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ON CATEGORIES OF INDECOMPOSABLE MODULES II

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We have studied the Krull-Remak-Schmidt-Azumaya's theorem from point of view of categories in [4], §3. In this note, we shall study further properties of those categories.

We have defined an additive category \mathfrak{A} induced from a family of completely indecomposable modules $\{M_\alpha\}$, namely whose objects consist of directsums of M_α and studied the quotient category $\mathfrak{A}/\mathfrak{I}$ with respect to the ideal \mathfrak{I} in \mathfrak{A} in [4].

In the first section, we characterize submodules M_0 in an object M in \mathfrak{A} , which is in \mathfrak{A} and $M_0 \equiv M \pmod{\mathfrak{I}}$, and show that every such M_0 coincides with M if and only if \mathfrak{I} is the Jacobson radical of \mathfrak{A} , (see [7] for the radical).

In the second section, we consider Conditions II and III defined in [4], which are related with exchange property defined in [1]. We change slightly the definition of exchange property in this note and show that every direct summand of objects M in \mathfrak{A} has the exchange property in M if and only if \mathfrak{I} is the Jacobson radical of \mathfrak{A} .

In the final section, we restrict ourselves to a case where M_α 's are projective. We shall show, in this case, that objects in \mathfrak{A} are closely related to semi-perfect modules defined in [9]. Especially we show that an object $P = \sum_I \oplus P_\alpha$ in \mathfrak{A} is perfect if and only if $\{P_\alpha\}$ is an elementwise T -nilpotent system defined in [4] and P is semi-perfect if and only if $\{P_\alpha\}$ is an elementwise semi- T -nilpotent system.

Let R be a ring with unit element and all modules in this note be unitary right R -modules. An R -module M is called *completely indecomposable* if $\text{End}_R(M) = S_M$ is a (non-commutative) local ring. We assume here that indecomposable modules mean completely indecomposable.

1. Dense submodules

Let M_0 be an R -module and assume $M_0 = \sum_I \oplus M_\alpha$, where M_α 's are indecomposable modules. We have defined an additive category \mathfrak{A} in [4] from the above decomposition as follows: The objects of \mathfrak{A} consist of some directsums of M_α 's and the morphisms of \mathfrak{A} consist of all R -homomorphisms. We denote

those morphisms by $[M, N']_R$ and we call \mathfrak{A} is induced from a family $\{M_\alpha\}$. Furthermore, we have defined an ideal \mathfrak{S} in \mathfrak{A} in [4] as follows: Let $N = \sum \oplus M'_\beta$, $N' = \sum \oplus M'_{\beta'}$ be in \mathfrak{A} . \mathfrak{S} consists of all morphisms f in $[N, N']_R$ such that $p_{\beta'} f i_\beta \in [M_\beta, M'_{\beta'}]_R$ is not isomorphic for any β and β' , where i_β , $p_{\beta'}$ are injection and projection, respectively and N, N' run through all objects in \mathfrak{A} . We know from Theorem 7 in [4], that \mathfrak{S} defines an ideal in \mathfrak{A} and $\mathfrak{A}/\mathfrak{S}$ is a C_3 -completely reducible abelian category, (see [2] for the definition of ideals).

When we consider object N and morphism f in $\mathfrak{A}/\mathfrak{S}$, we denote them by \bar{N} and \bar{f} . Furthermore, if N is in \mathfrak{A} and $N = N_1 \oplus N_2$ as R -modules, then \bar{N}_i means $\text{Im } \bar{e}_i$, where e_i is a projection of N to N_i . Let S be a ring. By $J(S)$ we denote the Jacobson radical of S and by \mathfrak{S}_M we denote $\mathfrak{S} \cap \text{End}_R(M)$ for an R -module M .

Let $M \supseteq N$ be objects in \mathfrak{A} and i the inclusion of N to M . If i is isomorphic modulo \mathfrak{S} , i.e. $\bar{M} = \bar{N}$, then we call N a *dense submodule* in M .

Proposition 1. *Every dense submodule of M is R -isomorphic to M .*

Proof. Since $\bar{M} = \bar{N}$, M and N have isomorphic direct summands by [4], Corollary 1 to Theorem 7.

Let $\{M_\alpha\}$ be a family of completely indecomposable modules and $\{f_{\alpha_i}\}_{i=1}^\infty$ any sequence of non isomorphic R -homomorphisms of M_{α_i} to $M_{\alpha_{i+1}}$ in $\{M_\alpha\}$ ($M_{\alpha_{i+1}}$ may be equal to M_{α_i}). If there exists n , which depends on the above sequence and for any element m in M_{α_1} , such that $f_{\alpha_n} f_{\alpha_{n-1}} \cdots f_{\alpha_1}(m) = 0$, then we call $\{M_\alpha\}$ a (*elementwise*) *T -nilpotent system* (cf. [4]). If the above condition is satisfied for any sequence $\{f_{\alpha_i}\}$ such that $M_{\alpha_i} \neq M_{\alpha_j}$ if $i \neq j$, then we call $\{M_\alpha\}$ a (*elementwise*) *semi- T -nilpotent system*. In general, a semi- T -nilpotent system is not T -nilpotent.

Let $M = \sum_I \oplus M_\alpha$ and J a subset of I , then we denote a submodule $\sum_J \oplus M_{\alpha'}$ by M_J .

Proposition 2. *Let M and P be in \mathfrak{A} and $\bar{M} \supseteq \bar{P}$. Then there exists a submodule P_0 in M satisfying the following conditions.*

- 1 P_0 is an object in \mathfrak{A} ; $P_0 = \sum_I \oplus M_\alpha$.
- 2 P_{0J} is a direct summand of M for any finite subset J of I , (if $\{M_{\alpha'}\}_{J'}$ is T -nilpotent system, J' need not be finite).
- 3 $\bar{P}_0 = \bar{P}$.

Furthermore, if $\bar{P} = \text{Im } \bar{e}$ and e is an idempotent in $S_M = \text{End}_R(M)$, then we can choose P_0 in $\text{Im } e$.

