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On the Equations of Evolution in a Banach Space

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§0. Introduction. In this paper, we again consider the equations of evolution

$$dx(t)/dt = A(t)x(t)+f(t) \tag{0.1}$$

and its associated homogeneous equation

$$dx(t)/dt = A(t)x(t) \tag{0.1'}$$

such as was treated in the previous papers [3] and [4]. However, we shall show that we can replace the strong continuous differentiability of $A(t)A(s)^{-1}$ by its Hölder continuity by means of a slight change of the proof. It is quite clear that the differentiability of $A(t)A(s)^{-1}$ is not necessary for the construction of the formal fundamental solution $U(t, s)$ of (0.1′). In the previous papers, however, we used the differentiability essentially in appearance when we proved that the formal fundamental solution was really the desired one. So it is in this part that the modification of the proof is required. The inhomogeneous equation (0.1) can be treated similarly. Next, we shall give a generalization of a theorem of Solomijak concerning a perturbed equation

$$dx(t)/dt = A(t)x(t)+B(t)x(t)+f(t)\,. \tag{0.2}$$

Evidently, it is absurd now to consider (0.2) under the same assumptions about $B(t)$ as in [3] and [4].

For the existence of the second derivative of $U(t, s)$, it is sufficient to assume that $A(t)A(s)^{-1}$ has a Hölder continuous derivative in t.

§1. The fundamental solution of (0.1′). We denote by $\sum$ a fixed closed sector which consists of those complex numbers λ satisfying $-\theta\leqq \arg\lambda\leqq\theta$, $\theta>\frac{\pi}{2}$ plus the origin. Throughout this paper, we assume

Assumptions 1°. For each t with $-\infty<a\leqq t\leqq b<\infty$, $A(t)$ is a closed operator with its domain dense in a Banach space $\mathfrak{X}$ and its range in $\mathfrak{X}$. The resolvent set $\rho(A(t))$ of $A(t)$, $a \leqq t \leqq b$ contains $\sum$ and the resolvent $(\lambda I - A(t))^{-1}$ satisfies

$$\|(\lambda I - A(t))^{-1}\| \leqq \frac{M}{|\lambda|+1} \tag{1.1}$$

for each $\lambda \in \sum$, where M is a positive constant independent of λ and t.

2°. The domain $\mathfrak{D}$ of $A(t)$ is independent of t and the bounded operator $A(t)A(s)^{-1}$ is Hölder continuous in t in the uniform operator topology for each fixed s:

$$\|A(t)A(s)^{-1} - A(r)A(s)^{-1}\| \leqq K|t-r|^{\rho}, \quad K > 0, \quad 0 < \rho \leqq 1. \tag{1.2}$$

By assumption 1°, $A(s)$ generates a semi-group $\exp(tA(s))$ of bounded operators by the formula

$$\exp(tA(s)) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda I - A(s))^{-1} d\lambda, \tag{1.3}$$

where Γ is any contour running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ in the sector $\sum$. $\exp(tA(s))$ and $tA(s)\exp(tA(s))$ is uniformly bounded in $0 < t < \infty$ and $a \leqq s \leqq b$:

$$\|\exp(tA(s))\| \leqq C(M, \theta) \tag{1.4}$$

$$\|A(s)\exp(tA(s))\| \leqq C(M, \theta)/t, \tag{1.5}$$

where the constant on the right depends only on the arguments in the bracket.

We shall use C to denote constants which depend only on M, θ, K, ρ and $b-a$ at the most.

The fundamental solution $U(t, s)$ of (0. 1′) is formally given by

$$\begin{cases} U(t, s) = \exp((t-s)A(s)) + W(t, s), \\ W(t, s) = \int_s^t \exp((t-\tau)A(\tau))R(\tau, s)d\tau, \end{cases} \tag{1.6}$$

where $R(t, s)$ is the solution of the integral equation

$$\begin{cases} R(t, s) - \int_s^t R_1(t, \tau)R(\tau, s)d\tau = R_1(t, s) \\ R_1(t, s) = (A(t) - A(s))\exp((t-s)A(s)). \end{cases} \tag{1.7}$$

The integral equation (1. 7) can be solved by a successive approximation method:

$$R(t, s) = \sum_{m=1}^{\infty} R_m(t, s) \tag{1.8}$$

where $R_m(t, s)=\int_s^t R_1(t, \tau)R_{m-1}(\tau, s)d\tau$, $m=2, 3, \cdots$.

