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THE HYPERBOLIC OPERATORS WITH THE CHARACTERISTICS VANISHING WITH THE DIFFERENT SPEEDS

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0. Introduction

Consider for the partial differential operator

$$P(t, D_t, D_x) := \sum_{j+|\alpha|\le m} a_{j,\alpha}(t) D_t^j D_x^\alpha, \qquad t \in [0, T], \quad x \in \mathbb{R}^n,$$

with coefficients $a_{j,\alpha}(t) \in C^1([0, T])$ the Cauchy problem with the data prescribed at $t = s$,

$$\begin{cases} P(t, D_t, D_x)u(t, x) = f(t, x), & t \in [0, T], \quad x \in \mathbb{R}^n, \\ D_t^j u(s, x) = u_j(x), & j = 0, \dots, m-1, \quad x \in \mathbb{R}^n, \end{cases} \tag{0.1}$$

where $s \in [0, T]$. For the principal symbol $P_m(t, \lambda, \xi)$ of the operator P defined by

$$P_m(t, \lambda, \xi) := \sum_{j+|\alpha|=m} a_{j,\alpha}(t) \lambda^j \xi^\alpha$$

we assume that for all $t \in [0, T]$, $\xi \in \mathbb{R}^n$, the following representation

$$\begin{cases} P_m(t, \lambda, \xi) = \displaystyle\prod_{j=1}^{m} (\lambda - \lambda_j(t, \xi)), \\ |\lambda_j(t, \xi) - \lambda_k(t, \xi)| \ge C\lambda_j(t)|\xi|, \quad j < k, \end{cases} \tag{0.2}$$

with the real-valued functions $\lambda_j(t, \xi)$, $j = 1, \dots, m$, and with non-negative continuous functions $\lambda_j(t)$, $j = 1, \dots, m$, $\lambda_j \in C([0, T])$, holds. Thus the operator $P(t, D_t, D_x)$ is a hyperbolic operator with the characteristics $\lambda_j(t, \xi)$, $j = 1, \dots, m$. We make also the assumptions

$$\begin{aligned} &|\partial_t^k a_{j,\alpha}(t)| \le C\lambda_1(t) \cdots \lambda_{|\alpha|}(t) \left(\frac{\lambda_{|\alpha|}(t)}{\Lambda_{|\alpha|}(t)} \right)^k, \\ &1 \le |\alpha|, \quad j + |\alpha| = m, \quad k = 0, 1, \end{aligned} \tag{0.3}$$

and $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_m(t)$. Here $\Lambda_j(t) := \int_0^t \lambda_j(s)ds$, $j = 1, \dots, m$. If $\lambda_1(t)$ vanishes at $t = 0$, $\lambda_1(0) = 0$, then the characteristics of operator vanish too. Thus if the characteristics are distinct for $t \neq 0$, then $P(t, D_t, D_x)$ is a hyperbolic operator with the characteristics of variable multiplicity. The solvability of the Cauchy problem in the space of C^∞ functions requires some conditions on the lower order terms of operator (see, e.g. [4]), while the consideration in the Gevrey spaces with the small exponent is free of these conditions (see, e.g. [2]). There is also an approach combining these two ones (see, e.g. [3] and the references therein) if the Gevrey exponent is not small. We will take initial data from the space of C^∞ functions therefore we will assume for the coefficients $a_{j,\alpha}(t)$ with $j + |\alpha| \leq m - 1$, that is for the lower order terms, the following estimates

$$(0.4) \qquad |\partial_t^k a_{j,\alpha}(t)| \leq C \left(\prod_{i=1}^{|\alpha|} \lambda_i(t)\right) \left(\prod_{l=|\alpha|+1}^{m-j} \frac{\lambda_l(t)|\log \lambda_l(t)|}{\Lambda_l(t)}\right) \left(\frac{\lambda_{m-j}(t)}{\Lambda_{m-j}(t)}\right)^k,$$
$$k = 0, 1,$$

$$(0.5) \qquad |\operatorname{Im} a_{j,\alpha}(t)| \leq c \left(\prod_{i=1}^{|\alpha|} \lambda_i(t)\right) \left(\frac{\lambda_{|\alpha|}(t)}{\Lambda_{|\alpha|}(t)}\right), \quad j + |\alpha| = m - 1.$$

These kind of conditions for the coefficients with $j + |\alpha| \leq m - 1$ are called *Levi conditions*. To make result more transparent we restrict ourselves to the case with $\lambda_i(t) := \lambda^{n_i}(t)$, $i = 1, \dots, m$, where n_j, are the non-negative numbers $1 = n_1 \leq n_2 \leq \cdots \leq n_m$, while function $\lambda = \lambda(t)$, $\lambda \in C^1([0, T])$, $\lambda(0) = 0$, $\lambda(t \neq 0) \neq 0$, $\lambda'(t) := d\lambda(t)/dt \geq 0$, satisfies

$$(0.6) \qquad c_1 \frac{\lambda(t)}{\Lambda(t)} \leq \frac{\lambda'(t)}{\lambda(t)} \leq \frac{\lambda(t)}{\Lambda(t)}, \qquad c_1 > \frac{m-1}{m}, \quad \Lambda(t) := \int_0^t \lambda(s)ds.$$

In our special case we can write conditions (0.3), (0.4), and (0.5) also as follows:

$$(0.7) \quad |\partial_t^k a_{j,\alpha}(t)| \leq C\,(\lambda(t))^{\sum_{i=1}^{|\alpha|} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}|\log \lambda(t)|\right)^{m-j-|\alpha|} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad k = 0, 1,$$

$$(0.8) \quad |\operatorname{Im} a_{j,\alpha}(t)| \leq c\,(\lambda(t))^{\sum_{i=1}^{|\alpha|} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}\right), \quad j + |\alpha| = m - 1.$$

To simplify the notations we group functions $\lambda_i(t)$, $i = 1, \dots, m$, in accordance with n_i:

$$(0.9) \quad 1 = n_1 = \cdots = n_{d_1} < n_{d_1+1} = \cdots = n_{d_1+d_2} < \cdots < n_{d_1+\cdots+d_{s-1}+1} = \cdots = n_{d_1+\cdots+d_s},$$

where

$$(0.10) \qquad d_1 + d_2 + \cdots + d_s = m, \quad d_s \geq 2, \quad s \geq 2.$$

(For the case with $s = 1$ see [17].) Thus, for sufficiently small T we have

$$\begin{aligned} \lambda_1(t) \equiv \cdots \equiv \lambda_{d_1}(t) &> \lambda_{d_1+1}(t) \equiv \cdots \equiv \lambda_{d_1+d_2}(t) \\ &> \cdots > \lambda_{d_1+\cdots+d_{s-1}+1}(t) \equiv \cdots \equiv \lambda_{d_1+\cdots+d_s}(t). \end{aligned} \tag{0.11}$$

We will call coefficient $a_{j,\alpha}$ a "coupling coefficient" if for some r, $r = 1, \ldots, s-1$, the inequalities $1 \le |\alpha| \le d_1 + d_2 + \cdots + d_r$ and $j \le m - d_1 - d_2 - \cdots - d_r - 2$ are fulfilled. The coefficients $a_{j,\alpha}$ with $j + |\alpha| \ge m-1$ are not coupling. In the next theorem we require more from the coupling coefficients, namely

$$\begin{aligned} &|\partial_t^k a_{j,\alpha}(t)| \le C\,(\lambda(t))^{\sum_{i=1}^{|\alpha|} n_i}\,|\log\lambda(t)|^{-|\alpha| + \sum_{l=1}^{r(j,\alpha)} d_l} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m-j-|\alpha|+k}, \\ &k = 0, 1. \end{aligned} \tag{0.12}$$

Here the number $r(j, \alpha)$ is defined by

$$r(j,\alpha) := \min\{r;\, 1 \le r \le s-1,\ 1 \le |\alpha| \le d_1 + d_2 + \cdots + d_r,\ j \le m - d_1 - d_2 - \cdots - d_r - 2\}.$$

To describe a propagation phenomena in the Cauchy problem we denote

$$\lambda_{\max} := \sup\left\{ |\lambda_j(t,\xi)|;\, j = 1, \ldots, m,\ t \in [0,T],\ \xi \in \mathbb{R}^n,\ |\xi| = 1 \right\}$$

and define *a hyperbolic cone of principal symbol* p_m by

$$\Gamma := \{(\lambda, \xi) \in \mathbb{R}^{n+1};\, \lambda > \lambda_{\max}|\xi|\}, \tag{0.13}$$

while Γ^* is a dual cone of Γ that is

$$\Gamma^* := \{(t,x) \in \mathbb{R}^{n+1};\, t\lambda + x\cdot\xi \ge 0 \quad \text{for all} \quad (\lambda,\xi) \in \Gamma\} \tag{0.14}$$

and will be called *a propagation cone of symbol* p_m. In the next theorem we use notations

$$D_0 := \bigcup_{i=0}^{m-1} \operatorname{supp} u_i, \qquad \Omega_0 := \operatorname{supp} f, \qquad K^+(\tau, y) := (\tau, y) + \Gamma^*.$$

The main result of this paper is the following theorem.

Theorem 0.1. *Assume* (0.2), (0.3), (0.6) *to* (0.8), *and* (0.12). *There exists a non-negative number* l *such that for every* $s \in [0, T]$ *and any real number* r *the Cauchy problem* (0.1) *for every*

$$f \in C\left([0,T]; H_{(r+l)}(\mathbb{R}^n)\right), \quad u_i \in H_{(r+l+m-i)}(\mathbb{R}^n), \quad i = 0, \ldots, m-1,$$

has a solution

$$u \in \bigcap_{j=0}^{m} C^j \left([0,T]; H_{(r+m-j)}(\mathbb{R}^n)\right).$$

This solution is unique and satisfies an a priori estimate

$$\sum_{j=0}^{m-1} \|D_t^j u(t)\|^2_{(m-1-j+r)} \le C_r \left(\sum_{j=0}^{m-1} \|D_t^j u_j\|^2_{(m+l+r-1-j)} + \left| \int_s^t \|f(\tau)\|^2_{(l+r)} d\tau \right| \right) \tag{0.15}$$

for all $t \in [0,T]$, with a constant C_r independent of s. For the support of the solution the following

$$\operatorname{supp} u \subset \left\{ \bigcup_{y \in D_0} K^+(0,y) \right\} \bigcup \left\{ \bigcup_{(\tau,y)\in\Omega_0} K^+(\tau,y) \right\}, \tag{0.16}$$

holds.

Thus according to this theorem solution propagates along the propagation cone.

DEFINITION 0.2. The Cauchy problem (0.1) is said to be well-posed if the statements of Theorem 0.1 hold.

EXAMPLE 0.3. For the second-order operator $P = D_t^2 - t^{2l} D_x^2 + t^k D_x$, $P_2 = (\lambda - t^l|\xi|)(\lambda + t^l|\xi|)$, $a_{0,1} = t^k$, condition (0.4) implies $k \ge l-1$.

The referee noted that in contrast to that example the conditions of the next ones cannot be simply derived from the necessary conditions given in the article by V. Ivrii and V. Petkov [4].

EXAMPLE 0.4. For the operator $P = P_3 + P_2 + P_1$, $P_s = \sum_{j+|\alpha|=s} a_{j,\alpha}(t)\lambda^j \xi^\alpha$, $s = 1, 2, 3$, with the principal symbol

$$P_3 = (\lambda - a_1 t^{n_1}|\xi|)(\lambda - a_2 t^{n_2}|\xi|)(\lambda - a_3 t^{n_3}|\xi|), \quad 1 \le n_1 \le n_2 \le n_3, \quad a_i \ne a_j,$$

condition (0.4) will be satisfied if

$$D_t^k a_{0,\alpha}(t) = \mathcal{O}\left(t^{n_1+n_2-1-k}\right), \quad |\alpha| = 2, \quad k = 0, 1,$$
$$D_t^k a_{1,\alpha}(t) = \mathcal{O}\left(t^{n_1-1-k}\right), \quad D_t^k a_{0,\alpha}(t) = \mathcal{O}\left(t^{n_1-2-k}\right), \quad |\alpha| = 1, \quad k = 0, 1.$$

In the forthcoming paper we will prove that these conditions as well as the conditions of the next example are also necessary conditions for the C^∞ well-posedness.

EXAMPLE 0.5. For the operator $P = P_3 + P_2 + P_1$, $P_s = \sum_{j+|\alpha|=s} a_{j,\alpha}(t)\lambda^j \xi^\alpha$, $s = 1, 2, 3$, with the principal symbol

$$
\begin{aligned}
&P_3 = (\lambda - a_1 \exp(-n_1\delta t^{-1})|\xi|)(\lambda - a_2 \exp(-n_2\delta t^{-1})|\xi|)(\lambda - a_3 \exp(-n_3\delta t^{-1})|\xi|),\\
&\mathbb{R} \ni a_i \neq a_j, \quad \delta > 0, \quad 1 = n_1 < n_2 \leq n_3,
\end{aligned}
$$

with T small. The conditions on the lower order terms are the following:

$$
\begin{aligned}
&|D_t^k a_{0,\alpha}(t)| \leq C_k t^{-3-2k} \exp(-(n_1 + n_2)\delta t^{-1}), \quad |\alpha| = 2, \quad k = 0, 1,\\
&|\operatorname{Im} a_{0,\alpha}(t)| \leq C t^{-2} \exp(-(n_1 + n_2)\delta t^{-1}), \quad |\alpha| = 2,\\
&|D_t^k a_{1,\alpha}(t)| \leq C_k t^{-3-2k} \exp(-n_1\delta t^{-1}), \quad |\alpha| = 1, \quad k = 0, 1,\\
&|D_t^k a_{0,\alpha}(t)| \leq C_k t^{-4-2k} \exp(-n_1\delta t^{-1}), \quad |\alpha| = 1, \quad k = 0, 1.
\end{aligned}
$$

For the scalar operators and for the systems with the double characteristics a microlocal energy method developed by the first authors and by T. Nishitani in [6], [7], [8], [12] allows to prove well-posedness and gives a complete picture of the propagation of singularities.

A. Nersesian and G. Oganesian [11] for the infinite degenerate hyperbolic equation under more restrictive (compare with (0.7), (0.8)) conditions, proved the existence and the uniqueness of the solution in the Sobolev spaces as well as the energy estimates.

The Cauchy problem for the operators with $\lambda(t) = t^{n_1}$ and coinciding non-negative integer n_j, $n_1 = n_2 = \cdots = n_m$, is investigated by K. Shinkai [13]. In that paper a fundamental solution is constructed as a sum of the Fourier integral operators. Using fundamental solution in [14] in the concept of the Stokes matrix a description of the propagation of the singularities is given.

More references for the operators with $\lambda = \lambda(t)$ $(n_1 = n_2 = \cdots = n_m)$ having zero of infinite order, can be found in [17]. In connections with this case we mention here a result on the well-posedness by S. Tarama [15].

F. Colombini and N. Orru [1] announced: if the Cauchy problem for the operator with coefficients $a_{j,\alpha} = a_{j,\alpha}(t)$ independent of $x \in \mathbb{R}$ and vanishing of finite order at $t = 0$, and without any lower order terms, $a_{j,\alpha}(t) \equiv 0$ $(j + |\alpha| < m)$, is C^∞ well-posed, then

$$
\sum_{j \neq k} \frac{\lambda_j(t, \xi)^2 + \lambda_k(t, \xi)^2}{(\lambda_j(t, \xi) - \lambda_k(t, \xi))^2} \leq C.
$$

The present paper is organised as follows. To make a presentation more transparent we consider a case of the x-independent coefficients. For the case of the x-dependent coefficients we obtain results by means of the Fourier integral operators with a symbols from the classes determined by the function $\lambda = \lambda(t)$. This approach developed in [16], [17] will be applied in the forthcoming paper. The main idea is based on the observation that if the principal part of the operator is assumed to have the smooth characteristic roots, then each zero of the complete symbol is smooth in some subdomain of the cotangent space (we follow [16], [17] and call these subdomains the *hyperbolic zones*) and can serve as a symbol. In that subdomain this zero is just a perturbation of the corresponding root of principal symbol and can be written as the series. The coefficients of this series are found by multiple logarithmic residues and are estimated along with their derivatives. In the remaining part of cotangent space (will be called the *pseudodifferential zones* [16], [17]) the exponential function of the operator is a pseudodifferential operator of the finite order.

