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On the Examples in the Classification of Open Riemann Surfaces (I)

By Yukinari Tôki

In the preceding paper[1] the author has given two examples on the classification of open Riemann surfaces, but more examples will be needed to complete the classification.

The following notations are customary in the theory of Riemann surfaces:

O_G	the class of Riemann surfaces without the Green's function.	
O_{HP}	the class of Riemann surfaces without any non-constant single-valued	positive harmonic functions.
O_{HB}	"	bounded harmonic functions.
O_{HD}	"	harmonic functions of finite Dirichlet integrals.
O_{AB}	"	bounded analytic functions.
O_{AD}	"	analytic functions of finite Dirichlet integrals.

The known inclusion-relations between them are

$$\begin{tikzcd}[column sep=small, row sep=small] & O_{HD} \arrow[dr, phantom, sloped, "\subseteq"] & \\ O_G \subseteq O_{HP} \subseteq O_{HB} \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & O_{AD} \\ & O_{AB} \arrow[ur, phantom, sloped, "\subseteq"] & \end{tikzcd} \quad \text{and} \quad O_G \subset O_{HB}.$$

In the present paper we shall show in §1 that we may put $\subset$ in place of $\subseteq$ in the above relations, and in §2 that there exists no inclusion-relation between O_{AB} and O_{HD}.

§1. In order to prove $O_G \subset O_{HP}$ it is sufficient to construct a Riemann surface, on which the Green's function exists but not any non-constant single-valued positive harmonic function. This surface is almost the same as the one that the author has shown in the previous work.

1) Y. Tôki, On the Classification of Open Riemann Surfaces, Osaka Math. J. 4, (1952), pp. 191-201.

We shall define a sequence $\{k_\mu\}$ $(\mu=1,2,3,\cdots)$ as follows:

By the cut α_μ along the positive real axis, we make out of the ring domain $-\left(\frac{1}{2}\right)^{2\mu+1}<\log|z|<-\left(\frac{1}{2}\right)^{2\mu+2}$ a simply connected domain D_μ.

Let $\omega(z, \alpha_\mu, D_\mu)$ be the harmonic measure of α_μ with respect to D_μ, and let C_μ be the circle $\log|z|=-\frac{3}{2}\left(\frac{1}{2}\right)^{2\mu+2}$.

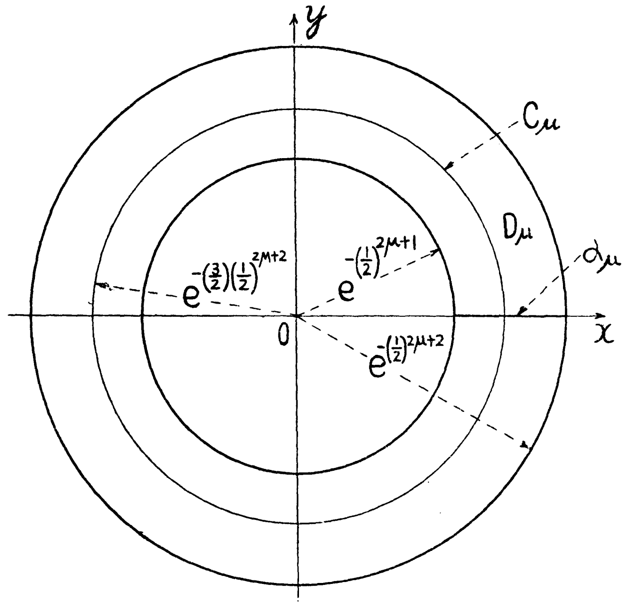


Fig. 1

Put

$$k_\mu=\underset{z\in C_\mu}{\mathrm{Min}}\,\omega(z, \alpha_\mu, D_\mu). \qquad (\mu=1,2,3,\cdots) \tag{1}$$

Then we have a sequence $\{k_\mu\}$, and see easily that $\lim\limits_{\mu\to\infty} k_\mu=0$.

On the other hand we shall define another sequence of positive integers $\{\tau_\mu\}$ $(\mu=1,2,3,\cdots)$ such that

$$\tau_1<\tau_2<\tau_3<\cdots,$$

and that

$$\underset{z\in\gamma_\mu}{\mathrm{Max}}\,\omega(z, \beta_\mu, R_\mu)\leq k_\mu^3, \tag{2}$$

where R_μ is the domain enclosed by four straight lines $x=-\left(\frac{1}{2}\right)^{2\mu}$, $x=-\left(\frac{1}{2}\right)^{2\mu+1}$, $y=0$, and $y=\frac{\pi}{2^{\tau_\mu}}$, and β_μ is the part of the boundary parallel to the imaginary axis and γ_μ is the part of the straight line $x = -\frac{3}{2}\left(\frac{1}{2}\right)^{2\mu+1}$ contained in R_μ.

