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Osaka Journal of Mathematics. 45(1) P.159-P.171

2008-03

https://doi.org/10.18910/9156

10.18910/9156
CARTAN MATRICES OF SYMMETRIC ALGEBRAS HAVING GENERALIZED STANDARD STABLE TUBES

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(Received June 12, 2006, revised February 26, 2007)

Abstract

We prove that the Cartan matrices of the symmetric artin algebras whose Auslander-Reiten quivers admit a generalized standard stable tube are singular and derive some consequences.

Introduction

In the paper, by an algebra is meant an artin algebra (associative, with an identity) over a commutative artin ring \( R \). For an algebra \( A \), we denote by \( \text{mod} A \) the category of finitely generated right \( A \)-modules, by \( \text{rad}(\text{mod} A) \) the Jacobson radical of \( \text{mod} A \), and by \( \text{rad}^\infty(\text{mod} A) \) the intersection of all powers \( \text{rad}^i(\text{mod} A) \), for \( i \geq 1 \), of \( \text{rad}(\text{mod} A) \). For an algebra \( A \), we denote by \( D: \text{mod} A \rightarrow \text{mod} A^{\text{op}} \) the standard duality \( \text{Hom}_R(\_ , I) \), where \( I \) is a minimal injective cogenerator in \( \text{mod} R \). Further, we denote by \( \Gamma_A \) the Auslander-Reiten quiver of \( A \), and by \( \tau_A \) the Auslander-Reiten translation \( D \text{Tr} \). We will not distinguish between an indecomposable module from \( \text{mod} A \) and the vertex of \( \Gamma_A \) corresponding to it. A component in \( \Gamma_A \) of the form \( \mathbb{Z}A_{\infty}/(\tau^r) \), \( r \geq 1 \), is called a stable tube of rank \( r \). Therefore, a stable tube of rank \( r \) in \( \Gamma_A \) is an infinite component consisting of \( \tau_A \)-periodic indecomposable \( A \)-modules having period \( r \). An algebra \( A \) is called selfinjective if the projective \( A \)-modules are injective. A distinguished class of selfinjective algebras is formed by the symmetric algebras for which \( A \cong D(A) \) as \( A \)-\( A \)-bimodules. We also mention that, for an arbitrary algebra \( B \), the trivial extension \( T(B) = B \ltimes D(B) \) of \( B \) by the injective cogenerator \( D(B) \) is a symmetric algebra.

The Auslander-Reiten quiver is an important combinatorial and homological invariant of the module category \( \text{mod} A \) of an algebra \( A \), and frequently we may recover \( A \) and \( \text{mod} A \) from the behaviour of distinguished components of \( \Gamma_A \) in the category \( \text{mod} A \). Following [21], a component \( \mathcal{C} \) of \( \Gamma_A \) is called generalized standard if \( \text{rad}^\infty(X,Y) = 0 \) for all modules \( X \) and \( Y \) in \( \mathcal{C} \). In the paper, we are concerned with the problem of describing the structure of selfinjective algebras \( A \) for which the Auslander-Reiten quiver \( \Gamma_A \) admits a generalized standard component, raised in [22, Problem 7].

2000 Mathematics Subject Classification. Primary 16D50, 16G70; Secondary 16D40, 16G10.

The research of the first two named authors supported by the Polish Scientific Grant KBN No. 1 P03A 018 27.
The structure of all selfinjective algebras $A$ with $\Gamma_A$ having a nonperiodic generalized standard component has been described completely in [27], [28]. On the other hand, the structure of selfinjective algebras $A$ with $\Gamma_A$ having a periodic generalized standard component is still only emerging (see [6], [7], [8], [9], [13], [15], [16], [25], [26] for some recent results in this direction).

In this paper, we are interested in the structure of symmetric algebras for which the Auslander-Reiten quiver admits a generalized standard stable tube. This is a wide class of symmetric algebras containing the trivial extensions $T(B)$ of all quasitilted algebras $B$ of canonical type over an algebraically closed field (see [1], [14], [15], [17]). We also note that an arbitrary basic finite dimensional algebra $B$ over a field is a factor algebra of a symmetric algebra $A$ with $\Gamma_A$ having a generalized standard stable tube (see [25]).