By Lemma 5.1 of [3], we have

Lemma 1.1. *The series* (1.8) *converges uniformly in the wider sense in* $a\leqq s<t\leqq b$ *in the uniform operator topology. The sum is strongly continuous in* $a\leqq s<t\leqq b$ *and satisfies* (1.7) *and*

$$\|R(t, s)\|\leqq C(t-s)^{\rho-1}. \tag{1.9}$$

Lemma 1.2. *For* $s<\tau<t$,

$$\|R(t, s)-R(\tau, s)\|\leqq C\left\{\frac{(t-\tau)^\rho}{t-s}+\frac{t-\tau}{(t-s)(\tau-s)^{1-\rho}}+\frac{(t-\tau)^\rho}{(t-s)^{1-\rho}}\log\frac{t-s}{t-\tau}\right\}. \tag{1.10}$$

Proof. As in Lemma 1.3 of [3], we get

$$\|R_1(t, s)-R_1(\tau, s)\|\leqq C\left\{\frac{(t-\tau)^\rho}{t-s}+\frac{t-\tau}{(t-s)(\tau-s)^{1-\rho}}\right\}. \tag{1.11}$$

On the other hand,

$$\int_s^t R_1(t, \sigma)R(\sigma, s)d\sigma-\int_s^\tau R_1(\tau, \sigma)R(\sigma, s)d\sigma$$
$$=\int_\tau^t R_1(t, \sigma)R(\sigma, s)d\sigma+\int_s^\tau (R_1(t, \sigma)-R_1(\tau, \sigma))R(\sigma, s)d\sigma.$$

The norm of the first term on the right hand side is dominated by

$$\begin{aligned}\left\|\int_\tau^t R_1(t, \sigma)R(\sigma, s)d\sigma\right\| &\leqq C\int_\tau^t (t-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma\\ &= C\int_\tau^t (t-\sigma)^{\rho-1}(\sigma-\tau)^{\rho-1}\frac{(\sigma-\tau)^{1-\rho}}{(\sigma-s)^{1-\rho}}d\sigma\\ &\leqq C\left(\frac{t-\tau}{t-s}\right)^{1-\rho}\int_\tau^t (t-\sigma)^{\rho-1}(\sigma-\tau)^{\rho-1}d\sigma\\ &= CB(\rho, \rho)(t-\tau)^\rho(t-s)^{\rho-1}.\end{aligned} \tag{1.12}$$

By (1.11), the second term is bounded in norm by

$$C\int_s^\tau\left\{\frac{(t-\tau)^\rho}{t-\sigma}+\frac{t-\tau}{(t-\sigma)(\tau-\sigma)^{1-\rho}}\right\}(\sigma-s)^{\rho-1}d\sigma.$$

We first estimate the integral

$$\int_s^\tau (t-\tau)(t-\sigma)^{-1}(\tau-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma.$$

If $\tau \geqq (t+s)/2$,

$$(1.13)\qquad \int_s^{(t+s)/2} (t-\sigma)^{-1}(\tau-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma \leqq \frac{2}{t-s}\int_s^{\tau}(\tau-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma$$
$$= 2B(\rho,\ \rho)\,(t-s)^{-1}(\tau-s)^{2\rho-1}\,,$$

and

$$(1.14)\qquad \int_{(t+s)/2}^{\tau}(t-\sigma)^{-1}(\tau-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma \leqq \left(\frac{t-s}{2}\right)^{\rho-1}\int_s^{\tau}(t-\sigma)^{-1}(\tau-\sigma)^{\rho-1}d\sigma$$
$$\leqq \frac{2^{1-\rho}}{\rho}(t-s)^{\rho-2}(\tau-s)^{\rho}+\frac{2^{1-\rho}}{\rho(1-\rho)}(t-s)^{\rho-1}(t-\tau)^{\rho-1}\,.$$

If $\tau<(t+s)/2$,

$$(1.15)\qquad \int_s^{\tau}(t-\sigma)^{-1}(\tau-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma \leqq 2B(\rho,\ \rho)\,(t-s)^{-1}(\tau-s)^{2\rho-1}\,.$$

Hence, noting $(t-\tau)(t-s)^{\rho-2}\leqq(t-\tau)^{\rho}(t-s)^{-1}$, we obtain

$$(1.16)\qquad \int_s^{\tau}\frac{t-\tau}{(t-\sigma)(\tau-\sigma)^{1-\rho}(\sigma-s)^{1-\rho}}d\sigma \leqq C\left\{\frac{t-\tau}{(t-s)(\tau-s)^{1-2\rho}}\right.$$
$$\left.+\frac{(t-\tau)^{\rho}(\tau-s)^{\rho}}{t-s}+\frac{(t-\tau)^{\rho}}{(t-s)^{1-\rho}}\right\}.$$

By a similar method, we get

$$(1.17)\qquad \int_s^{\tau}\frac{(t-\tau)^{\rho}}{(t-\sigma)(\sigma-s)^{1-\rho}}d\sigma \leqq C(\rho)\frac{(t-\tau)^{\rho}}{(t-s)^{1-\rho}}\left(\log\frac{t-s}{t-\tau}+1\right)$$

Combining (1.7), (1.11), (1.12), (1.16) and (1.17), we obtain (1.10).

