Title	Cartan matrices of symmetric algebras having generalized standard stable tubes
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Citation	Osaka Journal of Mathematics. 2008, 45(1), p. 159-171
Version Type	VoR
URL	https://doi.org/10.18910/9156
rights	
Note	

Białkowski, J., Skowroński, A. and Yamagata, K.
Osaka J. Math.
45 (2008), 159–171

CARTAN MATRICES OF SYMMETRIC ALGEBRAS HAVING GENERALIZED STANDARD STABLE TUBES

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(Received June 12, 2006, revised February 26, 2007)

Abstract
We prove that the Cartan matrices of the symmetric artin algebras whose Auslander-Reiten quivers admit a generalized standard stable tube are singular and derive some consequences.

Introduction

In the paper, by an algebra is meant an artin algebra (associative, with an identity) over a commutative artin ring R. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, for $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. For an algebra A, we denote by $D\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_R(-, I)$, where I is a minimal injective cogenerator in $\operatorname{mod} R$. Further, we denote by Γ_A the Auslander-Reiten quiver of A, and by τ_A the Auslander-Reiten translation $D\operatorname{Tr}$. We will not distinguish between an indecomposable module from $\operatorname{mod} A$ and the vertex of Γ_A corresponding to it. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_{\infty}/(\tau^r)$, $r \geq 1$, is called a *stable tube* of rank r. Therefore, a stable tube of rank r in Γ_A is an infinite component consisting of τ_A-periodic indecomposable A-modules having period r. An algebra A is called *selfinjective* if the projective A-modules are injective. A distinguished class of selfinjective algebras is formed by the *symmetric algebras* for which $A \cong D(A)$ as A-A-bimodules. We also mention that, for an arbitrary algebra B, the *trivial extension* $\mathrm{T}(B) = B \ltimes D(B)$ of B by the injective cogenerator $D(B)$ is a symmetric algebra.

The Auslander-Reiten quiver is an important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A, and frequently we may recover A and $\operatorname{mod} A$ from the behaviour of distinguished components of Γ_A in the category $\operatorname{mod} A$. Following [21], a component $\mathscr{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}^{\infty}(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$. In the paper, we are concerned with the problem of describing the structure of selfinjective algebras A for which the Auslander-Reiten quiver Γ_A admits a generalized standard component, raised in [22, Problem 7].

2000 Mathematics Subject Classification. Primary 16D50, 16G70; Secondary 16D40, 16G10.
The research of the first two named authors supported by the Polish Scientific Grant KBN No. 1 P03A 018 27.

The structure of all selfinjective algebras A with Γ_A having a nonperiodic generalized standard component has been described completely in [27], [28]. On the other hand, the structure of selfinjective algebras A with Γ_A having a periodic generalized standard component is still only emerging (see [6], [7], [8], [9], [13], [15], [16], [25], [26] for some recent results in this direction).

In this paper, we are interested in the structure of symmetric algebras for which the Auslander-Reiten quiver admits a generalized standard stable tube. This is a wide class of symmetric algebras containing the trivial extensions $\mathrm{T}(B)$ of all quasitilted algebras B of canonical type over an algebraically closed field (see [1], [14], [15], [17]). We also note that an arbitrary basic finite dimensional algebra B over a field is a factor algebra of a symmetric algebra A with Γ_A having a generalized standard stable tube (see [25]).

The paper is organized as follows. In the preliminary Section 1 we present some facts on generalized standard stable tubes needed in the proof of the main result. In Section 2 we prove the main result and derive some consequences. In Section 3 we present some relevant examples illustrating our considerations.

For basic background on the representation theory of algebras applied here we refer to [2], [3], [4].

1. Stable tubes

Let A be an algebra. A module X in $\operatorname{mod} A$ is said to be a *brick* if $\operatorname{End}_A(X)$ is a division algebra. Two modules X and Y in $\operatorname{mod} A$ with $\operatorname{Hom}_A(X, Y) = 0$ and $\operatorname{Hom}_A(Y, X) = 0$ are said to be *orthogonal*. For a stable tube $\mathcal{T}$ of Γ_A the unique τ_A-orbit of $\mathcal{T}$ formed by the modules having exactly one predecessor and exactly one successor is called the *mouth* of $\mathcal{T}$.

The following characterization of generalized standard stable tubes will be critical for our considerations.