The paper is organized as follows. In the preliminary Section 1 we present some facts on generalized standard stable tubes needed in the proof of the main result. In Section 2 we prove the main result and derive some consequences. In Section 3 we present some relevant examples illustrating our considerations.

For basic background on the representation theory of algebras applied here we refer to [2], [3], [4].

1. Stable tubes

Let $A$ be an algebra. A module $X$ in mod $A$ is said to be a brick if $\text{End}_A(X)$ is a division algebra. Two modules $X$ and $Y$ in mod $A$ with $\text{Hom}_A(X, Y) = 0$ and $\text{Hom}_A(Y, X) = 0$ are said to be orthogonal. For a stable tube $T$ of $\Gamma_A$ the unique $\tau_A$-orbit of $T$ formed by the modules having exactly one predecessor and exactly one successor is called the mouth of $T$.

The following characterization of generalized standard stable tubes will be critical for our considerations.

**Theorem 1.1.** Let $A$ be an algebra and $T$ be a stable tube of $\Gamma_A$. The following conditions are equivalent:

(i) $T$ is generalized standard.

(ii) The mouth of $T$ consists of pairwise orthogonal bricks.

Proof. This is a part of the characterization of generalized standard stable tubes given in [21, Corollary 5.3]. In fact, in [21] only the implication (ii) $\Rightarrow$ (i) was proved in details. Because in the proof of our main result the implication (i) $\Rightarrow$ (ii) is essentially needed, we give here its detailed proof (compare the proof of [23, Proposition 3.5]).

Assume $T$ is a generalized standard stable tube in $\Gamma_A$ and let $r$ be the rank of $T$. Denote by $E_1, E_2, \ldots, E_r$ the modules lying on the mouth of $T$. We may assume that $E_i = \tau_A E_{i+1}$ for any $i \in \{1, \ldots, r\}$, where $E_{r+1} = E_1$. Then, for any $i \in \{1, \ldots, r\}$,
we have in $T$ an infinite sectional path

$$E_i = E_i[1] \to E_i[2] \to \cdots \to E_i[j] \to E_i[j+1] \to \cdots$$

called the ray of $T$ starting at the mouth module $E_i$. Observe that every indecomposable module in $T$ is of the form $E_i[j]$, for some $i \in \{1, \ldots, r\}$ and some $j \geq 1$. Moreover, we have in mod $A$ almost split sequences

$$0 \to E_i[1] \to E_i[2] \to E_{i+1}[1] \to 0,$$

$$0 \to E_i[j] \to E_i[j+1] \oplus E_{i+1}[j-1] \to E_{i+1}[j] \to 0,$$

for $i \in \{1, \ldots, r\}$ and $j \geq 2$, where $E_{i+1}[j] = E_i[j]$. Then we may choose irreducible monomorphisms $u_{ij}: E_i[j-1] \to E_i[j]$ and irreducible epimorphisms $p_{ij}: E_i[j] \to E_{i+1}[j-1]$ such that $p_{i+1}u_{ij} \in \text{rad}^3$ in mod $A)$ and $p_{i+1}u_{ij} - u_{i+1}j p_{ij} \in \text{rad}^3$ in mod $A)$ for $i \in \{1, \ldots, r\}$ and $j \geq 2$. Observe also that, for any irreducible morphism $f: X \to Y$ with $X$ and $Y$ from $T$, there are automorphisms $b: X \to X$ and $c: Y \to Y$ such that

$$f^* b + \text{rad}^2(X, Y) = f + \text{rad}^2(X, Y) = cf^* + \text{rad}^2(X, Y),$$

where $f^*: X \to Y$ is the irreducible morphism of the form $u_{ij}$ or $p_{ij}$ chosen above. This follows from the fact that

$$\dim F(X) \text{rad}(X, Y)/\text{rad}^2(X, Y) = 1 \quad \text{and} \quad \dim \text{rad}(X, Y)/\text{rad}^2(X, Y)_{F(Y)} = 1$$

where $F(X) = \text{End}_A(X)/\text{rad}(\text{End}_A(X))$ and $F(Y) = \text{End}_A(Y)/\text{rad}(\text{End}_A(Y))$. We note also that a morphism $f: X \to Y$ between indecomposable modules in mod $A$ is irreducible if and only if $f \in \text{rad}(X, Y) \setminus \text{rad}^2(X, Y)$ (see [3, Proposition V.7.3]). Moreover, for any modules $X$ and $Y$ in mod $A$, there exists an integer $n$ such that $\text{rad}^n(X, Y) = \text{rad}^\infty(X, Y)$ (see [3, Lemma V.7.2]). Therefore, because the stable tube $T$ is generalized standard, any nonisomorphism $g: M \to N$ with $M$ and $N$ from $T$ is of the form $g = g_1 + \cdots + g_l$, where $g_1, \ldots, g_l$, (for some $t \geq 1$) are compositions of irreducible morphisms between indecomposable modules of the tube $T$.

