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## *On Wendt's Theorem of Knots, II*

By Shin'ichi KINOSHITA

1. Recently R. H. Fox introduced in his paper [1] an operation  $\tau$  called a single twist. Using this operation  $\tau$ , we introduce now a numerical knot<sup>1)</sup> invariant  $\bar{s}(k)$  defined as the minimal number of  $\tau^n$  which change the given knot  $k$  to the trivial one, where the natural number  $n$  is not fixed. By definition of  $\bar{s}(k)$  and  $s(k)$ <sup>2)</sup>

$$\bar{s}(k) \leq \bar{s}(k) \leq s(k).$$

Then the purpose of this note is to prove

$$(*) \quad e_g \leq (g-1)\bar{s}(k),$$

where  $e_g$  is the minimal number of essential generators of the 1-dimensional homology group of the  $g$ -fold cyclic covering space of  $S$ , branched along  $k$ . From the above inequality (\*) it follows that

$$e_g \leq (g-1)\bar{s}(k), \quad e_g \leq (g-1)s(k),$$

where the former is proved in [2] and the latter is due to H. Wendt [3].

2. Now we prove our inequality (\*). Let  $k$  be a knot. Suppose that  $k$  is deformed into  $k'$  by  $\tau^n$ . Then we are only to prove that

$$e_g(k') \leq e_g(k) + (g-1).$$

Let  $F(S-k)$  be the fundamental group of  $S-k$ . By [1] we may assume that

$$\begin{aligned} F(S-k) &= (a, b, A, B, x_1, x_2, \dots : \\ &\quad a = A, b = B, r_1 = 1, r_2 = 1, \dots), \\ F(S-k') &= (a, b, A, B, x_1, x_2, \dots : \\ &\quad a = A, b = A^n B, r_1 = 1, r_2 = 1, \dots). \end{aligned}$$

The 1-dimensional homology groups of  $S-k$  and  $S-k'$  are infinite cyclic. We denote by  $t$  a generator of either group. Then abelianization of  $F(S-k)$  or  $F(S-k')$  maps  $A$  into  $t^q$  and  $B$  into 1, where  $q$  is an integer. By usual methods the presentations of  $F(S-k)$  and  $F(S-k')$  can be transformed to the following one:

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1) A knot is a polygonal simple closed curve in the 3-sphere  $S$ .  
 2)  $\bar{s}(k)$  and  $s(k)$  are defined in [2].

$$\begin{aligned}
 F(S-k) &= (t, \bar{a}, \bar{b}, \bar{A}, \bar{B}, \bar{x}_1, \bar{x}_2, \dots : \\
 &\quad \bar{a} = \bar{A}, \bar{b} = \bar{B}, \bar{r}_1 = 1, \bar{r}_2 = 1, \dots, tf_1^{-1} = 1), \\
 F(S-k') &= (t, \bar{a}, \bar{b}, \bar{A}, \bar{B}, \bar{x}_1, \bar{x}_2, \dots : \\
 &\quad \bar{a} = \bar{A}, t^{nq}\bar{b} = (t^q\bar{A})^n\bar{B}, \bar{r}_1 = 1, \bar{r}_2 = 1, \dots, tf_2^{-1} = 1).
 \end{aligned}$$

Furthermore we may suppose that  $f_1 = f_2$ . Then the 1-dimensional homology group of the  $g$ -fold cyclic covering space of  $S$ , branched along  $k$ , is given by the matrix :

$$(1) \left( \begin{array}{c|cc|cc}
 \bar{B} & \bar{b} & \bar{A} & \bar{a} & \bar{x}_i \\
 \hline
 \left. \begin{array}{c} 1 \\ \dots \\ 1 \end{array} \right| \begin{array}{c} 1 \\ \dots \\ 1 \end{array} & & & 0 & \\
 \hline
 0 & & * & & 
 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} g \text{ rows}$$

$\underbrace{\hspace{10em}}_{g \text{ columns}}$

and that of  $S$ , branched along  $k'$ , is given by the following one :

$$(1') \left( \begin{array}{c|cc|cc}
 \bar{B} & \bar{b} & \bar{A} & \bar{a} & \bar{x}_i \\
 \hline
 \left. \begin{array}{c} 1 \\ \dots \\ 1 \end{array} \right| \begin{array}{c} -1 \\ \dots \\ -1 \end{array} & \begin{array}{c} a_0 \cdots a_{g-1} \\ a_{g-1} \cdots a_{g-2} \\ \dots \\ a_1 \cdots a_0 \end{array} & & 0 & \\
 \hline
 0 & & * & & 
 \end{array} \right).$$

Putting  $\sum_{i=0}^{g-1} a_i = \alpha$ , we can transform (1) and (1') to the following one, respectively :

$$(2) \left( \begin{array}{c|c}
 \left. \begin{array}{c} 1 \\ \dots \\ 1 \end{array} \right| & 0 \\
 \hline
 0 & *
 \end{array} \right),$$

$$(2') \left( \begin{array}{c|c|c|c} 11 \dots 1 & & \alpha \dots \alpha & \\ 1 \dots 1 & 0 & a_{g-1} \dots a_{g-2} & 0 \\ \dots & & \dots & \\ \dots 1 & & a_1 \dots a_0 & \\ \hline 0 & & * & \end{array} \right) .$$

(2') is equivalent to

$$(3) \left( \begin{array}{c|c|c|c} 1 & & 0 \dots 0 & \\ \dots & & a_{g-1} \dots a_{g-2} & 0 \\ \dots & 0 & \dots & \\ 1 & & a_1 \dots a_0 & \\ \hline 0 & & * & \end{array} \right) .$$

From (2) and (3) it is easy to see that

$$e_g(k') \leq e_g(k) + (g-1) .$$

Thus our proof is complete.

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**References**

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- [ 3 ] H. Wendt: Die gordische Auflösung von Knoten, Math. Z. **42**, 680-696 (1937).