Proof. Let $P = \sum_I \oplus M_\lambda$. Since $\bar{M} \supseteq \bar{P}$ and $\mathfrak{A}/\mathfrak{S}$ is completely reducible, there exist $i \in [P, M]_R$ and $p \in [M, P]_R$ such that $pi \equiv 1_P \pmod{\mathfrak{S}}$. Let J be a subset of I and i_J, p_J be inclusion and projection, respectively. Since $p_J p i_J$ is

isomorphic, $p_j p i i_j$ is R -isomorphic by [4], Lemma 8 and Theorem 8 if J is finite or $\{M_\lambda\}_J$ is a T -nilpotent system. Hence, $\alpha_j = i i_j$ splits, namely $\text{Im } \alpha_j$ is a direct summand of M . Therefore, i is R -monomorphic. We put $P_0 = \text{Im } i$ as R -module, then P_0 satisfies 1~3. If $P = \text{Im } e$, we have a relation $p e i \equiv p i \equiv 1_P \pmod{\mathfrak{F}}$. Hence, if we take $\alpha_j = e i i_j$, we know that $\text{Im } e i = P_0 \subseteq e M$.

The following theorem gives a special answer for Condition III in [4].

Theorem 1. *Let M be in \mathfrak{A} and $M = \sum_K \oplus N_\gamma$ as R -module. Then each N_γ contains a submodule P_γ such that P_γ is in \mathfrak{A} and $\sum \oplus P_\gamma$ is a dense submodule of M .*

Proof. Let π_γ be a projection of M to N_γ . Then from Proposition 2 we have P_γ in \mathfrak{A} such that $\bar{P}_\gamma = \text{Im } \pi_\gamma = \bar{N}_\gamma$. We shall show $\bar{M} = \sum_K \oplus \bar{P}_\gamma$. Let i_γ, i'_γ and i''_γ be inclusions of P_γ to M , of P_γ to N_γ and of N_γ to M such that $i_\gamma = i'_\gamma i''_\gamma$, respectively. Since $\bar{P}_\gamma = \text{Im } \pi_\gamma$, there exists an R -homomorphism p_γ of M to P_γ such that $i_\gamma p_\gamma = \pi_\gamma \pmod{\mathfrak{F}}$. Let $\{f_\gamma\}$ be an element in $\prod_\gamma [P_\gamma, N]_{\mathfrak{A}/\mathfrak{F}}$, where N is an object in \mathfrak{A} and $f_\gamma \in [P_\gamma, N]_R$. We put $f'_\gamma = f_\gamma p_\gamma i''_\gamma \in [N_\gamma, N]_R$ and $f = \prod_K f'_\gamma \in [M, N]_R$. Then we have $f i_\gamma = f i'_\gamma i''_\gamma = f'_\gamma i''_\gamma = f_\gamma p_\gamma i_\gamma$. Hence, $f i_\gamma \equiv f_\gamma \pmod{\mathfrak{F}}$. We shall show that f does not depend on a choice of representative f_γ . It is sufficient to show that if $f|_{P_\gamma} = f i_\gamma$ is in \mathfrak{F} for all γ , then f is in \mathfrak{F} for any f in $[M, N]_R$. Let $N = \sum_L \oplus M_\delta$; M_δ 's are indecomposable. If f is not in \mathfrak{F} , there exists an indecomposable direct summand T of M such that $p_\delta' f i_T$ is isomorphic, where $i_T: T \rightarrow M$, $p_\delta': N \rightarrow N_\delta$ are inclusion and projection, respectively. Since $\{\pi_\gamma\}_K$ is summable, $1_M = \sum_K \pi_\gamma$ and $f = \sum_K f \pi_\gamma$. Furthermore, since $\{p_\gamma' f \pi_\gamma i_T\}_K$ is summable and $p_\delta' f i_T = \sum p_\delta' f \pi_\gamma i_T$, there exists a finite subset K' in K such that $\sum_{K-K'} p_\delta' f \pi_\gamma i_T$ is not isomorphic, and $\sum_{K'} p_\delta' f \pi_\gamma i_T$ is isomorphic. Therefore, there exists γ in K' such that $p_\delta' f \pi_\gamma i_T$ is isomorphic. On the other hand $p_\delta' f \pi_\gamma i_T \equiv p_\delta' f i_\gamma i_T \equiv 0 \pmod{\mathfrak{F}}$, which is a contradiction. Conversely, we take a morphism $f \in [M, N]_{\mathfrak{A}/\mathfrak{F}}$ and $f \in [M, N]_R$. Put $f_\gamma = f i_\gamma$, then f_γ does not depend on a choice of f by Proposition 2 and [4], Lemma 5. Thus, we have shown that $[M, N]_{\mathfrak{A}/\mathfrak{F}} = \prod [P_\gamma, N]_{\mathfrak{A}/\mathfrak{F}}$.

We call such P_γ a *dense submodule* of N_γ .

Theorem 2. *Let M be in \mathfrak{A} induced from a family $\{M_\alpha\}$ of completely indecomposable modules M_α , and $N = \sum_I \oplus M_\rho'$ in \mathfrak{A} be a submodule of M . Then the following statements are equivalent.*

- 1 N is a dense submodule of M .
- 2 There exists a finite subset J of I , for any direct summand P of M , such that either $P \cap N_J \neq 0$ or $P \oplus N_J$ is not a direct summand of M .

3 N contains $\text{Im}(1-f)$ for some element f in \mathfrak{S}_M and i_N is monomorphic.

In those cases $N_{J'}$ is a direct summand of M for any finite subset J' of I . Furthermore, $\text{Im}(1-f)$ is always a dense submodule of M .

Proof. $1 \rightarrow 2$ Since every direct summand of M contains an indecomposable module by [4], Corollary 1 to Theorem 7, we may assume so is P . Since \bar{P} is a minimal object, \bar{P} is small (cf. [5], Theorem 1.4). Hence, $\bar{P} \subseteq \sum_J \oplus \bar{M}_{\alpha_i}$ for some finite set J . We assume that $P \cap N_J = 0$ and $P \oplus N_J$ is a direct summand of M . Let e, f and E be projections of M to P, N_J and $P \oplus N_J$, respectively. We may assume $E = e + f$ and $ef = fe = 0$. We denote the inclusion of submodules to M by i . Since $\bar{P} \subseteq \bar{N}_J$, there exists α in $[P, N_J]_R$ such that $i_J \alpha \equiv i_P \pmod{\mathfrak{S}}$. Since $\bar{I}_P = \bar{e} i_P$ and $\bar{f} \bar{e} = 0$, $\bar{I}_P = \bar{e} i_J \bar{\alpha} = \bar{0}$. Hence, $P = 0$, which is a contradiction.

