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ON ALMOST *M*-PROJECTIVES

Dedicated to Professor Teruo Kanzaki on his 60th birthday

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We have defined a new concept of almost M-projectives [7] and given several properties of them [4]. This paper is a continuous work of [4] and [7]. If a module M_0 is M_i-projective for a finite set of modules M_i, then M_0 is $\sum\oplus M_i$-projective [2]. However this fact is not ture for almost relative projectives [7]. As far as we know this is one of great differences between relative projectives and almost relative projectives. The main purpose of this paper is to fill this gap. Let R be a semiperfect ring whose Jacobson radical is nil. When M_0 is a local R-module and the M_i are R-modules whose endomorphism rings are local, we shall give a necessary and sufficient condition for M_0 to be almost $\sum\oplus M_i$-projective (Theorem 2), which is dual to [3], Theorem. We shall study this problem in [6] when R is right artinian.

First we take any ring R. Let M_0 be an R-module and M_1 an indecomposable and non-hollow R-module. Then we shall show, in §1, that M_0 is M_1-projective if M_0 is almost M_1-projective (Theorem 1). Next we shall assume that R is semiperfect. In §2 we study almost relative projectivity among local modules. From the results in this section we can understand differences between relative projectives and almost relative projectivces. Using those results, we shall give the main theorem above in §3.

1. Non cyclic modules

Throughout this paper we always assume that a ring R is a *semiperfect ring* with identity except in Theorem 1 and every module M is a unitary right R-module. We denote *the Jacobson radical, the length of* M by $\mathrm{J}(M)$ and $|M|$, respectively. e_i means always a primitive idempotent in R. We shall use the same terminologies in [4].

Let M and N be R-modules. For any exact sequence with K a submodule of M:

$$(1)\qquad \begin{array}{ccc} M & \xrightarrow{v} & M/K \to 0 \\ & & \uparrow h \\ & & N \end{array}$$

if either there exists $\tilde{h}: N \to M$ with $\nu\tilde{h}=h$ or there exist a non-zero direct summand M_1 of M and $\tilde{h}: M_1 \to N$ with $h\tilde{h}=\nu \mid M_1$, N is called *almost M-projective* [7]. (If we obtain only the first case, we call N *M-projective* [2].)

We note the following fact: when N is almost M-projective and M is indecomposable,

(#) if h in the diegrem (1) is not an epimorphism, there exists an $\tilde{h}: N \to M$ with $\nu\tilde{h} = h$.

The concept of almost relative projectives was introduced in [4] and [7] to study the structure of lifting module [8] and extending module [9]. We refer [4] and [7] for the details.

If every proper submodule of an R-module T is small in T, T is called *a hollow module.* In particular if T is a cyclic hollow, we call T *a local module.*

First we shall give the following theorem for any ring R.

Theorem 1. *Let R be any ring. Let M be a non-hollow and indecomposable R-module and M_0 an R-module. If M_0 is almost M-projective, then M_0 is M-projective.*

Proof. Take any diagram with row exact:

$$\text{(2)} \qquad \begin{array}{c} 0 \to K \to M \xrightarrow{\nu} M/K \to 0 \\ \qquad\quad\ \ \uparrow h \\ \qquad\quad\ \ M_0 \end{array}$$

First assume that K is not small in M. Then there exists a submodule K_1 in M such that $K_1 \neq M$ and $M=K_1+K$. Now we obtain a derived diagram from the above:

$$\begin{array}{c} 0 \to K_1 \cap K \to M \xrightarrow{\nu'} M/(K_1 \cap K) \approx M/K \oplus M/K_1 \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \uparrow h+0 \\ \qquad\qquad\qquad\qquad\qquad\qquad M_0 \end{array}$$

Since $M/K_1 \neq 0$, by assumption there exists $\tilde{h}$: $M_0 \to M$ such that $h=\pi\nu'\tilde{h}=\nu\tilde{h}'$, where π: $M/(K_1 \cap K) \to M/K$ is the projection, and hence $\pi\nu'=\nu$ from the construction. Therefore we may assume that K is small in M. Since M is not hollow, there exists two proper submodules K_1, K_2 of M with $M=K_1+K_2$. We may assume $K_1 \supset K$ for $i=1, 2$, since K is small in M. Then we obtain as above a derived diagram:

$$\begin{array}{c} M \xrightarrow{\nu} M/K \xrightarrow{\nu'} M/(K_1 \cap K_2) \approx M/K_1 \oplus M/K_2 \to 0 \\ \uparrow \nu'h \\ M_0 \end{array}$$