1. Logarithmic derivative of the Vandermonde matrix-valued functions. The correctors

By the Fourier transform against variable $x \in \mathbb{R}^n$ assuming that u and f belong to the corresponding spaces, we obtain from (0.1) the equation

$$P(t, D_t, \xi)u(t, \xi) = f(t, \xi), \tag{1.1}$$

for a new unknown function $u = u(t, \xi)$.

Our aim is to obtain an *a priori* estimate for the solutions of ordinary differential equation (1.1) with parameter $\xi \in \mathbb{R}^n$. To this end we write this equation as a system:

$$D_t U + \mathcal{A}(t, \xi)U + \mathcal{B}(t, \xi)U = \mathcal{F}(t, \xi),$$

where

$$U(t, \xi) := \begin{pmatrix} u(t) \\ D_t u(t) \\ \vdots \\ D_t^{m-1} u(t) \end{pmatrix}, \qquad \mathcal{F}(t, \xi) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(t, \xi) \end{pmatrix}. \tag{1.2}$$

Here $\mathcal{A}$ is a principal part, while $\mathcal{B}$ is a lower order term. The diagonalizer for matrix $\mathcal{A}(t, \xi)$ has a great importance for the following constructions, therefore we first of all consider its properties. As matter of fact a diagonalizer is the Vandermonde matrix constructed by the eigenvalues of $\mathcal{A}(t, \xi)$. Application of the diagonalizer leads to the derivative of the matrix multiplied on its inverse. This product will be called *Logarithmic derivative of the matrix*. Thus let $\mathcal{M}^{-1}(t, \xi)$ be the Vandermonde matrix corresponding to the system $\{\lambda_1(t, \xi), \ldots, \lambda_m(t, \xi)\}$,

$$\mathcal{M}^{-1}(t, \xi) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1(t, \xi) & \lambda_2(t, \xi) & \lambda_3(t, \xi) & \cdots & \lambda_m(t, \xi) \\ \lambda_1^2(t, \xi) & \lambda_2^2(t, \xi) & \lambda_3^2(t, \xi) & \cdots & \lambda_m^2(t, \xi) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{m-1}(t, \xi) & \lambda_2^{m-1}(t, \xi) & \lambda_3^{m-1}(t, \xi) & \cdots & \lambda_m^{m-1}(t, \xi) \end{pmatrix}$$

with the reciprocal matrix

$$\mathcal{M}(t,\xi) = \begin{pmatrix} \dfrac{\psi_{11}}{\varphi_1(\lambda_1)} & \dfrac{\psi_{12}}{\varphi_1(\lambda_1)} & \dfrac{\psi_{13}}{\varphi_1(\lambda_1)} & \cdots & \dfrac{\psi_{1m}}{\varphi_1(\lambda_1)} \\ \dfrac{\psi_{21}}{\varphi_2(\lambda_2)} & \dfrac{\psi_{22}}{\varphi_2(\lambda_2)} & \dfrac{\psi_{23}}{\varphi_2(\lambda_2)} & \cdots & \dfrac{\psi_{2m}}{\varphi_2(\lambda_2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\psi_{m1}}{\varphi_m(\lambda_m)} & \dfrac{\psi_{m2}}{\varphi_m(\lambda_m)} & \dfrac{\psi_{m3}}{\varphi_m(\lambda_m)} & \cdots & \dfrac{\psi_{mm}}{\varphi_m(\lambda_m)} \end{pmatrix},$$

where

$$\begin{aligned} \psi_{i,m-k} &= c_k + c_{k-1}\lambda_i + \cdots + c_0\lambda_i^k, \qquad i = 1,\dots,m, \quad k = 0,\dots,m-1, \\ c_k(t,\xi) &= \sum_{|\alpha|=k} a_{m-k,\alpha}(t)\xi^\alpha, \qquad k = 0,\dots,m-1, \\ \varphi_i(\tau) &= \prod_{k\neq i}(\tau-\lambda_k) = \sum_{k=1}^{m}\psi_{i,k}\tau^{k-1}, \qquad i = 1,\dots,m, \\ \mathcal{M}_{km}(t,\xi) &= \frac{\psi_{km}}{\varphi_k(\lambda_k)} = \frac{1}{\prod_{i\neq k}(\lambda_k(t,\xi)-\lambda_i(t,\xi))}. \end{aligned}$$

In particular,

$$|\mathcal{M}_{km}(t,\xi)| \le C|\xi|^{-m+1}\frac{1}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-k}}.$$

The derivative $\mathcal{M}_t^{-1}(t,\xi) := d\mathcal{M}^{-1}(t,\xi)/dt$ is easily calculated:

$$\mathcal{M}_t^{-1}(t,\xi) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \lambda_{1t}(t,\xi) & \lambda_{2t}(t,\xi) & \vdots & \lambda_{mt}(t,\xi) \\ 2\lambda_1\lambda_{1t} & 2\lambda_2\lambda_{2t} & \vdots & 2\lambda_m\lambda_{mt} \\ \vdots & \vdots & \vdots & \vdots \\ (m-1)\lambda_1^{m-2}\lambda_{1t} & (m-1)\lambda_2^{m-2}\lambda_{2t} & \cdots & (m-1)\lambda_m^{m-2}\lambda_{mt} \end{pmatrix}.$$

The matrix $\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)$ is a *logarithmic derivative* of the Vandermonde matrix $\mathcal{M}^{-1}(t,\xi)$. The following lemma gives the elements of this matrix.

Lemma 1.1. *For the logarithmic derivative of the Vandermonde matrix* $\mathcal{M}^{-1}(t,\xi)$*, that is for the matrix* $\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)$*, consisting of the elements*

$\big(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\big)_{kl}$ $(k, l = 1, \dots, m)$, *one has the following formula*

$$\big(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\big)_{kl} = \begin{cases} \lambda_{lt} \dfrac{1}{\prod_{i\neq k}(\lambda_k - \lambda_i)} \prod\limits_{i\neq k,l} (\lambda_l - \lambda_i) & \textit{when} \quad k \neq l, \\ \lambda_{kt} \sum\limits_{\substack{j=1 \\ j\neq k}}^{m} \dfrac{1}{(\lambda_k - \lambda_j)} & \textit{when} \quad k = l. \end{cases}$$

Proof. We have

$$\big(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\big)_{kl} = \sum_{i=2}^{m} \frac{\psi_{ki}}{\varphi_k(\lambda_k)}(i-1)\lambda_l^{i-2}\lambda_{lt} = \frac{1}{\varphi_k(\lambda_k)}\lambda_{lt}\left(\frac{\partial}{\partial\tau}\varphi_k(\tau)\right)_{\tau=\lambda_l}.$$

But

$$\left(\frac{\partial}{\partial\tau}\varphi_k(\tau)\right)_{\tau=\lambda_l} = \left(\frac{\partial}{\partial\tau}\prod_{i\neq k}(\tau-\lambda_i)\right)_{\tau=\lambda_l} = \left(\sum_{\substack{j=1\\ j\neq k}}^{m}\prod_{i\neq k,j}(\tau-\lambda_i)\right)_{\tau=\lambda_l} = \prod_{i\neq k,l}(\lambda_l-\lambda_i).$$

Hence,

$$\frac{1}{\varphi_k(\lambda_k)}\left(\frac{\partial}{\partial\tau}\varphi_k(\tau)\right)_{\tau=\lambda_l} = \frac{1}{\prod_{i\neq k}(\lambda_k-\lambda_i)}\prod_{i\neq k,l}(\lambda_l-\lambda_i).$$

Lemma is proved. □

Lemma 1.2. *Assume that conditions* (0.2), (0.3), (0.9), (0.10), *and* (0.11) *are satisfied. Then*

$$|\partial_t^k \lambda_j(t,\xi)| \le C\lambda_j(t)|\xi|\left(\frac{\lambda_j(t)}{\Lambda_j(t)}\right)^k, \tag{1.3}$$
$$j = 1, \dots, m, \quad k = 0, 1, \quad \textit{for all} \quad t \in [0, T].$$

Proof. First of all we set $|\xi| = 1$ and consider equation

$$\tau^m + \sum_{j=1,\dots,m} a_{m-j}(t,\xi)\tau^{m-j} = 0,$$

where

$$a_{m-j}(t,\xi) := \sum_{|\alpha|=j} a_{m-j,\alpha}(t)\xi^\alpha.$$

For these functions according to (0.3) the following estimate holds:

$$|\partial_t^k a_{m-j}(t,\xi)| \leq C_1 \lambda_1(t) \cdots \lambda_j(t) \left(\frac{\lambda_j(t)}{\Lambda_j(t)} \right)^k,$$
$$j = 1, \ldots, m, \quad k = 0, 1, \quad \text{for all} \quad t \in [0, T].$$

If we replace τ with $\lambda_1(t)\mu$, then the equation for μ,

$$\mu^m + \sum_{j=1,\ldots,m} \frac{a_{m-j}(t,\xi)}{\lambda_1^j(t)} \mu^{m-j} = 0,$$

has uniformly bounded coefficients. Hence its roots are also uniformly bounded, and we get

$$|\lambda_j(t,\xi)| \leq C_2 \lambda_1(t), \qquad j = 1, \ldots, m, \quad \text{for all} \quad t \in [0, T].$$

Further, fix point (t,ξ). At that point either $|\lambda_1(t,\xi)| \geq \lambda_1(t)C/4$ or $|\lambda_2(t,\xi)| \geq \lambda_1(t)C/4$ with C of (0.2). Let $|\lambda_1(t,\xi)| \geq \lambda_1(t)C/4$. Then we write

$$\tau^m + \sum_{j=1,\ldots,m} a_{m-j}(t,\xi)\tau^{m-j} = (\tau - \lambda_1(t,\xi)) \left(\tau^{m-1} + \sum_{j=1,\ldots,m-1} b_{m-1-j}(t,\xi)\tau^{m-1-j} \right).$$

Coefficients $b_{m-1-j}(t,\xi)$ are easily calculated:

$$b_0(t,\xi) = -\frac{a_0(t,\xi)}{\lambda_1(t,\xi)}, \quad b_1(t,\xi) = \frac{b_0(t,\xi) - a_1(t,\xi)}{\lambda_1(t,\xi)}, \ldots,$$
$$b_{m-2}(t,\xi) = \frac{b_{m-3}(t,\xi) - a_k(t,\xi)}{\lambda_1(t,\xi)}.$$

It follows

$$|b_0(t,\xi)| \leq C_3 \lambda_2(t) \cdots \lambda_m(t), \quad |b_1(t,\xi)| \leq C_3 \lambda_2(t) \cdots \lambda_{m-1}(t), \ldots,$$
$$|b_{m-2}(t,\xi)| \leq C_3 \lambda_2(t).$$

The constant C_3 is independent of (t,ξ). Thus, for the roots of the equation

$$\tau^{m-1} + \sum_{j=1,\ldots,m-1} b_{m-1-j}(t,\xi)\tau^{m-1-j} = 0,$$

that is for $\lambda_2(t,\xi), \lambda_3(t,\xi), \ldots, \lambda_m(t,\xi)$, by already used arguments we obtain

$$|\lambda_j(t,\xi)| \leq C_4 \lambda_2(t), \qquad j = 2, \ldots, m, \quad \text{for all} \quad t \in [0, T].$$

Step by step reducing the order of polynomials, we arrive at the estimate (1.3) with $k = 0$. From the implicit function theorem it follows that estimate with $k = 1$. Lemma is proved. $\square$

Lemma 1.3. *The elements* $\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kl}$ *of the logarithmic derivative of the Vandermonde matrix* $\mathcal{M}^{-1}(t,\xi)$ *can be estimated as follows*

$$\left|\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kl}\right| \leq \begin{cases} C\dfrac{\lambda'(t)}{\lambda(t)} \quad \text{if} \quad k=l, \quad \text{where} \quad l=1,\dots,m; \\ C\dfrac{\lambda'(t)}{\lambda(t)}\lambda^{n_{k+1}+\cdots+n_{l-1}+(m-l+1)n_l-n_k(m-k)}(t) \quad \text{if} \quad k<l-1, \\ \qquad \text{where} \quad l=3,\dots,m; \\ C\dfrac{\lambda'(t)}{\lambda(t)}\lambda^{(m-l+1)(n_l-n_{l-1})}(t) \quad \text{if} \quad k=l-1, \\ \qquad \text{where} \quad l=2,\dots,m; \\ C\dfrac{\lambda'(t)}{\lambda(t)}\lambda^{(m-l-1)n_l-n_{l+1}-\cdots-n_{k-1}-n_k(m-k)}(t) \quad \text{if} \quad k>l+1, \\ \qquad \text{where} \quad l=1,\dots,m-1; \\ C\dfrac{\lambda'(t)}{\lambda(t)}\lambda^{(m-l-1)(n_l-n_{l+1})}(t) \quad \text{if} \quad k=l+1, \\ \qquad \text{where} \quad l=1,\dots,m-1. \end{cases}$$

Proof. Due to Lemma 1.1, for $k=l$ we write

$$\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kk} = \lambda_{kt}\sum_{\substack{j=1\\ j<k}}^{m}\frac{1}{\left(\lambda_k-\lambda_j\right)} + \lambda_{kt}\sum_{\substack{j=1\\ j>k}}^{m}\frac{1}{\left(\lambda_k-\lambda_j\right)}.$$

It follows

$$\begin{aligned} &\left|\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kk}\right| \\ &\quad \leq C\lambda'(t)\lambda^{n_k-1}(t)|\xi|\sum_{j<k}\left|\frac{1}{\left(\lambda_k-\lambda_j\right)|\xi|}\right| + C\lambda'(t)\lambda^{n_k-1}(t)|\xi|\sum_{j>k}\left|\frac{1}{\left(\lambda_k-\lambda_j\right)|\xi|}\right| \\ &\quad \leq C\lambda'(t)\lambda^{n_k-1}(t)|\xi|\left|\frac{1}{\lambda^{n_{k-1}}(t)|\xi|}\right| + C\lambda'(t)\lambda^{n_k-1}(t)|\xi|\left|\frac{1}{\lambda^{n_k}(t)|\xi|}\right| \\ &\quad \leq C\frac{\lambda'(t)}{\lambda(t)}\frac{\lambda^{n_k}(t)}{\lambda^{n_{k-1}}(t)} + C\frac{\lambda'(t)}{\lambda(t)} \end{aligned}$$

with $n_k \geq n_{k-1}$. This proves the first statement of the lemma. If $k<l-1$ then

$$\begin{aligned} \left|\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kl}\right| &\leq \frac{\lambda'(t)}{\lambda(t)}\frac{\lambda_l(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k^{m-k}(t)}\left|\prod_{i\neq k,l}(\lambda_l-\lambda_i)\right| \\ &\leq \frac{\lambda'(t)}{\lambda(t)}\frac{\lambda_l(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k^{m-k}(t)}\lambda_1\lambda_2\cdots\lambda_{k-1}\lambda_{k+1}\cdots\lambda_{l-1}\lambda_l^{m-l} \end{aligned}$$

$$\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l(t)}{\lambda_k^{m-k}(t)} \lambda_{k+1} \cdots \lambda_{l-1} \lambda_l^{m-l}.$$

If $k = l - 1$, where $l = 2, \ldots, m$, then

$$\begin{aligned}
\left|\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kl}\right| &= \left| \lambda_{lt} \frac{1}{\prod_{i \neq l-1} (\lambda_{l-1} - \lambda_i)} \prod_{i \neq l-1,\, l} (\lambda_l - \lambda_i) \right| \\
&\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l(t)}{\lambda_1(t) \cdots \lambda_{l-2}(t) \lambda_{l-1}^{m-l+1}(t)} \lambda_1 \lambda_2 \cdots \lambda_{l-2} \lambda_l^{m-l} \\
&\leq \frac{\lambda'(t)}{\lambda(t)} \left(\frac{\lambda_l(t)}{\lambda_{l-1}(t)} \right)^{m-l+1}.
\end{aligned}$$

If $k > l + 1$, then

$$\begin{aligned}
\left|\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{kl}\right| &\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l(t)}{\lambda_1(t) \cdots \lambda_{k-1}(t) \lambda_k^{m-k}(t)} \left| \prod_{i \neq k,l} (\lambda_l - \lambda_i) \right| \\
&\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l(t)}{\lambda_1(t) \cdots \lambda_{k-1}(t) \lambda_k^{m-k}(t)} \lambda_1 \cdots \lambda_{l-1} \lambda_l^{m-l-1} \\
&\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l^{m-l-1}(t)}{\lambda_{l+1}(t) \cdots \lambda_{k-1}(t) \lambda_k^{m-k}(t)}.
\end{aligned}$$

If $k = l + 1$, where $l = 1, \ldots, m - 1$, then

$$\begin{aligned}
\left|\left(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\right)_{l+1,l}\right| &\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l(t)}{\lambda_1(t) \cdots \lambda_l(t) \lambda_{l+1}^{m-l-1}(t)} \left| \prod_{i \neq l+1,\, l} (\lambda_l - \lambda_i) \right| \\
&\leq \frac{\lambda'(t)}{\lambda(t)} \frac{\lambda_l(t)}{\lambda_1(t) \cdots \lambda_l(t) \lambda_{l+1}^{m-l-1}(t)} \lambda_1 \cdots \lambda_{l-1} \lambda_l^{m-l-1} \\
&\leq \frac{\lambda'(t)}{\lambda(t)} \left(\frac{\lambda_l(t)}{\lambda_{l+1}(t)} \right)^{m-l-1}.
\end{aligned}$$

The lemma is proved. □

The singularities at $t = 0$ of the elements of matrix $\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)$ belonging to the lower triangular part, and described by Lemma 1.3, are too strong, that is stronger than ones of the diagonal elements. Last ones have a "logarithmic derivative type singularity" like $\lambda'(t)/\lambda(t) = d \log \lambda(t)/dt$. At the same time the elements of the upper triangular part behave as some powers of $\lambda(t)$. This allows to "correct" singularities by means of the diagonal matrix, and finally reduce these singularities to the logarithmic derivative type. This is done in the next lemma where correcting matrix $\widetilde{\Lambda}$ is given explicitly. This matrix $\widetilde{\Lambda}$ will be called a "corrector" for $\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)$. The corrector $\widetilde{\Lambda}$ commutes with any diagonal matrix and this allows to preserve a diagonal structure of the principal part of operator, obtained after diagonalization.