Theorem 1.1. *Let A be an algebra and $\mathcal{T}$ be a stable tube of Γ_A. The following conditions are equivalent*:

(i) *$\mathcal{T}$ is generalized standard.*

(ii) *The mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks.*

Proof. This is a part of the characterization of generalized standard stable tubes given in [21, Corollary 5.3]. In fact, in [21] only the implication (ii) $\Rightarrow$ (i) was proved in details. Because in the proof of our main result the implication (i) $\Rightarrow$ (ii) is essentially needed, we give here its detailed proof (compare the proof of [23, Proposition 3.5]).

Assume $\mathcal{T}$ is a generalized standard stable tube in Γ_A and let r be the rank of $\mathcal{T}$. Denote by $E_1, E_2, \dots, E_r$ the modules lying on the mouth of $\mathcal{T}$. We may assume that $E_i = \tau_A E_{i+1}$ for any $i \in \{1, \dots, r\}$, where $E_{r+1} = E_1$. Then, for any $i \in \{1, \dots, r\}$, we have in $\mathcal{T}$ an infinite sectional path

$$E_i = E_i[1] \to E_i[2] \to \cdots \to E_i[j] \to E_i[j+1] \to \cdots$$

called the ray of $\mathcal{T}$ starting at the mouth module E_i. Observe that every indecomposable module in $\mathcal{T}$ is of the form $E_i[j]$, for some $i \in \{1, \ldots, r\}$ and some $j \geq 1$. Moreover, we have in $\operatorname{mod} A$ almost split sequences

$$0 \to E_i[1] \to E_i[2] \to E_{i+1}[1] \to 0,$$
$$0 \to E_i[j] \to E_i[j+1] \oplus E_{i+1}[j-1] \to E_{i+1}[j] \to 0,$$

for $i \in \{1, \ldots, r\}$ and $j \geq 2$, where $E_{r+1}[j] = E_1[j]$. Then we may choose irreducible monomorphisms $u_{ij} \colon E_i[j-1] \to E_i[j]$ and irreducible epimorphisms $p_{ij} \colon E_i[j] \to E_{i+1}[j-1]$ such that $p_{i2}u_{i2} \in \operatorname{rad}^3(\operatorname{mod} A)$ and $p_{i\,j+1}u_{i\,j+1} - u_{i+1\,j}p_{ij} \in \operatorname{rad}^3(\operatorname{mod} A)$ for $i \in \{1, \ldots, r\}$ and $j \geq 2$. Observe also that, for any irreducible morphism $f \colon X \to Y$ with X and Y from $\mathcal{T}$, there are automorphisms $b \colon X \to X$ and $c \colon Y \to Y$ such that

$$f^*b + \operatorname{rad}^2(X, Y) = f + \operatorname{rad}^2(X, Y) = cf^* + \operatorname{rad}^2(X, Y),$$

where $f^* \colon X \to Y$ is the irreducible morphism of the form u_{ij} or p_{ij} chosen above. This follows from the fact that

$$\dim_{F(X)} \operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y) = 1 \quad \text{and} \quad \dim \operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y)_{F(Y)} = 1$$

where $F(X) = \operatorname{End}_A(X)/\operatorname{rad}(\operatorname{End}_A(X))$ and $F(Y) = \operatorname{End}_A(Y)/\operatorname{rad}(\operatorname{End}_A(Y))$. We note also that a morphism $f \colon X \to Y$ between indecomposable modules in $\operatorname{mod} A$ is irreducible if and only if $f \in \operatorname{rad}(X, Y) \setminus \operatorname{rad}^2(X, Y)$ (see [3, Proposition V.7.3]). Moreover, for any modules X and Y in $\operatorname{mod} A$, there exists an integer n such that $\operatorname{rad}^n(X, Y) = \operatorname{rad}^\infty(X, Y)$ (see [3, Lemma V.7.2]). Therefore, because the stable tube $\mathcal{T}$ is generalized standard, any nonisomorphism $g \colon M \to N$ with M and N from $\mathcal{T}$ is of the form $g = g_1 + \cdots + g_t$, where $g_1, \ldots, g_t$ (for some $t \geq 1$) are compositions of irreducible morphisms between indecomposable modules of the tube $\mathcal{T}$.