Lemma 1.3. *For* $a\leqq s<t\leqq b$,

$$(1.18)\qquad \|A(t)\{\exp((t-s)A(t))-\exp((t-s)A(s))\}\|\leqq C|t-s|^{\rho-1}\,.$$

Proof. By (1.3) and a simple computation

$$(1.19)\qquad A(t)\{\exp((t-s)A(t))-\exp((t-s)A(s))\}$$
$$=\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda(t-s)}A(t)(\lambda I-A(t))^{-1}(A(t)-A(s))(\lambda I-A(s))^{-1}d\lambda\,.$$

By (1.1) and $A(t)(\lambda I-A(t))^{-1}=\lambda(\lambda I-A(t))^{-1}-I$, $A(t)(\lambda I-A(t))^{-1}$ is uniformly bounded in $a\leqq t\leqq b$ and $\lambda\in\sum$:

$$(1.20)\qquad \|A(t)(\lambda I-A(t))^{-1}\|\leqq M+1\,.$$

Thus, we have

(1. 21)
$$\begin{aligned}&\|(A(t)-A(s))(\lambda I-A(s))^{-1}\|\\ &\leqq \|(A(t)-A(s))A(s)^{-1}\|\ \|A(s)(\lambda I-A(s))^{-1}\|\\ &\leqq K(M+1)|t-s|^{\rho}.\end{aligned}$$

The assertion of the lemma follows immediately from (1. 19), (1. 20), (1. 21) and some elementary calculus.

We can operate $A(t)$ to each term of the right hand side of

$$\begin{aligned}W(t,\ s) = &\int_s^t \exp((t-\tau)A(t))d\tau R(t,\ s)\\ &+\int_s^t \{\exp((t-\tau)A(\tau))-\exp((t-\tau))A(t))\}R(\tau,\ s)d\tau+\\ &+\int_s^t \exp((t-\tau)A(t))(R(\tau,\ s)-R(t,\ s))d\tau\end{aligned}$$

by the previous lemmas. Thus, we get

(1. 21)
$$\begin{aligned}A(t)U(t,\ s) = &A(t)\exp((t-s)A(s))+\{\exp((t-s)A(t))-I\}R(t,\ s)\\ &+\int_s^t A(t)\{\exp((t-\tau)A(\tau))-\exp((t-\tau)A(t))\}R(\tau,\ s)d\tau\\ &+\int_s^t A(t)\exp((t-\tau)A(t))(R(\tau,\ s)-R(t,\ s))d\tau\,.\end{aligned}$$

It is easily seen that $A(t)W(t,\ s)$ satisfies

(1. 22)
$$\|A(t)W(t,\ s)\|\leqq C(t-s)^{\rho-1}.$$

Consequently, $A(t)U(t,\ s)$ satisfies

(1. 23)
$$\|A(t)U(t,\ s)\|\leqq C(t-s)^{-1}.$$

A similar formula and estimate can be proved for $(\partial/\partial t)U(t,\ s)$. However, by the fact that for any $x\in\mathfrak{X}$, $\{(\partial/\partial t)U_h(t,\ s)-A(t)U_h(t,\ s)\}x$ tends to 0 as $h\downarrow 0$ and the strong continuity of $A(t)U(t,\ s)$, we can obtain the estimate of $(\partial/\partial t)U(t,\ s)$ as well as the relation $(\partial/\partial t)U(t,\ s)=A(t)U(t,\ s)$. Hence, we have

Theorem 1. 1. *Under Assumptions* 1° *and* 2°, *the fundamental solution* $U(t,\ s)$ *of* (0, 1′) *exists.* $U(t,\ s)$ *is strongly continuous in* $a\leqq s\leqq t\leqq b$ *and satisfies*

(1. 24)
$$\left\|\frac{\partial}{\partial t}U(t,\ s)\right\| = \|A(t)U(t,\ s)\|\leqq\frac{C}{t-s}$$

with some constant C *depending only on* M, θ, K, ρ *and* $b-a$.

Next, we shall prove the uniqueness of the solution. For this purpose, it is sufficient to show that for each $x \in \mathfrak{D}$, $U(t, s)x$ is strongly continuously differentiable in s and it satisfies $-(\partial/\partial s)U(t, s)x = U(t, s)A(s)x$ (§1 of [4]). We showed in [4] that this is the case if $A(t)A(s)^{-1}$ is strongly continuously difierentiable in t.

Let $j(t)$ be an infinitely differentiable non-negative function defined in $-\infty < t < \infty$ such that $\int_{-\infty}^{\infty} j(t)dt = 1$ and $j(t) \equiv 0$ for $|t| > 1$. For any natural number n, we put $j_n(t) = nj(nt)$. Thus $j_n(t)$ determines a mollifier. We may and will assume that $A(t)$ is defined on the whole real line $-\infty < t < \infty$ and satisfies Assumptions 1° and 2° there with the same constants.