Lemma 1.4. *Let $\widetilde{\Lambda}$ be a diagonal matrix*

$$\widetilde{\Lambda} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \lambda^{a_2}(t) & \ddots & \\ 0 & 0 & & \lambda^{a_m}(t) \end{pmatrix}, \tag{1.4}$$

where the nonegative numbers $a_2, \dots, a_m$, are chosen by

$$a_k = -(m-2)n_1 + n_2 + \cdots + n_{k-1} + n_k(m-k) \geq 0, \quad k = 2, \dots, m. \tag{1.5}$$

Then the logarithmic derivative of the matrix $\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1}$ is estimated as follows:

$$\|\widetilde{\Lambda}\mathcal{M}(t,\xi)(\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1})_t\| \leq \|\widetilde{\Lambda}_t\widetilde{\Lambda}^{-1}\| + \|\widetilde{\Lambda}\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\widetilde{\Lambda}^{-1}\| \leq C_M \frac{\lambda'(t)}{\lambda(t)}$$

for all $t \in (0,T]$ and all $\xi \in \mathbb{R}^n$.

Proof. We have

$$\begin{aligned}
&(\widetilde{\Lambda}\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\widetilde{\Lambda}^{-1})_{kl} = \lambda^{a_k - a_l}(t)(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi))_{kl} \\
&\quad = \lambda^{-(m-2)n_1+n_2+\cdots+n_{k-1}+n_k(m-k)-[-(m-2)n_1+n_2+\cdots+n_{l-1}+n_l(m-l)]}(t)(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi))_{kl}.
\end{aligned}$$

If $k > l$ we have

$$\begin{aligned}
&(\widetilde{\Lambda}\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\widetilde{\Lambda}^{-1})_{kl} \\
&\quad = \frac{\lambda'(t)}{\lambda(t)}\lambda^{n_l+\cdots+n_{k-1}+n_k(m-k)-n_l(m-l)+(m-l-1)n_l-n_{l+1}-\cdots-n_{k-1}-n_k(m-k)}(t),
\end{aligned}$$

where $n_l + \cdots + n_{k-1} + n_k(m-k) - n_l(m-l) + (m-l-1)n_l = 0$. If $k < l$ we have

$$\begin{aligned}
&(\widetilde{\Lambda}\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\widetilde{\Lambda}^{-1})_{kl} \\
&\quad = \lambda^{-(m-2)n_1+n_2+\cdots+n_{k-1}+n_k(m-k)-[-(m-2)n_1+n_2+\cdots+n_{l-1}+n_l(m-l)]}(t) \\
&\qquad \times \frac{\lambda'(t)}{\lambda(t)}\lambda^{n_{k+1}+\cdots+n_{l-1}+(m-l+1)n_l-n_k(m-k)}(t).
\end{aligned}$$

The exponent of $\lambda(t)$ is non-negative:

$$\begin{aligned}
&-(m-2)n_1 + n_2 + \cdots + n_{k-1} + n_k(m-k) \\
&-[-(m-2)n_1 + n_2 + \cdots + n_{l-1} + n_l(m-l)] \\
&+n_{k+1} + \cdots + n_{l-1} + (m-l+1)n_l - n_k(m-k) = n_l - n_k \geq 0.
\end{aligned}$$

Lemma is proved. □

2. Micro-local approach. Properties of the homogeneous hyperbolic polynomials

In this section we consider the homogeneous symbols $P_m(t,\lambda,\xi)$ satisfying (0.2) with $\lambda_j(t)=\lambda^{n_j}(t)$, $1\le n_1\le n_2\le\cdots\le n_m$, so that condition (0.3) becomes

$$|D_t^k a_{m-|\alpha|,\alpha}(t)|\le c(\lambda(t))^{\sum_{i=1}^{|\alpha|}n_i}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k,\quad k=0,1, \tag{2.1}$$

with $|\alpha|\neq 0$. (In the next proposition n is not only a dimension of ξ!)

Proposition 2.1. *Assume that the zeros $\lambda_i(t,\xi)$, $i=1,2,\dots,m$, of the principal symbol, that is roots of*

$$\lambda^m+\sum_{k=1}^{m}\lambda^{m-k}\sum_{|\alpha|=k}a_{m-k,\alpha}(t)\xi^\alpha=0, \tag{2.2}$$

satisfy (0.2), *and that* (0.3) *is fulfilled. Then for n such that $\lambda_n(t)\neq\lambda_{n+1}(t)$ the roots $\tau_i(t,\xi)$, $i=1,2,\dots,n$, of "truncated" principal symbol, that is the roots of equation*

$$\tau^n+\sum_{k=1}^{n}\tau^{n-k}\sum_{|\alpha|=k}a_{m-k,\alpha}(t)\xi^\alpha=0 \tag{2.3}$$

for sufficiently small T inherit behaviour of the roots of principal symbol, that is

$$|D_t^k\tau_i(t,\xi)|\le c\lambda^{n_i}(t)|\xi|\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k,\quad k=0,1\quad(i=1,2,\dots,n), \tag{2.4}$$

$$|\tau_j(t,\xi)-\tau_k(t,\xi)|\ge\delta_1\lambda(t)^{n_j}|\xi|,\quad j<k, \tag{2.5}$$

for all $t\in(0,T]$ and all $\xi\in\mathbb{R}^n$, with some positive constant δ_1.

Proof. We have according to assumptions

$$|D_t^k a_{m-|\alpha|,\alpha}(t)|\le c\lambda_1(t)\cdots\lambda_{|\alpha|}(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k,\quad k=0,1\quad(|\alpha|=1,2,\dots,m). \tag{2.6}$$

Consider equation

$$\tau^m+(A_1+w_1)\tau^{m-1}+\cdots+(A_m+w_m)=0, \tag{2.7}$$

where

$$A_k=\sum_{|\alpha|=k}a_{m-k,\alpha}(t)\xi^\alpha,\qquad w=(w_1,\dots,w_m)\in\mathbb{C}^m,$$

w is a perturbation of the coefficients. If we suppose that the zeros $\lambda_1, \dots, \lambda_m$ of the polynomial $P(\lambda) = \lambda^m + A_1\lambda^{m-1} + \cdots + A_m$ are distinct, then in some neighbourhood of $0 \in \mathbb{C}^m$ they depend analytically on w:

$$\tau_i(t,\xi) = \lambda_i(t,\xi) + \sum_{|\alpha|>0} c_\alpha^{(i)}(t,\xi) w^\alpha, \tag{2.8}$$

$i = 1, \dots, n$. Consider first n roots $\tau_1(t,\xi)$, ..., $\tau_n(t,\xi)$. For applications we need a precise estimate for the domain of convergence of these series. To find that out for a fixed i, let us denote

$$F(w,z) = \frac{1}{P'_w(\lambda_i)} \left\{ (z+\lambda_i)^m + (A_1 + w_1)(z+\lambda_i)^{m-1} + \cdots + (A_m + w_m) \right\}, \tag{2.9}$$

where $z \in \mathbb{C}$, while P is a polynomial of the left-hand side of (2.2) and $P'(\lambda) := dP(\lambda)/d\lambda$ is a separating polynomial. Then, according to [17, (2.1.68)],

$$c_\alpha^{(i)} = \sum_{k=1}^{2|\alpha|-1} \frac{(-1)^k}{k!\alpha!} \left[\left(\partial_{w_1}^{\alpha_1} \cdots \partial_{w_m}^{\alpha_m}\right) \left(\frac{d}{dz}\right)^{k-1} (F(w,z)-z)^k \right]_{w=0,z=0}. \tag{2.10}$$

It is evident that $F(0,0) = 0$, while $F'_z(0,0) = 1$, and $\partial_{w_1}^{\alpha_1} \cdots \partial_{w_m}^{\alpha_m} (F(w,z)-z) = 0$ when $|\alpha| \geq 2$.

The next lemma gives the estimates for $c_\alpha^{(i)}(t,\xi)$, ($i = 1, \dots, n$), which allow to determine the radiuses of convergence of series (2.8).

Lemma 2.2. *There is a positive number c such that*

$$|c_\alpha^{(i)}(t,\xi)| \leq c^{|\alpha|}(\lambda_i(t)|\xi|)|\xi|^{-\sum_{l=1}^m l\alpha_l}(\lambda_1(t)\cdots\lambda_i(t))^{-|\alpha|}(\lambda_i(t))^{i|\alpha|-\sum_{l=1}^m l\alpha_l}, \tag{2.11}$$

for all $t \in (0,T]$ and all $\xi \in \mathbb{R}^n$.

Proof. We have

$$\begin{aligned} &\partial_{w_1}^{\alpha_1} \cdots \partial_{w_m}^{\alpha_m} \left(\frac{d}{dz}\right)^{k-1} (F(w,z)-z)^k \\ &\quad = \frac{k!}{(k-|\alpha|)!} \frac{1}{(P'(\lambda_i(t,\xi)))^{|\alpha|}} \sum_{j=k-1-m|\alpha|+\sum_{l=1}^m l\alpha_l}^{k-1} \frac{(k-1)!}{j!(k-1-j)!} \\ &\quad \times \left(\left(\frac{d}{dz}\right)^j (F(w,z)-z)^{k-|\alpha|} \right) \left(\left(\frac{d}{dz}\right)^{k-1-j} (z+\lambda_i(t,\xi))^{m|\alpha|-\sum_{l=1}^m l\alpha_l} \right), \end{aligned} \tag{2.12}$$

if $k \geq |\alpha|$, while $\partial_{w_1}^{\alpha_1} \cdots \partial_{w_m}^{\alpha_m} \left(d/dz\right)^{k-1} (F(w,z)-z)^k = 0$ if $|\alpha| > k$. Here $k-1-j \leq$

$m|\alpha| - \sum_{l=1}^m l\alpha_l$ and $0 \le j \le k-1$. Denote $P^{(l)}(\lambda) = (d^l P(\lambda))/(d\lambda^l)$, then

$$\left(\left(\frac{d}{dz}\right)^j (F(w,z)-z)^{k-|\alpha|} \right)_{w=0,\ z=0}$$
$$= \left(P'(\lambda_i(t,\xi))\right)^{|\alpha|-k} \sum_{\substack{|\beta|=j,\\ 2\le\beta_l\le m}} \frac{j!}{\beta!} P^{(\beta_1)}(\lambda_i(t,\xi)) \cdots P^{(\beta_{k-|\alpha|})}(\lambda_i(t,\xi)).$$

Thus

$$c_\alpha^{(i)}(t,\xi) = \sum_{k=|\alpha|}^{2|\alpha|-1} \frac{(-1)^k}{k!\alpha!} \frac{k!}{(k-|\alpha|)!} \frac{1}{(P'(\lambda_i(t,\xi))^k} \sum_{j=k-1-m|\alpha|+\sum_{l=1}^m l\alpha_l}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}$$
$$\times \frac{(m|\alpha| - \sum_{l=1}^m l\alpha_l)!}{(m|\alpha| - \sum_{l=1}^m l\alpha_l - k + 1 + j)!} (\lambda_i(t,\xi))^{m|\alpha| - \sum_{l=1}^m l\alpha_l - k+1+j}$$
$$\times \sum_{\substack{|\beta|=j,\\ 2\le\beta_l\le m}} \frac{j!}{\beta!} P^{(\beta_1)}(\lambda_i(t,\xi)) \cdots P^{(\beta_{k-|\alpha|})}(\lambda_i(t,\xi)).$$

with $k-1-j \le m|\alpha| - \sum_{l=1}^m l\alpha_l$, and $0 \le j \le k-1$. One has

$$|P^{(l)}(\lambda_i(t,\xi))| \le C\lambda_1(t)\cdots\lambda_{i-1}(t)(\lambda_i(t))^{m-l-i+1}|\xi|^{m-l}.$$

It follows

$$|c_\alpha^{(i)}(t,\xi)|$$
$$\le \sum_{k=|\alpha|}^{2|\alpha|-1} \frac{1}{\alpha!(k-|\alpha|)!} \frac{1}{(\delta\lambda_1(t)\cdots\lambda_{i-1}(t)(\lambda_i(t))^{m-i})^k} \frac{1}{|\xi|^{(m-1)k}}$$
$$\times \sum_{j=k-1-m|\alpha|+\sum_{l=1}^m l\alpha_l}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}$$
$$\times \frac{(m|\alpha| - \sum_{l=1}^m l\alpha_l)!}{(m|\alpha| - \sum_{l=1}^m l\alpha_l - k+1+j)!} (c\lambda_i(t)|\xi|)^{m|\alpha| - \sum_{l=1}^m l\alpha_l - k+1+j}$$
$$\times \sum_{\substack{|\beta|=j,\\ 2\le\beta_l\le m}} \frac{j!}{\beta!} c^{k-|\alpha|} (\lambda_1(t)\cdots\lambda_{i-1}(t))^{k-|\alpha|} (\lambda_i(t))^{m(k-|\alpha|)-(i-1)(k-|\alpha|)-j} |\xi|^{m(k-|\alpha|)-j}$$
$$\le \delta^{-2|\alpha|} c^{(m+1)|\alpha|} |\xi|^{1-\sum_{l=1}^m l\alpha_l} (\lambda_1(t)\cdots\lambda_{i-1}(t))^{-|\alpha|} (\lambda_i(t))^{1+(i-1)|\alpha| - \sum_{l=1}^m l\alpha_l}$$
$$\times \sum_{k=|\alpha|}^{2|\alpha|-1} \frac{1}{\alpha!(k-|\alpha|)!} \sum_{j=k-1-m|\alpha|+\sum_{l=1}^m l\alpha_l}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}$$
$$\times \frac{(m|\alpha| - \sum_{l=1}^m l\alpha_l)!}{(m|\alpha| - \sum_{l=1}^m l\alpha_l - k+1+j)!} \sum_{\substack{|\beta|=j,\\ 2\le\beta_l\le m}} \frac{j!}{\beta!}.$$

