $R$ be a (non-commutative) local ring such that $J(R)$ is $T$-nil-*

*potent. Let $S$ be the ring of column finite matrices over $R$ with any degree. Then every idempotent in $S/J(S)$ is lifted to $S$.*

Proof. Put $M=\sum_{I\ni\alpha}\oplus R_\alpha$; $R_\alpha\approx R$. Then $S=S_M$ and $J(S_M)=\mathfrak{J}_M$ by [4], Lemma 10.

## 2 Exchange property

We shall recall Condition II in [4]. Let $M=\sum_I\oplus M_\alpha=\sum_I\oplus N_\beta$ be decompositions of $M$ with indecomposable modules $M_\alpha$, $N_\beta$, Condition II in [4] says that for any subset $J$ of $I$, there exists a subset $J'$ of $I$ such that $M=\sum_{J'}M_\alpha \oplus\sum_J\oplus N_\beta$. However, this is a special case of exchange property defined in [1]. Furthermore, this property induces Condition III in [4], namely every direct summand of $M$ is in $\mathfrak{A}$. Therefore, we shall define a weaker exchange property than one in [1]. Let $M$ be as above (in $\mathfrak{A}$), and $M=\sum_{I\ni i}\oplus P_i$ be any direct decomposition as $R$-modules. We call a direct summand $N$ of $M$ has the $|I|$-exchange property in $M$ if $M=N\oplus\sum_{I\ni i}\oplus P_i'$ and $P_i'\subset P_i$ for any decomposition $M=\sum_{I\ni i}\oplus P_i$ with $|I|$-factors. If $N$ has the *$|I|$-exchange property in $M$* for any cardinal $|I|$, we call $N$ has the *exchange property in $M$*. It is clear that if $N$ has the exchange property in $M$, then $N$ is an object of $\mathfrak{A}$. P. Crawley and B. Jónsson have shown in [1], Theorem 7.1 (and [10], Theorem 1) that if $M$ is countably generated for all $\alpha$ in $I$, then Condition III is satisfied.

In the following we always assume that $M=\sum_I\oplus M_\alpha=N_1\oplus N_2$ with indecomposable modules $M_\alpha$.

**Lemma 2.** *If either $N_1$ is finitely generated or a dense submodule of $N_1$ is a $T$-nilpotent system, then $N_i$ is in $\mathfrak{A}$, ($i=1, 2$).*

Proof. If $N_1$ is finitely generated, then $N_1$ is a direct summand of $M_J$ for some finite subset $J$ of $I$. Hence, $N_1\approx M_{J'}$ for some $J'\subset J$ by Krull-Remak-Schmidt's theorem. Therefore, $M=N_1\oplus\sum\oplus M_{\varphi(\alpha)}$ by [4], Corollary 1 to Theorem 7 (Azumaya's theorem), and hence $N_2\approx\sum\oplus M_{\varphi(\nu)}$ is in $\mathfrak{A}$. Next, we assume that a dense submodule $N_0$ of $N_1$ is a $T$-nilpotent system. Then $N_0=N_1$ by Proposition 2. Hence $N_1$ is in $\mathfrak{A}$. Since $\mathfrak{A}/\mathfrak{J}$ is completely reducible, $\bar{M}=\bar{N}_1\oplus\sum_{K\ni\beta}\bar{M}_\beta=\sum_{I-K}\bar{M}_\alpha\oplus\sum_K\bar{M}_\beta$ for some $K$ in $I$. Let $p$ be a projection of $M$ to $\sum_{I-K}M_\alpha$. Since $\bar{p}\,|\,\bar{N}_1$ is isomorphic and $\{M_\alpha\}_{I-K}$ is a $T$-nilpotent system, $p\,|\,N_1$ is an $R$-isomorphism of $N_1$ to $\sum_{I-K}\oplus M_\alpha$. Therefore, $M=N_1\oplus\operatorname{Ker}p=N_1\oplus\sum_K M_\beta$ $=N_1\oplus N_2$. Hence, $N_2\approx\sum\oplus M_\beta$.

The following lemma is a special case of [1], Lemma 3.10 and [10], Proposition 1, however we shall give a proof from point of view of our categories.