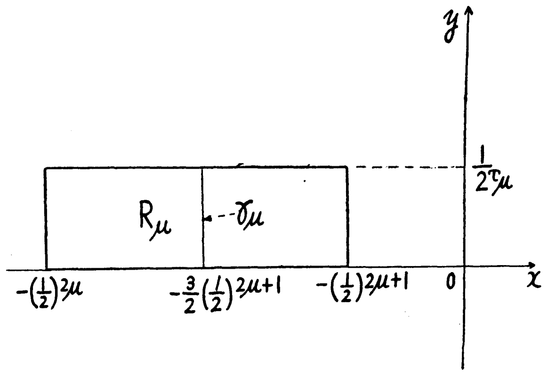


Fig. 2

Now we shall construct a Riemann surface with the Green's function but without any non-constant single-valued positive harmonic function. We consider the surface $\boldsymbol{F}$ cut along radial slits S_μ^ν ($\mu = 1, 2, \cdots$; $\nu = 1, 2, \cdots, 2^{\tau_\mu}$) on the unit-circle $|z| < 1$, where

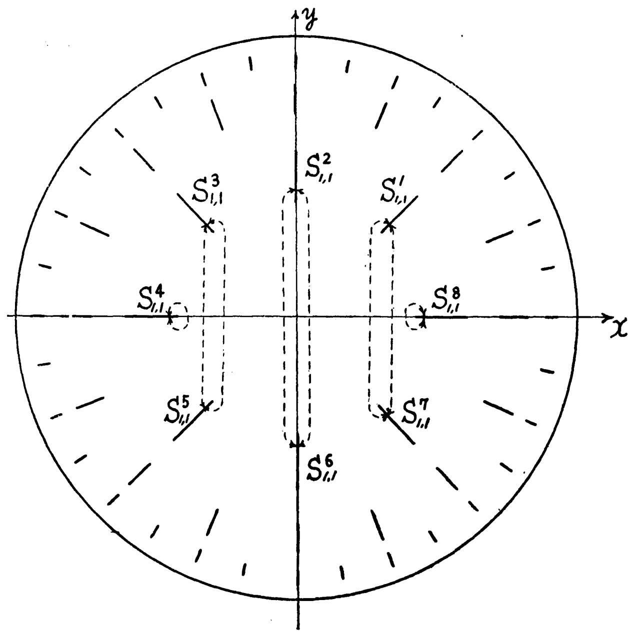
$$S_\mu^\nu ;\ z = re^{i\theta_\nu},\ -\left(\frac{1}{2}\right)^{2\mu} \leq \log r \leq -\left(\frac{1}{2}\right)^{2\mu+1},\ \theta_\nu = \frac{2\nu\pi}{2^{\tau_\mu}}.$$

By the relation $\mu = 2^{m-1}(2n-1)$ natural numbers μ correspond one-to-one to the pairs of two natural numbers (m, n). Therefore we shall denote the slits S_μ^ν by $S_{m,n}^\nu$. These slits $S_{m,n}^\nu$ ($m = 1, 2, \cdots$; $n = 1, 2, \cdots$; $\nu = 1, 2, \cdots, 2^{\tau_\mu}$) are symmetric with respect to the real axis.

m \ n	1	2	3	4	5	$\cdots$
1	1	3	5	7	9	$\cdots$
2	2	6	10	14	18	$\cdots$
3	4	12	20	28	36	$\cdots$
4	8	24	40	56	72	$\cdots$
$\vdots$	$\cdots$					

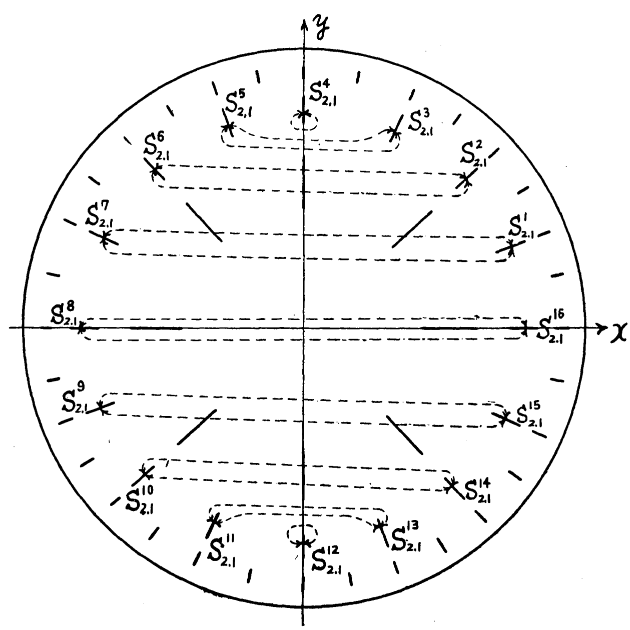
Let $T_1(z)$ be the indirectly conformal mapping such that each point z corresponds to the point $\bar{z}$. We shall identify each two sides of slits $S_{1,n}^\nu$ ($n = 1, 2, \cdots$; $\nu = 1, 2, \cdots, 2^{\tau_\mu}$) corresponding each other $T_1(z)$.

Let $T_2(z)$ be the mapping such that each point z corresponds to the symmetric point $\tilde{z}$ with respect to the imaginary axis. Then we shall identify two sides of slits $S_{2,n}^\nu$ ($n = 1, 2 \cdots$; $\nu = 1, 2, \cdots, 2^{\tau_\mu}$) corresponding each other by $T_2(z)$.



y
x
$S_{1,1}^{1}$
$S_{1,1}^{2}$
$S_{1,1}^{3}$
$S_{1,1}^{4}$
$S_{1,1}^{5}$
$S_{1,1}^{6}$
$S_{1,1}^{7}$
$S_{1,1}^{8}$