The sums of the last inequality contain terms with $k-1-j \le m|\alpha| - \sum_{l=1}^{m} l\alpha_l$, and $0 \le j \le k-1$, only. They can be estimated as follows

$$\sum_{k=|\alpha|}^{2|\alpha|-1} \frac{1}{\alpha!(k-|\alpha|)!} \sum_{j=k-1-m|\alpha|+\sum_{l=1}^{m} l\alpha_l}^{k-1} \frac{(k-1)!}{j!(k-1-j)!} \frac{(m|\alpha| - \sum_{l=1}^{m} l\alpha_l)!}{(m|\alpha| - \sum_{l=1}^{m} l\alpha_l - k + 1 + j)!}$$
$$\times \sum_{\substack{|\beta|=j,\\ 2\le\beta_l\le m\\ l=1,\dots,k-|\alpha|}} \frac{j!}{\beta!} \le c_1^{|\alpha|}.$$

Finally we obtain (2.11) with some positive constant c. Lemma is proved. □

Lemma 2.3. *For every given positive c there is $T > 0$ such that if $w \in \mathbb{C}^m$ satisfies*

$$w_1 = w_2 = \cdots = w_n = 0, \quad |w_j| \le c\lambda_1(t)\cdots\lambda_j(t)|\xi|^j, \qquad j = n+1, \dots, m,$$

with $\xi \in \mathbb{R}^n$ and $t \in [0, T]$, then series (2.8) *taken at* (t, ξ) *converge.*

Proof. Indeed, the sums in (2.8) include the terms with α such that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ only, and therefore

$$|w^\alpha| \le c^{|\alpha|}(\lambda_1(t)\cdots\lambda_{n+1}(t))^{|\alpha|}(\lambda_{n+2}(t))^{\alpha_{n+2}+\cdots+\alpha_m}\cdots(\lambda_m(t))^{\alpha_m}|\xi|^{\sum_{l=n+1}^{m} l\alpha_l}.$$

Hence

$$\begin{aligned} |c_\alpha^{(i)}(t,\xi)w^\alpha| \le\ & c_2^{|\alpha|}(\lambda_i(t)|\xi|)(\lambda_{i+1}(t)\cdots\lambda_n(t))^{|\alpha|}(\lambda_{n+1}(t))^{\sum_{l=n+1}^{m}\alpha_l} \\ & \times(\lambda_{n+2}(t))^{\sum_{l=n+2}^{m}\alpha_l}\cdots(\lambda_m(t))^{\alpha_m}(\lambda_i(t))^{i|\alpha|-\sum_{l=n+1}^{m} l\alpha_l} \\ \le\ & (\lambda_i(t)|\xi|)\left(c_2\frac{\lambda_{i+1}(t)}{\lambda_i(t)}\cdots\frac{\lambda_n(t)}{\lambda_i(t)}\right)^{|\alpha|}\left(\frac{\lambda_{n+1}(t)}{\lambda_i(t)}\right)^{\sum_{l=n+1}^{m}(l-n)\alpha_l} \\ \le\ & (\lambda_i(t)|\xi|)\left(c_3\frac{\lambda_{n+1}(t)}{\lambda_i(t)}\right)^{|\alpha|}, \qquad i = 1, \dots, n, \end{aligned} \tag{2.13}$$

with a positive constant c_3. This proves a convergence of the series (2.8) for small T, since for every ε there exists T such that

$$\sum_{|\alpha|>0}\left(c_3\frac{\lambda_{n+1}(t)}{\lambda_i(t)}\right)^{|\alpha|} \le \varepsilon \quad \text{for all} \quad t \in (0, T], \quad i = 1, \dots, n,$$

and $\lambda_{n+1}(t)/\lambda_n(t) \to 0$ as $t \to 0$. Lemma is proved. □

Completion of the proof of Proposition 2.1. It remains to choose $w_j = -A_j$ for $j = n+1, \dots, m$. Thus for $k = 0$ inequalities (2.4) and (2.5) are proved. The derivatives can be estimated by the formula of derivative of an implicit function. Proposition is proved. $\square$

Corollary 2.4. *Assume that the conditions of* Proposition 2.1 *are satisfied. Let $n < m - 1$, $\lambda_{n+1}(t) \neq \lambda_n(t)$. Then the roots $\tau_i(t, \xi)$, $i = 1, 2, \dots, n, n+1$, of equation*

$$\tau^{n+1} + \sum_{k=1}^{n+1} \tau^{n+1-k} \sum_{|\alpha|=k} a_{m-k,\alpha}(t)\xi^\alpha = 0$$

for sufficiently small T are real-valued and they inherit properties of the roots of principal symbol, that is (2.4) *and* (2.5), *and additionally*

$$\begin{aligned}
&|D_t^k \tau_{n+1}(t, \xi)| \le c\lambda_n(t)|\xi| \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad k = 0, 1, \quad \xi \in \mathbb{R}^n, \quad t \in (0, T],\\
&|\tau_j(t, \xi) - \tau_{n+1}(t, \xi)| \ge \delta_1' \lambda_j(t)|\xi|, \quad j \le n, \quad t \in [0, T], \quad \xi \in \mathbb{R}^n.
\end{aligned}$$

Proof. The first n roots of this equation

$$\tau^{n+1} + \sum_{k=1}^{n} \tau^{n+1-k} \sum_{|\alpha|=k} a_{m-k,\alpha}(t)\xi^\alpha = 0$$

coincide with the roots of (2.3), while the last one just vanish, $\tau_{n+1}(t, \xi) \equiv 0$. According to (2.8) and to (2.13), for every given $\varepsilon > 0$ and for sufficiently small T one has

$$|\tau_l(t, \xi) - \lambda_l(t, \xi)| \le \varepsilon\lambda_l(t)|\xi|, \quad l = 1, \dots, n, \quad t \in [0, T],$$

where $\lambda_i(t, \xi)$, $i = 1, 2, \dots, n$, are the zeros of the principal symbol. Hence

$$\begin{aligned}
&|\tau_l(t, \xi) - \tau_{n+1}(t, \xi)|\\
&\quad \ge |\lambda_l(t, \xi) - \lambda_{n+1}(t, \xi)| - |\lambda_{n+1}(t, \xi) - \tau_{n+1}(t, \xi)| - |\tau_l(t, \xi) - \lambda_l(t, \xi)|\\
&\quad \ge \delta_1\lambda_l(t)|\xi| - c\lambda_{n+1}(t)|\xi| - \varepsilon\lambda_l(t)|\xi|\\
&\quad \ge \delta_1'\lambda_l(t)|\xi|, \quad l = 1, \dots, n, \quad t \in [0, T].
\end{aligned}$$

If we turn now to perturbed equation

$$\mu^{n+1} + \sum_{k=1}^{n} \mu^{n+1-k} \sum_{|\alpha|=k} a_{m-k,\alpha}(t)\xi^\alpha + w_{n+1} = 0,$$

then by the arguments already used in the proof of the Proposition 2.1, one can check that ($\alpha = \alpha_{n+1} \in \mathbb{R}$)

$$\begin{aligned} |c_\alpha^{(i)}(t,\xi)| &\le c^{|\alpha|}(\lambda_i(t)|\xi|)|\xi|^{-(n+1)\alpha}(\lambda_i(t))^{-(n+1-i)\alpha}(\lambda_1(t)\cdots\lambda_i(t))^{-\alpha}, \ i \le n, \\ |c_\alpha^{(n+1)}(t,\xi)| &\le c^{|\alpha|}(\lambda_n(t)|\xi|)|\xi|^{-(n+1)\alpha}(\lambda_1(t)\cdots\lambda_n(t)\lambda_n(t))^{-\alpha}. \end{aligned}$$

To prove the corollary it is enough to consider $w_{n+1} \in \mathbb{R}$ and to require

$$|w_{n+1}| \le c\lambda_1(t)\cdots\lambda_n(t)\lambda_{n+1}(t)|\xi|^{n+1}.$$

Thus we are permitted to set $w_{n+1} = \sum_{|\alpha|=n+1} a_{m-n-1,\alpha}(t)\xi^\alpha$. The corollary is proved. □

3. Zones. Properties of inhomogeneous hyperbolic polynomials

For the given positive numbers N_l, $l = 1, 2, \ldots, m$, with $N_{l+1} \ge N_l$, we denote by $t_{\xi,l}$, $l = 1, 2, \ldots, m$, the roots of the equations

$$\Lambda_l(t)\langle\xi\rangle = N_l \log\langle\xi\rangle. \tag{3.1}$$

Here $\langle\xi\rangle := (e + |\xi|^2)^{1/2}$. If $\lambda_{l+1}(t) = \lambda_l(t)$, then we set $N_{l+1} = N_l$. Following [17] for $M \ge 1$ we define *hyperbolic zones* $Z_{h,l}(M, N_l)$, and *pseudodifferential zones* $Z_{pd,l}(M, N_l)$, $l = 1, \ldots, m$, by

$$\begin{aligned} Z_{h,l}(M,N_l) &:= \Big\{(t,\xi) \in [0,T] \times \mathbb{R}^n; \Lambda_l(t)\langle\xi\rangle \ge N_l \log\langle\xi\rangle, \quad \langle\xi\rangle \ge M\Big\}, \\ Z_{pd,l}(M,N_l) &:= \Big\{(t,\xi) \in [0,T] \times \mathbb{R}^n; \Lambda_l(t)\langle\xi\rangle \le N_l \log\langle\xi\rangle, \quad \langle\xi\rangle \ge M\Big\}. \end{aligned}$$

There are $2s$ different hyperbolic and pseudodifferential zones, and

$$Z_{h,l+1}(M,N_{l+1}) \subset Z_{h,l}(M,N_l), \ Z_{pd,l}(M,N_l) \subset Z_{pd,l+1}(M,N_{l+1}), \qquad l = 1, \ldots, m-1.$$

Proposition 3.1. *Suppose that either $n = m-1$ or $\lambda_{n+1}(t) \ne \lambda_n(t)$. Further, assume that additionally to the conditions of* Proposition 2.1 *the inequalities* (0.7) *hold for $\alpha \ne 0$, $m - j \ge |\alpha| + 1$ and all $t \in (0, T]$. Then the roots $\tau_i(t,\xi)$, $i = 1, 2, \ldots, n+1$, of "truncated" complete symbol, that is the roots of equation*

$$\tau^{n+1} + \sum_{k=1}^{n+1} \tau^{n+1-k} \sum_{|\alpha|\le k} a_{m-k,\alpha}(t)\xi^\alpha = 0, \tag{3.2}$$

in zone $Z_{h,n}(M, N_n)$, that is for all $(t,\xi) \in Z_{h,n}(M,N_n)$, possess the properties (2.4), (2.5), *and*

$$|D_t^k \tau_{n+1}(t,\xi)| \le c\lambda_n(t)|\xi| \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad k = 0, 1, \tag{3.3}$$

$$|\tau_{n+1}(t,\xi) - \tau_r(t,\xi)| \geq \delta_1 \lambda(t)^{n_r}|\xi|, \quad r < n+1, \quad \delta_1 > 0. \tag{3.4}$$

If additionally the inequalities (0.8) *hold for every* $t \in [0,T]$, *then for all* $(t,\xi) \in Z_{h,n}(M,N_n)$ *one has*

$$|\operatorname{Im}\tau_i(t,\xi)| \leq c\left\{\frac{\lambda(t)}{\Lambda(t)} + (m-2)\frac{\lambda(t)\log^2\lambda(t)}{\Lambda(t)\Lambda_n(t)\langle\xi\rangle}\right\}, \quad i = 1,\dots,n+1. \tag{3.5}$$

Proof. Taking into account Proposition 2.1 and Corollary 2.4 we restrict ourselves to the case $n = m-1$, so that $Z_{h,n}(M,N_n) = Z_{h,m}(M,N_m)$. Consider again equation (2.7) and series (2.8), which have coefficients $c_\alpha^{(i)}$ satisfying estimates (2.11). But now we choose the perturbations

$$w_j = \sum_{|\alpha|<j} a_{m-j,\alpha}(t)\xi^\alpha, \quad j = 1,\dots,m.$$

By means of (0.7) we conclude that for every positive ε there exists N_1 such that

$$\begin{aligned}
|w_j(t,\xi)| &\leq c + \sum_{0<|\alpha|<j} c\left(\prod_{i=1}^{j}\lambda_i(t)\right)\left(\prod_{l=|\alpha|+1}^{j}\frac{|\log\lambda(t)|}{\Lambda_l(t)}\right)|\xi|^{|\alpha|} \\
&\leq c + \lambda_1(t)\cdots\lambda_j(t)|\xi|^j \sum_{0<|\alpha|<j}\frac{1}{N_1}\prod_{l=|\alpha|+1}^{j}\frac{|\log\lambda(t)|N_l}{|\xi|\Lambda_l(t)} \\
&\leq \varepsilon\lambda_1(t)\cdots\lambda_j(t)|\xi|^j, \quad \forall(t,\xi)\in Z_{h,n}(M,N_n), \quad j = 1,\dots,m.
\end{aligned}$$

By repetition of the arguments have been used in the proof of Proposition 2.1, we derive (2.4), (2.5), (3.3), (3.4).

Now we use (0.7) and (0.5). To prove (3.5) we first of all note that since $\operatorname{Im}\ \lambda_i(t,\xi) \equiv 0$ and $\operatorname{Im}\ c_\alpha^{(i)}(t,\xi) \equiv 0$, we have

$$\operatorname{Im}\tau_i(t,\xi) = \sum_{j=1}^{m} c^{(i)}_{(0,\dots,0,\underset{j}{1},0,\dots,0)}(t,\xi)\operatorname{Im}w_j + \sum_{|\alpha|\geq 2} c_\alpha^{(i)}(t,\xi)\operatorname{Im}w^\alpha.$$

Further, $|\operatorname{Im}w_1(t,\xi)| \leq c$, and $|\operatorname{Im}w_2(t,\xi)| \leq c + c|\xi|\lambda_1^2(t)/\Lambda_1(t)$, while for $j = 3,\dots,m$ we have

$$\begin{aligned}
|\operatorname{Im}w_j(t,\xi)| &\leq c + \sum_{|\alpha|=j-1}|\operatorname{Im}a_{m-j,\alpha}(t)\xi^\alpha| + \sum_{0<|\alpha|\leq j-2}|\operatorname{Im}a_{m-j,\alpha}(t)\xi^\alpha| \\
&\leq c + \frac{c\lambda_1(t)\cdots\lambda_{j-1}(t)\lambda_{j-1}(t)|\xi|^{j-1}}{\Lambda_{j-1}(t)} \\
&\quad + c\lambda_1(t)\cdots\lambda_{j-1}(t)\lambda_j(t)|\xi|^j\sum_{l=1}^{j-2}\prod_{k=l+1}^{j}\frac{|\log\lambda(t)|}{\Lambda_k(t)|\xi|}.
\end{aligned}$$

Hence

$$\left|c^{(i)}_{(1,0,\dots,0)}(t,\xi)\,\mathrm{Im}\,w_1(t,\xi)\right| + \left|c^{(i)}_{(0,1,0,\dots,0)}(t,\xi)\,\mathrm{Im}\,w_2(t,\xi)\right| \le c\frac{\lambda(t)}{\Lambda(t)}, \tag{3.6}$$
$$i = 1,\dots,m,$$

and for $i = 1,\dots,m$ we obtain

$$\begin{aligned}\left|\sum_{j=3}^{m} c^{(i)}_{(0,\dots,0,\underset{j}{1},0,\dots,0)}(t,\xi)\,\mathrm{Im}\,w_j(t,\xi)\right| &\le \sum_{j=3}^{m} c(\lambda_i(t))^{i-j+1}|\xi|^{1-j}\frac{1}{\lambda_1(t)\cdots\lambda_i(t)}\\ &\quad\times\left\{\frac{1+\lambda_1(t)\cdots\lambda_{j-1}(t)\lambda_{j-1}(t)|\xi|^{j-1}}{\Lambda_{j-1}(t)}\right.\\ &\quad\left.+\lambda_1(t)\cdots\lambda_{j-1}(t)\lambda_j(t)|\xi|^j\sum_{l=1}^{j-2}\prod_{k=l+1}^{j}\frac{|\log\lambda(t)|}{\Lambda_k(t)|\xi|}\right\}\\ &\le c\left\{\frac{\lambda(t)}{\Lambda(t)}+(m-2)\frac{\lambda(t)\log^2\lambda(t)}{\Lambda(t)\Lambda_i(t)\langle\xi\rangle}\right\}.\end{aligned} \tag{3.7}$$