Let π_1 be the projection of $M/(K_1 \cap K_2)$ onto M/K_1 and i_1 the inclusion of M/K_1

into $M/(K_1\cap K_2)$. Then $i_1\pi_1 v' h\colon M_0\to M/(K_1\cap K_2)$ is not an epimorphism. Hence there exists $\tilde{h}\colon M_0\to M$ with $v' v\tilde{h}=i_1\pi_1 v' h$, and so $(h-v\tilde{h})(M_0)\subset\ker(v'-i_1\pi_1 v')=K_1/K$. Accordingly we have a diagram

$$\begin{array}{ccl} M & \xrightarrow{v} & M/K \to 0 \\ & & \uparrow h-v\tilde{h} \\ & & M_0 \end{array}$$

and $(h-v\tilde{h})(M_0)\subset K_1/K\subsetneqq M/K$. Hence there exists again $\tilde{h}_1\colon M_0\to M$ with $v\tilde{h}_1=h-v\tilde{h}$, and so $h=v(\tilde{h}+\tilde{h}_1)$. Therefore M_0 is M-projective.

Corollary 1. *Let R be semiperfect, and let M be an indecomposable module not isomorphic to any local modules eR/A and M_0 an R-module such that M_0J is small in M_0. Then M_0 is M-projective if M_0 is almost M-projective, where e is a primitive idempotent in R and J is the Jacobson radical of R.*

Proof. If M is not hollow, M_0 is M-projective by Theorem 1. Hence we assume that M is hollow. If the h in the proof of Theorem 1 is an epimorphism, $M/K\neq(M/K)J$ by assumption. Hence since M is hollow, M/K is local and so M is also local, which is a contradiction. Accordingly h is always not an epimorpism, and hence M_0 is M-projective.

Let Z be the ring of integers and p a prime. Then $E(Z/p)$, the injective hull of Z/p, is a uniserial Z_p-module and hence $E(Z/p)$ is almost $E(Z/p)$-projective. However $E(Z/p)$ is not $E(Z/p)$-projective. Hence we need the assumption on M_0 in Corollary 1.

From Corollary 1 if M_0 is almost M-projective, but not M-projective for an indecomposable module M, M must be a local module eR/B whenever M_0 is finitely generated, where e is a primitive idempotent.

Proposition 1. *Let M_0 be as in Corollary 1 and M_1 an indecomposable module. Assume that M_0 is almost M_1-projective. Then M_0 is M_1-projective if and only if either M_1 is not of a form eR/A or $M_1\approx eR/A$ and any homomorphism: $M_0/J(M_0)\to M_1/J(M_1)$ is liftable to an element $f\colon M_0\to M_1$.*

Proof. We assume "if" part. If $M_1\not\approx eR/A$, M_0 is M_1-projective by Corollary 1. Hence suppose $M_1=eR/A$ and put $\bar{M}_0=M_0/M_0J=\sum\oplus\overline{e_iR}$. If $e\not\approx e_i$ for all i, there are no epimorphisms $h'\colon M_0\to M_1/K_1$, where $K_1\subsetneqq M_1$. Hence M_0 is M_1-projective. Assume that $e\approx e_1$. Take a diagram:

$$\begin{array}{ccl} M_1 & \xrightarrow{v} & M_1/K_1 \to 0 \\ & & \uparrow h \\ & & M_0 \quad . \end{array}$$

If h is an epimorphism, then h induces an epimorphism $\bar{h}\colon M_0/J(M_0)\to M_1/J(M_1)$. By assumption there exists $h_1\colon M_0\to M_1$ such that $\bar{h}_1=\bar{h}$, *i.e.*, $(vh_1-h)(M_0)\subset$

$(M_1/K_1)J$. Hence there exists $\tilde{h}: M_0 \to M_1$ with $\nu\tilde{h}=h$ from (♯). If h is not an epimorphism, by (♯) we obtain always $\tilde{h}'$ similar to the above $\tilde{h}$. Hence M_0 is M_1-projective. "only if" part is clear.

Corollary 2. *Let M_0 and M_1 be as in Proposition 1. If M_0 is almost M_1-projective but not M_1-projective, there exists a homomorphism $\tilde{h}: M_1 \to M_0$ which induces a monomorphism of $M_1/\mathrm{J}(M_1)$ into $M_0/\mathrm{J}(M_0)$.*

Proof. Since M_0 is not M_1-projective, we have an epimorphism h: $M_0/\mathrm{J}(M_0) \to M_1/\mathrm{J}(M_1)$, which is not liftable, by Proposition 1. Hence there exists the desired homomorphism $\tilde{h}: M_1 \to M_0$.