For any $x \in \mathfrak{D}$, we define

$$A_n(t)x = \int_{-\infty}^{\infty} j_n(t-\tau)A(\tau)xd\tau\,. \tag{1.25}$$

By an elementary calculus we have

$$\begin{aligned} & x-(\lambda I - A_n(t))(\lambda I - A(t))^{-1}x \\ & = \int_{-\infty}^{\infty} j_n(t-\tau)(A(\tau)-A(t))(\lambda I - A(t))^{-1}xd\tau \end{aligned} \tag{1.26}$$

for any $x \in \mathfrak{X}$. However, for the value of t such that $j_n(t-\tau) \neq 0$ in the above integral

$$\begin{aligned} & \|(A(\tau)-A(t))(\lambda I - A(t))^{-1}\| \\ & \leqq K(M+1)n^{-\rho}\,. \end{aligned}$$

This implies

$$\|x-(\lambda I - A_n(t))(\lambda I - A(t))^{-1}x\| \leqq K(M+1)n^{-\rho}\|x\| \tag{1.27}$$

for any x, or

$$\begin{aligned} \left(1-\frac{K(M+1)}{n^\rho}\right)\|(\lambda I - A(t))y\| & \leqq \|(\lambda I - A_n(t))y\| \\ & \leqq \left(1+\frac{K(M+1)}{n^\rho}\right)\|(\lambda I - A(t))y\| \end{aligned} \tag{1.28}$$

for any $y \in \mathfrak{D}$. This shows that $A_n(t)$ is closed on $\mathfrak{D}$ if $K(M+1)n^{-\rho} < 1$. In fact, let $\{y_j\}$ be a sequence from $\mathfrak{D}$ such that $y_j \to y$ and $(\lambda I - A_n(t))y_j \to z$ as $j \to \infty$. Replacing y by $y_j - y_k$ in (1.28), we see that $(\lambda I - A(t))y_j$ also converges to some element, which together with the closedness of $A(t)$ implies $y \in \mathfrak{D}$ and $(\lambda I - A(t))y_j \to (\lambda I - A(t))y$. Finally, putting $y - y_j$ in place of y in (1.28) shows that $(\lambda I - A_n(t))y_j \to (\lambda I - A_n(t))y$. Thus, the closedness of $A_n(t)$ has been proved.

By (1.27), for any n so large that $K(M+1)n^{-\rho}<1$, $(\lambda I-A_n(t))\times(\lambda I-A(t))^{-1}$ has a bounded inverse. Hence, the range of $\lambda I-A_n(t)$ is the whole of $\mathfrak{X}$. More precisely, $\lambda I-A_n(t)$ maps $\mathfrak{D}$ onto $\mathfrak{X}$ in a one-to-one manner. As λ is any number of $\sum$, this implies $\rho(A_n(t))\supset\sum$.

By (1.27), we immediately obtain for any $\lambda\in\sum$

$$(1.29)\qquad \|(\lambda I-A_n(t))^{-1}\|\leqq\frac{M_n}{|\lambda|+1}$$

where $M_n=M\{1-K(M+1)n^{-\rho}\}^{-1}$ tends to M as $n\to\infty$, and

$$(1.30)\qquad \|(\lambda I-A(t))(\lambda I-A_n(t))^{-1}\|\leqq\frac{1}{1-K(M+1)n^{-\rho}}.$$

By a simple computation and (1.30) for $\lambda=0$, we also obtain

$$\|A_n(t)A_n(s)^{-1}-A_n(r)A_n(s)^{-1}\|\leqq K_n|t-r|^\rho$$

where $K_n=K\{1-K(M+1)n^{-\rho}\}^{-1}$ tends to K as $n\to\infty$. Thus, for the equation

$$(1.31)\qquad dx(t)/dt = A_n(t)x(t)$$

all the hypotheses in [4] are satisfied, therefore the fundamental solution $U_n(t, s)$ of (1.31) exists. $\|U_n(t, s)\|$ is bounded by a constant which depends continuously on M_n, θ, K_n, ρ and $t-s$. Hence $U_n(t, s)$ is uniformly bounded with respect to t, s and n if n is sufficiently large. We have also

$$(1.32)\qquad U_n(t, r)U_n(r, s) = U_n(t, s)\quad\text{for}\quad s\leqq r\leqq t,\quad\text{and}$$

$$(1.33)\qquad -\frac{\partial}{\partial s}U_n(t, s)x = U_n(t, s)A_n(s)x\quad\text{for}\quad x\in\mathfrak{D}.$$

Let x be any element in $\mathfrak{D}$. Then,

$$\begin{aligned}U(t, s)x-U_n(t, s)x &= \int_s^t\frac{\partial}{\partial\sigma}(U_n(t, \sigma)U(\sigma, s)x)\,d\sigma\\ &= \int_s^t U_n(t, \sigma)(A(\sigma)-A_n(\sigma))U(\sigma, s)x\,d\sigma\\ &= \int_s^t U_n(t, \sigma)(A(\sigma)-A_n(\sigma))A(\sigma)^{-1}A(\sigma)U(\sigma, s)A(s)^{-1}A(s)x\,d\sigma.\end{aligned}$$