Let E_i and E_k, with $i, k \in \{1, \ldots, r\}$, be two modules on the mouth of $\mathcal{T}$. We may assume that $i \leq k$, and hence $E_i = \tau_A^s E_k$ for $s = k - i \geq 0$. We will show that $\operatorname{rad}(E_i, E_k) = 0$. Observe that any nontrivial path in $\mathcal{T}$ from E_i to E_k is of length $2s + 2rl$ for some $l \geq 0$. In particular, we have

$$\operatorname{rad}(E_i, E_k) = \operatorname{rad}^{2s}(E_i, E_k), \quad \text{if} \quad i \neq k,$$
$$\operatorname{rad}(E_i, E_k) = \operatorname{rad}^{2r}(E_i, E_k), \quad \text{if} \quad i = k,$$

and

$$\operatorname{rad}^{2s+2rl+1}(E_i, E_k) = \operatorname{rad}^{2s+2r(l+1)}(E_i, E_k), \quad \text{for any} \quad l \geq 0.$$

Moreover, we have $\operatorname{rad}^m(E_i, E_k) = \operatorname{rad}^\infty(E_i, E_k) = 0$ for some $m \geq 1$. Therefore, it is enough to show that $\operatorname{rad}^p(E_i, E_k) \subseteq \operatorname{rad}^{p+1}(E_i, E_k)$ for any $p \in \{1, \dots, m-1\}$. Take $p \in \{1, \dots, m-1\}$. We may assume that $p \geq 2s$ (for $i \neq k$) or $p \geq 2r$ (for $i = k$). Let h be a nonzero morphism from $\operatorname{rad}^p(E_i, E_k)$. Observe that $\operatorname{rad}^p(\operatorname{mod} A)$ is a left ideal of $\operatorname{mod} A$ generated by the compositions of p irreducible morphisms in $\operatorname{mod} A$. Hence $h = h_1 + \cdots + h_d$, for some $d \geq 1$, where each h_t is the composition $h_t = h_{t\,q_t} \cdots h_{t\,2} h_{t\,1}$ of a sequence of irreducible morphisms

$$E_i = X_{t\,1} \xrightarrow{h_{t\,1}} X_{t\,2} \xrightarrow{h_{t\,2}} \cdots \to X_{t\,q_t} \xrightarrow{h_{t\,q_t}} X_{t\,q_t+1} = E_k$$

with $q_t \geq p$. Then, for each $t \in \{1, \dots, d\}$, there exists $j_t \in \{2, \dots, q_t\}$ such that $X_{t\,j_t} = E_i[j_t]$ and $X_{t\,j_t+1} = E_{i+1}[j_t - 1]$. Then there is an automorphism a_i of $E_i = E_i[1]$ such that

$$\begin{aligned} h_t + \operatorname{rad}^{p+1}(E_i, E_k) &= h_{t\,q_t} \cdots h_{t\,j_t+1}\, p_{i\,j_t} u_{i\,j_t} \cdots u_{i\,2} a + \operatorname{rad}^{p+1}(E_i, E_k) \\ &= \pm h_{t\,q_t} \cdots h_{t\,j_t+1} u_{i+1\,j_t-1} \cdots p_{i\,2} u_{i\,2} a + \operatorname{rad}^{p+1}(E_i, E_k) \\ &= 0 + \operatorname{rad}^{p+1}(E_i, E_k), \end{aligned}$$

because $p_{i\,2} u_{i\,2} \in \operatorname{rad}^3(\operatorname{mod} A)$. Hence $h_t \in \operatorname{rad}^{p+1}(E_i, E_k)$. This shows that $h = h_1 + \cdots + h_d \in \operatorname{rad}^{p+1}(E_i, E_k)$. Hence, by induction on p, we conclude that $\operatorname{rad}(E_i, E_k) = \operatorname{rad}^m(E_i, E_k) = \operatorname{rad}^\infty(E_i, E_k) = 0$. Therefore, the mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks. Hence (i) implies (ii). □

We mention also that if R is an algebraically closed field K, then a stable tube $\mathcal{T}$ of Γ_A is generalized standard if and only if $\mathcal{T}$ is standard in the sense of [19] (see [24, Lemma 1.3]), that is, the full subcategory of $\operatorname{mod} A$ given by the modules of $\mathcal{T}$ is equivalent to the mesh category $K(\mathcal{T})$ of $\mathcal{T}$.

We need also the following fact.