**Lemma 3.** *Let $M$ and $N_i$ be as above. If $N_1=\sum_{i=1}^{n}\oplus M_i'$ and $M_i'\approx M_{\alpha i}$ for all $i$, then $N_1$ has the exchange property in $M$.*

Proof. We assume that $M=N_1\oplus N_2=\sum_{I'}\oplus Q_\alpha$ as $R$-modules. Then $\bar{M}=\bar{N}_1\oplus\bar{N}_2=\sum_{I'}\oplus\bar{P}_\alpha$, where $P_\alpha=\sum_{J_\alpha}\oplus P_{\alpha j}$ is a dense submodule of $Q_\alpha$ and $P_{\alpha j}$'s are indecomposable. Since $\bar{N}_1=\sum_{i=1}^{n}\oplus\bar{M}_i'$ is a small object in $\mathfrak{A}/\mathfrak{J}$, there exist a finite subset $I''$ of $I$ and a finite subset $J_i'$ of $J_i$ for $i\in I''$ such that $\bar{N}_1\subseteq\sum_{I''\ni i}\sum_{J_i\ni j}\oplus\bar{P}_{ij}$. We know from Proposition 2, 2) that $\sum_i\sum_j\oplus P_{ij}=P$ is a direct summand of $M$. Since $I''$ and $J_i'$ are finite, $\mathfrak{J}_P=J(S_P)$ by [4], Lemma 8. Hence, $P$ contains a direct summand $N_1'$ such that $\bar{N}_1'=\bar{N}_1$, $(P=N_1'\oplus P')$. $M=N_1'\oplus P'\oplus\sum_{I''}\oplus Q_i'\oplus\sum_{I-I''}\oplus Q_\alpha$, where $Q_i=Q_i'\oplus\sum_{J_i-J_i'}\oplus P_{ij}$. Let $p_{N'_1}$ be a projection of $M$ to $N_1'$ in this decomposition. Since $\bar{N}_1=\bar{N}_1'$ and $\bar{N}_1\cap(\bar{P}'\oplus\sum_{I''}\oplus\bar{Q}_i'\oplus\sum_{I-I''}\oplus\bar{Q}_\alpha)=\bar{0}$, (see the proof of Theorem 1), $\bar{p}_{N'_1}\,|\,\bar{N}_1$ is isomorphic. Therefore, $p_{N'_1}|N_1$ is isomorphic as an $R$-module. Thus, we obtain that $M=N_1\oplus P'\oplus\sum_{I''}\oplus Q_1'\oplus\sum_{I-I''}\oplus Q_\alpha$.

**Theorem 4.** *Let $M=\sum_I\oplus M_\alpha$ with $M_\alpha$ completely indecomposable, and $N_1=\sum_{I'}\oplus M_\beta'$ be a direct summand of $M$; $M=N_1\oplus N_2$. If $I'$ is finite or $\{M_\beta'\}$ is a $T$-nilpotent system, then $N_i$ has the exchange property in $M$ for $i=1, 2$.*