2. Local modules

We shall study almost relative projectives among local modules. We recall here the definion of the lifting property of simple modules modulo radical (briefly 1.p.s.m) [5]. Let T_1 and T_2 be local modules. If for any simple submodule U in $T_1/\mathrm{J}(T_1) \oplus T_2/\mathrm{J}(T_2)$ there exists a direct summand T' of $T=T_1 \oplus T_2$ such that $T'+(\mathrm{J}(T_1)\oplus \mathrm{J}(T_2))/(\mathrm{J}(T_1)\oplus \mathrm{J}(T_2))=U$, then we say that T has *the* 1.p.s.m.. This is equivalent to the following: for every element f in $\mathrm{Hom}_R(T_1/\mathrm{J}(T_1), T_2/\mathrm{J}(T_2))$ is liftable to an element in $\mathrm{Hom}_R(T_1, T_2)$ or so is f^{-1} to an element in $\mathrm{Hom}_R(T_2, T_1)$, provided $|T_1|$ and $|T_2|$ are finite. Now in this paper we call the latter equivalent property *the* 1.p.s.m. even if $|T_i|$ is infinite.

Let A, B be right ideals in eR. If eR/B is epimorphic to eR/A, there exists a unit v in eRe such that $vB \subset A$ and $eR/B \approx eR/vB$. We denote this situation by $B \lesssim A$.

Proposition 2. *Let R be a semi-perfect ring and A, B right ideals in eR such that either eR/A or eR/B is noetherian. Then eR/A is almost eR/B-projective if and ony if $eJeA \subset B$ and $eR/A \oplus eR/B$ has the* 1.p.s.m.. *In this case if eR/A is not eR/B-projective, then eR/B is eR/A-projective.*

Proof. If eR/A is almost eR/B-projective, $eJeA \subset B$ by [4], Proposition 2 and $eR/A \oplus eR/B$ has the 1.p.s.m. by definition. Conversely if $eR/A \oplus eR/B$ has the 1.p.s.m., then 1) eR/B is epimorphic to eR/A or 2) eR/A is epimorphic to eR/B by definition. In either case we may asume 1) $A \supset B$ or 2) $A \subset B$ by the remark above (note that $veJe=eJe=eJev$).

Case 1) Assume $eJeA \subset B \subset A$. Take the diagram (2), where $M_0=eR/A$ and $M=eR/B$. If h is not an epimorphism, h is given by an element j in eJe. Since $jA \subset B$, h is liftable to an $\tilde{h}=j_l$: $eR/A \to eR/B$, where j_l is the left-sided multiplication of j. Next we assume that h is an epimorphism. Then h is given by a unit u in eRe. Since $eR/A \oplus eR/B$ has the 1.p.s.m. and either eR/A or eR/B is noetherian, there exists a unit u' in eRe such that $u^{-1} \equiv u'$ (mod eJe) and $u'B \subset A$. Put $u'=u^{-1}+j'$; $j' \in eJe$. Then $A \supset u'B=(u^{-1}+j')B$ and $j'B \subset j'A \subset B \subset A$.

Hence $u^{-1}B\subset A$. Putting $\tilde{h}=(u^{-1})_l$, $h\tilde{h}=v$. Hence eR/A is almost eR/B-projective.

Case 2) We can show in the same manner that eR/A is eR/B-projective. Finally assume that eR/A is not eR/B-projetive. Then we may assume $B\subset A$ by Corollary 2. Further $eJeB$ $(\subset eJeA\subset B)\subset A$. Hence eR/B is almost eR/A-projective by the first statement. While $B\subset A$ implies that eR/B is eR/A-projective by Corollary 2.

We shall apply the above proposition to a partucular case, e.g. an algebra over an algebraically closed field.

Lemma 1. *Let $M_0=eR/A$ and $M_1=eR/B$. Then M_0 is M_1-projective if and only if for any generator $a_0=a_0e$ of M_0 (resp. $a_1=a_1e$ of M_1), a mapping $a_0\rightarrow a_1$ gives us an epimorphism of M_0 onto M_1.*

Proof. Since $a_ie=a_i$ $(i=0, 1)$, a_i is a unit in eRe. The last statement of the lemma is equivalent to $\{x\in eR\,|\,a_0x\in A$, i.e. $x\in a_0^{-1}A\}\subset\{x\in eR\,|\,a_1x\in B$, i.e. $x\in a_1^{-1}B\}$. Hence $A\subset B$ by taking $a_0=a_1=e$ and $uA\subset B$ for any unit u in eRe by taking $a_0=u^{-1}$ and $a_1=e$. Let j be any element in eJe. Then $(e+j)A\subset B$ and $eA\subset B$ from the above. Hence $jA\subset B$. Therefore $eReA\subset B$, and so M_0 is M_1-projective by [1], p. 22, Exercise 4. The converse is clear from the above and [1].