Lemma 1.2. *Let A be a selfinjective algebra and $\mathcal{T}$ be a stable tube of Γ_A. Then the mouth of $\mathcal{T}$ contains at least one nonsimple module.*

Proof. We may assume that A is an indecomposable algebra. Let r be the rank of $\mathcal{T}$ and $E_1, \dots, E_r$ be the modules lying on the mouth of $\mathcal{T}$ with $E_i = \tau_A E_{i+1}$ for $i \in \{1, \dots, r\}$ and $E_{r+1} = E_1$. Assume that the modules $E_1, \dots, E_r$ are simple. For each $i \in \{1, \dots, r\}$, denote by P_i the projective cover of E_i in $\operatorname{mod} A$. Consider the syzygy functor Ω_A: $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, A$ on the stable category $\underline{\operatorname{mod}}\, A$ of $\operatorname{mod} A$, which assigns to any object M of $\underline{\operatorname{mod}}\, A$ the kernel $\Omega_A(M)$ of the projective cover $P(M) \to M$ of M in $\operatorname{mod} A$. Then Ω_A induces an automorphism of the stable Auslander-Reiten quiver Γ_A^s of A (see [3, Corollary X.1.10]). Hence the syzygies $\Omega_A(E_1) = \operatorname{rad} P_1, \dots, \Omega_A(E_r) = \operatorname{rad} P_r$ of the simple modules $E_1, \dots, E_r$ form the mouth of a stable tube of Γ_A^s with rad $P_i = \tau_A(\operatorname{rad} P_{i+1})$ for $i \in \{1, \dots, r\}$, and $P_{r+1} = P_1$. Applying now the shape of almost split sequences with the middle term having projective-injective direct summand (see [3, Proposition V.5.5]), we conclude that $P_i/\operatorname{soc} P_i \cong \operatorname{rad} P_{i+1}$ for all $i \in \{1, \dots, r\}$. This implies that $P_1, \dots, P_r$ are uniserial modules with the simple composition factors from the family $E_1, \dots, E_r$ of simple modules. Therefore, A is a selfinjective Nakayama algebra and $P_1, \dots, P_r$ is a complete set of pairwise nonisomorphic indecomposable projective A-module. In particular, A is of finite representation type. But this contradicts the fact that $\mathcal{T}$ is an infinite component of Γ_A. □

2. The main result

Let A be an algebra and $P_1, P_2, \dots, P_n$ be a complete set of representatives of isomorphism classes of indecomposable projective A-modules. Then $S_1 = P_1/\operatorname{rad} P_1$, $S_2 = P_2/\operatorname{rad} P_2, \dots, S_n = P_n/\operatorname{rad} P_n$ is a complete set of representatives of isomorphism classes of simple A-modules. For a module M in $\operatorname{mod} A$, denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Then $[S_1], [S_2], \dots, [S_n]$ is a $\mathbb{Z}$-basis of $K_0(A)$. Moreover, if M is a module in $\operatorname{mod} A$ and $[M] = m_1[S_1] + m_2[S_2] + \cdots + m_n[S_n]$ with $m_1, m_2, \dots, m_n \in \mathbb{Z}$, then $m_1, m_2, \dots, m_n$ are the multiplicities of the simple modules $S_1, S_2, \dots, S_m$ as composition factors of M. For $i, j \in \{1, \dots, n\}$, denote by c_{ij} the multiplicity of the simple module S_i as a composition factor of P_j. Then the integral $n \times n$-matrix $C_A = (c_{ij})$ is called the *Cartan matrix* of A (see [4, (1.7.9)]).

Theorem 2.1. *Let A be a symmetric algebra such that the Auslander-Reiten quiver Γ_A admits a generalized standard stable tube. Then the Cartan matrix C_A of A is singular.*

Proof. Let $\mathcal{T}$ be a generalized standard stable tube in Γ_A and r be the rank of $\mathcal{T}$. Let $E_1, \dots, E_r$ be the modules lying on the mouth of $\mathcal{T}$ with $E_i = \tau_A E_{i+1}$, for $i \in \{1, \dots, r\}$, $E_{r+1} = E_1$. Take the module $E = E_1 \oplus \cdots \oplus E_r$. Observe that $E = \tau_A E$. Since A is a symmetric algebra, we have $\tau_A E \cong \Omega_A^2 E$ (see [3, Proposition IV.3.8]). Therefore, we obtain an exact sequence

$$0 \to E \to P_1(E) \to P_0(E) \to E \to 0$$

where $P_0(E)$ is the projective cover of E and $P_1(E)$ is the injective envelope of E in $\operatorname{mod} A$. This leads to the equality

$$[P_1(E)] = [P_0(E)]$$

in the Grothendieck group $K_0(A)$.