By (1.27) and the uniform boundedness of $A(\sigma)U(\sigma, s)A(s)^{-1}$, we get

$$\begin{aligned}&\|(U(t, s)-U_n(t, s))x\|\\ &\leqq\sup_{\sigma,n}\|U_n(t, \sigma)\|K(M+1)n^{-\rho}\sup_\sigma\|A(\sigma)U(\sigma, s)A(s)^{-1}\|\,\|A(s)x\|(t-s).\end{aligned}$$

Thus, $U_n(t, s)$ tends to $U(t, s)$ on $\mathfrak{D}$, consequently on the whole of $\mathfrak{X}$. Letting $n\to\infty$ in (1. 32), we get

$$U(t, r)U(r, s) = U(t, s) \quad \text{for} \quad s\leqq r\leqq t, \quad \text{and} \tag{1. 34}$$

$$-\frac{\partial}{\partial s}U(t, s)x = U(t, s)A(s)x \quad \text{for} \quad x\in\mathfrak{D}. \tag{1. 35}$$

However, to show (1. 35), we must verify that the convergence of $U_n(t, s)A_n(s)x$ to $U(t, s)A(s)x$ is uniform with respect to s. If we want to avoid this nuisance we can do as follows. Using

$$\|R(t, s)A(s)^{-1}\|\leqq C(t-s)^{\rho} \tag{1. 36}$$

which can be easily proved, we can show

$$\begin{cases} \dfrac{U(s+h, s)-I}{h}x \to \quad A(s)x & \text{as} \quad h\downarrow 0 \\ \dfrac{U(s, s+h)-I}{h}x \to -A(s)x & \text{as} \quad h\uparrow 0 \end{cases} \tag{1. 37}$$

for $x\in\mathfrak{D}$. Combining this with (1. 34), we readily obtain (1. 35).

Theorem 1. 2. *Under Assumptions* 1° *and* 2°, *the solution* $x(t)$ *of* (0. 1) *is uniquely determined by the inhomogeneous term and the initial data. Moreover, the fundamental solution of* (0. 1′) *satisfies* (1. 34) *and* (1. 35).

REMARK. If there exists an operator valued function $U(t, s)$ with the property that $U(t, s)x$ gives the solution of (0. 1′) having the initial value x at $t=s$ for each $x\in\mathfrak{D}$ and that (1. 37) holds, then, the following three assertions are equivalent:

i) the uniqueness holds for (0. 1),
ii) (1. 34) holds,
iii) (1. 35) holds.

Next, we consider the inhomogeneous equation (0. 1).

Theorem 1. 3. *If* $f(t)$ *is Hölder continuous in* $a\leqq t\leqq b$:

$$\|f(t)-f(r)\|\leqq F|t-r|^{\gamma}, \quad F>0, \quad 0<\gamma\leqq 1, \tag{1. 38}$$

then

$$x(t) = U(t, s)x+\int_s^t U(t, \sigma)f(\sigma)d\sigma \tag{1. 39}$$

is the solution of (1. 1) *in* $s<t\leqq b$ *corresponding to the initial condition* $x(s)=x$.

Proof. By (1.22), we have

$$A(t)\int_s^t W(t,\ \sigma)f(\sigma)d\sigma = \int_s^t A(t)W(t,\ \sigma)f(\sigma)d\sigma .$$

As in the proof of Theorem 1.1, we can show

$$\begin{aligned} A(t)\int_s^t \exp((t-\sigma)A(\sigma))f(\sigma)d\sigma &= \{\exp((t-s)A(t))-I\}f(t) \\ &+\int_s^t A(t)\{\exp((t-\sigma)A(\sigma))-\exp((t-\sigma)A(t))\}f(\sigma)d\sigma+ \\ &+\int_s^t A(t)\exp((t-\sigma)A(t))(f(\sigma)-f(t))d\sigma . \end{aligned}$$

As $h\downarrow 0$,

$$\begin{aligned} &\frac{\partial}{\partial t}\int_s^{t-h} U(t,\ \sigma)f(\sigma)d\sigma - A(t)\int_s^{t-h} U(t,\ \sigma)f(\sigma)d\sigma - f(t) \\ &= U(t,\ t-h)f(t-h)-f(t)\to 0 , \end{aligned}$$

which implies the differentiability in t of $\int_s^t U(t,\ \sigma)f(\sigma)d\sigma$ and

$$\frac{\partial}{\partial t}\int_s^t U(t,\ \sigma)f(\sigma)d\sigma = A(t)\int_s^t U(t,\ \sigma)f(\sigma)d\sigma+f(t) . \qquad \text{(q.e.d.)}$$

Next, we consider a perturbed equation

$$dx(t)/dt = A(t)x(t)+B(t)x(t)+f(t) . \tag{1.40}$$

We denote by (1.40′) the associated homogeneous equation of (1.40). $B(t)$ is assumed to satisfy

4°. The bounded operator $B(t)A(s)^{-1}$ is strongly continuous in $a\leqq t\leqq b$ for every fixed s.