$2 \rightarrow 1$ If $\bar{M} \neq \bar{N}$, there exists an indecomposable module P such that $\bar{P} \oplus \bar{N}_J$ is a direct summand of \bar{M} for any finite subset J of I . Since $\bar{P} \oplus \bar{N}_J$ is a directsum of finite many of minimal objects, there exists a direct summand P_0 of M such that $P_0 \cap N_J = 0$ and $P_0 \oplus N_J$ is a direct summand of M (see the proof of Proposition 2).

$1 \rightarrow 3$ Let i be an inclusion of N to M . Since i is isomorphic, there exists j in $[M, N]_R$ such that $i j = \bar{I}_M$. Put $f = 1 - ij$, then $f \in \mathfrak{S}_M$ and $1 - f = ij$. Since i is monomorphic, $N \supseteq \text{Im}(1-f)$.

$3 \rightarrow 1$ First we shall show that $N' = \text{Im}(1-f)$ is a dense submodule of M . We know from the proof of Proposition 10 in [4] that $1-f$ is monomorphic and hence, N' is in \mathfrak{A} . Let i be an inclusion of N' to M and $1-f = i(1-f)'$; $(1-f)' \in [M, N']_R$. Since $1 \equiv 1-f \equiv i(1-f)'$ and $(1-f)$ is isomorphic, so is i . Therefore, N' is a dense submodule. Hence, $M \supseteq N \supseteq \text{Im}(1-f)$ implies that i_N is epimorphic. In order to get the last part we put $\bar{M} = \bar{N} = \bar{P}$ in Proposition 2, then $N_{J'}$ is a direct summand of M .

Corollary 1. Let M be an object in \mathfrak{A} and P a dense submodule of M . If for a direct summand $N = \sum_J \oplus M_{\alpha'}$ of M in \mathfrak{A} , J is finite or $\{M_{\alpha'}\}$ is a T -nilpotent system, there exists an automorphism σ of M such that $\sigma(N)$ is a direct summand of P .

Proof. P contains a submodule N_1' which is isomorphic to N_1 and is a direct summand of M by Proposition 2; say $M = N_1' \oplus N_2' = N_1 \oplus N_2$. Since $N_2' \approx N_2$, we obtain the corollary.

Corollary 2. Let $\{M_{\alpha}\}$ be a family of completely indecomposable modules and \mathfrak{A} the induced additive category from $\{M_{\alpha}\}$. Then the following conditions are equivalent.

- 1 $\{M_{\alpha}\}$ is an elementwise T -nilpotent system.
- 2 \mathfrak{S} is the Jacobson radical of \mathfrak{A} .
- 3 Every dense submodule of any M in \mathfrak{A} coincides with M .

Proof. $1 \leftrightarrow 2$ is obtained in [4], Theorem 8.

$1 \rightarrow 3$ Let N be a dense submodule of M . We know from 1 and Proposition 2 that N is a direct summand of M . Hence, $N=M$ by Theorem 2.

$3 \rightarrow 2$ Let f be in \mathfrak{F}_M and $N = \text{Im}(1-f)$. Since N is a dense submodule by Theorem 2, $N=M$. Therefore, $1-f$ is isomorphic, which implies \mathfrak{F}_M is equal to $J(S_M)$.

REMARK. Let $M = \sum_{i=1}^{\infty} \oplus M_i$ as in Theorem 2. We assume that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of monomorphisms but not epimorphisms f_i of M_i to M_{i+1} . Then for any finite set J of I there exists a dense submodule N in M such that $N \cap M_J = (0)$. Because, we make use of matrix representation of $[M, M]_R$ and by $\{e_{im}\}$ we denote a system of matrix unites. Put $f = \sum_i \sum_{k=1}^J f_{i+k-1} f_{i+k-2} \cdots f_i e_{i+k,i}$, then f is in \mathfrak{F} . Hence, $P = \text{Im}(1-f)$ is a dense submodule and $P \cap M_J = (0)$.

If we use the same argument for any set I , we can give an example in which for some subset J with $|J| \leq |I|$ there exists a dense submodule P in M such that $P \cap M_J = (0)$. Furthermore, we can give an example in which there exists a dense submodule P in $M = \sum_{i=1}^{\infty} \oplus M_i$ such that $P \cap M_i \neq (0)$ for all i and $P \neq M$.

In the above corollary, we have a situation $\mathfrak{F}_M = J(S_M)$. In this case we obtain a further result.

Lemma 1. *Let M be in \mathfrak{A} and $\mathfrak{F}_M = J(S_M)$. Then for every direct summand N of M we have $\mathfrak{F}_N = J(S_N)$.*

Proof. Since $\mathfrak{F}_M = J(S_M)$, N is in \mathfrak{A} by [4], Corollary 2 to Theorem 7. Put $N = eM$ for some idempotent e in S_M . Then it is clear that $e\mathfrak{F}_M e = \mathfrak{F}_N$, since \mathfrak{F} does not depend on decompositions of M by [4], Lemma 5. Furthermore, $J(S_N) = eJ(S_M)e$. Hence, $J(S_N) = \mathfrak{F}_M$.

Theorem 3. *Let P be in \mathfrak{A} and $\mathfrak{F}_P = J(S_P)$. Then every idempotent a in $S/J(S)$ is lifted to S .*

Proof. Let a be idempotent modulo \mathfrak{F}_P . Then there exist a module P_0 in \mathfrak{A} and $a' \in [P, P_0]_R$, $b' \in [P_0, P]_R$ such that $b'a' \equiv a$ and $a'b' \equiv 1_P \pmod{\mathfrak{F}}$. Since P_0 is isomorphic to a direct summand of P by [4], Theorem 7, $\mathfrak{F}_P = J(S_P)$ by Lemma 1. Hence b' is R -monomorphic and $\varepsilon' = a'b'$ is R -isomorphic on P_0 . We may regard P_0 as a direct summand of P via b' ; $P = \text{Im } b' \oplus Q$. We put $\varepsilon = b'\varepsilon'^{-1}b'^{-1} + 1_Q$, then $\varepsilon \equiv 1 \pmod{\mathfrak{F}}$. Put $e = \varepsilon b'a'$, then $e|P_0 = 1_P$ and $\text{Im } e = P_0$. Hence, e is idempotent in S_P and $e = \varepsilon b'a' \equiv a'b' \equiv a \pmod{\mathfrak{F}}$.

Corollary. *Let R be a (non-commutative) local ring such that $J(R)$ is T -nil-*

potent. Let S be the ring of column finite matrices over R with any degree. Then every idempotent in $S/J(S)$ is lifted to S .

Proof. Put $M = \sum_{I \ni \alpha} \oplus R_\alpha$; $R_\alpha \approx R$. Then $S = S_M$ and $J(S_M) = \mathfrak{J}_M$ by [4], Lemma 10.