Let $P_1, P_2, \dots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective A-modules, and

$$P_1(E) = m_1 P_1 \oplus \cdots \oplus m_n P_n, \quad P_0(E) = s_1 P_1 \oplus \cdots \oplus s_n P_n$$

be decompositions of $P_1(E)$ and $P_0(E)$ into direct sums of indecomposable modules, where, for a module M and $m \geq 0$, mM denotes the direct sum of m copies of M. Therefore, we obtain the equality

$$m_1[P_1] + \cdots + m_n[P_n] = s_1[P_1] + \cdots + s_n[P_n]$$

in $K_0(A)$.

Assume now that the Cartan matrix $C_A = (c_{ij})$ is nonsingular. Let $S_1, S_2, \ldots, S_n$ be the simple A-modules with $S_i = P_i/\mathrm{rad}\, P_i$ for any $i \in \{1, \ldots, n\}$. Observe that

$$[P_j] = c_{1j}[S_1] + c_{2j}[S_2] + \cdots + c_{nj}[S_n]$$

for any $j \in \{1, \ldots, n\}$. Because C_A is nonsingular, the columns

$$C_j = [c_{1j}, c_{2j}, \ldots, c_{nj}]^t, \quad j = 1, \ldots, n,$$

of C_A are independent in $\mathbb{Z}^n$, and consequently we obtain $m_1 = s_1, m_2 = s_2, \ldots, m_n = s_n$. Therefore, $P_1(E) \cong P_0(E)$ in mod A. This implies that $\mathrm{soc}\, E \cong E/\mathrm{rad}\, E = \mathrm{top}\, E$. It follows from Lemma 1.2 that there is $i \in \{1, \ldots, r\}$ such that E_i is not simple. Then $\mathrm{rad}\, E_i \neq 0$ and let S be a simple direct summand of $\mathrm{top}\, E_i$. Because $\mathrm{soc}\, E \cong \mathrm{top}\, E$, there exists $k \in \{1, \ldots, r\}$ such that S is a direct summand of $\mathrm{soc}\, E_k$. Then the composed morphism

$$E_i \to \mathrm{top}\, E_i \to S \to \mathrm{soc}\, E_k \to E_k$$

is a nonzero morphism in $\mathrm{rad}(E_i, E_k)$. On the other hand, by Theorem 1.1, the generalized standardness of the stable tube $\mathcal{T}$ implies that $E_1, \ldots, E_r$ are pairwise orthogonal bricks, or equivalently $\mathrm{rad}(E, E) = 0$. Therefore, the Cartan matrix C_A is singular. $\square$

By the remarkable theorem due to R. Brauer (see [4, Theorem 5.4.3]) the determinant of the Cartan matrix of a group algebra KG of a finite group G over a field K of characteristic $p > 0$ is a power of p. Therefore, we obtain the following fact.

Corollary 2.2. *Let K be a field of characteristic $p > 0$, G a finite group, $A = KG$ and $\mathcal{T}$ a stable tube of Γ_A. Then $\mathcal{T}$ is not generalized standard.*

We note that a group algebra KG of infinite representation type has many stable tubes (see [10], [11]).

Let B be an algebra with nonsingular Cartan matrix C_B. We may then consider the Coxeter matrix $\Phi_B = -C_B^t C_B^{-1}$ of B. We note that the Cartan matrix of any algebra of finite global dimension is nonsingular (see [2, Proposition II.3.10] or [19, p.70]).

Corollary 2.3. *Let B be an algebra with nonsingular Cartan matrix C_B and $\mathrm{T}(B) = B \ltimes D(B)$ be the trivial extension of B by $D(B)$. Assume that the Auslander-Reiten quiver $\Gamma_{\mathrm{T}(B)}$ of $\mathrm{T}(B)$ admits a generalized standard stable tube. Then 1 is an eigenvalue of the Coxeter matrix Φ_B of B.*