Proof. We know from the assumption and [4], Theorem 8 that $\mathfrak{J}_{N_1}=J(S_{N_1})$. Let $M=N_1\oplus N_2=\sum_J\oplus Q_\alpha$. Then $\bar{M}=\bar{N}_1\oplus\bar{N}_2=\sum\oplus\bar{P}_\alpha$, where $P_\alpha$ is a dense submodule of $Q_\alpha$. We put $P_\alpha=\sum_{J_\alpha\ni i}\oplus P_{\alpha i}$ $(\in\mathfrak{A})$. Since $\mathfrak{A}/\mathfrak{J}$ is completely reducible, $\bar{M}=\bar{N}_2\oplus\sum_J\sum_{J_\alpha'}\oplus\bar{P}_{\alpha i}$, where $J_\alpha'$ is a subset of $J_\alpha$. The fact $\bar{N}_1\approx\bar{P}=\sum_J\sum_{J_\alpha'}\oplus\bar{P}_{\alpha i}$ implies $\mathfrak{J}_P=J(S_P)$. Let $p_{N_1}$ be a projection of $M$ to $N_1$ with $\operatorname{Ker}p_{N_1}=N_2$. Then $\bar{p}_{N_1}|\bar{P}$ is isomorphic, and hence $p_{N_1}|P$ is isomorphic as an $R$-module. Therefore, $M=P\oplus N_2$ and $\sum_{T'_\alpha}\oplus P_{\alpha i}\subseteq Q_\alpha$. We have shown that $N_2$ has the exchange property. If $I'$ is finite, then $N_2$ has the exchange property from Lemma 3. Thus, we may assume that $\{M'_\beta\}$ is a $T$-nilpotent system. Noting that $N_2$ is an object of $\mathfrak{A}$ by Lemma 2, first we assume $N_1=\sum_K\oplus T_\alpha$, $N_2=\sum_{K'}\oplus T'_\beta$ and $T_\alpha\not\approx T'_\beta$ for any $\alpha$, $\beta$, where $T_\alpha$ and $T'_\beta$ are indecomposable. We make use of the same notation as above. Then $\bar{M}=\bar{N}_2\oplus\sum_J\oplus\bar{P}'_\alpha$ and $P'_\alpha\oplus P''_\alpha=P_\alpha$. Since $\sum\oplus P'_\alpha\approx N_1$, $\sum\oplus P'_\alpha$ is a direct summand of $M$ by Proposition 2, say $M=\sum(P_\alpha'\oplus P'''_\alpha)$ and $Q_\alpha=P_\alpha'\oplus P_\alpha'''$. Then $\sum_J\oplus P'''_\alpha$ is an object in $\mathfrak{A}$ by Lemma 2. Let $p$ be a projection of $M$ to $\sum_J\oplus P'_\alpha$ with $\operatorname{Ker} p=\sum_J\oplus P'''_\alpha$. Then $\bar{p}\,|\,\bar{N}_2$ is isomorphic, since $\bar{N}_1\cap\sum\oplus\bar{P}'''_\alpha=\bar{0}$ and $\bar{p}(\bar{N}_2)=\bar{0}$ by the assumption. Hence, $p\,|\,N_1$ is isomorphic as an $R$-module, which implies $M=N_1\oplus\operatorname{Ker} p=N_1\oplus\sum P'''_\alpha$. Hence, $N_1$ has the exchange property in $M$. In general case, we choose all direct components $T'_{\beta'}$ in $N_2$, which is isomorphic to some $T_\alpha$ in $N_1$ and put $N_2'=\sum\oplus T'_{\beta'}$; $N_2=N_2'\oplus N_2''$. Then, $N_1'=N_1\oplus N_2'$ satisfies the assumption in the first case. Therefore, $M=N_1'\oplus\sum_J\oplus P'''_\alpha$ and $Q_\alpha=P_\alpha'\oplus P'''_\alpha$. Then $M=N_2'\oplus N_1\oplus\sum_J\oplus P'''_\alpha$. Since $N_2'$ satisfies the assumption in the theorem, $N_1\oplus\sum\oplus P'''_\alpha$ has the exchange property from the beginning case. Therefore, $M=N_1\oplus\sum_J\oplus P'''_\alpha\oplus\sum_J\oplus P^{\mathrm{iv}}_\alpha$, and $Q_\alpha\supseteq P^{\mathrm{iv}}_\alpha$. Thus, we have proved that $N_1$ has the exchange property in $M$.

**Corollary.** *Let $\mathfrak{A}$ be as above. Then the following statements are equivalent.*

1 *Every direct summand of object $M$ in $\mathfrak{A}$ has the $\aleph_0$-exchange property in $M$.*

2 *Every direct summand of object $M$ in $\mathfrak{A}$ has the exchange property in $M$.*

3 *$\{M_\alpha\}$ is an elementwise T-nilpotent system.*

Proof. $1\rightarrow3$ Let $M=\sum_I\oplus M_\alpha=\sum_{I'}\oplus M'_{\beta'}\oplus\sum_{I''}\oplus M'_{\gamma'}$ be a direct decompositions with $|I'|\leqslant\aleph_0$. Since every direct summand of $\sum_{I'}\oplus M'_{\beta'}$ has the $\aleph_0$-exchange property in $M$, it has the $\aleph_0$-exchange property in $\sum_{I'}\oplus M'_{\beta'}$. Therefore, Condition II is satisfied for $\sum_{I'}\oplus M'_{\beta'}$, which implies 3 by [4], Lamma 9.

$3\rightarrow2$ It is clear from the theorem and Proposition 2.

$2\rightarrow1$ It is clear.