Let $E_i$ and $E_k$, with $i, k \in \{1, \ldots, r\}$, be two modules on the mouth of $T$. We may assume that $i \leq k$, and hence $E_i = \tau_A^i E_k$ for $s = k - i \geq 0$. We will show that $\text{rad}(E_i, E_k) = 0$. Observe that any nontrivial path in $T$ from $E_i$ to $E_k$ is of length $2s + 2rl$ for some $l \geq 0$. In particular, we have

$$\text{rad}(E_i, E_k) = \text{rad}^{2s}(E_i, E_k), \quad \text{if} \quad i \neq k,$$

$$\text{rad}(E_i, E_k) = \text{rad}^{2s}(E_i, E_k), \quad \text{if} \quad i = k,$$

and

$$\text{rad}^{2s+2rl+1}(E_i, E_k) = \text{rad}^{2s+2rl+1}(E_i, E_k), \quad \text{for any} \quad l \geq 0.$$
Moreover, we have \( \operatorname{rad}^m(E_i, E_k) = \operatorname{rad}^\infty(E_i, E_k) = 0 \) for some \( m \geq 1 \). Therefore, it is enough to show that \( \operatorname{rad}^p(E_i, E_k) \subseteq \operatorname{rad}^{p+1}(E_i, E_k) \) for any \( p \in \{1, \ldots, m - 1\} \). Take \( p \in \{1, \ldots, m - 1\} \). We may assume that \( p \geq 2s \) (for \( i \neq k \)) or \( p \geq 2r \) (for \( i = k \)). Let \( h \) be a nonzero morphism from \( \operatorname{rad}^p(E_i, E_k) \). Observe that \( \operatorname{rad}^p(\text{mod } A) \) is a left ideal of \( \text{mod } A \) generated by the compositions of \( p \) irreducible morphisms in \( \text{mod } A \). Hence \( h = h_1 + \cdots + h_d \), for some \( d \geq 1 \), where each \( h_i \) is the composition \( h_i = h_{1t_1} \cdots h_{2t_2} h_{1t_1} \) of a sequence of irreducible morphisms

\[
E_i = X_{t_11} \xrightarrow{h_{1t_1}} X_{t_21} \xrightarrow{h_{2t_2}} \cdots \xrightarrow{h_{1t_1}} X_{t_q1} \xrightarrow{h_{1t_2}} X_{1q+1} = E_k
\]

with \( q_i \geq p \). Then, for each \( t \in \{1, \ldots, d\} \), there exists \( j_t \in \{2, \ldots, q_t\} \) such that \( X_{t,j_t} = E_t[j_t] \) and \( X_{t,j_t+1} = E_t[j_t+1] \). Then there is an automorphism \( a_t \) of \( E_t = E_t[1] \) such that

\[
h_t + \operatorname{rad}^{p+1}(E_i, E_k) = h_{1t_1} + \cdots h_{1t_2} p_i_{1j_t} u_{i_{1j_t}} \cdots u_{12a} + \operatorname{rad}^{p+1}(E_i, E_k)
\]

\[
= \pm h_{1t_1} + \cdots h_{2t_2} u_{i_{1j_t+1}} p_i_{2u_{1j_t+1}} \cdots u_{12a} + \operatorname{rad}^{p+1}(E_i, E_k)
\]

\[
= 0 + \operatorname{rad}^{p+1}(E_i, E_k),
\]

because \( p_i_{1j_t} u_{i_{1j_t}} \in \operatorname{rad}^3(\text{mod } A) \). Hence \( h_t \in \operatorname{rad}^{p+1}(E_i, E_k) \). This shows that \( h = h_1 + \cdots + h_d \in \operatorname{rad}^{p+1}(E_i, E_k) \). Hence, by induction on \( p \), we conclude that \( \operatorname{rad}(E_i, E_k) = \operatorname{rad}^m(E_i, E_k) = \operatorname{rad}^\infty(E_i, E_k) = 0 \). Therefore, the mouth of \( T \) consists of pairwise orthogonal bricks. Hence (i) implies (ii). \( \square \)