Fig. 3



y
x
$S_{2,1}^{1}$
$S_{2,1}^{2}$
$S_{2,1}^{3}$
$S_{2,1}^{4}$
$S_{2,1}^{5}$
$S_{2,1}^{6}$
$S_{2,1}^{7}$
$S_{2,1}^{8}$
$S_{2,1}^{9}$
$S_{2,1}^{10}$
$S_{2,1}^{11}$
$S_{2,1}^{12}$
$S_{2,1}^{13}$
$S_{2,1}^{14}$
$S_{2,1}^{15}$
$S_{2,1}^{16}$

Fig. 4

Let $T_{3,1}(z)$ be the mapping such that each point z $\left(0 \leq \arg z \leq \frac{\pi}{2}\right.$ or $\left.\pi \leq \arg z \leq \frac{3}{2}\pi\right)$ corresponds to the symmetric point z with respect to the line $y = \left(\tan \frac{\pi}{4}\right) x$. Let $T_{3,2}(z)$ be the mapping such that each point $z\left(\frac{\pi}{2} \leq \arg z \leq \pi \text{ or } \frac{3}{2}\pi \leq \arg z \leq 2\pi\right)$ corresponds to the symmetric point $\tilde{z}$ with respect to line $y = \left(\tan \frac{3}{4}\pi\right) x$.

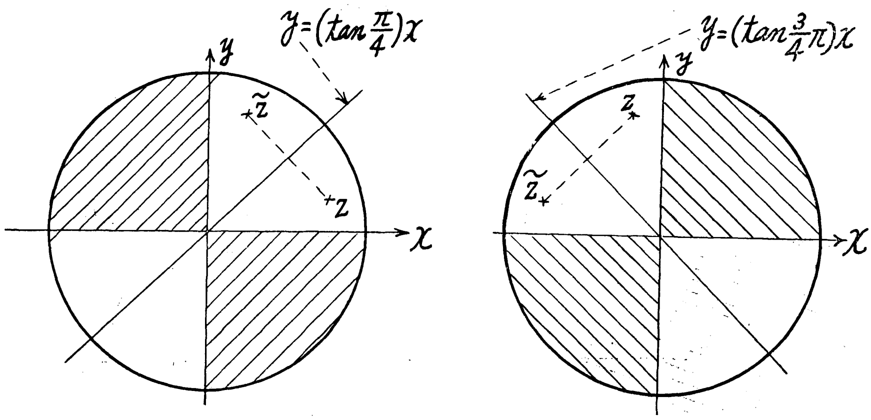


Fig. 5

Then we shall identify two sides of slits $S_{3,n}^{\nu}$ $(n = 1, 2, \cdots; \nu = 1, 2, \cdots, 2^{\tau_\mu})$ corresponding each other by $T_{3,1}(z)$ and $T_{3,2}(z)$.

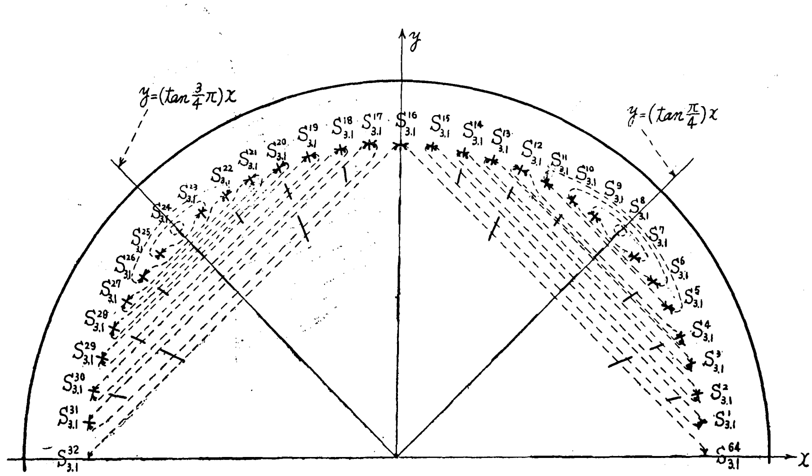


Fig. 6

Next we shall define the mapping $T_{4,1}(z)$, $T_{4,2}(z)$, $T_{4,3}(z)$, $T_{4,4}(z)$ as follows.

$T_{4,1}(z)$	the mapping such that each point z $\left(0 \leqq \arg z \leqq \frac{\pi}{4} \text{ or } \pi \leqq \arg z \leqq \left(\pi+\frac{\pi}{4}\right)\right)$	corresponds to the symmetric point $\tilde{z}$ with respect to $y=\left(\tan \frac{\pi}{8}\right) x$
$T_{4,2}(z)$	" $\left(\frac{\pi}{4} \leqq \arg z \leqq \frac{\pi}{2} \text{ or} \left(\pi+\frac{\pi}{4}\right) \leqq \arg z \leqq \frac{3}{2}\pi\right)$	" $y=\left(\tan \frac{3}{8}\pi\right) x$
$T_{4,3}(z)$	" $\left(\frac{\pi}{2} \leqq \arg z \leqq \frac{3}{4}\pi \text{ or } \left(\pi+\frac{\pi}{2}\right) \leqq \arg z \leqq \left(\pi+\frac{3}{4}\pi\right)\right)$	" $y=\left(\tan \frac{5}{8}\pi\right) x$
$T_{4,4}(z)$	" $\left(\frac{3}{4}\pi \leqq \arg z \leqq \pi \text{ or } \left(\pi+\frac{3}{4}\pi\right) \leqq \arg z \leqq (\pi+\pi)\right)$	" $y=\left(\tan \frac{7}{8}\pi\right) x$