**Proposition 3** ([1], [3], [6] and [10]). *Let $M$ be in $\mathfrak{A}$ and $M=N_1\oplus N_2$. If $N_1$ is countably generated, then $N_1$ is in $\mathfrak{A}$. If every $M_\alpha$ is countably generated, then every direct summand of $M$ is in $\mathfrak{A}$.*

Proof. We make use the argument of the proof of [1], Theorem 7.1. First, we note that for any element $x$ in $N_1$ there exists a direct summand $N_0$ of $N_1$ such that $x\in N_0$ and $N_0$ is in $\mathfrak{A}$. Because, there exists a finite set $J$ such that $M_J$ contains $x$. From Theorem 4 we have $M=M_J\oplus N_1'\oplus N_2'$, $N_1=N_1'\oplus N_1''$ and $N_2=N_2'\oplus N_2''$, where $N_1''=(M_J\oplus N_2')\cap N_1$ and $N_2''=(M_J\oplus N_1')\cap N_2$, and $x\in N_1''$. If we use the same argument in [6], then we obtain the proposition.

## 3 Semi-perfect modules

We shall study further properties of $\mathfrak{A}$ in a case of semi-perfect modules defined by $E$. Mares in [9]. She has shown that every semi-perfect module is a direct sum of completely indecomposable semi-perfect modules ([9], Corollary 4.4). Let $P$ be an $R$-module and $J(P)$ the radical of $P$. If $P$ is semi-perfect, then $J(P)$ is small in $P$, $[P/J(P),P/J(P)]_{R/J(R)}\approx S_P/J(S_P)$ and $J(P)=PJ(R)$, (see [9], §§ 2-5).

**Theorem 5.** *Let P be a directly indecomposable projective module. Then P is completely indecomposable if and only if P is semi-perfect,* (cf. [5], the proof of Theorem 2.8).

Proof. If $P$ is semi-perfect, then $P$ is completely indecomposable by [9], Corollary 4.4. Conversely, we assume that so is $P$. Since $P/J(P)$ is $R/J(R)$-projective, $J([P/J(P),\ P/J(P)])=0$. From an exact sequence $0\to[P,\ J(P)]_R\to S_P\to[P/J(P),\ P/J(P)]_{R/J(R)}\to 0$ we have $[P,\ J(P)]\supset J(S_P)$. On the other hand, $J(S_P)$ is a unique maximal ideal in $S_P$ and $[P,\ J(P)]\neq S_P$. Hence, $\mathfrak{J}_P=J(S_P)=[P,\ J(P)]_R$. Next, we shall show that $J(P)$ is small in $P$. Let $N$ be a submodule of $P$ such that $P=J(P)+N$. From the following row exact sequence

$$\begin{tikzcd} N \arrow[r] & N/N\cap J(P) \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\ & P/J(P) & \\ & P \arrow[u] \arrow[uul, dashed, "f"] & \end{tikzcd}$$

we have $f:P\to N$, which commutes the above diagram. If $N\neq P, f\in\mathfrak{J}_P$. Hence, $\operatorname{Im} f\subset N\cap J(P)$, which is a contradiction. Finally, we show that $J(P)$ is a unique maximal submodule in $P$. Put $\bar{P}=P/J(P)$, $\bar{R}=R/J(R)$ and $\bar{S}=S_P/\mathfrak{J}_P$. We define $\mu:\bar{P}\otimes_{\bar{R}}[\bar{P},\ \bar{R}]_{\bar{R}}\to\bar{S}$ by setting $\mu(p\otimes f)(p')=pf(p')$. Since $\bar{P}\neq 0$ and $\bar{R}$-projective, $\mu\neq 0$. Furthermore, $\bar{S}$ is a division ring, and hence, $\mu$ is isomorphic. $\bar{P}\tau(P)=\bar{P}$ implies that there exists $p$ in $\bar{P}$ such that $\mu(p\otimes[\bar{P},\ \bar{R}])\neq 0$, where $\tau$ is the trace map of $\bar{P}$. Hence, $\mu(p\otimes f)\bar{S}=\bar{S}$ for some $f$ in $[\bar{P},\ \bar{R}]$. Therefore, $\bar{P}=\bar{S}\bar{P}=\mu(p\otimes f)\bar{S}\bar{P}\subset pf(\bar{P})\subset p\bar{R}\subset\bar{P}$. Hence, $\bar{P}=p\bar{R}\approx\bar{e}\bar{R}$ for some idempotent $\bar{e}$ in $\bar{R}$. Since $\bar{e}\bar{R}\bar{e}$ is a division ring and $\bar{R}$ is semi-simple, $\bar{P}$ is $\bar{R}$-irreducible by [8], Proposition 1 in p. 65. Hence, $J(P)$ is unique maximal, since $J(P)$ is the radical of $P$. Thus we have proved that $P$ is semi-perfect by [9], Theorem 5.1.