2 Exchange property

We shall recall Condition II in [4]. Let $M = \sum_I \oplus M_\alpha = \sum_I \oplus N_\beta$ be decompositions of M with indecomposable modules M_α, N_β . Condition II in [4] says that for any subset J of I , there exists a subset J' of I such that $M = \sum_{J'} \oplus M_\alpha \oplus \sum_J \oplus N_\beta$. However, this is a special case of exchange property defined in [1]. Furthermore, this property induces Condition III in [4], namely every direct summand of M is in \mathfrak{A} . Therefore, we shall define a weaker exchange property than one in [1]. Let M be as above (in \mathfrak{A}), and $M = \sum_{I \ni i} \oplus P_i$ be any direct decomposition as R -modules. We call a direct summand N of M has the $|I|$ -exchange property in M if $M = N \oplus \sum_{I \ni i} \oplus P'_i$ and $P'_i \subset P_i$ for any decomposition $M = \sum_{I \ni i} \oplus P_i$ with $|I|$ -factors. If N has the $|I|$ -exchange property in M for any cardinal $|I|$, we call N has the *exchange property in M* . It is clear that if N has the exchange property in M , then N is an object of \mathfrak{A} . P. Crawley and B. Jónsson have shown in [1], Theorem 7.1 (and [10], Theorem 1) that if M is countably generated for all α in I , then Condition III is satisfied.

In the following we always assume that $M = \sum_I \oplus M_\alpha = N_1 \oplus N_2$ with indecomposable modules M_α .

Lemma 2. *If either N_1 is finitely generated or a dense submodule of N_1 is a T -nilpotent system, then N_i is in \mathfrak{A} , ($i=1, 2$).*

Proof. If N_1 is finitely generated, then N_1 is a direct summand of M_J for some finite subset J of I . Hence, $N_1 \approx M_{J'}$ for some $J' \subset J$ by Krull-Remak-Schmidt's theorem. Therefore, $M = N_1 \oplus \sum \oplus M_{\varphi(\omega)}$ by [4], Corollary 1 to Theorem 7 (Azumaya's theorem), and hence $N_2 \approx \sum \oplus M_{\varphi(\omega)}$ is in \mathfrak{A} . Next, we assume that a dense submodule N_0 of N_1 is a T -nilpotent system. Then $N_0 = N_1$ by Proposition 2. Hence N_1 is in \mathfrak{A} . Since $\mathfrak{A}/\mathfrak{J}$ is completely reducible, $\bar{M} = \bar{N}_1 \oplus \sum_{K \ni \beta} \bar{M}_\beta = \sum_{I-K} \bar{M}_\alpha \oplus \sum_K \bar{M}_\beta$ for some K in I . Let p be a projection of M to $\sum_{I-K} M_\alpha$. Since $\bar{p}| \bar{N}_1$ is isomorphic and $\{M_\alpha\}_{I-K}$ is a T -nilpotent system, $p|N_1$ is an R -isomorphism of N_1 to $\sum_{I-K} \oplus M_\alpha$. Therefore, $M = N_1 \oplus \text{Ker } p = N_1 \oplus \sum_K \oplus M_\beta = N_1 \oplus N_2$. Hence, $N_2 \approx \sum \oplus M_\beta$.

The following lemma is a special case of [1], Lemma 3.10 and [10], Proposition 1, however we shall give a proof from point of view of our categories.

Lemma 3. *Let M and N_i be as above. If $N_1 = \sum_{i=1}^n \oplus M_i'$ and $M_i' \approx M_{\alpha i}$ for all i , then N_1 has the exchange property in M .*

Proof. We assume that $M = N_1 \oplus N_2 = \sum_I \oplus Q_{\alpha}$ as R -modules. Then $M = \bar{N}_1 \oplus \bar{N}_2 = \sum_{I'} \oplus \bar{P}_{\alpha}$, where $P_{\alpha} = \sum_j \oplus P_{\alpha j}$ is a dense submodule of Q_{α} and $P_{\alpha j}$'s are indecomposable. Since $\bar{N}_1 = \sum_{i=1}^n \oplus \bar{M}_i'$ is a small object in $\mathfrak{A}/\mathfrak{S}$, there exist a finite subset I'' of I and a finite subset J_i' of J_i for $i \in I''$ such that $\bar{N}_1 \subseteq \sum_{I'' \ni i} \sum_{J_i \ni j} \oplus \bar{P}_{ij}$. We know from Proposition 2, 2) that $\sum_i \sum_j \oplus P_{ij} = P$ is a direct summand of M . Since I'' and J_i' are finite, $\mathfrak{S}_P = J(S_P)$ by [4], Lemma 8. Hence, P contains a direct summand N_1' such that $\bar{N}_1' = \bar{N}_1$, ($P = N_1' \oplus P'$). $M = N_1' \oplus P' \oplus \sum_{I''} \oplus Q_i' \oplus \sum_{I-I''} \oplus Q_{\alpha}$, where $Q_i = Q_i' \oplus \sum_{J_i - J_i'} \oplus P_{ij}$. Let $p_{N_1'}$ be a projection of M to N_1' in this decomposition. Since $\bar{N}_1 = \bar{N}_1'$ and $\bar{N}_1 \cap (\bar{P}' \oplus \sum_{I''} \oplus \bar{Q}_i' \oplus \sum_{I-I''} \oplus \bar{Q}_{\alpha}) = \bar{0}$, (see the proof of Theorem 1), $\bar{p}_{N_1'} | \bar{N}_1$ is isomorphic. Therefore, $p_{N_1'} | N_1$ is isomorphic as an R -module. Thus, we obtain that $M = N_1 \oplus P' \oplus \sum_{I''} \oplus Q_i' \oplus \sum_{I-I''} \oplus Q_{\alpha}$.