Proposition 3. *Let M be an R-module and $M_0=eR/A$. Then M_0 is M-projective if and only if for any $m=me$ in M and any generator $a_0=a_0e$ of M_0, a mapping $a_0\rightarrow m$ gives us an epimorphism of M_0 onto mR.*

Proof. If M_0 is M-projective, then M_0 is N-projective for any submodule N of M by defintion. Hence we obtain "only if" part from Lemma 1, since $mR\approx eR/B$ for some B. Conversely take $m=me$ in M with $h(e+A)=v(m)$ in the diagram (2). Since there exists $\tilde{h}$: $M_0\rightarrow mR(\subset M)$ with $\tilde{h}(e+A)=m$ by assumption, $v\tilde{h}=h$.

From the above result we shall define a new concept. Let $M_0=eR/A$ be a local module. An R-module N is called *locally generated* by M_0 if every cyclic submodule nR of N with $ne=n$ is a homomorphic image of M_0.

Now we assume that eJ/B is locally generated by eR/A. For any element x in eJe we obtain an epimorphism f: $eR/A\rightarrow(xR+B)/B\subset eR/B$. Then $f(e+A)=xr+B$ and r is a unit in eRe and there exists y in eRe such that $yA\subset B$ and $y\equiv xr$ (mod B). Put $y=xr+b$; $b\in B$. Then $B\supset yA=(xr+b)A$. Hence since $bA\subset B$, $xrA\subset B$. Therefore eJ/B is locally generated by eR/A if and only if

(3) for any element x in eJe, there exists a unit u_x in eRe such that $xu_xA\subset B$.

If $eJeA\subset B$, (3) is trivially satisfied.

Lemma 2. *Let R be a right artinian ring and assume that $eR/A\oplus eR/A$ has the* l.p.s.m.. *Then* 1): *for $B\subset eR$ eR/A is almost eR/B-projective if and only* *if i) eJ/B is locally generated by eR/A and ii) $A\lesssim B$ or $A\gtrsim B$. 2): For an R-module M eR/A is M-projective if and only if M is locally generated by eR/A.*

Proof. 1) We assume that eR/A is almost eR/B-projective. Then i) and ii) are clear from Proposition 2 and the remark after (3). Conversely we assume i) and ii). We shall show $eJ^i eA\subset B$ for each i by induction on i. Assume $eJ^{i+1}eA\subset B$ and take an element x in $eJ^i e - eJ^{i+1} e$. Then from (3) there exists a unit r in eRe such that $xrA\subset B$. By assumption; l.p.s.m.

(4) $r = u+j; u$ is a unit in eRe with $uA = A$ and $j\in eJe$.

Then $B\subset xrA=(xu+xj)A$ and $xj\in eJ^{i+1}e$. Hence $xA=xuA\subset B$ by induction hypothesis, and so $eJeA\subset B$ by taking $i=1$. From ii) we may assume $A\subset B$ or $A\supset B$. Hence it is clear that $eR/A\oplus eR/B$ has the l.p.s.m. for $eR/A\oplus eR/A$ does. Therefore eR/A is almost eR/B-projective by Proposition 2.

2) Assume that M is locally generated by eR/A. Let m be an element in M with $me=m$. Then $mR\approx eR/B$ for some B. Now we shall show that eR/A is eR/B-projective. Since eR/B is locally generated by eR/A, (3) holds for any element in eRe from the argument given before (3). Hence the observation after (4) shows $eReA\subset B$ and hence eR/A is eR/B-projective by [1]. Accordingly eR/A is M-projective by Lemma 1 and Proposition 3. The converse is clear from Proposition 3.

Proposition 4. *Let R be a right artinian ring and M an R-module. We assume that $eR/A\oplus eR/A$ has the* l.p.s.m.. *Then eR/A is almost M-projective if and only if for any element $m=me$ in M, we obtain one of the following:*

1) *If mR is not a direct summand of M, then mR is a homomorphic image of eR/A.*

2) *If mR is a direct summand of M, then either mR is a homomorphic image of eR/A or eR/A is that of mR.*

Proof. Assume that eR/A is almost M-projective. Let $m=me$ be in M and mR not a direct summand of M. We shall show that eR/A is mR-projective. Consider a diagram with K a submodule of mR:

$$\begin{array}{ccc} mR & \xrightarrow{\nu} & mR/K \rightarrow 0 \\ & & \uparrow h \\ & & eR/A \end{array}$$