Now let $\{P_\alpha\}$ be a family of completely indecomposable projective modules, and $\mathfrak{A}$ the induced additive category from $\{P_\alpha\}$. Let $P=\sum\oplus P_\alpha$ and $P'=\sum\oplus P_\beta'$ be in $\mathfrak{A}$ and $f$ in $[P,\ P']_R$. If $f_{\alpha\beta}=p_\alpha f i_\beta$ is epimorphic, then $f_{\alpha\beta}$ splits and hence $f_{\alpha\beta}$ is isomorphic. Since $J(P_\alpha')$ is unique maximal, $\operatorname{Im} f_{\alpha\beta}$

$\subseteq J(P_\alpha')$ is $f_{\alpha\beta}$ is not isomorphic. Hence, if $f$ is in $\mathfrak{J}$, then $\mathrm{Im} f \subseteq \sum \oplus J(P_\beta')$ $= J(P')$. Conversely, if $\mathrm{Im} f \subseteq J(P')$, then $f$ is in $\mathfrak{J}$. Therefore, $[P, P']_R \cap \mathfrak{J}$ $= [P, J(P')]_R$. Furthermore, $0 \to [P, J(P')]_R \to [P, P']_R \to [P/J(P), P'/J(P')]_{R/J(R)}$ $\to 0$ is exact. Thus, for any object $P$ in $\mathfrak{A}$, many arguments in $\mathfrak{A}/\mathfrak{J}$ concerned with $\bar{P}$ coincide with those as $R/J(R)$-modules. From this reason, we make use of terminologies in $\mathfrak{A}/\mathfrak{J}$, instead of ones as $R/J(R)$-modules, if there are no confusions.

**Theorem 6.** *Let $\mathfrak{A}$ be an induced category from a family of completely indecomposable projective modules $\{P_\alpha\}$. Then an object $P = \sum_I \oplus P_\gamma$ in $\mathfrak{A}$ is perfect if and only if $\{P_\gamma\}_I$ is an elementwise T-nilpotent system.*

Proof. Let $\mathfrak{J}'$ be a full subcategory in $\mathfrak{J}$ which is induced from $\{P_\gamma\}_I$. If $P$ is perfect, then every object in $\mathfrak{A}'$ is semi-perfect. Hence, $\mathfrak{J}'$ is equal to the Jacobson radical in $\mathfrak{A}'$ by the above remark and [9], Theorem 2.4. Therefore, $\{P_\gamma\}_I$ is a $T$-nilpotent system by [4], Theorem 8. Conversely, we assume that $\{P_\gamma\}_I$ is a $T$-nilpotent system. Then $\mathfrak{J}_P = J(S_P)$ for every object $P$ in $\mathfrak{A}'$. We shall show that $J(P)$ is small in $P$ for every object $P$ in $\mathfrak{A}'$. Let $P = Q + J(P)$ for some submodule $Q$ and $p_1$ a projection of $P$ to $P_1$, where $P = \sum_{I'} \oplus P_\gamma$. Since $p_1(J(P))$ $\subset J(P_1)$ and $J(P_1)$ is small by Theorem 5, $p_1(Q) = P_1$. Hence, there exists $f$ in $[P_1, Q]_R$ such that $p_1 f = 1_{P_1}$. Therefore, $Q$ contains an object in $\mathfrak{A}'$ which is a direct summand of $P$. Let $T$ be the set of such objects in $Q$ and define a pertially order in $T$ by the inclusion. We take a totally ordered subset $Q_1 \subset Q_2 \subset \cdots$ in $T$. Put $Q_0 = \cup Q_i$, then $Q_0 = \sum \oplus N_\beta$; $N_\beta \approx P_{\pi(\beta)}$ by Lemma 2. Furthermore, the inclusion $i_\beta$: $N_\beta \to P$ is not zero modulo $\mathfrak{J}$, since $Q_j$ is a direct summand of $P$. Hence, $Q_0$ is a direct summand of $P$ by the proof of Proposition 2,2. Thus, we have a maximal element $P_0$ in $T$. $P = P_0 \oplus U$ and $Q = P_0$ $\oplus Q \cap U$. Since $P = Q + J(P)$ and $J(P) = J(P_0) \oplus J(U)$, $U = J(U) + U \cap Q$. $U$ is also in $\mathfrak{A}'$ by Lemma 2. If $U \neq 0$, $U \cap Q$ contains an object in $\mathfrak{A}'$ which is a direct summand of $U$ and hence of $P$. Which contradicts to the maximality of $P_0$. Therefore, $P = P_0 = Q$. Thus, every object in $\mathfrak{A}'$ is semi-perfect by Theorem 5 and [9], Theorem 5.2.