On the other hand there is an estimate $|w_j(t,\xi)| \le c\lambda_1(t)\cdots\lambda_j(t)|\xi|^{j-1}|\log\lambda(t)|/\Lambda_j(t)$ which leads to

$$\begin{aligned}|w^\alpha(t,\xi)| &\le c^\alpha(\lambda_1(t))^{\sum_{k=1}^m\alpha_k}(\lambda_2(t))^{\sum_{k=2}^m\alpha_k}\cdots(\lambda_m(t))^{\alpha_m}|\xi|^{\sum_{l=1}^m(l-1)\alpha_l}\\ &\quad\times\left(\frac{|\log\lambda(t)|}{\Lambda_1(t)}\right)^{\alpha_1}\cdots\left(\frac{|\log\lambda(t)|}{\Lambda_m(t)}\right)^{\alpha_m}.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\sum_{|\alpha|\ge 2} c^{(i)}_\alpha(t,\xi)\,\mathrm{Im}\,w(t,\xi)^\alpha\right|\\ &\quad\le \sum_{|\alpha|=2}^{\infty} c^{|\alpha|}|\xi|^{1-|\alpha|}\frac{(\lambda_i(t))^{1+i|\alpha|-\sum_{l=1}^m l\alpha_l}}{(\lambda_1(t)\cdots\lambda_i(t))^{|\alpha|}}(\lambda_1(t))^{\sum_{k=1}^m\alpha_k}(\lambda_2(t))^{\sum_{k=2}^m\alpha_k}\cdots(\lambda_m(t))^{\alpha_m}\\ &\quad\times\left(\frac{|\log\lambda(t)|}{\Lambda_1(t)}\right)^{\alpha_1}\cdots\left(\frac{|\log\lambda(t)|}{\Lambda_m(t)}\right)^{\alpha_m}.\end{aligned} \tag{3.8}$$

We have also

$$\begin{aligned}&\frac{(\lambda_1(t))^{\sum_{k=1}^m\alpha_k}\cdots(\lambda_i(t))^{\sum_{k=i}^m\alpha_k}}{(\lambda_1(t)\cdots\lambda_i(t))^{|\alpha|}}\left(\frac{|\log\lambda(t)|}{\Lambda_1(t)}\right)^{\alpha_1}\cdots\left(\frac{|\log\lambda(t)|}{\Lambda_i(t)}\right)^{\alpha_i}\\ &\quad\le(\lambda_i(t))^{\sum_{k=1}^{i-1}(k-i)\alpha_k}\left(\frac{|\log\lambda(t)|}{\Lambda_i(t)}\right)^{\sum_{k=1}^{i}\alpha_k},\end{aligned} \tag{3.9}$$

$$(\lambda_{i+1}(t))^{\sum_{k=i+1}^m \alpha_k} \cdots (\lambda_m(t))^{\alpha_m} \left(\frac{|\log\lambda(t)|}{\Lambda_{i+1}(t)}\right)^{\alpha_{i+1}} \cdots \left(\frac{|\log\lambda(t)|}{\Lambda_m(t)}\right)^{\alpha_m}$$
$$\le (\lambda_i(t))^{\sum_{k=i+1}^m (k-i)\alpha_k} \left(\frac{|\log\lambda(t)|}{\Lambda_i(t)}\right)^{\sum_{k=i+1}^m \alpha_k}. \tag{3.10}$$

From (3.8) to (3.10) we derive

$$\left|\sum_{|\alpha|\ge 2} c_\alpha^{(i)}(t,\xi)\,\mathrm{Im}\,w(t,\xi)^\alpha\right| \le \sum_{|\alpha|=2}^{\infty} \frac{\lambda_i(t)}{|\xi|}\left(\frac{|\log\lambda(t)|}{\Lambda_i(t)}\right)^2 \frac{1}{N_1^{|\alpha|-2}}$$
$$\le c\,\frac{\lambda(t)\log^2\lambda(t)}{\Lambda(t)\Lambda_i(t)|\xi|}, \qquad (x,\xi)\in Z_{h,n}(M,N_n). \tag{3.11}$$

The estimates (3.6), (3.7), and (3.11) prove (3.5). Proposition is proved. □

4. Well-posedness of the Cauchy problem

In this section we complete a proof of Theorem 0.1. To this end we are going to establish for the ordinary differential operator

$$Lu = D_t^m u(t,\xi) + \sum_{j+|\alpha|\le m,\ \ j<m} a_{j,\alpha}(t)\xi^\alpha D_t^j u(t,\xi) \tag{4.1}$$

with the parameter $\xi\in\mathbb{R}^n$, the following estimates

$$E(s_2,u) \le C(1+|\xi|^2)^l \left(E(s_1,u) + \left|\int_{s_1}^{s_2} |Lu(\tau,\xi)|\,d\tau^2\right|\right) \tag{4.2}$$

with some positive number l, for all $s_1, s_2\in[0,T]$. Here

$$E(t,u) := |u(t,\xi)|^2 + |D_t u(t,\xi)|^2 + \cdots + |D_t^{m-1}u(t,\xi)|^2.$$

First of all we note that for ξ chosen from arbitrary fixed compact set of $\mathbb{R}^n$ the estimate (4.2) is evident for all $s_1, s_2\in[0,T]$. Consideration of the case when ξ belongs to the remaining part, differs in each pseudodifferential and hyperbolic zones.
First pseudodifferential zone $t\le t_{\xi,1}$. Denote by $\rho(t,\xi)$ the positive root of the following equation

$$\rho^m - 1 - \langle\xi\rangle\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m-1}\log^{m-1}\langle\xi\rangle = 0. \tag{4.3}$$

Condition (0.6) implies $\rho_t(t,\xi)\ge 0$ for all t and ξ. The choice of function $\rho=\rho(t,\xi)$ is done precisely by the upper bound of (0.4) for the lower order terms of operator with $j = 0$, $|\alpha| = 1$. Then we write for

$$U(t,\xi) := \begin{pmatrix} u(t) \\ D_t u(t) \\ \vdots \\ D_t^{m-1} u(t) \end{pmatrix}, \qquad \mathcal{F}(t,\xi) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(t,\xi) \end{pmatrix}, \qquad f(t,\xi) := Lu(t,\xi),$$

a system

$$D_t U + A(t,\xi)U = \mathcal{F}(t,\xi).$$

By means of the nonsingular matrix

$$N^{-1}(t,\xi) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \rho(t,\xi) & 0 & \cdots & 0 \\ 0 & 0 & \rho^2(t,\xi) & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \rho^{m-1}(t,\xi) \end{pmatrix}$$

with the reciprocal matrix $N(t,\xi)$ we make a change of unknown function, $V = NU$, $U = N^{-1}V$, and arrive at the system

$$D_t V = -NAN^{-1}V + iNN_t^{-1}V + NF.$$

In the first pseudodifferential zone, $t \le t_{\xi,1}$, the lower order terms of operator with $j = 0$, $|\alpha| = 1$, dominate in the sense that their representative, function $\rho = \rho(t,\xi)$, dominates. Indeed, one can easily check

$$\|N(t,\xi)A(t,\xi)N^{-1}(t,\xi)\| \le c\rho(t,\xi) \quad \text{for all } 0 \le t \le t_{\xi,1},$$
$$\|N(t,\xi)N_t^{-1}(t,\xi)\| \le c\frac{\rho_t(t,\xi)}{\rho(t,\xi)} \quad \text{for all } 0 \le t \le t_{\xi,1},$$
$$\left| \int_{s_1}^{s_2} \left(\rho(t,\xi) + \frac{\rho_t(t,\xi)}{\rho(t,\xi)} \right) dt \right| \le C \log|\xi| \quad \text{for all} \quad 0 \le s_1, s_2 \le t_{\xi,1}.$$

The only nontrivial is the first inequality. To prove it we begin with the remark that the elements of matrix $N(t,\xi)A(t,\xi)N^{-1}(t,\xi)$ are either $\rho(t,\xi)$ or

$$\rho(t,\xi)^{-(m-1-j)} \sum_{|\alpha| \le m-j} a_{j,\alpha}(t)\xi^\alpha, \qquad j = 0, \dots, m-1.$$

Further, we use condition (0.7) to estimate each term of the sum:

$$\begin{aligned} &\rho(t,\xi)^{-(m-1-j)} |a_{j,\alpha}(t)\xi^\alpha| \\ &\quad \le C\rho(t,\xi)^{-(m-1-j)} (\lambda(t))^{\sum_{i=1}^{|\alpha|} n_i} \left(\frac{\lambda(t)}{\Lambda(t)} |\log\lambda(t)| \right)^{m-j-|\alpha|} |\xi|^{|\alpha|}. \end{aligned}$$

It remains to prove for $t \leq [0, t_{\xi,1}]$ the estimate

$$(\lambda(t))^{\sum_{i=1}^{s} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-j-s} |\xi|^s \leq C\rho(t,\xi)^{m-j},$$
$$j = 0, \dots, m-1,\ 0 < s \leq m-j.$$

For $j = 0$, $s = 1$ this is the following inequality

$$\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-1} |\xi| \leq C\left(1 + \langle\xi\rangle\lambda(t)\Big(\frac{\lambda(t)}{\Lambda(t)}\Big)^{m-1}\log^{m-1}\langle\xi\rangle\right).$$

Function of the left hand side is increasing due to (0.6), so that one can find a point t_a as a solution to

$$\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-1} |\xi| = 1.$$

Hence for all $t \in [0, t_a]$ desired inequality holds evidently. At the same time again due to (0.6) there is a positive number C_a such that

$$C_a \log\langle\xi\rangle \geq |\log\lambda(t_a)| \geq |\log\lambda(t)| \quad \text{for all} \quad t \in [t_a, t_{\xi,1}].$$

This completes the proof for the case with $j = 0$, $s = 1$. To handle the remaining terms we write

$$\begin{aligned}
&(\lambda(t))^{\sum_{i=1}^{s} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-j-s} |\xi|^s \\
&\quad = (\lambda(t))^{-1+\sum_{i=1}^{s} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{1-j-s} |\xi|^{s-1}\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-1} |\xi| \\
&\quad \leq C\,(\lambda(t))^{-1+\sum_{i=1}^{s} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{1-j-s} |\xi|^{s-1}\rho(t,\xi)^m, \qquad t \in [0, t_{\xi,1}].
\end{aligned}$$

Taking into account consideration of the terms with $j = 0$, $s = 1$, we can restrict ourselves to the proof of the estimate

$$(\lambda(t))^{-1+\sum_{i=1}^{s} n_i} \left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{1-j-s} |\xi|^{s-1}\rho(t,\xi)^j \leq C, \qquad t \in [0, t_{\xi,1}].$$

The function of the left hand side is non-decreasing so that it is enough to estimate it at point $t = t_{\xi,1}$. At that point for $s > 0$ we have to prove

$$\big(\lambda(t_{\xi,1})\big)^{-j-s+\sum_{i=1}^{s} n_i} \Lambda(t_{\xi,1})^{j+s-1}|\log\lambda(t_{\xi,1})|^{1-j-s}|\xi|^{s-1}\rho(t_{\xi,1},\xi)^j \leq C,$$

or an equivalent inequality

$$\big(\lambda(t_{\xi,1})\big)^{m(-j-s+\sum_{i=1}^{s} n_i)} \Lambda(t_{\xi,1})^{mj}|\log\lambda\langle\xi\rangle|^{-jm}\rho(t_{\xi,1},\xi)^{mj} \leq C.$$

The last one is a consequence of the evident inequality

$$\left(\lambda(t_{\xi,1})\right)^{m(-j-s+\sum_{i=1}^{s} n_i)} \Lambda(t_{\xi,1})^{mj} |\log\lambda\langle\xi\rangle|^{-jm} \lambda(t_{\xi,1})^{mj} \Lambda(t_{\xi,1})^{-(m-1)j} |\log\lambda\langle\xi\rangle|^{(m-1)j} \langle\xi\rangle^{j} \leq C$$

since

$$\left(\lambda(t_{\xi,1})\right)^{m(-j-s+\sum_{i=1}^{s} n_i)} \Lambda(t_{\xi,1})^{mj} |\log\lambda\langle\xi\rangle|^{-jm} \leq C.$$

Then for the "microenergy" $E_V(t,\xi) := \left(|V_1(t,\xi)|^2 + \cdots + |V_1(t,\xi)|^2\right)/2$ we derive

$$\frac{d}{dt} E_V(t,\xi) = -\sum_{k=1}^{m} \operatorname{Im}\left(\overline{V_k(t,\xi)}(-NAN^{-1}V + iNN_t^{-1}V + NF)_k\right).$$

It follows

$$\left|\frac{d}{dt} E_V(t,\xi)\right| \leq \left(\rho(t,\xi) + \frac{\rho_t(t,\xi)}{\rho(t,\xi)}\right) E_V(t,\xi) + |f(t,\xi)|^2, \qquad t \in [0, t_{\xi,1}].$$

The Gronwall inequality implies

$$E_V(s_2,u) \leq C(1+|\xi|^2)^{l_0} \left(E_V(s_1,u) + \left|\int_{s_1}^{s_2} |Lu(\tau,\xi)|^2 \, d\tau\right|\right)$$

with some positive number l_0 for all $s_1, s_2 \in [0, t_{\xi,1}]$. Then the inequalities

$$|V(t,\xi)| \leq |U(t,\xi)|, \qquad |U(t,\xi)| \leq \rho^{m-1}(t,\xi)|V(t,\xi)| \leq C(1+|\xi|)^{m-1}|V(t,\xi)|$$

prove claimed estimate (4.2) for all $s_1, s_2 \in [0, t_{\xi,1}]$.
Intersection of the first hyperbolic with the second pseudodifferential zone: $t \in [t_{\xi,1}, t_{\xi,d_1+1}]$. Now we denote

$$\rho_2(t,\xi) := \frac{\lambda(t)}{\Lambda(t)} + \left(|\xi|\lambda_{d_1+1}(t)\frac{\lambda_{d_1+2}(t)}{\Lambda_{d_1+2}(t)} \cdots \frac{\lambda_m(t)}{\Lambda_m(t)} \log^{m-d_1-1}\langle\xi\rangle\right)^{1/(m-d_1)}.$$

In the proof of the next proposition we will use the first summand of the function $\rho_2 = \rho_2(t,\xi)$ to estimate the coupling coefficients, while the second one allows to estimate the remaining coefficients.

Proposition 4.1. *The function $\rho_2 = \rho_2(t,\xi)$ and the coefficients $a_{j,\alpha}(t)$ satisfy*

$$\int_{t_{\xi,1}}^{t_{\xi,d_1+1}} \rho_2(t,\xi)dt \leq C\log|\xi|, \quad \left|\frac{\rho_{2t}(t,\xi)}{\rho_2(t,\xi)}\right| \leq C\frac{\lambda(t)}{\Lambda(t)} \tag{4.4}$$

for all $t \in [t_{\xi,1}, t_{\xi,d_1+1}]$,

$$\left|\sum_{|\alpha|\le m-j} a_{j,\alpha}(t)\xi^\alpha\right| \le C(\rho_2(t,\xi))^{m-d_1-j}\lambda_1(t)\cdots\lambda_{d_1}(t)|\xi|^{d_1}, \tag{4.5}$$
$$j = 0,\dots,m-d_1-2,$$

when $t \in [t_{\xi,1}, t_{\xi,d_1+1}]$.