5°. There exist positive constants C_0 and $\alpha\leqq 1$ such that

$$\|B(t)\exp(\tau A(s))\|\leqq C_0\tau^{-\alpha} \tag{1.41}$$

for $a\leqq t,\ s\leqq b$ and $\tau>0$.

Then the formal fundamental solution $U(t,\ s)$ of (1.40′) is given by

$$U(t,\ s) = \textstyle\sum_{m=0}^{\infty} U_m(t,\ s) , \tag{1.42}$$

where $U_0(t,\ s)$ is the fundamental solution of (0.1′) and $U_m(t,\ s) = \int_s^t U_0(t,\ \sigma)B(\sigma)U_{m-1}(\sigma,\ s)d\sigma = \int_s^t U_{m-1}(t,\ \sigma)B(\sigma)U_0(\sigma,\ s)d\sigma,\ m=1,\ 2,\ 3,\ \cdots$.

From (1.41), it follows that there is a constant C_2 such that

$\|B(t)U_0(t, s)\| \leqq C_2(t-s)^{-\alpha}$. If $\|U_0(t, s)\| \leqq C_1$ in $a \leqq s \leqq t \leqq b$, then for any m,

$$\|U_m(t, s)\| \leqq \frac{C_1 C_2^m \Gamma(1-\alpha)^m}{\Gamma(m(1-\alpha)+1)} (t-s)^{m(1-\alpha)}. \tag{1.43}$$

In general, we do not know whether $U(t, s)$ defined by (1.42) is really the fundamental solution of (1.40′). However, the assumptions above are sufficient for the validity of

$$-(\partial/\partial s)U(t, s)x = U(t, s)(A(s)+B(s))x$$

for $x \in \mathfrak{D}$ (see [4]). Hence, we have

Theorem 1.4. *The solution $x(t)$ of* (1.40) *which is continuous in $s \leqq t \leqq b$ and satisfies* (1.40) *in $s < t \leqq b$ is uniquely determined by the initial data at $t=s$ and the inhomogeneous part $f(t)$, $s \leqq t \leqq b$.*

The following theorem is a generalization of Theorem 6 of Solomijak [1].

Theorem 1.5. *If there exist positive constants α and δ, $\alpha+\delta<1$, such that $B(t)A(t)^{-\alpha}$ and $A(t)^{\delta}B(t)A(t)^{-\alpha-\delta}$ are bounded operators and strongly continuous in $a \leqq t \leqq b$, then $U(t, s)$ defined by* (1.42) *is the fundamental solution of* (1.40′) *and there is a constant C' such that*

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\| \leqq \frac{C'}{t-s}, \quad \|A(t)U(t, s)\| \leqq \frac{C'}{t-s} \quad \text{and} \tag{1.44}$$

$$\|B(t)U(t, s)\| \leqq \frac{C'}{(t-s)^{\alpha}}.$$

If $f(t)$ is Hölder continuous, then the solution $x(t)$ of (1.40) *corresponding to the initial condition $x(s)=x$ is given by*

$$x(t) = U(t, s)x + \int_s^t U(t, \sigma)f(\sigma)d\sigma.$$

Proof. We have only to prove the first half of the theorem. Let N be a positive constant with which we have $\|B(t)A(t)^{-\alpha}\| \leqq N$ and $\|A(t)^{\delta}B(t)A(t)^{-\alpha-\delta}\| \leqq N$. Assumptions 3°–5° are all implied by the assumptions of the theorem. First we prove the following lemma.

Lemma 1.4. *There is a constant C_3 such that*

$$\left\|\frac{\partial}{\partial t}U_1(t, s)\right\| \leqq \frac{C_3}{(t-s)^{\alpha}}. \tag{1.45}$$

Proof. It is easy to see

$$(1.46)\quad \frac{\partial}{\partial t}\int_s^t \exp((t-\sigma)A(\sigma))B(\sigma)U_0(\sigma,\ s)d\sigma = B(t)U_0(t,\ s)$$
$$+\int_s^t A(\sigma)^{1-\delta}\exp((t-\sigma)A(\sigma))A(\sigma)^{\delta}B(\sigma)A(\sigma)^{-\alpha-\delta}A(\sigma)^{\alpha+\delta}U_0(\sigma,\ s)d\sigma\,.$$

Using

$$(1.47)\quad \|A(\sigma)^{\alpha+\delta}U_0(\sigma,\ s)\| \leqq C_4(\sigma-s)^{-\alpha-\delta}\,,$$

$$(1.48)\quad \|A(\sigma)^{1-\delta}\exp((t-\sigma)A(\sigma))\| \leqq C_5(t-\sigma)^{\delta-1}\,,$$

we see that the norm of the second term on the right of (1.46) is bounded by

$$\int_s^t C_5(t-\sigma)^{\delta-1}NC_4(\sigma-s)^{-\alpha-\delta}d\sigma = NC_4C_5B(\delta,\ 1-\alpha-\delta)(t-s)^{-\alpha}\,.$$

Combining this with $\|B(t)U_0(t,\ s)\| \leqq C_2(t-s)^{-\alpha}$, we obtain

$$\left\|\frac{\partial}{\partial t}\int_s^t \exp((t-\sigma)A(\sigma))B(\sigma)U_0(\sigma,\ s)d\sigma\right\| \leqq C_6(t-s)^{-\alpha}\,.$$

The above inequality, together with the easily verifiable inequality

$$\left\|\frac{\partial}{\partial t}\int_s^t W(t,\ \sigma)B(\sigma)U_0(\sigma,\ s)d\sigma\right\| \leqq C_7(t-s)^{-\alpha+\rho}\,,$$

implies (1.45).

