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A Remark on Spectral Measures of the Flow of Brownian Motion.

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1. The purpose of this note is to remark that any spectral measure appearing in the flow of Brownian motion is absolutely continuous up to a trivial jump at the origin. The converse of this statement, that is, the fact that any absolutely continuous measure appears in the set of spectral measures of the flow of Brownian motion, was shown by N. WIENER as a result of his work on the theory of Brownian motion.[1]

On the other hand K. ITO proved that the flow of strictly stationary stochastic process $x(t, \omega)$ of Gaussian type is strongly mixing, provided that the correlation function

$$\varphi(t)=\int x(t+s, \omega)\, x(s, \omega)\, d\omega=\int_{-\infty}^{\infty} e^{it\lambda}\, dF(\lambda)$$

has the property that $\lim_{t\to\infty}\varphi(t)=0$.[2] But as is well-known, there exists a positive definite function $\varphi(t)$ with a singular spectral measure $F(\lambda)$, having the property that $\lim_{t\to\infty}\varphi(t)=0$. Hence K. ITO's result shows the existence of a strongly mixing flow which contains a singular spectral measure.

So far as the author knows, it has been an open question, whether the spectral types of strongly mixing flows are unique or not. Our remark, combined with the result of K. ITO, shows that this question is answered

1) H. WIENER, Generalized harmonic analysis, Acta Math. 55 (1930) p. 214.
E. HOPF, Ergodentheorie, Berlin, 1937, p 61.
In fact, for this purpose, it is sufficient to consider the invariant subspace of $L_2(\Omega)$, consisting of the functions of the form; $\xi^*(\omega)=\int_{-\infty}^{\infty}\xi(t)\, dx(t, \omega)$, where $\xi(t) \in L_2(-\infty, \infty)$. Ω is the measure space of Brownian motion, and the integral is in the sense of N. WIENER.

2) K. ITO, On the ergodicity of a certain stationary process, Proc. Imp. Acad., Tokyo, Vol. 20 (1944), p. 54.

negatively.

This note is a result of conversations between S. KAKUTANI and the author, but it is needless to say that Professor KAKUTANI is not responsible for this note.

2. By Brownian motion $\{x(A, \omega), \omega \varepsilon \Omega\}$, we understand a temporally homogeneous differential process $x(A, \omega)$, having a Gaussian distribution:

$$(1) \qquad Pr\left\{\omega \mid a < x(A, \omega) < b\right\} = \frac{1}{\sqrt{2.\pi \mid A \mid}} \int_a^b e^{-\frac{x^2}{2\mid A\mid}} dx,$$

where A is an interval[3]) on the infinite line, $\mid A \mid$ denotes the length of A; $x(A, \omega)$ in (1) is to be understood as the value of the interval function $x(A, \omega)$ for the interval A. It is a well-known fact that the flow T_t on Ω, defined by

$$(2) \qquad x(A, T_t\omega) = x(A + t, \omega),$$

is strongly mixing, where $A + t$ denotes the translation of A by t.

Let $\mathfrak{M}$ be the set of functions belonging to $L_2(\Omega)$, of which the means are zero: $f(\omega) \varepsilon \mathfrak{M} \rightleftarrows \int f(\omega)\, d\omega = 0$. Obviously $\mathfrak{M}$ is the orthocomplement of the trivial one-dimensional subspace of constant functions in $L_2(\Omega)$.

THEOREM. *For any $f(\omega) \varepsilon \mathfrak{M}$, the positive definite function $\varphi(t) = \int f(T_t\omega)\overline{f(\omega)}\, d\omega$ has an absolutely continuous spectral measure.*

PROOF: We denote by $\mathfrak{F}$ the subset of functions of $\mathfrak{M}$, which are defined by conditions depending only on finite sums of intervals. We shall first show that the assertion of the Theorem is true for $f(\omega) \varepsilon \mathfrak{F}$. For any $f(\omega) \varepsilon \mathfrak{F}$, there exists a positive number C such that, if $t > C$, $f(T_t\omega)$ and $f(\omega)$ are stochastically independent. Therefore we have

$$\begin{aligned}\varphi(t) = \int f(T_t\omega)\overline{f(\omega)}\, d\omega &= \int f(T_t\omega)\, d\omega \int \overline{f(\omega)}\, d\omega \\ &= 0, \quad \text{for} \quad \mid t \mid > C.\end{aligned}$$

It is obvious that

$$(3) \qquad g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixz}\varphi(z)\, dz = \frac{1}{2\pi}\int_{-C}^{C} e^{-ixz}\varphi(z)\, dz$$

[3]) We mean by "interval" always finite interval.

is a positive function, since $\varphi(z)$ is a positive definite function. Since $\varphi(z)$ is a uniformly continuous function, we obtain

$$\lim_{A\to\infty}\int_{-A}^{A} e^{itx}\, g(x)\, dx = \lim_{A\to\infty}\frac{1}{2\pi}\int_{-C}^{C}\left(\int_{-A}^{A} e^{i(t-z)x}\, dx\right)\varphi(z)\, dz$$

$$= \lim_{A\to\infty}\frac{1}{\pi}\int_{-C}^{C}\frac{\sin A(t-z)}{t-z}\,\varphi(z)\, dz = \varphi(t).$$

This shows that

$$(4)\qquad \varphi(t) = \int_{-\infty}^{\infty} e^{itx}\, g(x)\, dx.$$

In particular, putting $t=0$ in (4), we have

$$(5)\qquad \int_{-\infty}^{\infty} g(x)\, dx = \varphi(0) = 1.$$

By (3), (4) and (5), we conclude that $\varphi(t)$ has the absolutely continuous measure $\int_{-\infty}^{\lambda} g(x)\, dx$.

Let us denote by $E(\lambda)$ the resolution of identity of the one-parameter unitary group acting on $L_2(\Omega)$, which corresponds to the flow T_t on Ω. Then to a Borel set B on the infinite line, there corresponds a projective operator $E(B)$, and for any $f(\omega) \varepsilon L_2(\Omega)$, $(E(B)f, f)$ is the spectral measure of the positive definite function $\varphi(t) = \int f(T_t\,\omega)\,\overline{f(\omega)}\, d\omega$:

$$\varphi(t) = \int f(T_t\,\omega)\,\overline{f(\omega)}\, d\omega = \int_{-\infty}^{\infty} e^{it\lambda}\,(E(d\lambda)f, f).$$

Let $f(\omega)$ be a function of $\mathfrak{M}$. Since $\mathfrak{F}$ is dense in $\mathfrak{M}$, there exists a sequence of functions $f_n(\omega) \varepsilon \mathfrak{F}$, such that $\| f_n - f \| \to 0$, as $n\to\infty$. Therefore for any Borel set B, we have

$$(6)\qquad (E(B)f, f) = \lim_{n\to\infty}(E(B)f_n, f_n)$$

Since we have shown that, for any $f_n \varepsilon \mathfrak{F}$, $(E(B)f_n, f_n)$ is an absolutely continuous measure, it follows from (6) immediately that $(E(B)f, f)$ is an absolutely continuous measure. This completes the proof of the Theorem.

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