Then we obtain a derived diagram

$$\begin{array}{ccc} M & \xrightarrow{\nu_M} & M/K \rightarrow 0 \\ \cup & & \cup \\ mR & \xrightarrow{\nu} & mR/K \rightarrow 0 \\ & & \uparrow h \\ & & eR/A\,. \end{array}$$

Since eR/A is almost M-projective, a) there exists $\tilde{h}$: $eR/A \to M$ with $\nu_M \tilde{h} = h$ or b) there exist a direct summand M_1 of M and $\tilde{h}$: $M_1 \to eR/A$ with $h\tilde{h} = \nu_M | M_1$. Assume b). Since $\nu_M(M_1) \subset h(eR/A) \subset \nu(mR)$ and $K \subset mR$, $M_1 \subset mR$. Hence $M_1 = mR$ for mR is hollow, which contradicts the initial assumption. Therefore, if mR is not a direct summand of M, we always obtain the case a). Then since $\nu_M(\tilde{h}(eR/A)) \subset h(eR/A) \subset \nu(mR)$, $\tilde{h}(eR/A) \subset mR$. Hence eR/A is mR-projective, whence mR is a homomorphic image of eR/A by Proposition 3. Next we assume that mR is a direct summand of M. Then eR/A is almost mR-projective by definition. Then we obtain 2) from Lemma 2–1)-ii). Conversely assume 1) and 2). Take any diagram with $K \subset M$:

$$\begin{array}{ccc} M & \xrightarrow{\nu} & M/K \to 0 \\ & & \uparrow h \\ & & eR/A \end{array}$$

and put $h(\tilde{e}) = \nu(m)$ for some $m = me \in M$, where $\tilde{e} = e + A$ in eR/A. Assume that mR is not a direct summand of M. Then since mR is hollow, $m'R$ is not a direct summand of M for any m' $(=m'e)$ in mR. Accordingly mR is locally generated by eR/A from 1) and so eR/A is mR-projective by Lemma 2–2). Hence there exists a homomorphism $\tilde{h}$: $eR/A \to mR \subset M$ with $\tilde{h}(\tilde{e}) = m$ by Lemma 1. Therefore $\nu\tilde{h} = h$. Assume the case 2). Since mR is a local module, any proper submodule of mR is not a direct summand of M. Hence eR/A is almost mR-projective by 1) and Lemma 2–1). Take the derived diagram from the above one

$$\begin{array}{ccc} mR & \xrightarrow{\nu} & mR/(K \cap mR) \to 0 \\ & & \uparrow h \\ & & eR/A \end{array}$$

Since eR/A is almost mR-projective, we obtain an $\tilde{h}$: $eR/A \to mR$ (or $mR \to eR/A$) which makes the above diagram commutative. Noting that mR is a direct summand of M, we know that eR/A is almost M-projective.

REMARK. We don not need Lemma 2 in the first half of the proof of Proposition 4, and hence it shows the following fact: Let R be semiperfect and eR/A almost M-projective. Then for $m = me$ in M such that mR is not a direct summand of M, eR/A is mR-projective.

We note that if R is an algebra over an algebraically closed field of finite dimension, $eR/A \oplus eR/A$ has always the l.p.s.m.. Further Lemma 2 and Proposition 4 are not true without the assumption: l.p.s.m. of $eR/A \oplus eR/A$ (see the next examples).

EXAMPLE 1. Let $L \supset K$ be fields with $L = aK \oplus bK$. Put $R_1 = L \oplus uL$, a trivial extension with $\mathrm{J}(R_1) = 0 \oplus uL$ and $V = xL \oplus yL$, a vector space over L.

Set

$$R=\begin{pmatrix}R_1 & V\\ 0 & R\end{pmatrix}$$

with $(ud)x=yd$ and $uy=0$; $d\in L$. Put $A=(0\ x(aK)\oplus yL)$ and $B=(0\ y(aK))$. Then for $e=e_{11}$ $eJe=J(R_1)$, $eJeA\not\subset B$ and for any $c'=cu(\neq 0)$ in eJe $c'c^{-1}A\subset B$ ((3)), and hence eJ/B is locally generated by eR/A. Further $eR/A\oplus eR/B$ has the l.p.s.m.. However eR/A is not almost eR/B-projective for $eJeA\not\subset B$.

2. Put $A'=(0\ x(aK)\oplus y(aK))$. Since $eJeB=0$ and $A'\supset B$, eR/A' is locally generated by eR/B. However eR/B is not eR/A'-projectiev.