We mention also that if \( R \) is an algebraically closed field \( K \), then a stable tube \( T \) of \( \Gamma_A \) is generalized standard if and only if \( T \) is standard in the sense of [19] (see [24, Lemma 1.3]), that is, the full subcategory of \( \text{mod } A \) given by the modules of \( T \) is equivalent to the mesh category \( K(T) \) of \( T \).

We need also the following fact.

**Lemma 1.2.** Let \( A \) be a selfinjective algebra and \( T \) be a stable tube of \( \Gamma_A \). Then the mouth of \( T \) contains at least one nonsimple module.

**Proof.** We may assume that \( A \) is an indecomposable algebra. Let \( r \) be the rank of \( T \) and \( E_1, \ldots, E_r \) be the modules lying on the mouth of \( T \) with \( E_i = \tau^A E_{i+1} \) for \( i \in \{1, \ldots, r\} \) and \( E_{r+1} = E_1 \). Assume that the modules \( E_1, \ldots, E_r \) are simple. For each \( i \in \{1, \ldots, r\} \), denote by \( P_i \) the projective cover of \( E_i \) in \( \text{mod } A \). Consider the syzygy functor \( \Omega_A \colon \text{mod } A \to \text{mod } A \) on the stable category \( \text{mod } A \) of \( \text{mod } A \), which assigns to any object \( M \) of \( \text{mod } A \) the kernel \( \Omega_A(M) \) of the projective cover \( P(M) \to M \) of \( M \) in \( \text{mod } A \). Then \( \Omega_A \) induces an automorphism of the stable Auslander-Reiten quiver \( \Gamma^\wedge_A \) of \( A \) (see [3, Corollary X.1.10]). Hence the syzygies \( \Omega_A(E_i) = \operatorname{rad} P_1, \ldots, \Omega_A(E_r) = \operatorname{rad} P_r \) of the simple modules \( E_1, \ldots, E_r \) form the mouth of a stable tube of \( \Gamma^\wedge_A \) with
rad \( P_i = \tau_A(\text{rad} \ P_{i+1}) \) for \( i \in \{1, \ldots, r\} \), and \( P_{r+1} = P_1 \). Applying now the shape of almost split sequences with the middle term having projective-injective direct summand (see [3, Proposition V.5.5]), we conclude that \( P_i/\text{soc} \ P_i \cong \text{rad} \ P_{i+1} \) for all \( i \in \{1, \ldots, r\} \). This implies that \( P_1, \ldots, P_r \) are uniserial modules with the simple composition factors from the family \( E_1, \ldots, E_r \) of simple modules. Therefore, \( A \) is a selfinjective Nakayama algebra and \( P_1, \ldots, P_{r} \) is a complete set of pairwise nonisomorphic indecomposable projective \( A \)-module. In particular, \( A \) is of finite representation type. But this contradicts the fact that \( T \) is an infinite component of \( \Gamma_A \).

\[ \square \]

2. The main result

Let \( A \) be an algebra and \( P_1, P_2, \ldots, P_n \) be a complete set of representatives of isomorphism classes of indecomposable projective \( A \)-modules. Then \( S_1 = P_1/\text{rad} \ P_1, S_2 = P_2/\text{rad} \ P_2, \ldots, S_n = P_n/\text{rad} \ P_n \) is a complete set of representatives of isomorphism classes of simple \( A \)-modules. For a module \( M \) in mod \( A \), denote by \( [M] \) the image of \( M \) in the Grothendieck group \( K_0(A) \) of \( A \). Then \( \{[S_1], [S_2], \ldots, [S_n]\} \) is a \( \mathbb{Z} \)-basis of \( K_0(A) \). Moreover, if \( M \) is a module in mod \( A \) and \( [M] = m_1[S_1] + m_2[S_2] + \cdots + m_n[S_n] \) with \( m_1, m_2, \ldots, m_n \in \mathbb{Z} \), then \( m_1, m_2, \ldots, m_n \) are the multiplicities of the simple modules \( S_1, S_2, \ldots, S_n \) as composition factors of \( M \). For \( i, j \in \{1, \ldots, n\} \), denote by \( c_{ij} \) the multiplicity of the simple module \( S_j \) as a composition factor of \( P_j \). Then the integral \( n \times n \)-matrix \( C_A = (c_{ij}) \) is called the Cartan matrix of \( A \) (see [4, (1.7.9)]).

