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Author(s)	Kinoshita, Shin'ichi
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Kinoshita, Shin'ichi
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On Wendt's Theorem of Knots, II

By Shin'ichi KINOSHITA

1. Recently R. H. Fox introduced in his paper [1] an operation τ called a single twist. Using this operation τ, we introduce now a numerical knot[1] invariant $\bar{\bar{s}}(k)$ defined as the minimal number of τ^n which change the given knot k to the trivial one, where the natural number n is not fixed. By definition of $\bar{s}(k)$ and $s(k)$[2]

$$\bar{\bar{s}}(k) \leqq \bar{s}(k) \leqq s(k) .$$

Then the purpose of this note is to prove

$$(*) \qquad e_g \leqq (g-1)\bar{\bar{s}}(k) ,$$

where e_g is the minimal number of essential generators of the 1-dimensional homology group of the g-fold cyclic covering space of S, branched along k. From the above inequality $(*)$ it follows that

$$e_g \leqq (g-1)\bar{s}(k) , \qquad e_g \leqq (g-1)s(k) ,$$

where the former is proved in [2] and the latter is due to H. Wendt [3].

2. Now we prove our inequality $(*)$. Let k be a knot. Suppose that k is deformed into k' by τ^n. Then we are only to prove that

$$e_g(k') \leqq e_g(k) + (g-1) .$$

Let $F(S-k)$ be the fundamental group of $S-k$. By [1] we may assume that

$$\begin{aligned} F(S-k) &= (a,\ b,\ A,\ B,\ x_1,\ x_2, \cdots : \\ &\quad a = A,\ b = B,\ r_1 = 1,\ r_2 = 1, \cdots), \\ F(S-k') &= (a,\ b,\ A,\ B,\ x_1,\ x_2, \cdots : \\ &\quad a = A,\ b = A^n B,\ r_1 = 1,\ r_2 = 1, \cdots). \end{aligned}$$

The 1-dimensional homology groups of $S-k$ and $S-k'$ are infinite cyclic. We denote by t a generator of either group. Then abelianization of $F(S-k)$ or $F(S-k')$ maps A into t^q and B into 1, where q is an integer. By usual methods the presentations of $F(S-k)$ and $F(S-k')$ can be transfomed to the following one:

1) A knot is a polygonal simple closed curve in the 3-sphere S.
2) $\bar{s}(k)$ and $s(k)$ are defined in [2].

$$\begin{aligned} F(S-k) = (&t,\ \bar a,\ \bar b,\ \bar A,\ \bar B,\ \bar x_1,\ \bar x_2, \cdots : \\ &\bar a = \bar A,\ \bar b = \bar B,\ \bar r_1 = 1,\ \bar r_2 = 1, \cdots,\ t\bar f_1^{-1} = 1), \\ F(S-k') = (&t,\ \bar a,\ \bar b,\ \bar A,\ \bar B,\ \bar x_1,\ \bar x_2, \cdots : \\ &\bar a = \bar A,\ t^{nq}\bar b = (t^q \bar A)^n \bar B,\ \bar r_1 = 1,\ \bar r_2 = 1, \cdots,\ t\bar f_2^{-1} = 1). \end{aligned}$$

Furthermore we may suppose that $f_1 = f_2$. Then the 1-dimensional homology group of the g-fold cyclic covering space of S, branched along k, is given by the matrix:

$$(1) \qquad \begin{array}{c} \\ \\ \end{array} \begin{array}{cc|c|ccc} & \bar B & \bar b & \bar A & \bar a & \bar x_i \end{array}$$

$$(1) \qquad \left(\begin{array}{c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & \multicolumn{2}{c}{*} \end{array}\right) \begin{matrix} \} \ g \text{ rows} \\ \\ \end{matrix}$$

$\underbrace{\qquad\qquad}_{g \text{ columns}}$

and that of S, branched along k', is given by the following one:

$$\begin{array}{ccccc} \bar B & \bar b & \bar A & \bar a & \bar x_i \end{array}$$

$$(1') \qquad \left(\begin{array}{c|c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & \begin{matrix} a_0 \cdots\cdots a_{g-1} \\ a_{g-1} \cdots a_{g-2} \\ \cdots\cdots\cdots \\ a_1 \cdots\cdots a_0 \end{matrix} & 0 \\ \hline 0 & \multicolumn{3}{c}{*} \end{array}\right).$$

Putting $\sum_{i=0}^{g-1} a_i = \alpha$, we can transform (1) and (1′) to the following one, respectively:

$$(2) \qquad \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & * \end{array}\right),$$

$$
(2') \quad \left(\begin{array}{c|c|c|c}
\begin{matrix} 11\cdots 1 \\ 1\cdots 1 \\ \cdots \\ \cdots 1 \end{matrix} & 0 & \begin{matrix} \alpha\cdots\cdots\alpha \\ a_{g-1}\cdots a_{g-2} \\ \cdots\cdots \\ a_1\cdots\cdots a_0 \end{matrix} & 0 \\
\hline
0 & \multicolumn{3}{c}{*}
\end{array}\right).
$$

(2′) is equivalent to

$$
(3) \quad \left(\begin{array}{c|c|c|c}
\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 & \begin{matrix} 0\cdots\cdots 0 \\ a_{g-1}\cdots a_{g-2} \\ \cdots\cdots \\ a_1\cdots\cdots a_0 \end{matrix} & 0 \\
\hline
0 & \multicolumn{3}{c}{*}
\end{array}\right).
$$

From (2) and (3) it is easy to see that

$$e_g(k') \leqq e_g(k) + (g-1).$$

Thus our proof is complete.

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References

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