3. Direct sums

Let M_0, M_1 and M_2 be indecomposable modules and let M_0 be almost M_i-projective for $i=1, 2$. In this section we shall study a condition under which M_0 is almost $M_1\oplus M_2$-projective, when M_0 is cyclic. This is dual to [3], Theorem. We note that if M_0 is almost $M_1\oplus M_2$-projective, then M_0 is almost M_i-projective for $i=1, 2$ by definition. If $\mathrm{End}_R(M)$ is a local ring, we say M is an *l.e. module*.

Proposition 5. *Let M_0 be a finitely generated R-module and let M_1 be a local and* l.e. *module e_1R/A_1 and M_2 an* l.e. *module. Assume that i) M_0 is almost $M_1\oplus M_2$-projective, but M_0 is not M_1-projective, and ii) for any $m(\neq 0)$ in M_2 with $me_1=m$ we take any isomorphism f: $M_1/M_1J\approx mR/mJ$. Then f (or f^{-1} if $M_2=mR$) is liftable to f': $M_1\to M_2$ (or f': $M_2\to M_1$).*

Proof. Since M_0 is almost M_1-projective but not M_1-projective, there exist a maximal submodule B of M_0 and an isomorphism g: $M_0/B\to M_1/\mathrm{J}(M_1)$ which is not liftable to an element: $M_0\to M_1$ (cf. the proof of Proposition 1). Let f: $M_1/\mathrm{J}(M_1)\to mR/mJ$ be the given isomorphism and take a diagram:

$$\begin{array}{ccc} M_1\oplus M_2 & \xrightarrow{\nu_1+\nu_2} & M_1/J(M_1)\oplus M_2/mJ\to 0\\ & & \uparrow h\\ & & M_0/B\\ & & \uparrow \nu_0\\ & & M_0\ , \end{array}$$

where $h=g+fg$. Since M_0 is almost $M_1\oplus M_2$-projective, either there exists $\tilde{h}$: $M_0\to M_1\oplus M_2$ with $(\nu_1+\nu_2)\tilde{h}=h\nu_0$ or there exist a non-zero direct summand N of $M_1\oplus M_2$ and $\tilde{h}$: $N\to M_0$ with $h\,\nu_0\,\tilde{h}=(\nu_1+\nu_2)\,|\,N$. If the former occurs, taking the projection of $M_1\oplus M_2$ onto M_1, we have a contradiction to the choice of g. Hence we should obtain the latter. We may assume that N is an indecomposable module. Since N has the exchange property by assumption

$$M_1 \oplus M_2 = N \oplus M_1 \text{ or } = N \oplus M_2\,.$$

The first case: Let x_2 be any element in M_2. Then

$$x_2 = n + x_1;\ n \in N,\ x_1 \in M_1 \quad \text{and} \quad n = y_1 + y_2,\ y_i \in M_i\,.$$

Hence $x_2 = y_2$ and $x_1 = -y_1$. Put $z = \nu_0 \tilde{h}(n)$, and $\nu_1(y_1) = g(z)$, $\nu_2(y_2) = fg(z)$, i.e., $\nu_2(x_2) = f(\nu_1(-x_1))$. Then $M_2/mJ = f(M_1/J(M_1)) = mR/mJ$. Accordingly, $M_2 = mR$ and $-\pi \mid M_2 \colon M_2 \to M_1$ is a lifted element of f^{-1}, where $\pi \colon N \oplus M_1 \to M_1$ is the projection. We obtain a similar result for the second case.

Lemma 3. *Let $\{M_i\}_{i=1}^n$ be a set of indecomposable R-modules and let N and M_0 be R-modules. Assume that M_0 is almost M_i-projective for all i and N-projective. Take a diagram with row exact:*

$$\begin{array}{ccccccc} 0 \to K \to (\sum \oplus M_i) \oplus N & \xrightarrow{\nu} & H \to 0 \\ & & \uparrow h \\ & & M_0 \end{array}$$

If there exists a small submodule T in $\sum_i \oplus M_i$ such that $h(M_0) \subset \nu(T \oplus N)$, then there exists $\tilde{h} \colon M_0 \to (\sum_i \oplus M_i) \oplus N$ with $\nu \tilde{h} = h$.