Theorem 4. *Let $M = \sum_I \oplus M_{\alpha}$ with M_{α} completely indecomposable, and $N_1 = \sum_{I'} \oplus M_{\beta}'$ be a direct summand of M ; $M = N_1 \oplus N_2$. If I' is finite or $\{M_{\beta}'\}$ is a T -nilpotent system, then N_i has the exchange property in M for $i=1, 2$.*

Proof. We know from the assumption and [4], Theorem 8 that $\mathfrak{S}_{N_1} = J(S_{N_1})$. Let $M = N_1 \oplus N_2 = \sum_J \oplus Q_{\alpha}$. Then $\bar{M} = \bar{N}_1 \oplus \bar{N}_2 = \sum \oplus \bar{P}_{\alpha}$, where P_{α} is a dense submodule of Q_{α} . We put $P_{\alpha} = \sum_{J_{\alpha} \ni i} \oplus P_{\alpha i}$ ($\in \mathfrak{A}$). Since $\mathfrak{A}/\mathfrak{S}$ is completely reducible, $\bar{M} = \bar{N}_2 \oplus \sum_J \sum_{J_{\alpha}'} \oplus \bar{P}_{\alpha i}$, where J_{α}' is a subset of J_{α} . The fact $\bar{N}_1 \approx \bar{P} = \sum_J \sum_{J_{\alpha}'} \oplus \bar{P}_{\alpha i}$ implies $\mathfrak{S}_P = J(S_P)$. Let p_{N_1} be a projection of M to N_1 with $\text{Ker } p_{N_1} = N_2$. Then $\bar{p}_{N_1} | \bar{P}$ is isomorphic, and hence $p_{N_1} | P$ is isomorphic as an R -module. Therefore, $M = P \oplus N_2$ and $\sum_{I'} \oplus P_{\alpha i} \subseteq Q_{\alpha}$. We have shown that N_2 has the exchange property. If I' is finite, then N_2 has the exchange property from Lemma 3. Thus, we may assume that $\{M_{\beta}'\}$ is a T -nilpotent system. Noting that N_2 is an object of \mathfrak{A} by Lemma 2, first we assume $N_1 = \sum_K \oplus T_{\alpha}$, $N_2 = \sum_{K'} \oplus T'_{\beta}$ and $T_{\alpha} \neq T'_{\beta}$ for any α, β , where T_{α} and T'_{β} are indecomposable. We make use of the same notation as above. Then $\bar{M} = \bar{N}_2 \oplus \sum_J \oplus \bar{P}'_{\alpha}$ and $P'_{\alpha} \oplus P''_{\alpha} = P_{\alpha}$. Since $\sum \oplus P'_{\alpha} \approx N_1$, $\sum \oplus P'_{\alpha}$ is a direct sum-

mand of M by Proposition 2, say $M = \sum (P'_\alpha \oplus P'''_\alpha)$ and $Q_\alpha = P'_\alpha \oplus P'''_\alpha$. Then $\sum_j \oplus P'''_\alpha$ is an object in \mathfrak{A} by Lemma 2. Let p be a projection of M to $\sum_j \oplus P'_\alpha$ with $\text{Ker } p = \sum_j \oplus P'''_\alpha$. Then $\bar{p}|_{\bar{N}_2}$ is isomorphic, since $\bar{N}_1 \cap \sum_j \oplus \bar{P}'''_\alpha = \bar{0}$ and $\bar{p}(\bar{N}_2) = \bar{0}$ by the assumption. Hence, $p|_{N_1}$ is isomorphic as an R -module, which implies $M = N_1 \oplus \text{Ker } p = N_1 \oplus \sum_j \oplus P'''_\alpha$. Hence, N_1 has the exchange property in M . In general case, we choose all direct components $T_{\beta'}$ in N_2 , which is isomorphic to some T_α in N_1 and put $N'_2 = \sum \oplus T_{\beta'}$; $N_2 = N'_2 \oplus N''_2$. Then, $N'_1 = N_1 \oplus N'_2$ satisfies the assumption in the first case. Therefore, $M = N'_1 \oplus \sum_j \oplus P'''_\alpha$ and $Q_\alpha = P'_\alpha \oplus P'''_\alpha$. Then $M = N'_2 \oplus N_1 \oplus \sum_j \oplus P'''_\alpha$. Since N'_2 satisfies the assumption in the theorem, $N_1 \oplus \sum_j \oplus P'''_\alpha$ has the exchange property from the beginning case. Therefore, $M = N_1 \oplus \sum_j \oplus P'''_\alpha \oplus \sum_j \oplus P^{\text{iv}}_\alpha$, and $Q_\alpha \supseteq P^{\text{iv}}_\alpha$. Thus, we have proved that N_1 has the exchange property in M .

Corollary. *Let \mathfrak{A} be as above. Then the following statements are equivalent.*

- 1 *Every direct summand of object M in \mathfrak{A} has the \aleph_0 -exchange property in M .*
- 2 *Every direct summand of object M in \mathfrak{A} has the exchange property in M .*
- 3 *$\{M_\alpha\}$ is an elementwise T -nilpotent system.*

Proof. $1 \rightarrow 3$ Let $M = \sum_I \oplus M_\alpha = \sum_{I'} \oplus M'_{\beta'} \oplus \sum_{I''} \oplus M'_{\gamma'}$ be a direct decompositions with $|I'| \leq \aleph_0$. Since every direct summand of $\sum_{I'} \oplus M'_{\beta'}$ has the \aleph_0 -exchange property in M , it has the \aleph_0 -exchange property in $\sum_{I'} \oplus M'_{\beta'}$. Therefore, Condition II is satisfied for $\sum_{I'} \oplus M'_{\beta'}$, which implies 3 by [4], Lemma 9.

$3 \rightarrow 2$ It is clear from the theorem and Proposition 2.

$2 \rightarrow 1$ It is clear.

Proposition 3 ([1], [3], [6] and [10]). *Let M be in \mathfrak{A} and $M = N_1 \oplus N_2$. If N_1 is countably generated, then N_1 is in \mathfrak{A} . If every M_α is countably generated, then every direct summand of M is in \mathfrak{A} .*

Proof. We make use the argument of the proof of [1], Theorem 7.1. First, we note that for any element x in N_1 there exists a direct summand N_0 of N_1 such that $x \in N_0$ and N_0 is in \mathfrak{A} . Because, there exists a finite set J such that M_J contains x . From Theorem 4 we have $M = M_J \oplus N'_1 \oplus N'_2$, $N_1 = N'_1 \oplus N''_1$ and $N_2 = N'_2 \oplus N''_2$, where $N''_1 = (M_J \oplus N'_2) \cap N_1$ and $N''_2 = (M_J \oplus N'_1) \cap N_2$, and $x \in N''_1$. If we use the same argument in [6], then we obtain the proposition.

3 Semi-perfect modules

We shall study further properties of \mathfrak{A} in a case of semi-perfect modules defined by E. Mares in [9]. She has shown that every semi-perfect module is a direct sum of completely indecomposable semi-perfect modules ([9], Corollary 4.4). Let P be an R -module and $J(P)$ the radical of P . If P is semi-perfect, then $J(P)$ is small in P , $[P/J(P), P/J(P)]_{R/J(R)} \approx S_P/J(S_P)$ and $J(P) = PJ(R)$, (see [9], §§ 2-5).