Proof. Since $\mathrm{T}(B)$ is a symmetric algebra, applying Theorem 2.1, we conclude that the Cartan matrix $C_{\mathrm{T}(B)}$ of $\mathrm{T}(B)$ is singular. On the other hand, it has been observed in [12, Proposition 8.2] that

$$\det C_{\mathrm{T}(B)} = (-1)^n \det C_B \det(\Phi_B - I_n),$$

where n is the rank of $K_0(A)$ and I_n is the identity matrix of degree n. Hence, we obtain $\det(\Phi_B - I_n) = 0$, and consequently 1 is an eigenvalue of Φ_B. □

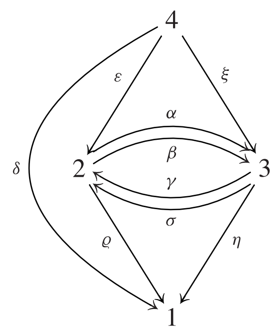
The problem of describing the selfinjective algebras with the Auslander-Reiten quiver having a generalized standard stable tube is strongly related to the problem (see [22, Problem 3]) of describing the algebras with the Auslander-Reiten quiver having a faithful generalized standard stable tube. Namely, if $\mathcal{T}$ is a generalized standard stable tube of an Auslander-Reiten quiver Γ_A and $\operatorname{ann}\mathcal{T}$ is the annihilator of $\mathcal{T}$ in A (the intersection of the annihilators of all modules in $\mathcal{T}$) then $\mathcal{T}$ is a faithful generalized standard stable tube of $\Gamma_{A/\operatorname{ann}\mathscr{C}}$. We also note that all modules in a faithful generalized standard stable tube $\mathcal{T}$ of an Auslander-Reiten quiver Γ_A have the projective dimension one and the injective dimension one (see [21, Lemma 5.9]).

3. Examples

The aim of this section is to present some examples relevant to considerations in Section 2.

We first exhibit an algebra C having singular Cartan matrix and a generalized standard stable tube in the Auslander-Reiten quiver $\Gamma_{\mathrm{T}(C)}$ of its trivial extension $\mathrm{T}(C)$.

EXAMPLE 3.1. Let K be an algebraically closed field. Consider the bound quiver algebra $C = KQ/I$ where Q is the quiver



and I is the ideal in the path algebra KQ of Q generated by the elements $\gamma\alpha$, $\alpha\gamma$, $\beta\sigma$, $\sigma\beta$, $\alpha\sigma - \beta\gamma$, $\sigma\alpha - \gamma\beta$, and $\varepsilon\alpha\sigma\varrho - \xi\gamma\beta\eta$. Then C is a generalized canonical algebra in the sense of [24, Section 2]. Indeed, let B_0 be the path algebra $K\Delta^{(0)}$ of the quiver

$$\Delta^{(0)}\colon 4 \xrightarrow{\ \delta\ } 1$$

and $B_1 = K\Delta^{(1)}/I^{(1)}$, where $\Delta^{(1)}$ is the quiver obtained from Q by deleting the arrow δ and $I^{(1)}$ is the ideal in $K\Delta^{(1)}$ generated by the same elements as I. Consider also the path algebra $H = K\Sigma$, where Σ is the Kronecker quiver

$$\begin{tikzcd} 2 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 3. \end{tikzcd}$$

Then the trivial extension algebra $B = \mathrm{T}(H)$ of H is the bound quiver algebra $K\Gamma/J$ where Γ is the quiver

$$\begin{tikzcd} 2 \arrow[r, bend left=40, "\alpha"] \arrow[r, bend left=20, "\beta"'] & 3 \arrow[l, bend left=20, "\gamma"'] \arrow[l, bend left=40, "\sigma"] \end{tikzcd}$$

and J is the ideal in $K\Gamma$ generated by $\gamma\alpha$, $\alpha\gamma$, $\beta\sigma$, $\sigma\beta$, $\alpha\sigma - \beta\gamma$, $\sigma\alpha - \gamma\beta$. Following notation of [24, Corollary 2.5] consider the one-point extension

$$B' = B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}$$

of B by the faithful B-module $M = B_B$. Then B' is a basic connected algebra having an indecomposable projective faithful module Q with $\operatorname{rad} Q \cong M$. Then the algebra B_1 (defined above) is the one-point coextension

$$B_1 = B'' = [Q]B' = \begin{bmatrix} B' & D(Q) \\ 0 & K \end{bmatrix},$$

the indecomposable projective B_1-module $P(4)$ at the vertex 4 is faithful and coincides with the indecomposable injective B_1-module $I(1)$ at the vertex 1. Then, by [24, Corollary 2.5], C is the generalized canonical algebra obtained from the algebras B_0 and B_1 by glueing their bound quivers at the vertices 1 and 4. Then it follows from [24, Theorem 2.1] that Γ_C admits an infinite family of pairwise orthogonal faithful (generalized) standard stable tubes. Then, by [16, Corollary 1.3], $\Gamma_{\mathrm{T}(C)}$ also admits an infinite family of pairwise orthogonal (generalized) standard stable tubes. Observe also that the