In the above argument, we have used only facts that $P_i$ are semi-perfect and $\mathfrak{J}_P = J(S_P)$. Hence, from Lemma 1, [4], Corollary 2 to Theorem 7 and [9], Theorem 2.3 we have

**Proposition 4.** *Let $P = \sum \oplus P_\alpha$ and $P_\alpha$ semi-perfect. Then $P$ is semi-perfect if and only if $\mathfrak{J}_P = J(S_P)$.*

**Theorem 7.** *Let $P$ be an object in $\mathfrak{A}$ induced from projective, completely indecomposable modules $P_\alpha$. Then we have the following equivalent conditions.*

1 *$P$ is semi-perfect.*

2 $\mathfrak{F}_P = J(S_P)$.

3 *Every dense submodule of P coincides with P.*

4 $P = \sum_I \oplus P_\alpha$ *in* $\mathfrak{A}$, *then* $\{P_\alpha\}$ *is a semi-T-nilpotent system.*

5 *P satisfies the Condition* II *in* [4].

Proof. $1 \leftrightarrow 2$ is proved in Proposition 4.

$1 \rightarrow 3$ Let $N$ be a dense submodule of $P$. Then $N \supseteq \mathrm{Im}\,(1\text{-}f)$ for some $f \in \mathfrak{F}_P$. Hence, $P \subseteq N + f(P) \subseteq N + J(P) \subseteq P$ by the remark before Theorem 6. Therefore, $P = N + J(P)$ implies $P = N$, since $J(P)$ is small.

$3 \rightarrow 4$ Let $\{f_i\}$ be a family of non isomorphisms of $P_{\alpha_i}$ to $P_{\alpha_{i+1}}$ ($P_{\alpha_i} \neq P_{\alpha_{i+1}}$). Put $f = \sum(-e_{i+1\,i} f_i)$, where $\{e_{ij}\}$ is a system of matrix units in $S_P$. Then $\mathrm{Im}\,(1-f)$ is a dense submodule of $P$. From the assumtion and the argument of Lemma 9 in [4], we know that $\{f_i\}$ is a $T$-nilpotent sequence.

$2 \rightarrow 5$ is proved in [4], Corollary 2 to Theorem 7.

$5 \rightarrow 4$ is proved in [4], Lemma 9.

$4 \rightarrow 1$. Let

$$P = \sum_{K \ni \alpha} \sum_{I_\alpha \ni \beta} \oplus M_{\alpha\beta} \cdots (*),$$

where $M_{\alpha\beta}$'s are indecomposable and $M_{\alpha\beta} \approx M_{\alpha\beta'}$, $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. First we assume that the cardinal $\lambda_\alpha$ of $|I_\alpha|$ is finite for all $\alpha$ in $K$. We put $P_{\alpha(n)\,n} = \sum_{\beta=1}^{n} \oplus M_{\alpha\beta}$, where $n = \lambda_\alpha$, and show that $J(P)$ is small in $P$. We assume $P = N + J(P)$ for some submodule $N$ of $P$. Let $p_{\alpha(n)\,n}$ be a projection of $P$ to $P_{\alpha(n)\,n}$. Since $\lambda_\alpha$ is finite, $J(P_{\alpha(\,)\,n})$ is small in $P_{\alpha(n)\,n}$. Hence, $p_{\alpha(n)\,n} | N$ is epimorphic, and there exists $g \in [P_{\alpha(n)\,n}, N]_R$ such that $(p_{\alpha(n)\,n} | N) g = 1_{P_{\alpha(n)\,n}}$. Put $P'_{\alpha(n)\,n} = \mathrm{Im}\, g$. Since $\mathrm{Ker}\, p_{\alpha(n)\,n} = \sum_{\alpha \neq \alpha(n)} \oplus P_{\alpha\beta}$,

$$P = P'_{\alpha(n)\,n} \oplus \sum_{\alpha \neq \alpha(n)} \oplus P_{\alpha\beta} \cdots (**)\,.$$