Theorem 5. *Let P be a directly indecomposable projective module. Then P is completely indecomposable if and only if P is semi-perfect, (cf. [5], the proof of Theorem 2.8).*

Proof. If P is semi-perfect, then P is completely indecomposable by [9], Corollary 4.4. Conversely, we assume that so is P . Since $P/J(P)$ is $R/J(R)$ -projective, $J([P/J(P), P/J(P)]) = 0$. From an exact sequence $0 \rightarrow [P, J(P)]_R \rightarrow S_P \rightarrow [P/J(P), P/J(P)]_{R/J(R)} \rightarrow 0$ we have $[P, J(P)] \supset J(S_P)$. On the other hand, $J(S_P)$ is a unique maximal ideal in S_P and $[P, J(P)] \neq S_P$. Hence, $\mathfrak{F}_P = J(S_P) = [P, J(P)]_R$. Next, we shall show that $J(P)$ is small in P . Let N be a submodule of P such that $P = J(P) + N$. From the following row exact sequence

$$\begin{array}{ccccc} N & \longrightarrow & N/N \cap J(P) & \longrightarrow & 0 \\ & \nwarrow f & \uparrow \wr & & \\ & & P/J(P) & & \\ & & \uparrow & & \\ & & P & & \end{array}$$

we have $f: P \rightarrow N$, which commutes the above diagram. If $N \neq P$, $f \in \mathfrak{F}_P$. Hence, $\text{Im } f \subset N \cap J(P)$, which is a contradiction. Finally, we show that $J(P)$ is a unique maximal submodule in P . Put $\bar{P} = P/J(P)$, $\bar{R} = R/J(R)$ and $\bar{S} = S_P/\mathfrak{F}_P$. We define $\mu: \bar{P} \otimes_{\bar{R}} [\bar{P}, \bar{R}]_{\bar{R}} \rightarrow \bar{S}$ by setting $\mu(p \otimes f)(p') = pf(p')$.

Since $\bar{P} \neq 0$ and \bar{R} -projective, $\mu \neq 0$. Furthermore, \bar{S} is a division ring, and hence, μ is isomorphic. $\bar{P}\tau(P) = \bar{P}$ implies that there exists p in \bar{P} such that $\mu(p \otimes [\bar{P}, \bar{R}]) \neq 0$, where τ is the trace map of \bar{P} . Hence, $\mu(p \otimes f)\bar{S} = \bar{S}$ for some f in $[\bar{P}, \bar{R}]$. Therefore, $\bar{P} = \bar{S}\bar{P} = \mu(p \otimes f)\bar{S}\bar{P} \subset pf(\bar{P}) \subset p\bar{R} \subset \bar{P}$. Hence, $\bar{P} = p\bar{R} \approx \bar{e}\bar{R}$ for some idempotent \bar{e} in \bar{R} . Since $\bar{e}\bar{R}\bar{e}$ is a division ring and \bar{R} is semi-simple, \bar{P} is \bar{R} -irreducible by [8], Proposition 1 in p. 65. Hence, $J(P)$ is unique maximal, since $J(P)$ is the radical of P . Thus we have proved that P is semi-perfect by [9], Theorem 5.1.

Now let $\{P_\alpha\}$ be a family of completely indecomposable projective modules, and \mathfrak{A} the induced additive category from $\{P_\alpha\}$. Let $P = \sum \oplus P_\alpha$ and $P' = \sum \oplus P'_\beta$ be in \mathfrak{A} and f in $[P, P']_R$. If $f_{\alpha\beta} = p_\alpha f'_\beta$ is epimorphic, then $f_{\alpha\beta}$ splits and hence $f_{\alpha\beta}$ is isomorphic. Since $J(P'_\alpha)$ is unique maximal, $\text{Im } f_{\alpha\beta}$

$\subseteq J(P_\alpha')$ is $f_{\alpha\beta}$ is not isomorphic. Hence, if f is in \mathfrak{F} , then $\text{Im } f \subseteq \sum \oplus J(P_\beta') = J(P')$. Conversely, if $\text{Im } f \subseteq J(P')$, then f is in \mathfrak{F} . Therefore, $[P, P']_R \cap \mathfrak{F} = [P, J(P')]_R$. Furthermore, $0 \rightarrow [P, J(P')]_R \rightarrow [P, P']_R \rightarrow [P/J(P), P'/J(P')]_{R/J(R)} \rightarrow 0$ is exact. Thus, for any object P in \mathfrak{A} , many arguments in $\mathfrak{A}/\mathfrak{F}$ concerned with \bar{P} coincide with those as $R/J(R)$ -modules. From this reason, we make use of terminologies in $\mathfrak{A}/\mathfrak{F}$, instead of ones as $R/J(R)$ -modules, if there are no confusions.

Theorem 6. *Let \mathfrak{A} be an induced category from a family of completely indecomposable projective modules $\{P_\alpha\}$. Then an object $P = \sum_I \oplus P_\gamma$ in \mathfrak{A} is perfect if and only if $\{P_\gamma\}_I$ is an elementwise T -nilpotent system.*

Proof. Let \mathfrak{F}' be a full subcategory in \mathfrak{F} which is induced from $\{P_\gamma\}_I$. If P is perfect, then every object in \mathfrak{A}' is semi-perfect. Hence, \mathfrak{F}' is equal to the Jacobson radical in \mathfrak{A}' by the above remark and [9], Theorem 2.4. Therefore, $\{P_\gamma\}_I$ is a T -nilpotent system by [4], Theorem 8. Conversely, we assume that $\{P_\gamma\}_I$ is a T -nilpotent system. Then $\mathfrak{F}_P = J(S_P)$ for every object P in \mathfrak{A}' . We shall show that $J(P)$ is small in P for every object P in \mathfrak{A}' . Let $P = Q + J(P)$ for some submodule Q and p_1 a projection of P to P_1 , where $P = \sum_I \oplus P_\gamma$. Since $p_1(J(P)) \subseteq J(P_1)$ and $J(P_1)$ is small by Theorem 5, $p_1(Q) = P_1$. Hence, there exists f in $[P_1, Q]_R$ such that $p_1 f = 1_{P_1}$. Therefore, Q contains an object in \mathfrak{A}' which is a direct summand of P . Let T be the set of such objects in Q and define a partial order in T by the inclusion. We take a totally ordered subset $Q_1 \subset Q_2 \subset \dots$ in T . Put $Q_0 = \cup Q_i$, then $Q_0 = \sum \oplus N_\beta$; $N_\beta \approx P_{\pi(\beta)}$ by Lemma 2. Furthermore, the inclusion $i_\beta: N_\beta \rightarrow P$ is not zero modulo \mathfrak{F} , since Q_j is a direct summand of P . Hence, Q_0 is a direct summand of P by the proof of Proposition 2.2. Thus, we have a maximal element P_0 in T . $P = P_0 \oplus U$ and $Q = P_0 \oplus Q \cap U$. Since $P = Q + J(P)$ and $J(P) = J(P_0) \oplus J(U)$, $U = J(U) + U \cap Q$. U is also in \mathfrak{A}' by Lemma 2. If $U \neq 0$, $U \cap Q$ contains an object in \mathfrak{A}' which is a direct summand of U and hence of P . Which contradicts to the maximality of P_0 . Therefore, $P = P_0 = Q$. Thus, every object in \mathfrak{A}' is semi-perfect by Theorem 5 and [9], Theorem 5.2.