**Theorem 2.1.** Let \( A \) be a symmetric algebra such that the Auslander-Reiten quiver \( \Gamma_A \) admits a generalized standard stable tube. Then the Cartan matrix \( C_A \) of \( A \) is singular.

**Proof.** Let \( T \) be a generalized standard stable tube in \( \Gamma_A \) and \( r \) be the rank of \( T \). Let \( E_1, \ldots, E_r \) be the modules lying on the mouth of \( T \) with \( E_i = \tau_A E_{i+1} \), for \( i \in \{1, \ldots, r\} \), \( E_{r+1} = E_1 \). Take the module \( E = E_1 \oplus \cdots \oplus E_r \). Observe that \( E = \tau_A E \).

Since \( A \) is a symmetric algebra, we have \( \tau_A E \cong \Omega_A^2 E \) (see [3, Proposition IV.3.8]). Therefore, we obtain an exact sequence

\[ 0 \to E \to P_1(E) \to P_0(E) \to E \to 0 \]

where \( P_0(E) \) is the projective cover of \( E \) and \( P_1(E) \) is the injective envelope of \( E \) in mod \( A \). This leads to the equality

\[ [P_1(E)] = [P_0(E)] \]

in the Grothendieck group \( K_0(A) \).

Let \( P_1, P_2, \ldots, P_n \) be a complete set of pairwise nonisomorphic indecomposable projective \( A \)-modules, and

\[ P_1(E) = m_1 P_1 \oplus \cdots \oplus m_n P_n, \quad P_0(E) = s_1 P_1 \oplus \cdots \oplus s_n P_n \]
be decompositions of $P_1(E)$ and $P_0(E)$ into direct sums of indecomposable modules, where, for a module $M$ and $m \geq 0$, $mM$ denotes the direct sum of $m$ copies of $M$. Therefore, we obtain the equality

$$m_1[P_1] + \cdots + m_n[P_n] = s_1[P_1] + \cdots + s_n[P_n]$$

in $K_0(A)$.

Assume now that the Cartan matrix $C_A = (c_{ij})$ is nonsingular. Let $S_1, S_2, \ldots, S_n$ be the simple $A$-modules with $S_i = P_i / \text{rad } P_i$ for any $i \in \{1, \ldots, n\}$. Observe that

$$[P_j] = c_{1j}[S_1] + c_{2j}[S_2] + \cdots + c_{nj}[S_n]$$

for any $j \in \{1, \ldots, n\}$. Because $C_A$ is nonsingular, the columns

$$C_j = [c_{1j}, c_{2j}, \ldots, c_{nj}]^t, \quad j = 1, \ldots, n,$$

of $C_A$ are independent in $\mathbb{Z}^n$, and consequently we obtain $m_1 = s_1, m_2 = s_2, \ldots, m_n = s_n$. Therefore, $P_1(E) \cong P_0(E)$ in mod $A$. This implies that $\text{soc } E \cong E / \text{rad } E = \text{top } E$. It follows from Lemma 1.2 that there is $i \in \{1, \ldots, r\}$ such that $E_i$ is not simple. Then $\text{rad } E_i \neq 0$ and let $S$ be a simple direct summand of $\text{top } E_i$. Because $\text{soc } E \cong \text{top } E$, there exists $k \in \{1, \ldots, r\}$ such that $S$ is a direct summand of $\text{soc } E_k$. Then the composed morphism

$$E_i \rightarrow \text{top } E_i \rightarrow S \rightarrow \text{soc } E_k \rightarrow E_k$$

is a nonzero morphism in $\text{rad}(E_i, E_k)$. On the other hand, by Theorem 1.1, the generalized standardness of the stable tube $T$ implies that $E_1, \ldots, E_r$ are pairwise orthogonal bricks, or equivalently $\text{rad}(E, E) = 0$. Therefore, the Cartan matrix $C_A$ is singular. □

By the remarkable theorem due to R. Brauer (see [4, Theorem 5.4.3]) the determinant of the Cartan matrix of a group algebra $KG$ of a finite group $G$ over a field $K$ of characteristic $p > 0$ is a power of $p$. Therefore, we obtain the following fact.