Now, we assume $N \not\subset M_{\alpha(n)\,i_1}$ and $x_1 \in M_{\alpha(n)\,i_1} - N$. Then $x_1 = x' + \sum y_i$ from $(**)$, where $x' \in P'_{\alpha(n)\,n}$, $y_i \in P_{\alpha\beta}$. From the assumption there exists some $y_i \notin N$, since $P'_{\alpha(n)\,n} \subset N$. Hence, there exists $x_2$ in $M_{\alpha_i,\,i_2} - N$ such that $y_i = x_2 + \sum z_j$, $z_j \in M_{\alpha_i j}$ $(j \neq i_2)$. If we replace $(*)$ by $(**)$, we can find $x_3$ in $M_{\alpha_j k} - N$ and $P = P'_{\alpha(n)\,n} \oplus P'_{\alpha(m)\,m} \oplus \sum \oplus P_{\alpha\beta}$. Repeating this argument, we have a sequence $\{x_j\}$ so that $x_i \in M_{\alpha_i k_i} - N$, and $f_i(x_i) = x_{i+1}$, where $f_i$ is a projection of $P$ to $M_{\alpha_{i+1} k_{i+1}}$, which is a contradiction. Therefore, $J(P)$ is small. Finally, we shall consider a general case. Let $P = \sum_{\lambda_\alpha \geqslant \aleph_0} \sum_\beta \oplus M_{\alpha\beta} \oplus \sum_{\lambda_\alpha < \aleph_0} \sum_i \oplus M_{\alpha_i}$ and put $P_1 = \sum_{\lambda_\alpha \geqslant \aleph_0} \sum_\beta M_{\alpha\beta}$ and $P_2 = \sum_{\lambda_\alpha < \aleph_0} \sum_i \oplus M_{\alpha_i}$. We know from the first case $P_2$ is semi-perfect. If $\lambda_\alpha \geqslant \aleph_0$ for $\alpha$, the fact that $\{M_{\alpha\beta}\}$ is semi-$T$-nilpotent implies from the definition that $\{M_{\alpha\beta}\}$ is a $T$-nilpotent system. Hence, $P_1$ is perfect by Theorem 6. Therefore, $P$ is semi-perfect from [9], Corollary 5.3 (see Proposition 6 below).

**Corollary 1.** *Let $P$ be projective and artinian, then $P$ is perfect. Furthermore, if $P'$ is a directsum of artinian submodules and $P'$ is semi-perfect, then $P'$ is perfect.*

Proof. If $P$ is artinian and projective, then $P$ is in $\mathfrak{A}$ and $\mathfrak{J}_P$ is nilpotent ideal by [5], Theorem 2.8. Hence, for any directsum $M$ of any copies of $P$, we have $\mathfrak{J}_M=J(S_M)$, since $\mathfrak{J}_M$ is nilpotent. Therefore, $M$ is semi-perfect, and $P$ is perfect. Let $P'=\sum \oplus P_i$; $P_i$'s are artinian and $P'$ be semi-perfect. Then $\{P_i\}$ is a semi-T-nilpotent system from Theorem 7. Furthermore, since $J_{P_i}$ is nilpotent, $\{P_i\}$ is a T-nilpotent system. Hence, $P'$ is perfect from Theorem 6.

**Corollary 2.** *Let $P$ be a semi-perfect module. Then there exists a maximal one among submodules which are perfect and direct summand of $P$. Those maximal perfect submodules are isomorphic each other.*