Proof. Inequalities of (4.4) easily verified. Further, if $|\alpha| \le d_1$, $l = 1,\dots,m-d_1-1$, then we use condition (0.12) with $r(j,\alpha) = 1$:

$$\begin{aligned}
&\left|a_{l-1,\alpha}(t)\xi^\alpha\right|(\rho_2(t,\xi))^{-(m-d_1-l)}\frac{1}{\lambda_1(t)\cdots\lambda_{d_1}(t)|\xi|^{d_1}}\\
&\quad\le c\lambda_1(t)\cdots\lambda_{|\alpha|}(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m-(l-1)-|\alpha|}|\log\lambda(t)|^{d_1-|\alpha|}|\xi|^{|\alpha|}(\rho_2(t,\xi))^{-(m-d_1-l)}\\
&\qquad\times\frac{1}{\lambda_1(t)\cdots\lambda_{d_1}(t)|\xi|^{d_1}}\\
&\quad\le c\lambda_1(t)\cdots\lambda_{|\alpha|}(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m-(l-1)-|\alpha|}|\log\lambda(t)|^{d_1-|\alpha|}|\xi|^{|\alpha|}\left(\frac{\lambda(t)}{\Lambda(t)}\right)^{-(m-d_1-l)}\\
&\qquad\times\frac{\lambda(t)^{-d_1}(t)}{|\xi|^{d_1}}\\
&\quad\le C\left(\frac{|\log\lambda(t)|}{\Lambda(t)|\xi|}\right)^{d_1-|\alpha|}\frac{\lambda(t)}{\Lambda(t)}\\
&\quad\le C\rho_2(t,\xi).
\end{aligned}$$

If $|\alpha| \ge d_1+1$, $l = 1,\dots,m-d_1-1$, then we use condition (0.4):

$$\begin{aligned}
&\left|a_{l-1,\alpha}(t)\xi^\alpha\right|(\rho_2(t,\xi))^{-(m-d_1-l)}\frac{1}{\lambda_1(t)\cdots\lambda_{d_1}(t)|\xi|^{d_1}}\\
&\le c\rho_2(t,\xi)\lambda_{d_1+1}(t)\cdots\lambda_{|\alpha|}(t)\left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-(l-1)-|\alpha|}|\xi|^{|\alpha|-d_1}\\
&\quad\times\left\{\frac{\lambda(t)}{\Lambda(t)}+\left(|\xi|\lambda_{d_1+1}(t)\frac{\lambda_{d_1+2}(t)}{\Lambda_{d_1+2}(t)}\cdots\frac{\lambda_m(t)}{\Lambda_m(t)}\log^{m-d_1-1}\langle\xi\rangle\right)^{1/(m-d_1)}\right\}^{-(m-d_1-l)-1}\\
&\le c\rho_2(t,\xi)\lambda_{d_1+1}(t)\cdots\lambda_{|\alpha|}(t)\left(\frac{\lambda(t)}{\Lambda(t)}|\log\lambda(t)|\right)^{m-(l-1)-|\alpha|}|\xi|^{|\alpha|-d_1}\\
&\quad\times\left(|\xi|\lambda_{d_1+1}(t)\frac{\lambda_{d_1+2}(t)}{\Lambda_{d_1+2}(t)}\cdots\frac{\lambda_m(t)}{\Lambda_m(t)}\log^{m-d_1-1}\langle\xi\rangle\right)^{-\{(m-d_1-l)+1\}/(m-d_1)}\\
&\le c\rho_2(t,\xi)\lambda_{d_1+1}(t)\cdots\lambda_{|\alpha|}(t)\lambda_{d_1+1}(t)^{-\{(m-d_1-l)+1\}/(m-d_1)-|\alpha|+d_1+\{(m-d_1-l)+1\}/(m-d_1)}\\
&\quad\times\left(\frac{\Lambda_{d_1+1}(t)|\xi|}{|\log\lambda(t)|}\right)^{|\alpha|-d_1-\{(m-d_1-l)+1\}/(m-d_1)}
\end{aligned}$$

$$\le c\rho_2(t,\xi)\lambda(t)^{n_{d_1+1}+\cdots+n_{|\alpha|}+n_{d_1+1}(-|\alpha|+d_1)}$$
$$\le c\rho_2(t,\xi).$$

Proposition is proved. □

Now we use a matrix $Q(t,\xi)$ with the reciprocal

$$Q^{-1}(t,\xi) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \rho_2(t,\xi) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \rho_2^2(t,\xi) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \rho_2^{m-d_1-2}(t,\xi) & 0 \\ 0 & 0 & 0 & \cdots & 0 & M^{-1}(t,\xi) \end{pmatrix},$$

where

$$M^{-1}(t,\xi) = \rho_2^{m-d_1-1}(t,\xi)\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tau_1(t,\xi) & \tau_2(t,\xi) & \cdots & \tau_{d_1+1}(t,\xi) \\ \tau_1^2(t,\xi) & \tau_2^2(t,\xi) & \cdots & \tau_{d_1+1}^2(t,\xi) \\ \vdots & \vdots & \vdots & \vdots \\ \tau_1^{d_1}(t,\xi) & \tau_2^{d_1}(t,\xi) & \cdots & \tau_{d_1+1}^{d_1}(t,\xi) \end{pmatrix},$$

to make a change of the unknown function, $W = QU$, while $U = Q^{-1}W$. We note that this is a last zone where we do not need the correctors constructed by Lemma 1.4. Decompose the matrix $A(t,\xi)$ as follows:

$$A(t,\xi) = A_1(t,\xi) + A_2(t,\xi),$$

where

$$A_2(t,\xi) :=$$
$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \sum\limits_{|\alpha|\le m} a_{0,\alpha}(t)\xi^\alpha & \sum\limits_{|\alpha|\le m-1} a_{1,\alpha}(t)\xi^\alpha & \cdots & \cdots & \sum\limits_{|\alpha|\le d_1+2} a_{m-d_1-2,\alpha}(t)\xi^\alpha & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $QAQ^{-1} = QA_1Q^{-1} + QA_2Q^{-1} = \mathcal{D}_1 + B_1$, where

$$\mathcal{D}_1 = \begin{pmatrix} \begin{matrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \tau_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_{d_1+1} \end{matrix} \end{pmatrix}, \quad \|B_1(t,\xi)\| \le C\rho_2(t,\xi). \tag{4.6}$$

Here $\{\tau_i\}_1^{d_1+1}$ are the roots of the equation (3.2) with $n = d_1$. The last inequality of (4.6) holds due to Proposition 4.1. By use of (4.4), Propositions 3.1, 4.1, and of the estimates

$$\|M(t,\xi)\| \le C, \quad \|M(t,\xi)^{-1}\| \le C\langle\xi\rangle^{d_1}$$

we prove (4.2) in $Z_{h,d_1}(M, N_{d_1}) \setminus Z_{h,d_1+1}(M, N_{d_1+1})$.

Intersection of the second hyperbolic with the third pseudodifferential zone: $t \in [t_{\xi,d_1+d_2}, t_{\xi,d_1+d_2+1}]$. We first time need the corrector constructed by Lemma 1.4 in this zone. By means of the zone by zone microlocal consideration one can find an analogy of the logarithmic derivative of the Vandermonde matrix below with the Leray-Volevich's systems [5]. We denote

$$\rho_3(t,\xi) := \frac{\lambda(t)}{\Lambda(t)} + \left(|\xi|\lambda_{d_1+d_2+1}(t)\frac{\lambda_{d_1+d_2+2}(t)}{\Lambda_{d_1+d_2+2}(t)}\cdots\frac{\lambda_m(t)}{\Lambda_m(t)}\log^{m-d_1-d_2-1}\langle\xi\rangle\right)^{1/(m-d_1-d_2)}.$$

Now we use a matrix $Q(t,\xi)$ with the reciprocal

$$Q^{-1}(t,\xi) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \rho_3(t,\xi) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \rho_3^2(t,\xi) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \rho_3^{m-d_1-d_2-2}(t,\xi) & 0 \\ 0 & 0 & 0 & \cdots & 0 & M^{-1}(t,\xi)\widetilde{\Lambda}^{-1}(t) \end{pmatrix},$$

where

$$M^{-1}(t,\xi) = \rho_3^{m-d_1-d_2-1}(t,\xi)\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tau_1(t,\xi) & \tau_2(t,\xi) & \cdots & \tau_{d_1+d_2+1}(t,\xi) \\ \tau_1^2(t,\xi) & \tau_2^2(t,\xi) & \cdots & \tau_{d_1+d_2+1}^2(t,\xi) \\ \vdots & \vdots & \vdots & \vdots \\ \tau_1^{d_1+d_2}(t,\xi) & \tau_2^{d_1+d_2}(t,\xi) & \cdots & \tau_{d_1+d_2+1}^{d_1+d_2}(t,\xi) \end{pmatrix},$$

while $\widetilde{\Lambda}^{-1}(t)$ is a corrector constructed with accordance to Lemma 1.4 where we set:

$$a_k = -(d_1+d_2-1)n_1 + n_2 + \cdots + n_{k-1} + n_k(d_1+d_2+1-k) \ge 0, \quad k = 2,\ldots,d_1+d_2+1.$$

Lemma 4.2. *For all α and all $l = 1, 2, \dots, m - d_1 - d_2 - 1$, an estimate*

$$\left| \sum_{|\alpha|=1} a_{l-1,\alpha}(t)\xi^\alpha \right| \rho_3^{l-(m-d_1-d_2)}(t,\xi) \frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \le C\rho_3(t,\xi)$$

holds for all $t \in [t_{\xi,d_1+d_2}, t_{\xi,d_1+d_2+1}]$.

Proof. We consider two cases. To estimate the coupling coefficients we use the first summand of the function $\rho_3 = \rho_3(t,\xi)$. The second one helps to handle non-coupling coefficients.

For $|\alpha| \ne 0$, $|\alpha| \le d_1 + d_2$, $(j = l - 1)$ $l = 1, 2, \dots, m - d_1 - d_2 - 1$, $l - 1 - (m - d_1 - d_2) \le 0$ we use condition (0.12) with $r = 2$ and obtain

$$\begin{aligned}
&\left| a_{l-1,\alpha}(t)\xi^\alpha \right| \rho_3^{l-(m-d_1-d_2)}(t,\xi) \frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\le C\rho_3(t,\xi) \left(\prod_{i=1}^{|\alpha|} \lambda_i(t) \right) \left(\prod_{j=|\alpha|+1}^{m-(l-1)} \frac{\lambda_j(t)}{\Lambda_j(t)} \right) |\log\lambda(t)|^{d_1+d_2-|\alpha|} |\xi|^{|\alpha|} \Bigg\{ \frac{\lambda(t)}{\Lambda(t)} \\
&\quad + \left(|\xi|\lambda_{d_1+d_2+1}(t) \frac{\lambda_{d_1+d_2+2}(t)}{\Lambda_{d_1+d_2+2}(t)} \cdots \frac{\lambda_m(t)}{\Lambda_m(t)} \log^{m-d_1-d_2-1}\langle\xi\rangle \right)^{1/(m-d_1-d_2)} \Bigg\}^{l-1-(m-d_1-d_2)} \\
&\quad \times \frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\le C\rho_3(t,\xi) \left(\prod_{i=1}^{|\alpha|} \lambda_i(t) \right) \left(\prod_{j=|\alpha|+1}^{m-(l-1)} \frac{\lambda_j(t)}{\Lambda_j(t)} \right) |\log\lambda(t)|^{d_1+d_2-|\alpha|} |\xi|^{|\alpha|} \\
&\quad \times \left(\frac{\lambda(t)}{\Lambda(t)} \right)^{l-1-(m-d_1-d_2)} \frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\le C\rho_3(t,\xi) \left(\prod_{i=1}^{|\alpha|} \lambda_i(t) \right) |\log\lambda(t)|^{d_1+d_2-|\alpha|} \\
&\quad \times \left(\frac{\lambda_2(t)}{\Lambda_2(t)} \right)^{d_1+d_2-|\alpha|} \frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2-|\alpha|}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\le C\rho_3(t,\xi) \left(\prod_{i=1}^{|\alpha|} \lambda_i(t) \right) \frac{\lambda^{a_k+n_2(d_1+d_2-|\alpha|)}(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \left(\frac{|\log\lambda(t)|}{|\xi|\Lambda_2(t)} \right)^{d_1+d_2-|\alpha|} \\
&\le C\rho_3(t,\xi) \frac{\lambda(t)^{a_k+n_2(d_1+d_2-|\alpha|)+n_1+\cdots+n_{|\alpha|}}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}}.
\end{aligned}$$

This completes the proof since an exponent of $\lambda(t)$ is non-negative:

$$a_k + n_2(d_1 + d_2 - |\alpha|) + n_1 + \cdots + n_{|\alpha|} - n_1 - \cdots - n_{k-1} - n_k(d_1 + d_2 - k + 1)$$

$$
\begin{aligned}
&= -(d_1+d_2)n_1 + n_2(d_1+d_2-|\alpha|) + n_1 + \cdots + n_{|\alpha|} \\
&\geq -n_1|\alpha| + n_1 + \cdots + n_{|\alpha|} \geq 0.
\end{aligned}
$$

If $|\alpha| \geq d_1+d_2+1$ and $j = 0, \ldots, m-d_1-d_2-2$, then the coefficient $a_{j,\alpha}(t)$ is not coupling and we use condition (0.4):

$$
\begin{aligned}
&\left|a_{l-1,\alpha}(t)\xi^{\alpha}\right|\rho_3^{l-(m-d_1-d_2)}(t,\xi)\frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\leq C\rho_3(t,\xi)\left(\prod_{i=1}^{|\alpha|}\lambda_i(t)\right)\left(\prod_{j=|\alpha|+1}^{m-(l-1)}\frac{\lambda_j(t)}{\Lambda_j(t)}\right)|\log\lambda(t)|^{d_1+d_2-|\alpha|}|\xi|^{|\alpha|}\left\{\frac{\lambda(t)}{\Lambda(t)}\right. \\
&\quad\left.+\left(|\xi|\lambda_{d_1+d_2+1}(t)\frac{\lambda_{d_1+d_2+2}(t)}{\Lambda_{d_1+d_2+2}(t)}\cdots\frac{\lambda_m(t)}{\Lambda_m(t)}\log^{m-d_1-d_2-1}\langle\xi\rangle\right)^{1/(m-d_1-d_2)}\right\}^{l-1-(m-d_1-d_2)} \\
&\quad\times\frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\leq C\rho_3(t,\xi)\left(\prod_{i=1}^{|\alpha|}\lambda_i(t)\right)\left(\prod_{j=|\alpha|+1}^{m-(l-1)}\frac{\lambda_j(t)}{\Lambda_j(t)}\right)|\log\lambda(t)|^{d_1+d_2-|\alpha|}|\xi|^{|\alpha|} \\
&\quad\times\left(|\xi|\lambda_{d_1+d_2+1}(t)\frac{\lambda_{d_1+d_2+2}(t)}{\Lambda_{d_1+d_2+2}(t)}\cdots\frac{\lambda_m(t)}{\Lambda_m(t)}\log^{m-d_1-d_2-1}\langle\xi\rangle\right)^{\{l-1-(m-d_1-d_2)\}/(m-d_1-d_2)} \\
&\quad\times\frac{\lambda^{a_k}(t)}{|\xi|^{d_1+d_2}\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\leq C\rho_3(t,\xi)\left(\prod_{i=1}^{|\alpha|}\lambda_i(t)\right)\left(\prod_{j=|\alpha|+1}^{m-(l-1)}\frac{\lambda_j(t)}{\Lambda_j(t)}\right)|\log\lambda(t)|^{d_1+d_2-|\alpha|} \\
&\quad\times|\xi|^{|\alpha|-(d_1+d_2)+\{l-1-(m-d_1-d_2)\}/(m-d_1-d_2)} \\
&\quad\times\left(\lambda_{d_1+d_2+1}(t)\frac{\lambda_{d_1+d_2+2}(t)}{\Lambda_{d_1+d_2+2}(t)}\cdots\frac{\lambda_m(t)}{\Lambda_m(t)}\log^{m-d_1-d_2-1}\langle\xi\rangle\right)^{\{l-1-(m-d_1-d_2)\}/(m-d_1-d_2)} \\
&\quad\times\frac{\lambda^{a_k}(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\leq C\rho_3(t,\xi)\left(\prod_{i=1}^{|\alpha|}\lambda_i(t)\right)\left(\lambda_{d_1+d_2+1}(t)\right)^{[l-1-(m-d_1-d_2)]/(m-d_1-d_2)} \\
&\quad\times\frac{\lambda^{a_k}(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\quad\times\left(\frac{\lambda_{d_1+d_2+1}(t)}{\Lambda_{d_1+d_2+1}(t)}\right)^{m-(l-1)-|\alpha|+\{[l-1-(m-d_1-d_2)][(m-d_1-d_2)-1]\}/(m-d_1-d_2)} \\
&\quad\times|\log\lambda(t)|^{(l-1)-m+d_1+d_2+m-(l-1)-|\alpha|+\{[l-1-(m-d_1-d_2)][m-d_1-d_2-1]\}/(m-d_1-d_2)} \\
&\quad\times|\xi|^{|\alpha|-(d_1+d_2)-1+(l-1)/(m-d_1-d_2)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\rho_3(t,\xi)\lambda(t)^{n_1+\cdots+n_{|\alpha|}+n_{(d_1+d_2+1)}\left\{\frac{[l-1-(m-d_1-d_2)]}{(m-d_1-d_2)}+m-(l-1)-|\alpha|+\frac{[l-1-(m-d_1-d_2)][(m-d_1-d_2)-1]}{(m-d_1-d_2)}\right\}} \\
&\quad\times\frac{\lambda^{a_k}(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}} \\
&\quad\times\left(\frac{|\xi|\Lambda_{d_1+d_2+1}(t)}{|\log\lambda(t)|}\right)^{|\alpha|-(d_1+d_2)-1+(l-1)/(m-d_1-d_2)} \qquad |\log\lambda(t)|^{(l-1)-m+d_1+d_2} \\
&\leq C\rho_3(t,\xi)\lambda(t)^{n_1+\cdots+n_{|\alpha|}+n_{(d_1+d_2+1)}(-|\alpha|+(d_1+d_2))}\frac{\lambda^{a_k}(t)}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{d_1+d_2-k+1}}.
\end{aligned}
$$