**Corollary 2.2.** Let $K$ be a field of characteristic $p > 0$, $G$ a finite group, $A = KG$ and $T$ a stable tube of $\Gamma_A$. Then $T$ is not generalized standard.

We note that a group algebra $KG$ of infinite representation type has many stable tubes (see [10], [11]).

Let $B$ be an algebra with nonsingular Cartan matrix $C_B$. We may then consider the Coxeter matrix $\Phi_B = -C_B^{-1}C_B^{-1}$ of $B$. We note that the Cartan matrix of any algebra of finite global dimension is nonsingular (see [2, Proposition II.3.10] or [19, p.70]).
**Corollary 2.3.** Let $B$ be an algebra with nonsingular Cartan matrix $C_B$ and $T(B) = B \ltimes D(B)$ be the trivial extension of $B$ by $D(B)$. Assume that the Auslander-Reiten quiver $\Gamma_{T(B)}$ of $T(B)$ admits a generalized standard stable tube. Then 1 is an eigenvalue of the Coxeter matrix $\Phi_B$ of $B$.

Proof. Since $T(B)$ is a symmetric algebra, applying Theorem 2.1, we conclude that the Cartan matrix $C_{T(B)}$ of $T(B)$ is singular. On the other hand, it has been observed in [12, Proposition 8.2] that

$$\det C_{T(B)} = (-1)^n \det C_B \det(\Phi_B - I_n),$$

where $n$ is the rank of $K_0(A)$ and $I_n$ is the identity matrix of degree $n$. Hence, we obtain $\det(\Phi_B - I_n) = 0$, and consequently 1 is an eigenvalue of $\Phi_B$. 

The problem of describing the selfinjective algebras with the Auslander-Reiten quiver having a generalized standard stable tube is strongly related to the problem (see [22, Problem 3]) of describing the algebras with the Auslander-Reiten quiver having a faithful generalized standard stable tube. Namely, if $T$ is a generalized standard stable tube of an Auslander-Reiten quiver $\Gamma_A$ and $\text{ann} T$ is the annihilator of $T$ in $A$ (the intersection of the annihilators of all modules in $T$) then $T$ is a faithful generalized standard stable tube of $\Gamma_A/\text{ann} T$. We also note that all modules in a faithful generalized standard stable tube $T$ of an Auslander-Reiten quiver $\Gamma_A$ have the projective dimension one and the injective dimension one (see [21, Lemma 5.9]).

### 3. Examples

The aim of this section is to present some examples relevant to considerations in Section 2.

We first exhibit an algebra $C$ having singular Cartan matrix and a generalized standard stable tube in the Auslander-Reiten quiver $\Gamma_{T(C)}$ of its trivial extension $T(C)$.

**Example 3.1.** Let $K$ be an algebraically closed field. Consider the bound quiver algebra $C = K Q / I$ where $Q$ is the quiver

![Diagram of quiver](image)
and $I$ is the ideal in the path algebra $KQ$ of $Q$ generated by the elements $\gamma\alpha, \alpha\gamma, \beta\sigma, \sigma\beta, \alpha\sigma - \beta\gamma, \sigma\alpha - \gamma\beta$, and $\epsilon\sigma\sigma\sigma = \xi\gamma\beta\eta$. Then $C$ is a generalized canonical algebra in the sense of [24, Section 2]. Indeed, let $B_0$ be the path algebra $K\Delta^{(0)}$ of the quiver

$$\Delta^{(0)}: 4 \xrightarrow{\delta} 1$$

and $B_1 = K\Delta^{(1)}/I^{(1)}$, where $\Delta^{(1)}$ is the quiver obtained from $Q$ by deleting the arrow $\delta$ and $I^{(1)}$ is the ideal in $K\Delta^{(1)}$ generated by the same elements as $I$. Consider also the path algebra $H = K\Sigma$, where $\Sigma$ is the Kronecker quiver

$$2 \xrightarrow{\alpha} 3 \xleftarrow{\beta} 3,$$

Then the trivial extension algebra $B = T(H)$ of $H$ is the bound quiver algebra $K\Gamma/J$ where $\Gamma$ is the quiver