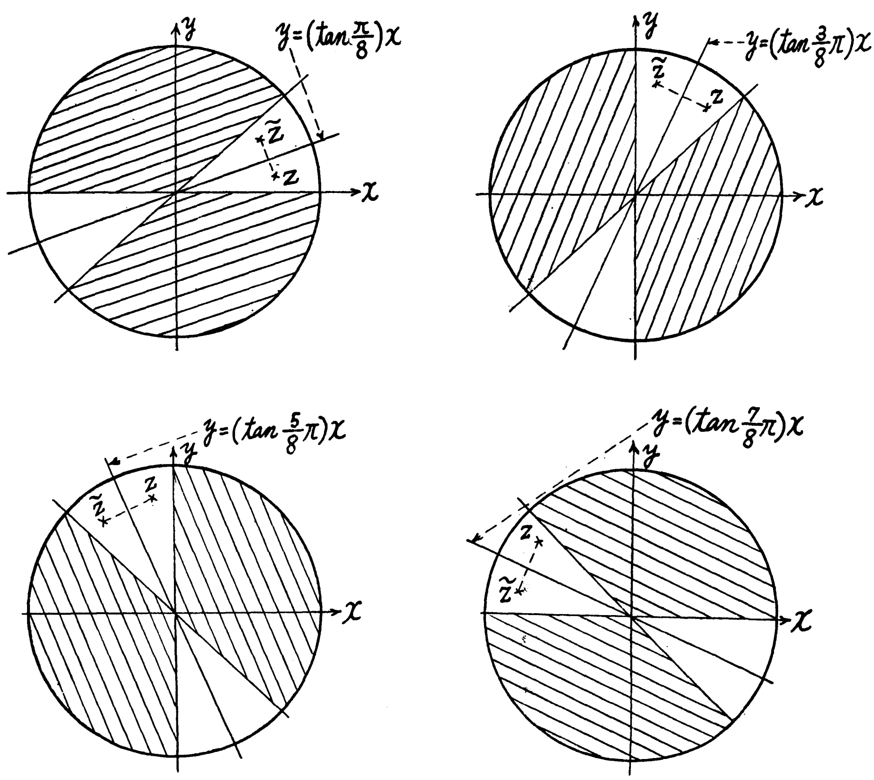


Fig. 7

Then we shall identify each sides of slits $S_{4,n}^{\nu}$ $(n=1,2,\cdots;\ \nu=1,2,\cdots,2^{\tau_{\mu}})$ corresponding each other by $T_{4,1}(z)$, $T_{4,2}(z)$, $T_{4,3}(z)$ and $T_{4,4}(z)$.

Proceeding in this way we can construct a Riemann surface $\hat{\boldsymbol{F}}$. We shall prove $\hat{\boldsymbol{F}}$ is just the required Riemann surface.

Lemma 1. Let D_μ' be a simply connected domain enclosed by two circles $\log|z| = -\left(\frac{1}{2}\right)^{2\mu+1}$, $\log|z| = -\left(\frac{1}{2}\right)^{2\mu+2}$, and a Jordan arc α_μ' connecting the two circles.

Then $2k_\mu' > k_\mu$ where $k_\mu' = \underset{z\in C_\mu}{\operatorname{Min}}\, \omega(z, \alpha_\mu', D_\mu')$ $(\mu = 1, 2, \cdots)$.

Proof. Let $k_\mu' = \omega(z_0, \alpha_\mu', D_\mu')$ at z_0 on C_μ. We may suppose without loss of generality that z_0 is on the real axis. We shall denote by

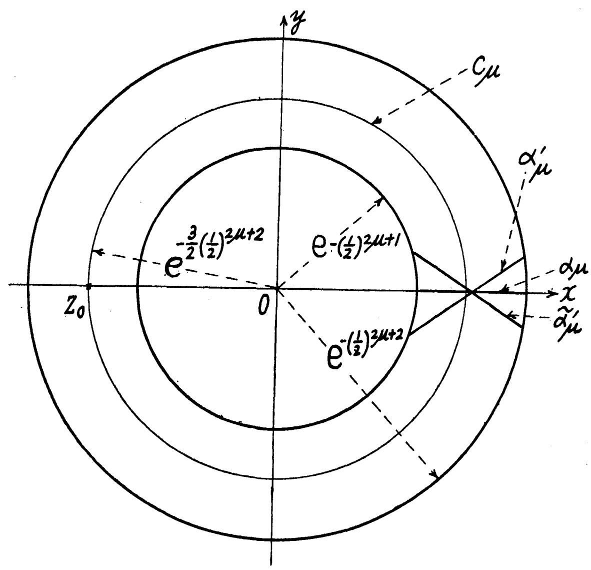


Fig. 8

$\tilde{D}_\mu'$ and $\tilde{\alpha}_\mu'$ the symmetric domain and arc of D_μ and α_μ respectively with respect to the real axis, and denote by D_μ'' the component of $D_\mu'\cdot\tilde{D}_\mu'$ containing the point z_0.