Proof. Let $P=\sum_{\lambda_\alpha<\aleph_0}\sum\oplus M_{\alpha i}\oplus\sum_{\lambda_\alpha\geqslant\aleph_0}\sum M_{\alpha\beta}$ as in the above proof. If $\mathfrak{J}_{M_{\alpha i}}$ is elementwise $T$-nilpotent, then $\{M_{\alpha i}, M_{\alpha\beta}\}$ is $T$-nilpotent, since it is semi-$T$-nilpotent. Hence, if we chooseevery $M_{\alpha i}$ whose ideal $\mathfrak{J}_{M_{\alpha i}}$ is $T$-nilpotent, $P_1=\sum'_{\lambda_\alpha<\aleph_0}\sum\oplus M_{\alpha i}\oplus\sum_{\lambda_\alpha\geqslant\aleph_0}\sum\oplus M_{\alpha\beta}$ is a direct summand of $M$ and perfect, where $\sum'$ runs through all $M_{\alpha i}$ in the above. Put $P=P_1\oplus P_2$. If $P=Q_1\oplus Q_2$, $Q_1\supseteq P_1$ and $Q_1$ is perfect, then $P=Q_1'\oplus P_1\oplus Q_2$ and $Q_2=Q_2'\oplus P_1$. Since $P_2\approx Q_1'\oplus Q_2$, $Q_1'=(0)$ by the assumption. Hence, $P_1$ is a desired perfect submodule. Let $T_1$ be a maximal element as in Corollary 2; $P=T_1\oplus T_2$, then $T_2$ is in $\mathfrak{A}$. It is clear that $T_1$, $P_1$ and $T_2$, $P_2$ have the isomorphic direct components, respectively. Hence, $P_1\approx T_1$.

Finally, we shall give some results concerned with ones obtained in [9].

First we shall give another proof of [9], Theorem 5.5.

**Proposition 5** ([9]). *Let $\mathfrak{A}$ be as above and $P$ a direct summand of an object $M$ in $\mathfrak{A}$. If $J(P)$ is small in $P$, then $P$ is in $\mathfrak{A}$.*

Proof. Let $M=P\oplus P_1$ and $P=eM$ for some idempotent $e$. $P$ contains a dense submodule $P_0$ with inclusion $i$ such that $if\equiv e \pmod{\mathfrak{J}}$ for some $f$ in $[M, P_0]_R$. Put $e=if+x$, $x\in\mathfrak{J}$. Then $P=P_0+x(P)$ and $x(P)\subset P\cap J(M)=J(P)$. Hence, $P=P_0$.

**Proposition 6** ([9], Corollary 5.3). *Let $\{P\}_1^n$ be a finite set of semi-perfect modules. Then $\sum_1^n\oplus P_i$ is semi-perfect.*

Proof. Since $\mathfrak{J}_{P_i}=J(S_{P_i})$ for every $i$, we can show $\mathfrak{J}_P=J(S_P)$ by using fundamental transformations of matrices (see [4], Lemma 8). Hence, $P$ is semi-perfect from Propostion 4.

**Proposition 7** ([9], Theorem 7.2). *If $J(R)$ is right T-nilpotent, then every semi-perfect modules is perfect.*

Proof. Let $P=\sum_I \oplus P_\alpha$ be semi-perfect. Then $\mathfrak{F}_P=[P, J(P)]=[P, PJ(R)]$. Hence, for any $f\in[P_\alpha, P_\beta]\cap\mathfrak{F}$ and $x_\alpha\in P_\alpha, f(x_\alpha)=\sum x_{\beta_i}a_{\beta_i}$, $a_{\beta_i}\in J(R)$. Therefore, $\{P_\alpha\}$ is $T$-nilpotent system by the assumption. Hence, $P$ is perfect from Theorem 6.

REMARK. [9], Theorem 5.1 is a special case of Theorem 3.

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### References

[1] P. Crawley and B. Jónnson: *Refinements for infinite direct decomposition of algebraic systems*, Pacific J. Math. **14** (1964), 797-855.
[2] C. Ehresmann: Catégories et Structures, Dunod, Paris, 1965.
[3] S. Elliger: *Zu dem Satz von Krull-Remak-Schmidt-Azumaya*, Math. Z. **115** (1970), 227-230.
[4] M. Harada and Y. Sai: *On categories of indecomposable moduls* I, Osaka J. Math. **7** (1970), 323-344.
[5] M. Harada: *On semi-simple categories*, Osaka J. Math. **7** (1970), 89-95.
[6] I. Kaplansky: *Projective modules*, Ann. of Math. **68** (1958), 372-377.
[7] G.M. Kelly: *On the radical of a category*, J. Austral. Math. Soc. **4** (1964), 299-307.
[8] N. Jacobson: Structure of Rings, Amer. Math. Soc., 1956.
[9] E. Mares: *Semi-perfect modules*, Math. Z. **83** (1963), 347-360.
[10] R.B. Warfield Jr.: A *Krull-Schmidt theorem for infinite sums of moduels*, Proc. Amer. Math. Soc. **22** (1969), 460-465.