Proof. Put $M^* = \sum_i \oplus M_i \oplus N$ and $\pi_1 \colon M^* \to \sum_i \oplus M_i$, $\pi_2 \colon M^* \to N$ the projections. Further put $K^i = \pi_i(M^*)$ for $i = 1, 2$. We can derive the following diagram (cf. [4]):

$$\begin{array}{ccc} \sum_i \oplus M_i & \xrightarrow{\nu'} & (\sum_i \oplus M_i)/K^1 \to 0 \\ & & \uparrow \pi_1' h \\ & & M_0 \end{array}$$

where $\pi_1' \colon H \xrightarrow{\nu^*} M^*/(K^1 \oplus K^2) \to (\sum_i \oplus M_i)/K^1$ is the projection (we note that $K \subset (K^1 \oplus K^2)$ and $H = M^*/K$, and so we obtain the natural epimorphism ν^*). From the assumption $\pi_1' h(M_0)$ is small in $(\sum_i \oplus M_i)/K^1$. Hence there exists $\tilde{h}_1 \colon M_0 \to \sum_i \oplus M_i$ with $\nu' \tilde{h}_1 = \pi_1' h$ by [4], Lemma 1. Since M_0 is N-projective, we obtain the desired homomorphism from the remark before [4], Lemma 1.

The following theorem is dual to [3], Theorem and will be generalized in [6] to a case where M_0 is a finitely generated module, when R is right artinian.

Theorem 2. *Assume that R is a semiperfect ring and J is nil. Let $\{M_i\}_{i=1}^n$ be a set of* l.e. *modules and M_0 a local module e_0R/A_0. Then the following are equivalent:*

1) *M_0 is almost $\sum_{i=1}^n \oplus M_i$-projective.*
2) *The following are fulfiled:*
i) *M_0 is almost M_i-projective for all $i \geqslant 1$.*
ii) *If M_0 is not M_k-projective for $k = i$ and j, then $M_i \oplus M_j$ has the* l.p.s.m. *(in this case $M_i \approx e_0R/A_i$, $M_j \approx e_0R/A_j$).*

Proof. 2)$\to$1) We may assume that there exists an integer m such that M_0 is M_i-projective for all $i>m$ and M_0 is not M_j-projective for all $j\leqslant m$ and hence all M_j $(j\leqslant m)$ are local modules e_0R/A_j by Corollary 1. Take a diagram with row exact:

$$(5)\qquad \begin{array}{c} 0\to K\to M=\sum\oplus M_i \xrightarrow{\nu} M/K\to 0 \\ \qquad\qquad\qquad\qquad \uparrow h \\ \qquad\qquad\qquad\qquad M_0=eR/A\,. \end{array}$$

Let $h(\tilde{e}_0)=(\sum a_i)+K$; $a_i\in M_i$, where $\tilde{e}_0=e_0+A$ in M_0. We may assume $a_i\,e_0=a_i$. We show that

there exists $\tilde{h}$: $M_0\to M$ (or there exist a non-zero direct summand N of M and a homomorphism $\tilde{h}$; $N\to M_0$) such that $\nu\tilde{h}=h$ (or $h\tilde{h}=\nu\,|\,N$).

If $a_i\in J(M_i)$ for all $(m\geqslant)\ i\geqslant 1$, there exists $\tilde{h}$: $M_0\to M$ such that $\nu\tilde{h}=h$ by Lemma 3. Hence we assume that there exists an integer k such that $a_j\in \mathrm{J}(M_j)$ for $(m\geqslant)\,j>k$ and $a_{j'}\notin \mathrm{J}(M_{j'})$ for $1\leqslant j'\leqslant k$. Then $a_{j'}$ is a generator of $M_{j'}$, since $M_{j'}$ is local. Now M_0 is not M_t-projective for $t=1, s\leqslant k$, and so $M_1\oplus M_s$ has the l.p.s.m. by assumption. Hence there exists f: $M_1\to M_s$ (or $M_s\to M_1$) such that $f(a_1)=a_s+a_s\,j_s$ (or $f(a_s)=a_1+a_1\,j_s$) for some $j_s\in J$. We take a new decomposition $M=M_1(f)\oplus M_s\oplus\sum_{i\neq 1,s}\oplus M_i$ (or $M_1\oplus M_s(f)\oplus\sum_{i\neq 1,s}\oplus M_i$), where $M_1(f)=\{x+f(x)\,|\,x\in M_1\}\subset M_1\oplus M_s$. Then $a_1+a_s=(a_1+f(a_1))+(a_s-f(a_1))=(a_1+f(a_1))-a_s\,j_s$ and $(a_1+f(a_1))\in M_1(f)$, $a_s\,j_s\in J(M_s)$ (similar for another case). Hence iterating this argument, we remain ourselves a case $k=1$, i.e., M_0 is not M_1-projective and $a_t\in \mathrm{J}(M_t)$ for all $(m\geqslant)t>1$. Since $a_tR\subset \mathrm{J}(M_t)$ for $1<t\leqslant m$, there exists

$$(6)\qquad \tilde{h}_t\colon M_0\to M_t \quad\text{such that}\quad \tilde{h}_t(\tilde{e}_0)=a_t,\ (n\geqslant t>1)$$