Cartan matrix of C is of the form

$$\begin{bmatrix} 1 & 4 & 4 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in the natural ordering $P(1)$, $P(2)$, $P(3)$, $P(4)$ of indecomposable projective C-modules, and is singular. We also mention that C is of infinite global dimension, because the simple modules $S(2)$ and $S(3)$ at the vertices 2 and 3 are of infinite projective dimension.

EXAMPLE 3.2. Let B be a concealed generalized canonical algebra over an algebraically closed field K, introduced in [16, Section 3]. Recall that B is an algebra of the form $\mathrm{End}_C(T)$, where C is a generalized canonical algebra, defined in [24, Section 2], and T is a tilting C-module cogenerated by the canonical family $\mathcal{T}^C$ of pairwise orthogonal faithful (generalized) standard stable tubes of Γ_C. Then, by [16, Theorem 1.1], the Auslander-Reiten quiver Γ_B of B admits a canonical family $\mathcal{T}^B$ of pairwise orthogonal faithful (generalized) standard stable tubes. Consider a selfinjective algebra A of the form $A = \hat{B}/(\varphi\nu_{\hat{B}})$, where $\hat{B}$ is the repetitive algebra of B, $\nu_{\hat{B}}$ is the Nakayama automorphism of $\hat{B}$ and φ is a positive automorphism of $\hat{B}$. We note that the induced Galois covering $\hat{B} \to \hat{B}/(\varphi\nu_{\hat{B}}) = A$ is a positive Galois covering in the sense of [29]. It has been proved in [16, Theorem 1.2] that the Auslander-Reiten quiver Γ_A of A admits an infinite family of pairwise orthogonal (generalized) standard stable tubes. We also note that A is symmetric if and only if $A \cong \mathrm{T}(B)$ (see [18, Theorem 2]). Therefore, if φ is a strictly positive automorphism of $\hat{B}$, then A is a nonsymmetric selfinjective algebra with Γ_A having an infinite family of (generalized) standard stable tubes. We refer to [18], [26] and [29] for more details on selfinjective orbit algebras of repetitive algebras.

Assume now that B is of finite global dimension, and hence the Coxeter matrix Φ_B is defined. We note that this is the case if the generalized canonical algebra C is of finite global dimension. We know from [16, Section 4] that Γ_B admits a faithful generalized standard stable tube $\mathcal{T}$ of rank one. Then $\mathrm{Hom}_B(\mathcal{T}, B_B) = 0$ and $\mathrm{pd}_B X \leq 1$ for any module X in $\mathcal{T}$ (see [21, Lemma 5.9]). Take a module X in $\mathcal{T}$. Since B is a basic algebra, $[X]$ is the dimension vector $\underline{\dim}\, X$ of X, under the canonical identification $K_0(A) = \mathbb{Z}^n$. Applying now [2, Corollary IV.2.9], we conclude that $\underline{\dim}\, X = \underline{\dim}\, \tau_A X = \Phi_B\, \underline{\dim}\, X$, and consequently $\underline{\dim}\, X$ is an eigenvector of Φ_B with eigenvalue 1.

For each $m \geq 2$, consider the selfinjective orbit algebra $\Lambda_B^{(m)} = \hat{B}/(\nu_{\hat{B}}^m)$. It follows from [12, Proposition 8.2] that the determinant of $\Lambda_B^{(m)}$ is of the form

$$(-1)^{mn}(\det C_B)^m \prod_{r=1}^{m} \det(\Phi_B - \varepsilon_r I_n)$$

where $\varepsilon_1, \dots, \varepsilon_m$ are distinct m-th roots of unity, and I_n is the identity matrix of degree n.

Therefore, for any concealed generalized canonical algebra B of finite global dimension, the algebras $\Lambda_B^{(m)}$, $m \geq 2$, are nonsymmetric selfinjective algebras with singular Cartan matrices and the Auslander-Reiten quivers having generalized standard stable tubes.