As $h\downarrow 0$

$$\frac{\partial}{\partial t}\int_s^{t-h} U_0(t,\ \sigma)B(\sigma)U_0(\sigma,\ s)d\sigma - A(t)\int_s^{t-h} U_0(t,\ \sigma)B(\sigma)U_0(\sigma,\ s)d\sigma$$
$$= U_0(t,\ t-h)B(t-h)U_0(t-h,\ s)$$

tends to $B(t)U_0(t,\ s)$. As $A(t)$ is closed, this implies that $A(t)\int_s^t U_0(t,\ \sigma)$ $\times B(\sigma)U_0(\sigma,\ s)d\sigma$ is a bounded operator and that

$$(1.49)\quad (\partial/\partial t)U_1(t,\ s) = A(t)U_1(t,\ s)+B(t)U_0(t,\ s)\,.$$

By induction, for any $m\geqq 1$, we obtain

$$(1.50)\quad (\partial/\partial t)U_m(t,\ s) = A(t)U_m(t,\ s)+B(t)U_{m-1}(t,\ s)\,.$$

Again, by induction, we get

$$(1.51)\quad \left\|\frac{\partial}{\partial t}U_m(t,\ s)\right\| \leqq \frac{C_2^{m-1}C_3\Gamma(1-\alpha)^m}{\Gamma(m(1-\alpha))}(t-s)^{m(1-\alpha)-1}\,,$$

$$(1.52)\quad \|B(t)U_m(t,\ s)\| \leqq \frac{C_2^{m+1}\Gamma(1-\alpha)^{m+1}}{\Gamma((m+1)(1-\alpha))}(t-s)^{-\alpha+m(1-\alpha)}\,.$$

Combining (1.50), (1.51) and (1.52), we obtain the theorem.

§2. Successive derivative of the solution. We shall show that if $A(t)A(s)^{-1}$ has a Hölder continuous derivative $A'(t)A(s)^{-1}$, $U(t, s)$ is twice continuously differentiable. The computation is tedious and troublesome, so we will only sketch the proof. We assume that

$$\|A'(t)A(s)^{-1}-A'(r)A(s)^{-1}\| \leqq K|t-r|^{\rho}. \tag{2.1}$$

$(\partial/\partial t)W(t, s)$ can be written follows:

$$\begin{aligned}\frac{\partial}{\partial t}W(t, s) &= \int_{s_1}^{t}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\exp((t-\tau)A(\tau))\cdot R(\tau, s)d\tau \\ &\quad -R(t, s)+\exp((t-s_1)A(s_1))R(s_1, s) \\ &\quad +\int_{s_1}^{t}\exp((t-\tau)A(\tau))\frac{\partial}{\partial \tau}R(\tau, s)d\tau+\int_{s}^{s_1}A(\tau)\exp((t-\tau)A(\tau))R(\tau, s)d\tau\end{aligned} \tag{2.2}$$

where $s_1=(t+s)/2$. If we put

$$F(t, \tau, r) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda(t-\tau)}(\lambda I-A(r))^{-1}A'(r)(\lambda I-A(r))^{-1}d\lambda, \tag{2.3}$$

then $(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))=F(t, \tau, \tau)$ and, as is easily proved,

$$\|A(t)(F(t, \tau, t)-F(t, \tau, \tau))\| \leqq C(t-\tau)^{\rho-1}. \tag{2.4}$$

Consequently, the norm of each term of the right member of

$$\begin{aligned}A(t)\int_{s_1}^{t}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\exp((t-\tau)A(\tau))\cdot R(\tau, s)d\tau &= \int_{s_1}^{t}A(t)F(t, \tau, t)d\tau\cdot R(t, s) \\ &\quad +\int_{s_1}^{t}A(t)(F(t, \tau, \tau)-F(t, \tau, t))R(\tau, s)d\tau+ \\ &\quad +\int_{s_1}^{t}A(t)F(t, \tau, t)(R(\tau, s)-R(t, s))d\tau\end{aligned} \tag{2.5}$$

is uniformly bounded. Next, for $s<\tau<t$, as is easily proved,

$$\left\|\frac{\partial}{\partial t}R(t, s)-\frac{\partial}{\partial \tau}R(\tau, s)\right\| \leqq C\left\{(t-\tau)^{\rho}+\frac{t-\tau}{t-s}\right\}\left(\frac{1}{t-s}+\log\frac{t-s}{t-\tau}\right). \tag{2.6}$$