Exponent of $\lambda(t)$ is non-negative:

$$
\begin{aligned}
&n_1+\cdots+n_{|\alpha|}+n_{(d_1+d_2+1)}(-|\alpha|+(d_1+d_2))-(d_1+d_2-1)n_1 \\
&+n_2+\cdots+n_{k-1}+n_k(d_1+d_2+1-k)-\Big(n_1+\cdots+n_{k-1}+n_k(d_1+d_2-k+1)\Big) \\
&\qquad\geq\ n_1+\cdots+n_{d_1+d_2}-(d_1+d_2)n_1\geq 0.
\end{aligned}
$$

This completes the proof of lemma. □

Consideration of the remaining zones is similar to already given ones for the first and second zones. For $r=1,\ldots,s-1$ we set

$$
\begin{aligned}
(4.7)\quad \rho_{r+1}(t,\xi) &:= \frac{\lambda(t)}{\Lambda(t)} \\
&+\left(|\xi|\lambda_{d_1+d_2+\cdots+d_r+1}(t)\left(\frac{\lambda(t)}{\Lambda(t)}\log\langle\xi\rangle\right)^{m-d_1-d_2-\cdots-d_r-1}\right)^{1/(m-d_1-d_2-\cdots-d_r)}.
\end{aligned}
$$

Lemma 4.3. *The function $\rho_s=\rho_s(t,\xi)$ satisfies*

$$
\begin{aligned}
&\int_{t_{\xi,d_1+\cdots+d_{s-1}}}^{t_{\xi,d_1+\cdots+d_{s-1}+1}}\rho_s(t,\xi)dt\leq C\log|\xi|, \\
&\left|\frac{\rho_{s\,t}(t,\xi)}{\rho_s(t,\xi)}\right|\leq C\frac{\lambda(t)}{\Lambda(t)}\qquad \textit{for all}\quad t\in[t_{\xi,d_1+\cdots+d_{s-1}},t_{\xi,d_1+\cdots+d_{s-1}+1}].
\end{aligned}
$$

Proof. It is quite repetition of the proof of the first part of Proposition 4.1. □

The last pseudodifferential zone: $(t,\xi)\in Z_{h,d_1+\cdots+d_{s-1}}(M,N_{d_1+\cdots+d_{s-1}})\setminus Z_{h,m}(M,N_m)$.

If $\mathcal{M}^{-1}(t,\xi)$ is the Vandermonde matrix corresponding to the system $\{\tau_1(t,\xi),\ldots,\tau_{d_1+\cdots+d_{s-1}+1}(t,\xi)\}$ with $(t,\xi)\in Z_{h,d_1+\cdots+d_{s-1}}(M,N_{d_1+\cdots+d_{s-1}})$, then it is a "partial diagonalizer" for $\mathcal{A}(t,\xi)$. Thus, if we set

$$\mathcal{M}^{-1}(t,\xi)=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tau_1(t,\xi) & \tau_2(t,\xi) & \cdots & \tau_{d_1+\cdots+d_{s-1}+1}(t,\xi) \\ \tau_1^2(t,\xi) & \tau_2^2(t,\xi) & \cdots & \tau^2_{d_1+\cdots+d_{s-1}+1}(t,\xi) \\ \vdots & \vdots & \cdots & \vdots \\ \tau_1^{d_1+\cdots+d_{s-1}}(t,\xi) & \tau_2^{d_1+\cdots+d_{s-1}}(t,\xi) & \cdots & \tau^{d_1+\cdots+d_{s-1}}_{d_1+\cdots+d_{s-1}+1}(t,\xi) \end{pmatrix},$$

then

$$\mathcal{M}(t,\xi)=\begin{pmatrix} \dfrac{\psi_{11}}{\varphi_1(\tau_1)} & \dfrac{\psi_{12}}{\varphi_1(\tau_1)} & \cdots & \dfrac{\psi_{1,d_1+\cdots+d_{s-1}+1}}{\varphi_1(\tau_1)} \\ \dfrac{\psi_{21}}{\varphi_2(\tau_2)} & \dfrac{\psi_{22}}{\varphi_2(\tau_2)} & \cdots & \dfrac{\psi_{2,d_1+\cdots+d_{s-1}+1}}{\varphi_2(\tau_2)} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\psi_{d_1+\cdots+d_{s-1}+1,1}}{\varphi_m(\tau_{d_1+\cdots+d_{s-1}+1})} & \dfrac{\psi_{d_1+\cdots+d_{s-1}+1,2}}{\varphi_m(\tau_{d_1+\cdots+d_{s-1}+1})} & \cdots & \dfrac{\psi_{d_1+\cdots+d_{s-1}+1,d_1+\cdots+d_{s-1}+1}}{\varphi_m(\tau_{d_1+\cdots+d_{s-1}+1})} \end{pmatrix},$$

where

$$\begin{aligned} \psi_{i,d_1+\cdots+d_{s-1}+1-k} &= c_k+c_{k-1}\tau_i+\cdots+c_0\tau_i^k, \\ & \qquad i=1,\dots,d_1+\cdots+d_{s-1}+1, \quad k=0,\dots,d_1+\cdots+d_{s-1}, \\ c_k(t,\xi) &= \sum_{|\alpha|\le k} a_{m-k,\alpha}(t)\xi^\alpha, \qquad k=0,\dots,d_1+\cdots+d_{s-1}, \\ \varphi_i(\tau) &= \prod_{k\ne i}(\tau-\tau_k)=\sum_{k=1}^{d_1+\cdots+d_{s-1}+1}\psi_{i,k}\tau^{k-1}, \ \ i=1,\dots,d_1+\cdots+d_{s-1}+1. \end{aligned}$$

We have for the logarithmic derivative

Lemma 4.4. *Consider* $(t,\xi)\in Z_{h,d_1+\cdots+d_{s-1}}(M,N_{d_1+\cdots+d_{s-1}})\setminus Z_{h,m}(M,N_m)$. *Then*

$$\big(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\big)_{kl}=\begin{cases} \tau_{lt}(t,\xi)\dfrac{1}{\prod_{i\ne k}(\tau_k(t,\xi)-\tau_i(t,\xi))}\displaystyle\prod_{i\ne k,\ l}(\tau_l(t,\xi)-\tau_i(t,\xi)) \\ \qquad \text{if} \quad k\ne l, \\ \tau_{kt}(t,\xi)\displaystyle\sum_{\substack{j=1\\ j\ne k}}^{d_1+\cdots+d_{s-1}+1}\dfrac{1}{\big(\tau_k(t,\xi)-\tau_j(t,\xi)\big)} \quad \text{if} \quad k=l. \end{cases}$$

Proof. It is completely similar to the proof of Lemma 1.1 and we omit it. ☐

Lemma 4.5. *Let $\widetilde{\Lambda}$ be a diagonal matrix*

$$\widetilde{\Lambda} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \lambda^{a_2}(t) & \ddots & \\ 0 & 0 & & \lambda^{a_{d_1+\cdots+d_{s-1}+1}}(t) \end{pmatrix}, \tag{4.8}$$

where the nonegative numbers $a_2, \ldots, a_{d_1+\cdots+d_{s-1}+1}$, are chosen by

$$a_k = -(d_1 + \cdots + d_{s-1} + 1 - 2)n_1 + n_2 + \cdots + n_{k-1} + n_k(d_1 + \cdots + d_{s-1} + 1 - k) \geq 0,$$

with $k = 2, \ldots, d_1 + \cdots + d_{s-1} + 1$. Then for the logarithmic derivative of the matrix $\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1}(t)$ the following estimate

$$\|\widetilde{\Lambda}(t)\mathcal{M}(t,\xi)(\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1}(t))_t\| \leq C_M \frac{\lambda'(t)}{\lambda(t)}$$

holds for all $(t,\xi) \in Z_{h,d_1+\cdots+d_{s-1}}(M, N_{d_1+\cdots+d_{s-1}}) \setminus Z_{h,m}(M, N_m)$.

Proof. It follows from Lemma 4.4. □

Decompose the matrix $A(t,\xi)$ as follows: $A(t,\xi) = A_1(t,\xi) + A_2(t,\xi)$, where

$$A_2(t,\xi) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_1(t,\xi) & a_2(t,\xi) & \cdots & \cdots & a_{d_s-1}(t,\xi) & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$a_k(t,\xi) := \sum_{|\alpha| \leq m-k+1} a_{k-1,\alpha}(t)\xi^\alpha, \qquad k = 1, \ldots, d_s - 1.$$

Next we use a matrix $Q(t,\xi)$ with the reciprocal

$$Q^{-1}(t,\xi) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \rho_s(t,\xi) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \rho_s^2(t,\xi) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \rho_s^{d_s-2}(t,\xi) & 0 \\ 0 & 0 & 0 & \cdots & 0 & M^{-1}(t,\xi) \end{pmatrix}, \tag{4.9}$$

where $M^{-1}(t,\xi) = \rho_s^{d_s-1}(t,\xi)\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1}(t)$, matrix $\widetilde{\Lambda}^{-1}(t)$ is from (4.8), while $\rho_s(t,\xi)$ is chosen by (4.7) with $r = s - 1$. Then $QAQ^{-1} = QA_1Q^{-1} + QA_2Q^{-1} =$

$\mathcal{D}_{s-1} + B_1,$ where

$$\mathcal{D}_{s-1} = \begin{pmatrix} \begin{matrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \tau_{d_1+\cdots+d_{s-1}+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_1 \end{matrix} \end{pmatrix}.$$

Here $\{\tau_i\}_1^{d_1+\cdots+d_{s-1}+1}$ are the roots of the equation (3.2) with $n = d_+ \cdots + d_{s-1}$. We claim:

$$\|B_1(t,\xi)\| \le C\rho_s(t,\xi) \quad \text{for all} \quad (t,\xi) \in Z_{h,d_1+\cdots+d_{s-1}}(M, N_{d_1+\cdots+d_{s-1}}) \setminus Z_{h,m}(M, N_m).$$

To prove this we write

$$(QA_2Q^{-1})_{kl} = \sum_{k_1=1}^{d_s-1} \sum_{l_1=1}^{d_s-1} Q_{kk_1}(A_2)_{k_1l_1}(Q^{-1})_{l_1l}.$$

Consider the terms with $k = 1, \ldots, d_s - 1$:

$$\sum_{k_1=1}^{d_s-1} \sum_{l_1=1}^{d_s-1} Q_{k,k_1}(A_2)_{k_1l_1}(Q^{-1})_{l_1l} = \begin{cases} 0 & \text{if} \quad l \ne k+1, \\ \rho_s(t,\xi) & \text{if} \quad l = k+1 \le d_s. \end{cases}$$

Then we calculate the terms with $k = d_s, \ldots, m$:

$$\begin{aligned} &Q_{k,k_1}(A_2)_{k_1l_1}(Q^{-1})_{l_1l} \\ &\quad = \begin{cases} \rho_s(t,\xi)^{1-d_s}(\widetilde{\Lambda}(t))_{k-d_s+1,k-d_s+1}\mathcal{M}_{k-d_s+1,m-d_s+1}(t,\xi)a_l(t,\xi)\rho_s(t,\xi)^{l-1} \\ \qquad \text{if} \quad l \le d_s - 1, \\ 0, \quad \text{if} \quad l \ge d_s. \end{cases} \end{aligned}$$

We estimate therefore $M_{k,m-d_s+1}a_l(t,\xi)\rho_s(t,\xi)^{l-1}$ with $k = 1, \ldots, m - d_s + 1$ and $l \le d_s - 1$:

$$\begin{aligned} &|\rho_s(t,\xi)^{1-d_s}(\widetilde{\Lambda}(t))_{kk}\mathcal{M}_{k,m-d_s+1}(t,\xi)a_l(t,\xi)\rho_s(t,\xi)^{l-1}| \\ &\le \rho_s(t,\xi)^{l-d_s} \frac{\lambda(t)^{a_k}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-d_s-k+1}} |\xi|^{-(d_1+\cdots+d_{s-1})} \left| \sum_{|\alpha|\le m-l+1} a_{l-1,\alpha}(t)\xi^\alpha \right|, \end{aligned}$$

where a_k are given by Lemma 1.4: $a_1 = 0$ and

$$a_k = -(m - d_s - 1)n_1 + n_2 + \cdots + n_{k-1} + n_k(m - d_s + 1 - k) \ge 0, \quad k = 2, \ldots, m - d_s + 1.$$

Lemma 4.6. *For all $k \le m - d_s + 1$ and all $l \le d_s - 1$, the following estimate*

$$\rho_s(t,\xi)^{l-d_s} \frac{\lambda(t)^{a_k}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-d_s-k+1}} |\xi|^{-(d_1+\cdots+d_{s-1})} |a_{l-1,\alpha}(t)||\xi|^{|\alpha|} \le C\rho_s(t,\xi)$$

holds for all $(t,\xi) \in Z_{h,d_1+\cdots+d_{s-1}}(M, N_{d_1+\cdots+d_{s-1}}) \setminus Z_{h,m}(M, N_m)$.