Then we have

$$\omega(z, \alpha_\mu', D_\mu') + \omega(z, \tilde{\alpha}_\mu', \tilde{D}_\mu') > \omega(z, \alpha_\mu' + \tilde{\alpha}_\mu', D_\mu'') \qquad z \in D_\mu''$$

$$\omega(z_0, \alpha_\mu', D_\mu') = \omega(z_0, \tilde{\alpha}_\mu', \tilde{D}_\mu') = k_\mu'$$

$$2k_\mu' > \omega(z_0, \alpha_\mu' + \tilde{\alpha}_\mu', D_\mu'') \geq \omega(z_0, \alpha_\mu, D_\mu) \geq k_\mu$$

i. e.

$$2k_\mu' > k_\mu$$

Now let $u(z)$ be a non-constant single-valued positive harmonic function on $\hat{F}$ and let M_μ be the maximum value of $u(z)$ on the circle $\log|z| = -\left(\frac{1}{2}\right)^{2\mu+1}$.

i) When $\overline{\lim\limits_{\mu\to\infty}}\, k_\mu^2 M_\mu > 1$, there exists a sequence of positive integers such that

$$\mu_1 < \mu_2 < \mu_3 < \cdots,$$

and

$$\frac{1}{k_{\mu_j}^2} < M_{\mu_j}, \qquad (j = 1, 2, \cdots).$$

Then for every μ_j there exists a Jordan arc α_{μ_j} connecting two circles $\log|z| = -\left(\frac{1}{2}\right)^{2\mu_j+1}$ and $\log|z| = -\left(\frac{1}{2}\right)^{2\mu_j+2}$ such that $u(z) > \frac{1}{k_{\mu_j}}$, $z \in C_{\mu_j}$.

By the Lemma 1 we have

$$\operatorname*{Min}_{z \in C_{\mu_j}} \omega(z, \alpha'_{\mu_j}, D'_{\mu_j}) > \frac{1}{2} k_{\mu_j},$$

consequently

$$\operatorname*{Min}_{z \in C_{\mu_j}} u(z) \geq \operatorname*{Min}_{z \in C_{\mu_j}} M_{\mu_j} \omega(z, \alpha'_{\mu_j}, D'_{\mu_j}) > \frac{1}{k_{\mu_j}^2} \cdot \frac{1}{2} k_{\mu_j} = \frac{1}{2} \frac{1}{k_{\mu_j}}.$$

Let $j \to \infty$, then $k_{\mu_j} \to 0$, $\lim\limits_{j\to\infty} \frac{1}{2k_{\mu_j}} = \infty$.

Thus $u(z)$ must be reduced to constant infinity, which is a contradiction.

ii) When $\overline{\lim\limits_{\mu\to\infty}}\, k_\mu^2 M_\mu \leq 1$, we can find a number N such that for $\mu > N$

$$M_\mu < \frac{2}{k_\mu^2}.$$

We shall denote k_μ, and M_μ, respectively by $k_{m,n}$, and $M_{m,n}$, where the pairs of two natural numbers (m, n) correspond one-to-one to μ by the relation $\mu = 2^{m-1}(2n-1)$.

Put

$$u_1(z) = \frac{1}{2}\left[u(z) - u(T_1(z))\right],$$

then $u_1(z)$ is a single-valued harmonic function on $\hat{F}$ and vanishes on $S_{1,n}^\nu$ $(n = 1, 2, \cdots;\ \nu = 1, 2, \cdots, 2^{\tau_\mu})$ and $|u_1(z)| \leq M_{1,n}$.

In view of (2) we therefore obtain for $\mu > N$ and $\mu = 2n-1$

$$\max_{|z|=\exp-\left(\frac{1}{2}\right)^{2\mu+1}} u_1(z) \leq M_{1,n} \cdot k_{1,n}^2 < \frac{2}{k_{1,n}^2} k_{1,n}^3 = 2k_{1,n}.$$

Let $\mu \to \infty$, then $k_{1,n} \to 0$, and $u_1(z) \equiv 0$ on $\hat{\boldsymbol{F}}$. Therefore

(3) $$u(z) = u(T_1(z)) \text{ on } \hat{\boldsymbol{F}} \text{ and } \frac{\partial u}{\partial n} = 0,$$

where $\frac{\partial u}{\partial n}$ is the normal derivative with respect to the real axis or to $S_{1,n}^{\nu}$ $(n = 1, 2, \cdots;\ \nu = 1, 2, \cdots 2^{\tau_\mu})$.

Put $u_2(z) = \frac{1}{2}[u(z) - u(T_2(z))]$, then in the same way we have the next inequality

$$\max_{|z|=\exp-\left(\frac{1}{2}\right)^{\mu+1}} u_2(z) < 2k_{2,n} \text{ for } \mu > N \text{ and } \mu = 2(2n-1).$$

Let $\mu \to \infty$, then $k_{2,n} \to 0$, and $u_2(z) \equiv 0$ on $\hat{\boldsymbol{F}}$. Therefore

(4) $$u(z) = u(T_2(z)) \text{ on } \hat{\boldsymbol{F}} \text{ and } \frac{\partial u}{\partial n} = 0,$$

where $\frac{\partial u}{\partial n}$ is the normal derivative with respect to the imaginary axis or to $S_{2,n}^{\nu}$ $(n = 1, 2, \cdots;\ \nu = 1, 2, \cdots, 2^{\tau_\mu})$.

Next we divide $\hat{\boldsymbol{F}}$ into two components by the cuts on $S_{1,n}^{\nu}$ and by the cuts, say C', on the real axis but not on S_μ^ν. Let $\boldsymbol{F}_1$ be the component on the upper half plane. Then we shall identify two sides of slits $S_{1,n}^{\nu}$ on $\boldsymbol{F}_1$ corresponding by $T_2(z)$ and shall identify the cuts C' on $\boldsymbol{F}_1$ corresponding by $T_2(z)$. Thus we have a new Riemann surface, say $\hat{\boldsymbol{F}}_1$.