$$
\begin{array}{c}
2 \\
\alpha \quad \beta
\end{array}
\quad
\begin{array}{c}
3
\end{array}
\quad
\begin{array}{c}
\gamma
\end{array}

and $J$ is the ideal in $K\Gamma$ generated by $\gamma\alpha, \alpha\gamma, \beta\sigma, \sigma\beta, \alpha\sigma - \beta\gamma, \sigma\alpha - \gamma\beta$. Following notation of [24, Corollary 2.5] consider the one-point extension

$$B' = B[M] = \begin{bmatrix}
K & M \\
0 & B
\end{bmatrix}$$

of $B$ by the faithful $B$-module $M = B_B$. Then $B'$ is a basic connected algebra having an indecomposable projective faithful module $Q$ with $\text{rad} Q \cong M$. Then the algebra $B_1$ (defined above) is the one-point coextension

$$B_1 = B'' = \begin{bmatrix}
B' \\
0
\end{bmatrix} \begin{bmatrix}
D(Q) \\
K
\end{bmatrix},$$

the indecomposable projective $B_1$-module $P(4)$ at the vertex 4 is faithful and coincides with the indecomposable injective $B_1$-module $I(1)$ at the vertex 1. Then, by [24, Corollary 2.5], $C$ is the generalized canonical algebra obtained from the algebras $B_0$ and $B_1$ by gluing their bound quivers at the vertices 1 and 4. Then it follows from [24, Theorem 2.1] that $\Gamma_C$ admits an infinite family of pairwise orthogonal faithful (generalized) standard stable tubes. Then, by [16, Corollary 1.3], $\Gamma_{\Gamma_C}$ also admits an infinite family of pairwise orthogonal (generalized) standard stable tubes. Observe also that the
Cartan matrix of $C$ is of the form
\[
\begin{bmatrix}
1 & 4 & 4 & 2 \\
0 & 2 & 2 & 4 \\
0 & 2 & 2 & 4 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
in the natural ordering $P(1), P(2), P(3), P(4)$ of indecomposable projective $C$-modules, and is singular. We also mention that $C$ is of infinite global dimension, because the simple modules $S(2)$ and $S(3)$ at the vertices 2 and 3 are of infinite projective dimension.

**Example 3.2.** Let $B$ be a concealed generalized canonical algebra over an algebraically closed field $K$, introduced in [16, Section 3]. Recall that $B$ is an algebra of the form $\text{End}_C(T)$, where $C$ is a generalized canonical algebra, defined in [24, Section 2], and $T$ is a tilting $C$-module cogenerated by the canonical family $T^C$ of pairwise orthogonal faithful (generalized) standard stable tubes of $\Gamma_C$. Then, by [16, Theorem 1.1], the Auslander-Reiten quiver $\Gamma_B$ of $B$ admits a canonical family $T^B$ of pairwise orthogonal faithful (generalized) standard stable tubes. Consider a selfinjective algebra $A$ of the form $A = \hat{B}/(\varphi v_{\hat{B}})$, where $\hat{B}$ is the repetitive algebra of $B$, $v_{\hat{B}}$ is the Nakayama automorphism of $\hat{B}$ and $\varphi$ is a positive automorphism of $\hat{B}$. We note that the induced Galois covering $\hat{B} \to \hat{B}/(\varphi v_{\hat{B}}) = A$ is a positive Galois covering in the sense of [29]. It has been proved in [16, Theorem 1.2] that the Auslander-Reiten quiver $\Gamma_A$ of $A$ admits an infinite family of pairwise orthogonal (generalized) standard stable tubes. We also note that $A$ is symmetric if and only if $A \cong T(B)$ (see [18, Theorem 2]). Therefore, if $\varphi$ is a strictly positive automorphism of $\hat{B}$, then $A$ is a nonsymmetric selfinjective algebra with $\Gamma_A$ having an infinite family of (generalized) standard stable tubes. We refer to [18], [26] and [29] for more details on selfinjective orbit algebras of repetitive algebras.