by Lemma 1 and Remark in §2. On the other hand, consider f_1: $M_0/\mathrm{J}(M_0)\approx M_1/\mathrm{J}(M_1)$ $(f_1(e_0+\mathrm{J}(M_0))=a_1+\mathrm{J}(M_1))$. Since $M_0\oplus M_1$ has the l.p.s.m. by assumption i) and Proposition 2, there exists $\tilde{h}_1$: $M_1\to M_0$ (or $M_0\to M_1$) such that $\tilde{h}_1(a_1)\equiv\tilde{e}_0$ (mod $\mathrm{J}(M_0)$) (or $\tilde{h}_1(\tilde{e}_0)\equiv a_1$ (mod $\mathrm{J}(M_1)=a_1J$)), i.e.,

$$(7)\qquad \tilde{h}_1(a_1)=\tilde{e}_0+\tilde{e}_0\,j_0;\,j_0\in J\,,\quad\text{or}$$

$$(7')\qquad \tilde{h}_1(\tilde{e}_0)=a_1+a_1\,j_1;\,j_1\in J$$

Case (7'): Put $g=\sum_{t=1}^{n}\tilde{h}_t$: $M_0\to M$ and $h'=h-\nu g$. Then $h'(\tilde{e}_0)=\nu(a_1\,j_1)$ and $a_1\,j_1\in \mathrm{J}(M_1)$. Hence there exists h^*: $M_0\to M$ such that $\nu h^*=h'$ by Lemma 3 and so $h=\nu(g+h^*)$.

Case (7): Now put $g=(\sum_{t\geqslant 2}\tilde{h}_t)\,\tilde{h}_1$: $M_1\to\sum_{t\geqslant 2}\oplus M_t$. Then $g(a_1)=\sum_{t\geqslant 2}\tilde{h}_t(\tilde{e}_0)+\sum_{t\geqslant 2}\tilde{h}_t(\tilde{e}_0)\,j_0=\sum_{t\geqslant 2}a_t+\sum_{t\geqslant 2}a_t\,j_0$. Taking a decomposition $M=M_1(g)\oplus\sum_{t\geqslant 2}\oplus M_t$, $\sum_{t=1}^{n}a_t=a_1+g(a_1)-\sum_{t\geqslant 2}a_t\,j_0$ and $a_1+g(a_1)\in M_1(g)$, $a_t\,j_0\in M_t(t>1)$. Similarly to (6), we obtain by Lemma 1 and Remark in §2.

(8) $\quad \tilde{h}'_t: M_0 \rightarrow M_t \quad \text{with} \quad \tilde{h}'_t(\tilde{e}) = a_t j_0 \ (n \geqslant t > 1)$.

While since $M_1 \approx M_1(g)$ $(a_1 \leftrightarrow a_1 + g(a_1))$, from (7) we obtain $\tilde{h}'_1: M_1(g) \rightarrow M_0$ with

(9) $\quad \tilde{h}'_1(a_1 + g(a_1)) = \tilde{e}_0 + \tilde{e} j_0$.

Put $g' = (\sum_{t \geq 2} \tilde{h}'_t) \tilde{h}'_1$, and $\sum_{t=1}^n \oplus M_t = (M_1(g))(g') \oplus \sum_{t \geq 2} \oplus M_t$, $\sum_{s+1}^n a_t = (a_1 + g(a_1) + g'(a_1 + g(a_1)) - \sum_{t \geq 2} a_t j_0^2$. Repeating this procedure we obtain the final decomposition $M = M'_1 \oplus M_2 \oplus \cdots \oplus M_n$ and $\sum_{i=1}^n a_i \in M'_1 \approx M_1$, since J is nil. Thus we have derived the following diagram from (5):

$$\begin{array}{ccc} M'_1 & \xrightarrow{\nu} \nu(M'_1) & \rightarrow 0 \\ & \uparrow h & \\ & M_0 & \end{array}$$

Therefore there exists $\tilde{h}: M_0 \rightarrow M'_1$ (or $\tilde{h}: M'_1 \rightarrow M_0$) such that $\nu\tilde{h} = h$ (or $h\tilde{h} = \nu | M'_1$).

1)$\rightarrow$2) (cf. [7]). M_0 is almost M_t-projective for all t by definition. Let M_i and M_j be as in 2)-ii). Since M_0 is almost $M_1 \oplus M_2$-projective as above, $M_1 \oplus M_2$ has the l.p.s.m. by Proposition 5.

We shall show in [6] that Theorem 2 is useful when we characterize right Nakayama rings in terms of almost relative projectives.

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