Proof. We have to consider two different cases. If $|\alpha| \le m - d_s$ for all $k \le m - d_s + 1$ and all $l \le d_s - 1$ we use (0.4) and obtain

$$\begin{aligned}
&\rho_s(t,\xi)^{l-d_s} \frac{\lambda(t)^{a_k}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-d_s-k+1}} |\xi|^{-(d_1+\cdots+d_{s-1})} |a_{l-1,\alpha}(t)||\xi|^{|\alpha|} \\
&\le \rho_s(t,\xi)\rho_s(t,\xi)^{l-1-d_s} \frac{\lambda(t)^{-(m-d_s-1)n_1+n_2+\cdots+n_{k-1}+n_k(m-d_s+1-k)}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-d_s-k+1}} |\xi|^{-(d_1+\cdots+d_{s-1})} \\
&\quad \times \left(\prod_{i=1}^{m-(l-1)} \lambda_i(t)\right) \left(\prod_{p=|\alpha|+1}^{m-(l-1)} \frac{|\log\lambda_p(t)|}{\Lambda_p(t)}\right) |\xi|^{|\alpha|} \\
&\le \rho_s(t,\xi)\rho_s(t,\xi)^{l-1-d_s} \frac{\lambda(t)^{-(m-d_s-1)n_1+n_2+\cdots+n_{k-1}+n_k(m-d_s+1-k)}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-d_s-k+1}} |\xi|^{-(d_1+\cdots+d_{s-1})} \\
&\quad \times \left(\prod_{i=1}^{|\alpha|} \lambda_i(t)\right) \left(\prod_{p=|\alpha|+1}^{m-(l-1)} \frac{\lambda_p(t)}{\Lambda_p(t)}\right) |\log\lambda(t)|^{m-(l-1)-|\alpha|} |\xi|^{|\alpha|} \\
&\le \rho_s(t,\xi)\rho_s(t,\xi)^{l-1-d_s} \lambda(t)^{n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)} \\
&\quad \times \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m-(l-1)-|\alpha|} |\log\lambda(t)|^{m-(l-1)-|\alpha|} |\xi|^{|\alpha|-(d_1+\cdots+d_{s-1})} \\
&\le \rho_s(t,\xi) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{l-1-d_s} \lambda(t)^{n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)} \\
&\quad \times \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m-(l-1)-|\alpha|} |\log\lambda(t)|^{m-(l-1)-|\alpha|} |\xi|^{|\alpha|-(m-d_s)} \\
&\le \rho_s(t,\xi)\lambda(t)^{-|\alpha|+n_1+n_2+\cdots+n_{|\alpha|}} \left(\frac{|\log\lambda(t)|}{|\xi|\Lambda(t)}\right)^{m-d_s-|\alpha|} \\
&\le C\rho_s(t,\xi).
\end{aligned}$$

If $|\alpha| \ge m - d_s + 1$, then we use condition (0.4) and the second term of $\rho_s(t,\xi)$:

$$\begin{aligned}
&\rho_s(t,\xi)^{l-d_s} \frac{\lambda(t)^{a_k}}{\lambda_1(t)\cdots\lambda_{k-1}(t)\lambda_k(t)^{m-d_s-k+1}} |\xi|^{-(d_1+\cdots+d_{s-1})} |a_{l-1,\alpha}(t)||\xi|^{|\alpha|} \\
&\le C\rho_s(t,\xi) \left\{ |\xi|\lambda_m(t) \left(\frac{\lambda(t)}{\Lambda(t)} |\log\lambda(t)|\right)^{d_s-1} \right\}^{(l-1-d_s)/d_s} \lambda(t)^{n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\prod_{p=|\alpha|+1}^{m-(l-1)} \frac{\lambda_p(t)}{\Lambda_p(t)} \right) |\log \lambda(t)|^{m-(l-1)-|\alpha|} |\xi|^{|\alpha|-(d_1+\cdots+d_{s-1})} \\
&\leq C\rho_s(t,\xi)(\lambda_m(t))^{(l-1-d_s)/d_s} \lambda(t)^{n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)} |\xi|^{|\alpha|-(d_1+\cdots+d_{s-1})+(l-1-d_s)/d_s} \\
&\times \left(\frac{\lambda_m(t)}{\Lambda_m(t)} \right)^{m-(l-1)-|\alpha|+(d_s-1)(l-1-d_s)/d_s} |\log \lambda(t)|^{m-(l-1)-|\alpha|+(d_s-1)(l-1-d_s)/d_s} \\
&\leq C\rho_s(t,\xi)\lambda(t)^{n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)} (\lambda_m(t))^{m-d_s-|\alpha|} \left(\frac{|\xi|\Lambda_m(t)}{|\log \lambda(t)|} \right)^{|\alpha|-(m-d_s)+(l-1-d_s)/d_s} \\
&\leq C\rho_s(t,\xi)\lambda(t)^{n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)} (\lambda_m(t))^{m-d_s-|\alpha|} \\
&\leq C\rho_s(t,\xi),
\end{aligned}
$$

since for all $|\alpha| \geq m-d_s+1$ one has $n_1+n_2+\cdots+n_{|\alpha|}-(m-d_s)+n_m(m-d_s-|\alpha|) \geq 0$. The lemma is proved. $\square$

Then we make a change of the unknown function, $W = QU$, $U = Q^{-1}W$, and follow the approach used in previous zones.

The last hyperbolic zone: $(t,\xi) \in Z_{h,m}(M,N_m)$. We write for U and $\mathcal{F}$ defined by (1.2) system

$$
D_t U + \mathcal{A}(t,\xi)U = \mathcal{F}(t,\xi).
$$

If $\mathcal{M}^{-1}(t,\xi)$ is the Vandermonde matrix corresponding to $\{\tau_1(t,\xi),\ldots,\tau_m(t,\xi)\}$ when $(t,\xi) \in Z_{h,m}(M,N_m)$, then it is a diagonalizer for $\mathcal{A}(t,\xi)$. Thus

$$
\mathcal{M}^{-1}(t,\xi) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\tau_1(t,\xi) & \tau_2(t,\xi) & \tau_3(t,\xi) & \cdots & \tau_m(t,\xi) \\
\tau_1^2(t,\xi) & \tau_2^2(t,\xi) & \tau_3^2(t,\xi) & \cdots & \tau_m^2(t,\xi) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\tau_1^{m-1}(t,\xi) & \tau_2^{m-1}(t,\xi) & \tau_3^{m-1}(t,\xi) & \cdots & \tau_m^{m-1}(t,\xi)
\end{pmatrix}
$$

while

$$
\mathcal{M}(t,\xi) = \begin{pmatrix}
\dfrac{\psi_{11}}{\varphi_1(\tau_1)} & \dfrac{\psi_{12}}{\varphi_1(\tau_1)} & \dfrac{\psi_{13}}{\varphi_1(\tau_1)} & \cdots & \dfrac{\psi_{1m}}{\varphi_1(\tau_1)} \\
\dfrac{\psi_{21}}{\varphi_2(\tau_2)} & \dfrac{\psi_{22}}{\varphi_2(\tau_2)} & \dfrac{\psi_{23}}{\varphi_2(\tau_2)} & \cdots & \dfrac{\psi_{2m}}{\varphi_2(\tau_2)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\dfrac{\psi_{m1}}{\varphi_m(\tau_m)} & \dfrac{\psi_{m2}}{\varphi_m(\tau_m)} & \dfrac{\psi_{m3}}{\varphi_m(\tau_m)} & \cdots & \dfrac{\psi_{mm}}{\varphi_m(\tau_m)}
\end{pmatrix},
$$

where

$$
\psi_{i,m-k} = c_k + c_{k-1}\tau_i + \cdots + c_0\tau_i^k, \qquad i = 1,\ldots,m, \quad k = 0,\ldots,m-1,
$$

$$c_k(t,\xi) = \sum_{|\alpha|\le k} a_{m-k,\alpha}(t)\xi^\alpha, \qquad k = 0,\dots,m-1,$$

$$\varphi_i(\tau) = \prod_{k\neq i}(\tau-\tau_k) = \sum_{k=1}^{m}\psi_{i,k}\tau^{k-1}, \qquad i = 1,\dots,m.$$

Lemma 4.7. *Consider* $(t,\xi)\in Z_{h,m}(M,N_m)$*. Then*

$$\big(\mathcal{M}(t,\xi)\mathcal{M}_t^{-1}(t,\xi)\big)_{kl} = \begin{cases} \tau_{lt}(t,\xi)\dfrac{1}{\prod_{i\neq k}(\tau_k(t,\xi)-\tau_i(t,\xi))}\displaystyle\prod_{i\neq k,l}(\tau_l(t,\xi)-\tau_i(t,\xi)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad k\neq l, \\ \tau_{kt}(t,\xi)\displaystyle\sum_{\substack{j=1\\ j\neq k}}^{m}\dfrac{1}{\big(\tau_k(t,\xi)-\tau_j(t,\xi)\big)} \quad \textit{if} \quad k=l. \end{cases}$$

Proof. It is quite similar to the proof of the Lemma 1.1 and therefore we omit it. □

Lemma 4.8. *Consider* $(t,\xi)\in Z_{h,m}(M,N_m)$*. Let* $\widetilde{\Lambda}$ *be a diagonal matrix* (1.4)*, where the nonegative numbers* $a_2,\ \dots,\ a_m$*, are chosen by* (1.5)*. Then for the logarithmic derivative of the matrix* $\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1}$ *the following estimate*

$$\|\widetilde{\Lambda}(t)\mathcal{M}(t,\xi)(\mathcal{M}^{-1}(t,\xi)\widetilde{\Lambda}^{-1}(t))_t\| \le C_M\frac{\lambda'(t)}{\lambda(t)}$$

holds for all $(t,\xi)\in Z_{h,m}(M,N_m)$*.*

Proof. See proof of Lemma 1.4. □

We reduce the problem to the one for the equivalent equation

$$D_tW + \widetilde{\mathcal{D}}(t,\xi)W + \widetilde{\Lambda}\mathcal{M}(t,\xi)(D_t\mathcal{M}^{-1}(t,\xi))\widetilde{\Lambda}^{-1}W + \widetilde{\Lambda}(D_t\widetilde{\Lambda}^{-1})W = \widetilde{\mathcal{F}}(t,\xi)$$

for the unknown function $W = \widetilde{\Lambda}(t,\xi)\mathcal{M}(t,\xi)U$, where $U = \mathcal{M}(t,\xi)^{-1}\widetilde{\Lambda}(t,\xi)^{-1}W$, and

$$\widetilde{\mathcal{D}}(t,\xi) := \mathcal{M}(t,\xi)\mathcal{A}(t,\xi)\mathcal{M}^{-1}(t,\xi), \qquad \widetilde{\mathcal{F}}(t,\xi) := \widetilde{\Lambda}(t,\xi)\mathcal{M}(t,\xi)\mathcal{F}(t,\xi).$$

It is easy to see that $\widetilde{\mathcal{D}}$ is a diagonal matrix,

$$\widetilde{\mathcal{D}}(t,\xi) = \begin{pmatrix} \tau_1(t,\xi) & & 0 \\ & \ddots & \\ 0 & & \tau_m(t,\xi) \end{pmatrix},$$

where according to Proposition 3.1 (with $n = m-1$)

$$|\operatorname{Im}\tau_i(t,\xi)| \le c\left\{\frac{\lambda(t)}{\Lambda(t)} + (m-2)\frac{\lambda(t)\log\lambda(t)}{\Lambda(t)\Lambda_m(t)\langle\xi\rangle}\right\}, \quad i = 1,\dots,m,$$

for all $(t,\xi)\in Z_{h,m}(M,N_m)$. Then inequality

$$\int_{(t,\xi)\in Z_{h,m}(M,N_m)}\left\{\frac{\lambda(t)}{\Lambda(t)} + (m-2)\frac{\lambda(t)\log\lambda(t)}{\Lambda(t)\Lambda_m(t)\langle\xi\rangle}\right\}dt \le K\log\langle\xi\rangle, \qquad \xi\in\mathbb{R}^n,$$

gives desired estimate in the last hyperbolic zone $Z_{h,m}(M,N_m)$ for the microenergy $E(t,\xi) := \|W(t,\xi)\|^2$ of solution W. It follows the *a priori* estimate (0.15).

Proof of the finite propagation speed property. Existence and uniqueness are direct consequences of the *a priori* estimate. Thus it remains to prove existence of the cone of dependence, that is, if

$$f\,\Big|_{K_\gamma(x^0,t^0)} = 0, \quad u_i\,\Big|_{K_\gamma(x^0,t^0)\cap\{t=s\}} = 0, \quad i = 0,\dots,m-1, \tag{4.10}$$

then

$$u\,\Big|_{K_\gamma(x^0,t^0)} = 0, \tag{4.11}$$

provided that $|\gamma| \ge \sup\{|\lambda_l(t,x,\xi)|\,;t\in[0,T], x\in\mathbb{R}^n, \xi\in\mathbb{R}^n, |\xi| = 1, l = 1,\dots,m\}$.

Further, if the Cauchy data are given at $t = s$ with $s > 0$ then the problem possesses a cone of dependence. Indeed, an operator for $t > 0$ is strictly hyperbolic; hence if $s > 0$ and $t^0 > 0$ then $u(t,s,x) = 0$ for all $(t,x)\in K_\gamma(x^0,t^0)$, and all $t > 0$. (For the strictly hyperbolic case see, for instance, §12, Ch. 6 of [10].) At the same time the values of the solution $u(0,x)$, $(0,x)\in K_\gamma(x^0,t^0)$ can be obtained as limit of the values at $K_\gamma(x^0,t^0)\bigcap\{t>0\}$, so that $u(0,x)$ vanishes.

Consider the case $s = 0$, $t^0 > 0$, $\gamma > 0$, and assume that (4.10) holds. Define the set of the operators $P_\varepsilon(t,D_t,D_x)$, $0\le\varepsilon\le\varepsilon_0$, by means of the operator $P(t,D_t,D_x)$ as follows

$$P_\varepsilon(t,D_t,D_x) := P_\varepsilon(t+\varepsilon,D_t,D_x), \quad t\in[0,T-\varepsilon_0],\ x\in\mathbb{R}^n,$$

while $\varepsilon_0 < T - t^0$, provided that the fixed number ε_0 is small enough. Then, for these operators consider a family of Cauchy problems

$$\begin{cases} P_\varepsilon v_\varepsilon = f & \text{on} \quad [0,T-\varepsilon_0]\times\mathbb{R}^n,\\ D_t^j v_\varepsilon\,|_{t=0} = \psi_j, & \quad j = 0,\dots,m-1. \end{cases}$$

It is evident that $v_\varepsilon(t,x) = 0$ for all $(t,x)\in K_\gamma(x^0,t^0)$. According to the already proved statements of the theorem for every $r \in \mathbb{R}$ the estimate

$$\sum_{j=0}^{m-1} \|D_t^j v_\varepsilon(t)\|^2_{(m-1-j+r)} \leq C_r \left(\sum_{j=0}^{m-1} \|D_t^j \psi_j\|^2_{(m+l+r-1-j)} + \left| \int_0^t \|f(\tau)\|^2_{(l+r)} d\tau \right| \right)$$

holds for all $t \in [0, T - \varepsilon_0]$, with constants C_r and l independent of ε. Further, we have

$$\begin{cases} P_{\varepsilon_1}(v_{\varepsilon_1} - v_{\varepsilon_2}) = (P_{\varepsilon_1} - P_{\varepsilon_2})v_{\varepsilon_2} & \text{on} \quad [0, T - \varepsilon_0] \times \mathbb{R}^n, \\ D_t^j(v_{\varepsilon_1} - v_{\varepsilon_2})\,|_{t=0} = 0, & j = 0, \dots, m-1. \end{cases}$$

Substituting $r - l - 1$ for r in the a priori estimate we obtain

$$\begin{aligned} &\sum_{j=0}^{m-1} \|D_t^j(v_{\varepsilon_1} - v_{\varepsilon_2})(t)\|^2_{(m+r-2-l-j)} \\ &\leq C_r \int_0^t \|(P_{\varepsilon_1} - P_{\varepsilon_2})v_{\varepsilon_2}(\tau)\|^2_{(r-1)} d\tau \\ &\leq C_r \max_{j,\alpha} \left(\sup_{t\in[0,T]} \left| \int_{t+\varepsilon_1}^{t+\varepsilon_2} D_t a_{j,\alpha}(\tau) d\tau \right| \right)^2 \int_0^t \sum_{j+|\alpha|\leq m,\ j<m} \left\| D_t^j D_x^\alpha v_{\varepsilon_2}(\tau) \right\|^2_{(r-1)} d\tau. \end{aligned}$$

Then the right-hand side of the last inequality is dominated by by $C|\varepsilon_2 - \varepsilon_1|^2$, where the constant C is independent of ε. Thus, the sequence of the functions v_{ε_k}, $k = 1, 2, \dots$, corresponding to the sequence $\varepsilon_k \to 0$, is fundamental in the space

$$\bigcap_{j=0}^{m-1} C^{m-j-1}\left([0, T_1]; H_{(r+m-2-l-j)}(\mathbb{R}^n)\right), \qquad T_1 > 0.$$

In view of the uniqueness of the solution we have $u = \lim_{k\to\infty} v_{\varepsilon_k}$ in that space and consequently, in $\mathcal{D}'(K_\gamma(x^0, t^0))$. In particular,

$$\langle u, \varphi \rangle = \lim_{k\to\infty} \langle v_{\varepsilon_k}, \varphi \rangle = 0 \quad \text{for every test function } \varphi \in C_0^\infty(K_\gamma(x^0, t^0)),$$

implies (4.11) and completes the proof of the theorem. □

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