Let us define a function $u_3(z)$ on $\hat{\boldsymbol{F}}_1$ by the value of $u(z)$ on $\boldsymbol{F}_1$. Then by (3) and (4) $u_3(z)$ is harmonic on $\hat{\boldsymbol{F}}_1$.

Put

$$v_3(z) = \frac{1}{2}[u_3(z) - u_3(T_2(z^2))],$$

then in the same way $u_3(z) = u_3(T_2(z^2))$ on $\hat{\boldsymbol{F}}_1$. Therefore

(5) $u(z)$ is symmetric with respect to the lines $y = \left(\tan\frac{\pi}{4}\right)x$, $y = \left(\tan\frac{3\pi}{4}\right)x$, and $\frac{\partial u}{\partial n} = 0$,

where $\frac{\partial u}{\partial n}$ is the normal derivative with respect to the lines

$$y=\left(\tan\frac{\pi}{4}\right)x,\ y=\left(\tan\frac{3\pi}{4}\right)x, \text{ and to } S_{3,n}^{\vee} \text{ on } \hat{F}_1.$$

And next we divide $\hat{F}$ into four components by the cuts on $S_{1,n}^{\vee}$ and on $S_{2,n}^{\vee}$ and by the cuts, say C^2, on the real and imaginary axis but not on $S_{\mu}^{\vee}$. Let $\hat{F}_2$ be the component containing a point $z\left(\arg z=\frac{\pi}{4}\right)$. Now we shall identify two sides of slits $S_{1,n}^{\vee}$ and $S_{2,n}^{\vee}$ on F_2 corresponding by $T_2(z^2)$ and shall identify the cuts C^2 on F_2 corresponding by $T_2(z^2)$.

Thus we have a new Riemann surface, say $\hat{F}_2$. Let us define a function $u_4(z)$ on $\hat{F}_2$ by the value of $u(z)$ F_2. Then by (3), (4) and (5) $u_4(z)$ is harmonic on $\hat{F}_2$.

Put

$$v_4(z)=\frac{1}{2}\left[u_4(z)-u_4(T_2(z^{2^2}))\right],$$

then in the same way

$$u_4(z)=u_4(T_2(z^{2^2})) \text{ on } \hat{F}_2.$$

Therefore

(6) $u(z)$ is symmetric with respect to the lines $y=\left(\tan\frac{\pi}{8}\right)x$, $y=\left(\tan\frac{3\pi}{8}\right)x$, $y=\left(\tan\frac{5}{8}\pi\right)x$, $y=\left(\tan\frac{7}{8}\pi\right)x$ and $\frac{\partial u}{\partial n}=0$, where $\frac{\partial u}{\partial n}$ is the normal derivative with respect to above four lines or to $S_{4,n}^{\vee}$ on $\hat{F}_2$.

In the same way we can prove that $u(z)$ is symmetric with respect to the lines $y=\left(\tan\frac{2m+1}{2^n}\pi\right)x$, where $n=1, 2, \cdots$ and $m=1, 2, \cdots$.

Therefore $u(z)$ must be a constant on $\hat{F}$, which is a contradiction.

On the other hand let us define a function $G(p)$ on $\hat{F}$ by $\log\frac{1}{|z|}$ at the point p corresponding to z. Then it is clear that $G(p)$ is the Green's function on $\hat{F}$. Therefore the Riemann surface F is just the required one.

It is clear that the surface after extracting a point p from the surface $\hat{F}$ does not belong to O_{HP}, but belongs to O_{HB}.

Thus we have proved that $O_G \subset O_{HP} \subset O_{HB}$.

§ 2. Now we shall construct a Riemann surface with a single-valued bounded analytic fuuction, but with no harmonic function of finite Dirichlet integral.

We shall consider the surface $\boldsymbol{F}_0$ cut along the radial slits S_μ^ν $(\mu = 1, 2, \cdots ; \nu = 1, 2, \cdots, 2^{2\mu})$ on the unit-circle $|z| < 1$ as follows:

$$S_\mu^\nu ; \quad z = re^{i\theta_\nu}, \quad -\left(\frac{1}{2}\right)^{2\mu} \leq \log r \leq -\left(\frac{1}{2}\right)^{2\mu+1}, \quad \theta_\nu = \frac{2\nu\pi}{2^{2\mu}}$$

Let $\boldsymbol{F}(h)$ and $\hat{\boldsymbol{F}}(h)$ $(h = 1, 2, \cdots)$ be one-sheeted covering surfaces without any relative boundaries over the basic surface $\boldsymbol{F}_0$. We shall denote the slits S_μ^ν by $S_{m,n}^\nu$, where m and n are natural numbers with the relation $\mu = 2^{m-1}(2n-1)$.