In the above argument, we have used only facts that P_i are semi-perfect and $\mathfrak{F}_P = J(S_P)$. Hence, from Lemma 1, [4], Corollary 2 to Theorem 7 and [9], Theorem 2.3 we have

Proposition 4. *Let $P = \sum \oplus P_\alpha$ and P_α semi-perfect. Then P is semi-perfect if and only if $\mathfrak{F}_P = J(S_P)$.*

Theorem 7. *Let P be an object in \mathfrak{A} induced from projective, completely indecomposable modules P_α . Then we have the following equivalent conditions.*

- 1 P is semi-perfect.

- 2 $\mathfrak{F}_P = J(S_P)$.
- 3 Every dense submodule of P coincides with P .
- 4 $P = \sum_i \oplus P_{\alpha}$ in \mathfrak{A} , then $\{P_{\alpha}\}$ is a semi- T -nilpotent system.
- 5 P satisfies the Condition II in [4].

Proof. $1 \leftrightarrow 2$ is proved in Proposition 4.

$1 \rightarrow 3$ Let N be a dense submodule of P . Then $N \supseteq \text{Im}(1-f)$ for some $f \in \mathfrak{F}_P$. Hence, $P \subseteq N + f(P) \subseteq N + J(P) \subseteq P$ by the remark before Theorem 6. Therefore, $P = N + J(P)$ implies $P = N$, since $J(P)$ is small.

$3 \rightarrow 4$ Let $\{f_i\}$ be a family of non isomorphisms of P_{α_i} to $P_{\alpha_{i+1}}$ ($P_{\alpha_i} \neq P_{\alpha_{i+1}}$). Put $f = \sum (-e_{i+1} f_i)$, where $\{e_{ij}\}$ is a system of matrix units in S_P . Then $\text{Im}(1-f)$ is a dense submodule of P . From the assumption and the argument of Lemma 9 in [4], we know that $\{f_i\}$ is a T -nilpotent sequence.

$2 \rightarrow 5$ is proved in [4], Corollary 2 to Theorem 7.

$5 \rightarrow 4$ is proved in [4], Lemma 9.

$4 \rightarrow 1$. Let

$$P = \sum_{K \ni \alpha} \sum_{I \ni \beta} \oplus M_{\alpha\beta} \cdots (*),$$

where $M_{\alpha\beta}$'s are indecomposable and $M_{\alpha\beta} \approx M_{\alpha\beta'}$, $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. First we assume that the cardinal λ_{α} of $|I_{\alpha}|$ is finite for all α in K . We put $P_{\alpha^{(n)}n} = \sum_{\beta=1}^n \oplus M_{\alpha\beta}$, where $n = \lambda_{\alpha}$, and show that $J(P)$ is small in P . We assume $P = N + J(P)$ for some submodule N of P . Let $p_{\alpha^{(n)}n}$ be a projection of P to $P_{\alpha^{(n)}n}$. Since λ_{α} is finite, $J(P_{\alpha^{(n)}n})$ is small in $P_{\alpha^{(n)}n}$. Hence, $p_{\alpha^{(n)}n}|N$ is epimorphic, and there exists $g \in [P_{\alpha^{(n)}n}, N]_R$ such that $(p_{\alpha^{(n)}n}|N)g = 1_{P_{\alpha^{(n)}n}}$. Put $P'_{\alpha^{(n)}n} = \text{Im } g$. Since $\text{Ker } p_{\alpha^{(n)}n} = \sum_{\alpha \neq \alpha^{(n)}} \oplus P_{\alpha\beta}$,

$$P = P'_{\alpha^{(n)}n} \oplus \sum_{\alpha \neq \alpha^{(n)}} \oplus P_{\alpha\beta} \cdots (**).$$

Now, we assume $N \not\subseteq M_{\alpha^{(n)}i_1}$ and $x_1 \in M_{\alpha^{(n)}i_1} - N$. Then $x_1 = x' + \sum y_i$ from (**), where $x' \in P'_{\alpha^{(n)}n}$, $y_i \in P_{\alpha\beta}$. From the assumption there exists some $y_i \notin N$, since $P'_{\alpha^{(n)}n} \subset N$. Hence, there exists x_2 in $M_{\alpha i_2} - N$ such that $y_i = x_2 + \sum z_j$, $z_j \in M_{\alpha j}$ ($j \neq i_2$). If we replace (*) by (**), we can find x_3 in $M_{\alpha j k} - N$ and $P = P'_{\alpha^{(n)}n} \oplus P'_{\alpha^{(m)}m} \oplus \sum \oplus P_{\alpha\beta}$. Repeating this argument, we have a sequence $\{x_j\}$ so that $x_i \in M_{\alpha i k_i} - N$, and $f_i(x_i) = x_{i+1}$, where f_i is a projection of P to $M_{\alpha_{i+1} k_{i+1}}$, which is a contradiction. Therefore, $J(P)$ is small. Finally, we shall consider a general case. Let $P = \sum_{\lambda_{\alpha} \geq \aleph_0} \sum_{\beta} \oplus M_{\alpha\beta} \oplus \sum_{\lambda_{\alpha} < \aleph_0} \sum_i \oplus M_{\alpha i}$ and put $P_1 = \sum_{\lambda_{\alpha} \geq \aleph_0} \sum_{\beta} M_{\alpha\beta}$ and $P_2 = \sum_{\lambda_{\alpha} < \aleph_0} \sum_i \oplus M_{\alpha i}$. We know from the first case P_2 is semi-perfect. If $\lambda_{\alpha} \geq \aleph_0$ for α , the fact that $\{M_{\alpha\beta}\}$ is semi- T -nilpotent implies from the definition that $\{M_{\alpha\beta}\}$ is a T -nilpotent system. Hence, P_1 is perfect by Theorem 6. Therefore, P is semi-perfect from [9], Corollary 5.3 (see Proposition 6 below).