In the final example we show that there exist nonsymmetric selfinjective algebras with nonsingular Cartan matrices for which the Auslander-Reiten quiver admits a generalized standard stable tube. This will show that the symmetricity assumption is necessary for the validity of Theorem 2.1.

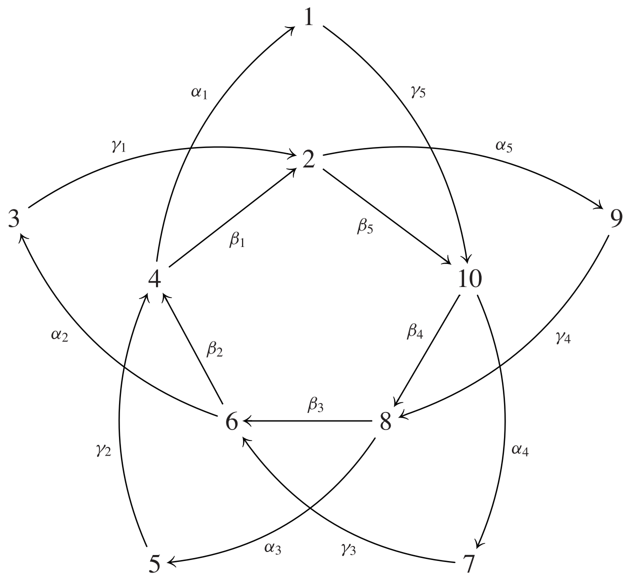
EXAMPLE 3.3. Let K be an algebraically closed field. Consider the bound quiver algebra $B = KQ/I$ where Q is the quiver

$$\begin{tikzcd}
7 \arrow[dr, "\gamma_3"] & & 8 \arrow[dl, "\beta_3"'] \arrow[dr, "\alpha_3"] & \\
& 6 \arrow[dl, "\alpha_2"] \arrow[dr, "\beta_2"] & & 5 \arrow[dl, "\gamma_2"] \\
3 \arrow[dr, "\gamma_1"] & & 4 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_1"] & \\
& 2 & & 1
\end{tikzcd}$$

and I is the ideal in the path algebra KQ of Q generated by the elements $\gamma_3\beta_2\alpha_1$, $\alpha_2\gamma_1 - \beta_2\beta_1$, $\beta_3\beta_2 - \alpha_3\gamma_2$. Then B is the exceptional (in the sense of [20, (3.2)]) tubular algebra B_4 of tubular type (3, 3, 3) presented in [7, Theorem 2.2]. It follows from [7, Section 3] that the Nakayama automorphism $\nu_{\hat{B}}$ of $\hat{B}$ admits a 4-root φ. For each $i \geq 1$, consider the selfinjective orbit algebra $\Omega_B^{(i)} = \hat{B}/(\varphi^i)$, and note that $\Omega_B^{(4)} = \hat{B}/(\nu_{\hat{B}}) = \mathrm{T}(B)$. It follows from the theory of selfinjective algebras of tubular type (see [6], [17], [20]) that, for $i \geq 4$, the Auslander-Reiten quiver of $\Omega_B^{(i)}$ admits an infinite family of generalized standard stable tubes. On the other hand, from the description of the determinants of Cartan matrices of selfinjective algebras of tubular type given in [5, Theorem], we know that, in the considered case, we have

$$\det C_{\Omega_B^{(i)}} = \begin{cases} 6 & \text{if} \quad i \equiv \pm 1 \pmod{6} \\ 12 & \text{if} \quad i \equiv \pm 2 \pmod{12} \\ 0 & \text{in other case} \end{cases}.$$

In particular, taking $A = \Omega_B^{(5)}$, we conclude that A is a nonsymmetric selfinjective algebra with $\det C_A = 6$ and the Auslander-Reiten quiver Γ_A having an infinite family of generalized standard stable tubes. In fact, A is the bound quiver algebra $K\Delta/J$ where Δ is the quiver



and J is the ideal in $K\Delta$ generated by the elements $\gamma_{i+2}\beta_{i+1}\alpha_i$, $\alpha_{i+1}\gamma_i - \beta_{i+1}\beta_i$, for $i \in \{1, \dots, 5\}$ with $\alpha_6 = \alpha_1$, $\beta_6 = \beta_1$, $\gamma_7 = \gamma_2$. Moreover, the Cartan matrix C_A is of the form (in the canonical numbering of indecomposable projective A-modules)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

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