Using (2.6), we can easily prove

$$\left\|A(t)\int_{s_1}^{t}\exp((t-\tau)A(\tau))\frac{\partial}{\partial \tau}R(\tau, s)d\tau\right\| \leqq \frac{C}{t-s}. \tag{2.7}$$

Combining (2.2), (2.5) and (2.7), we obtain

(2.8)
$$\left\|A(t)\frac{\partial}{\partial t}W(t, s)\right\| \leqq \frac{C}{t-s} \quad \text{and}$$
$$\left\|A(t)\frac{\partial}{\partial t}U(t, s)\right\| \leqq \frac{C}{(t-s)^2}.$$

As $U(t, s)$ satisfies (0.1′), we see that $A(t)^2 U(t, s)$ and $(\partial/\partial t)^2 U(t, s)$ both exist and are bounded by $C(t-s)^{-2}$ in norm. Hence, we have

Theorem 2.1. *If $A(t)A(s)^{-1}$ has a Hölder continuous derivative in t, $(\partial/\partial t)^2 U(t, s)$, $A(t)(\partial/\partial t)U(t, s)$ and $A(t)^2 U(t, s)$ are all bounded operators whose norms are dominated by $C(t-s)^{-2}$.*

We could also treat the inhomogeneous equation, but we omit it because it is tedious and of little interest.

§3. **Example.** As an example, we consider a partial differential equation

(3.1)
$$\frac{\partial}{\partial t}u(t, x) = \sum_{|\alpha| \leqq 2m} a_\alpha(t, x)\left(\frac{\partial}{\partial x}\right)^\alpha u(t, x) + f(t, x)$$

in a bounded cylindrical domain $[a, b] \times G$. We assume that the boundary ∂G of G is sufficiently smooth. We will consider (3.1) in $L_p(G)$, $1 < p < \infty$.

We denote by $W_p^l(G)$ the set of all the complex-valued functions defined in G whose distribution derivatives $D^\alpha u$ belong to $L_p(G)$ for any α with $0 \leqq |\alpha| \leqq l$, and by $\mathring{W}_p^l(G)$ the set of all the complex-valued functions in $W_p^l(G)$ whose distribution derivatives of order at most $l-1$ all vanish on ∂G in the usual generalized sense.

We assume that for each t, $A(t) = \sum_{|\alpha| \leqq 2m} a_\alpha(t, x)(\partial/\partial x)^\alpha$ is uniformly strongly elliptic and the coefficients $a_\alpha(t, x)$ are so smooth that the assumptions of Theorem 4 of Solomijak [2] hold good if we add a sufficiently large positive number to $A(t)$ if necessary. Hence we may assume that those assumptions hold for $A(t)$ itself uniformly in $a \leqq t \leqq b$. Therefore, there exist a positive number M and a sector $\sum$ of the type mentioned in §1 such that for any $\lambda \in \sum$ we have

$$\|(\lambda I - A(t))^{-1}\|_p \leqq \frac{M}{|\lambda| + 1}.$$

Next, we also assume that each $a_\alpha(t, x)$ is Hölder continuous in t uniformly in $[a, b] \times \bar{G}$:

$$|a_\alpha(t, x) - a_\alpha(\tau, x)| \leqq K_\alpha |t-\tau|^\rho, \quad K_\alpha > 0, \quad 0 < \rho \leqq 1.$$

Then, Assumptions 1° and 2° are satisfied by $A(t)$. Hence the fundamental solution $U(t, s)$ of (3.1) exists. It is also easily seen that for any integer l with $0 \leqq l \leqq 2m$, we have

$$\left\|\left(\frac{\partial}{\partial x}\right)^l U(t, s)\right\| \leqq \frac{C}{(t-s)^{l/2m}}$$

with some positive constant C.

If each $a_\alpha(t, x)$ has a bounded derivative in t which is uniformly Hölder continuous in t or if the coefficients of the formal adjoint of $A(t)$ are all uniformly Hölder continuous in t, then the adjoint operator $U(t, s)^*$ of $U(t, s)$ is the fundamental solution of the adjoint equation (§ 1 of [3] and [4]):

$$-\frac{d}{ds} v(s) = A(s)^* v(s) .$$

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Bibliography

[1] M, Z. Solomijak: The application of the semi-group theory to the study of differential equations in Banach spaces, Doklady Akad. Nauk, SSSR **122** (1958), 766–769.

[2] M. Z. Solomijak: Analytic semi-groups generated by elliptic differential operators, Doklady Akad. Nauk. SSSR **127** (1959), 37–39.

[3] H. Tanabe: A class of the equations of evolution in a Banach space, Osaka Math. J. **11** (1959), 121–145.

[4] H. Tanabe: Remarks on the equations of evolution in a Banach space, Osaka Math. J. **12** (1960), 145–165.