Assume now that $B$ is of finite global dimension, and hence the Coxeter matrix $\Phi_B$ is defined. We note that this is the case if the generalized canonical algebra $C$ is of finite global dimension. We know from [16, Section 4] that $\Gamma_B$ admits a faithful generalized standard stable tube $T$ of rank one. Then $\text{Hom}_B(T, B_B) = 0$ and $\text{pd}_B X \leq 1$ for any module $X$ in $T$ (see [21, Lemma 5.9]). Take a module $X$ in $T$. Since $B$ is a basic algebra, $[X]$ is the dimension vector $\dim X$ of $X$, under the canonical identification $K_0(B) = \mathbb{Z}^n$. Applying now [2, Corollary IV.2.9], we conclude that $\dim X = \dim \tau_A X = \Phi_B \dim X$, and consequently $\dim X$ is an eigenvector of $\Phi_B$ with eigenvalue 1.

For each $m \geq 2$, consider the selfinjective orbit algebra $\Lambda^{(m)}_B = \hat{B}/(v_{\hat{B}}^m)$. It follows from [12, Proposition 8.2] that the determinant of $\Lambda^{(m)}_B$ is of the form
\[
(-1)^m (\det C_B)^m \prod_{r=1}^{m} \det(\Phi_B - \varepsilon_r I_n)
\]
where \( \varepsilon_1, \ldots, \varepsilon_m \) are distinct \( m \)-th roots of unity, and \( I_n \) is the identity matrix of degree \( n \).

Therefore, for any concealed generalized canonical algebra \( B \) of finite global dimension, the algebras \( \Lambda_B^{(m)}, m \geq 2 \), are nonsymmetric selfinjective algebras with singular Cartan matrices and the Auslander-Reiten quivers having generalized standard stable tubes.

In the final example we show that there exist nonsymmetric selfinjective algebras with nonsingular Cartan matrices for which the Auslander-Reiten quiver admits a generalized standard stable tube. This will show that the symmetricity assumption is necessary for the validity of Theorem 2.1.

**Example 3.3.** Let \( K \) be an algebraically closed field. Consider the bound quiver algebra \( B = KQ/I \) where \( Q \) is the quiver

![Quiver Diagram](image)

and \( I \) is the ideal in the path algebra \( KQ \) of \( Q \) generated by the elements \( \gamma_3\beta_2\alpha_1, \alpha_3\gamma_1 - \beta_3\beta_1, \beta_3\beta_2 - \alpha_3\gamma_2 \). Then \( B \) is the exceptional (in the sense of [20, (3.2)]) tubular algebra \( B_4 \) of tubular type \( (3, 3, 3) \) presented in [7, Theorem 2.2]. It follows from [7, Section 3] that the Nakayama automorphism \( \gamma_B \) of \( \hat{B} \) admits a 4-root \( \varphi \). For each \( i \geq 1 \), consider the selfinjective orbit algebra \( \Omega_B^{(i)} = \hat{B}/(\varphi^i) \), and note that \( \Omega_B^{(i)} = \hat{B}/(\gamma_B) = T(B) \). It follows from the theory of selfinjective algebras of tubular type (see [6], [17], [20]) that, for \( i \geq 4 \), the Auslander-Reiten quiver of \( \Omega_B^{(i)} \) admits an infinite family of generalized standard stable tubes. On the other hand, from the description of the determinants of Cartan matrices of selfinjective algebras of tubular type given in [5, Theorem], we know that, in the considered case, we have

\[
\det C_{\Omega_B^{(i)}} = \begin{cases} 
6 & \text{if } i \equiv \pm 1 \pmod{6} \\
12 & \text{if } i \equiv \pm 2 \pmod{12} \\
0 & \text{in other case}
\end{cases}.
\]

In particular, taking \( A = \Omega_B^{(5)} \), we conclude that \( A \) is a nonsymmetric selfinjective algebra with \( \det C_A = 6 \) and the Auslander-Reiten quiver \( \Gamma_A \) having an infinite family of
generalized standard stable tubes. In fact, $A$ is the bound quiver algebra $K\Delta/J$ where $\Delta$ is the quiver

and $J$ is the ideal in $K\Delta$ generated by the elements $\gamma_{i+2}\beta_{i+1}\alpha_i$, $\alpha_{i+1}\gamma_i - \beta_{i+1}\beta_i$, for $i \in \{1, \ldots, 5\}$ with $\alpha_6 = \alpha_1$, $\beta_6 = \beta_1$, $\gamma_6 = \gamma_2$. Moreover, the Cartan matrix $C_A$ is of the form (in the canonical numbering of indecomposable projective $A$-modules)

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

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