Corollary 1. *Let P be projective and artinian, then P is perfect. Furthermore, if P' is a directsum of artinian submodules and P' is semi-perfect, then P' is perfect.*

Proof. If P is artinian and projective, then P is in \mathfrak{A} and \mathfrak{S}_P is nilpotent ideal by [5], Theorem 2.8. Hence, for any directsum M of any copies of P , we have $\mathfrak{S}_M = J(S_M)$, since \mathfrak{S}_M is nilpotent. Therefore, M is semi-perfect, and P is perfect. Let $P' = \sum \oplus P_i$; P_i 's are artinian and P' be semi-perfect. Then $\{P_i\}$ is a semi-T-nilpotent system from Theorem 7. Furthermore, since J_{P_i} is nilpotent, $\{P_i\}$ is a T-nilpotent system. Hence, P' is perfect from Theorem 6.

Corollary 2. *Let P be a semi-perfect module. Then there exists a maximal one among submodules which are perfect and direct summand of P . Those maximal perfect submodules are isomorphic each other.*

Proof. Let $P = \sum_{\lambda_\alpha < \aleph_0} \sum \oplus M_{\alpha i} \oplus \sum_{\lambda_\alpha > \aleph_0} \sum M_{\alpha \beta}$ as in the above proof. If $\mathfrak{S}_{M_{\alpha i}}$ is elementwise T-nilpotent, then $\{M_{\alpha i}, M_{\alpha \beta}\}$ is T-nilpotent, since it is semi-T-nilpotent. Hence, if we choose every $M_{\alpha i}$ whose ideal $\mathfrak{S}_{M_{\alpha i}}$ is T-nilpotent, $P_1 = \sum'_{\lambda_\alpha < \aleph_0} \sum \oplus M_{\alpha i} \oplus \sum_{\lambda_\alpha > \aleph_0} \sum \oplus M_{\alpha \beta}$ is a direct summand of M and perfect, where \sum' runs through all $M_{\alpha i}$ in the above. Put $P = P_1 \oplus P_2$. If $P = Q_1 \oplus Q_2$, $Q_1 \supseteq P_1$ and Q_1 is perfect, then $P = Q_1' \oplus P_1 \oplus Q_2$ and $Q_2 = Q_2' \oplus P_1$. Since $P_2 \approx Q_1' \oplus Q_2$, $Q_1' = (0)$ by the assumption. Hence, P_1 is a desired perfect submodule. Let T_1 be a maximal element as in Corollary 2; $P = T_1 \oplus T_2$, then T_2 is in \mathfrak{A} . It is clear that T_1 , P_1 and T_2 , P_2 have the isomorphic direct components, respectively. Hence, $P_1 \approx T_1$.

Finally, we shall give some results concerned with ones obtained in [9].

First we shall give another proof of [9], Theorem 5.5.

Proposition 5 ([9]). *Let \mathfrak{A} be as above and P a direct summand of an object M in \mathfrak{A} . If $J(P)$ is small in P , then P is in \mathfrak{A} .*

Proof. Let $M = P \oplus P_1$ and $P = eM$ for some idempotent e . P contains a dense submodule P_0 with inclusion i such that $if \equiv e \pmod{\mathfrak{S}}$ for some f in $[M, P_0]_R$. Put $e = if + x$, $x \in \mathfrak{S}$. Then $P = P_0 + x(P)$ and $x(P) \subset P \cap J(M) = J(P)$. Hence, $P = P_0$.

Proposition 6 ([9], Corollary 5.3). *Let $\{P_i\}_1^n$ be a finite set of semi-perfect modules. Then $\sum_1^n \oplus P_i$ is semi-perfect.*

Proof. Since $\mathfrak{S}_{P_i} = J(S_{P_i})$ for every i , we can show $\mathfrak{S}_P = J(S_P)$ by using

fundamental transformations of matrices (see [4], Lemma 8). Hence, P is semi-perfect from Proposition 4.

Proposition 7 ([9], Theorem 7.2). *If $J(R)$ is right T -nilpotent, then every semi-perfect module is perfect.*

Proof. Let $P = \sum_I \oplus P_\alpha$ be semi-perfect. Then $\mathfrak{F}_P = [P, J(P)] = [P, PJ(R)]$. Hence, for any $f \in [P_\alpha, P_\beta] \cap \mathfrak{F}$ and $x_\alpha \in P_\alpha$, $f(x_\alpha) = \sum x_{\beta_i} a_{\beta_i}$, $a_{\beta_i} \in J(R)$. Therefore, $\{P_\alpha\}$ is T -nilpotent system by the assumption. Hence, P is perfect from Theorem 6.

REMARK. [9], Theorem 5.1 is a special case of Theorem 3.

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References

- [1] P. Crawley and B. Jónnson: *Refinements for infinite direct decomposition of algebraic systems*, Pacific J. Math. **14** (1964), 797-855.
- [2] C. Ehresmann: *Catégories et Structures*, Dunod, Paris, 1965.
- [3] S. Elliger: *Zu dem Satz von Krull-Remak-Schmidt-Azumaya*, Math. Z. **115** (1970), 227-230.
- [4] M. Harada and Y. Sai: *On categories of indecomposable modules I*, Osaka J. Math. **7** (1970), 323-344.
- [5] M. Harada: *On semi-simple categories*, Osaka J. Math. **7** (1970), 89-95.
- [6] I. Kaplansky: *Projective modules*, Ann. of Math. **68** (1958), 372-377.
- [7] G.M. Kelly: *On the radical of a category*, J. Austral. Math. Soc. **4** (1964), 299-307.
- [8] N. Jacobson: *Structure of Rings*, Amer. Math. Soc., 1956.
- [9] E. Mares: *Semi-perfect modules*, Math. Z. **83** (1963), 347-360.
- [10] R.B. Warfield Jr.: *A Krull-Schmidt theorem for infinite sums of modules*, Proc. Amer. Math. Soc. **22** (1969), 460-465.