We shall construct the covering surface W over the unit-circle connecting the surfaces $\{\boldsymbol{F}(h)\}$ and $\{\hat{\boldsymbol{F}}(h)\}$ as follows:

<table>
<tr><td rowspan="16">We shall connect crosswise</td><td>$\boldsymbol{F}(\boldsymbol{k}+1)$ and $\hat{\boldsymbol{F}}(\boldsymbol{k}+1)$</td><td>on each slit over $S_{1,n}^\nu$</td></tr>
<tr><td>$\boldsymbol{F}(2\boldsymbol{k}+1)$ and $\hat{\boldsymbol{F}}(2\boldsymbol{k}+2)$</td><td rowspan="2">$S_{2,n}^\nu$</td></tr>
<tr><td>$\boldsymbol{F}(2\boldsymbol{k}+2)$ and $\hat{\boldsymbol{F}}(2\boldsymbol{k}+1)$</td></tr>
<tr><td>$\boldsymbol{F}(2^2\boldsymbol{k}+1)$ and $\hat{\boldsymbol{F}}(2^2\boldsymbol{k}+3)$</td><td rowspan="4">$S_{3,n}^\nu$</td></tr>
<tr><td>$\boldsymbol{F}(2^2\boldsymbol{k}+2)$ and $\hat{\boldsymbol{F}}(2^2\boldsymbol{k}+4)$</td></tr>
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<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+1)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+5)$</td><td rowspan="8">$S_{4,n}^\nu$</td></tr>
<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+2)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+6)$</td></tr>
<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+3)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+7)$</td></tr>
<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+4)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+8)$</td></tr>
<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+5)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+1)$</td></tr>
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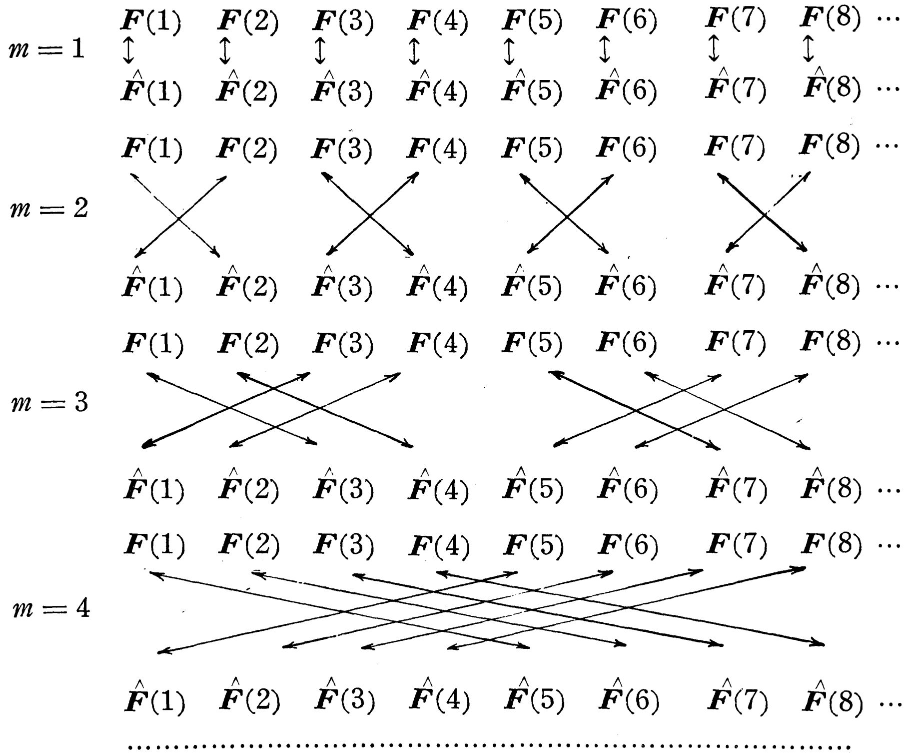
<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+7)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+3)$</td></tr>
<tr><td>$\boldsymbol{F}(2^3\boldsymbol{k}+8)$ and $\hat{\boldsymbol{F}}(2^3\boldsymbol{k}+4)$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td></tr>
<tr><td colspan="3">where $\boldsymbol{k} = 0, 1, 2, \cdots, \boldsymbol{n} = 1, 2, 3, \cdots$
$\nu = 1, 2, \cdots, 2^{2\mu}$.</td></tr>
</table>

We shall show the above correspondence among $\{\boldsymbol{F}(h)\}$ and $\{\hat{\boldsymbol{F}}(h)\}$ by diagrams:



Thus we can construct the Riemann surface W.

Lemma 2. Let D be two-sheeted covering surfaces over the stripe-domain $-\frac{1}{2} < x < -\frac{1}{16}$ in the z-$(=x+iy)$ plane, having its all branch points over the points $-\frac{1}{4}+\frac{n\pi}{2}i$ and $-\frac{1}{8}+\frac{n\pi}{2}i$ $(n=0, \pm 1, \pm 2, \cdots)$. Consider in D all the harmonic functions $u(z)$ that possess the zeros at $-\frac{1}{4}+\frac{n\pi}{2}i$ and $-\frac{1}{8}+\frac{n\pi}{2}i$ $(n=0, \pm 1, \pm 2, \cdots)$ respectively and statisfy $|u(z)| \leq M$ on the boundaries over $x=-\frac{1}{2}$ and $x=-\frac{1}{16}$. Then there exists a constant $0 < a < 1$, independent of u, such that

(1) $$|u(z)| \leq aM$$

holds on the straight line $x=-\frac{3}{16}$.

Proof. If (1) were not true on the segment $L: x=-\frac{3}{16}$, $0 \leq y \leq 2\pi$, there would exist a sequence $\{u_n(z)\}$

$$\lim_{n\to\infty} \max_{z\in L} |u_n(z)| = M.$$

A subsequence, say again $\{u_n(z)\}$, would converge towards a function $u(z)$, harmonic and bounded, $|u(z)| \leq M$, in D. The points z_n where $u_n(z)$ takes its maximum on L accumulate at least to one point z_0 on L. It follows from the continuity of $u(z)$ and the uniform convergence of $\{u_n(z)\}$ on L that $|u(z_0)| = M$. But $|u(z)|$ can not be identically M, since $u(z)$ really has the zero-points. This contradicts the maximum principle.

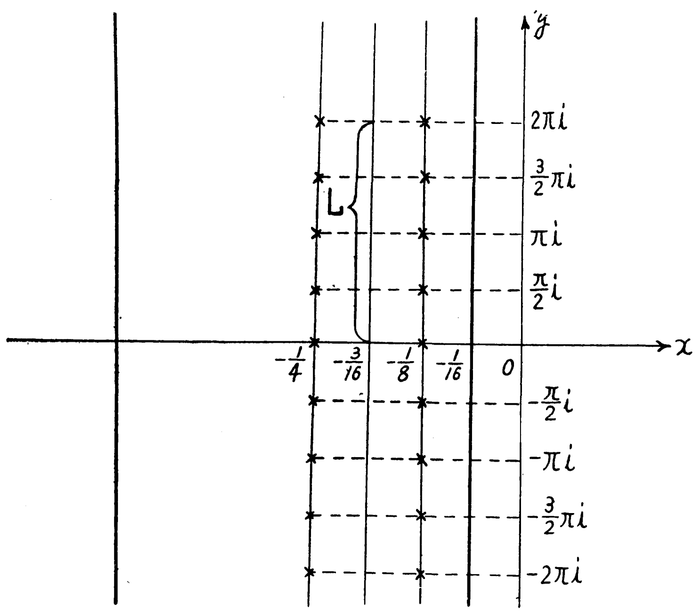


Fig. 9

Therefore (1) is true for L. By the transformations $T_m(z) = z + 2m\pi i$ $(m = \pm 1, \pm 2, \cdots)$ the segments L_m; $x = -\dfrac{3}{16}$, $2m\pi i \leq y \leq 2(m+1)\pi i$, are mapped on the segment L, consequently (1) is true for the whole straight line $x = -\dfrac{3}{16}$.

Then we shall prove that W is just the required Riemann surface. Let $u(p)$ be an arbitrary single-valued bounded harmonic function on W. We may assume $|u(p)| < 1$ without loss of generality. Let $W_{m,n}$ $(m = 1, 2, \cdots ; n = 1, 2, \cdots)$ be the covering subsurfaces of W over the ring-domains $R_{m,n}$ respectively, where

$$R_{m,n}, \quad -\left(\frac{1}{2}\right)^{2\mu-1} < \log|z| < -\left(\frac{1}{2}\right)^{2\mu+2} \quad \mu = -2^{m-1}(2n-1).$$

It is clear that each component of W_m is a two-sheeted covering surface over $R_{m,n}$.

Let $T_m(p)$ $(m=1, 2, \cdots)$ be the conformal mappings of W onto itself as follows:

$T_1(p)$ is the mapping	by which a point on $F(k+1)$ over z	corresponds to a point on $\hat{F}(k+1)$ over the same point z
$T_2(p)$	$F(2k+1)$	$\hat{F}(2k+2)$
	$F(2k+2)$	$\hat{F}(2k+1)$
$T_3(p)$	$F(2^2k+1)$	$\hat{F}(2^2k+3)$
	$F(2^2k+2)$	$\hat{F}(2^2k+4)$
	$F(2^2k+3)$	$\hat{F}(2^2k+1)$
	$F(2^2k+4)$	$\hat{F}(2^2k+2)$
$\vdots$	$\vdots$	$\vdots$

Put

$$u_m(p)=\frac{1}{2}\left[u(p)-u(T_m(p))\right], \qquad (m=1, 2, \cdots).$$

Then $u_m(p)$ are single-valued harmonic functions which vanish on the branch points over the end-points of $S^{\nu}_{m,n}$ $(n=1, 2, \cdots; \nu=1, 2, \cdots, 2^{2\mu})$ and $|u_m(p)|<1$.

Application of Lemma 2 after suitable auxiliary transformations implies that the inequality

$$|u_m(p)|<a<1$$

holds for all points p over the circles $\log|z|=-\frac{3}{4}\left(\frac{1}{2}\right)^{\mu}$ $(\mu=2^{m-1}(2n-1)$, $n=1, 2, \cdots)$. Then we see

$|u_m(p)|<a^n$ for all points over the circle $\log|z|=-\frac{3}{4}\left(\frac{1}{2}\right)^{2m-1}$. Let $n\to\infty$, then $a^n\to 0$. Therefore all functions $u_m(p)$ $(m=1, 2, \cdots)$ are identically zero on W. So $u(p)$ takes the same value on every points on $F(h)$ $(h=1, 2, \cdots)$ over a point z on the unit-circle. This fact means that $u(p)$ has no finite Dirichlet integral on W. Therefore by Virtanen's[2)] theorem there is no harmonic function with finite Dirichlet integral on W.

On the other hand if we put $w(p)=z$ for all p over z, then $w(p)$ is a single-valued bounded analytic function on W.

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2) K. I. Virtanen, Über die Existenz von beschränkten harmonischen Funktionen auf offenen Riemannschen Flächen, Ann. Acad. Scient. Fenn., A.I. 75 1950.