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MIXED PROBLEMS FOR THE WAVE EQUATION IN A QUARTER SPACE WITH A FIRST ORDER BOUNDARY CONDITION

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Introduction

Let us consider the mixed problem

$$
(0.1)\qquad \begin{cases} \square u \equiv (D_t^2 - D_x^2 - \sum_{j=1}^{n-1} D_{y_j}^2)u = f(x, y, t) \quad \text{in } \boldsymbol{R}_+^n \times (0, \infty), \\ Bu \equiv (D_x + \sum_{j=1}^{n-1} b_j(y, t) D_{y_j} + b_0(y, t) D_t + c(y, t))u|_{x=0} \\ \qquad\qquad\qquad\qquad = g(y, t) \quad \text{on } \boldsymbol{R}^{n-1} \times (0, \infty), \\ D_t u|_{t=0} = u_1(x, y) \quad \text{on } \boldsymbol{R}_+^n, \\ u|_{t=0} = u_0(x, y) \quad \text{on } \boldsymbol{R}_+^n, \end{cases}
$$

where $D_t = -i\dfrac{\partial}{\partial t}$, $D_x = -i\dfrac{\partial}{\partial x}$, $\cdots$ and $b_0(y, t), \cdots, b_{n-1}(y, t), c(y, t) \in \mathscr{B}^\infty(\boldsymbol{R}^{n-1} \times \overline{\boldsymbol{R}_+^1})$[1]. We say that (0.1) is C^∞ well-posed when there exists a unique solution $u(x, y, t)$ in $C^\infty(\overline{\boldsymbol{R}_+^n} \times \overline{\boldsymbol{R}_+^1})$ for any $(f, g, u_1, u_0) \in C^\infty(\overline{\boldsymbol{R}_+^n} \times \overline{\boldsymbol{R}_+^1}) \times C^\infty(\boldsymbol{R}^{n-1} \times \overline{\boldsymbol{R}_+^1}) \times C^\infty(\overline{\boldsymbol{R}_+^n}) \times C^\infty(\overline{\boldsymbol{R}_+^n})$ satisfying the compatibility condition of infinite order and it has a finite propagation speed.

When the coefficients of B are all constant, by Sakamoto [9] we know a necessary and sufficient condition for C^∞ well-posedness. Agemi and Shirota [2] studied the mixed problem (0.1) precisely when $n=2$, $c=0$ and b_j is real constant. Tsuji [12, 13] treats the case where $b_0, \cdots, b_{n-1}$, c are variable and $b_0, \cdots, b_{n-1}$ real-valued. Ikawa [3,4,5] investigates the C^∞ well-posedness in other domains than a half space.

In the present paper we shall study a sufficient condition for the mixed problem (0.1) to be C^∞ well-posed and measure the propagation speed when $b_0(y, t), \cdots, b_{n-1}(y, t)$ are real-valued. Furthermore, we shall give a necessary condition for the C^∞ well-posedness when $b_0(y, t), \cdots, b_{n-1}(y, t)$ are not all real-valued.

Let $b_0(y,t)$ and $b'(y,t) = (b_1(y,t), \cdots, b_{n-1}(y,t))$ be real-valued. Then, we have

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[1] $\mathscr{B}^\infty(M) = \{h(z) \in C^\infty(M);\ |h|_m = \sum_{|\alpha| \leq m} |D_z^\alpha h(z)| < \infty \text{ for } m = 0, 1, \cdots\}$.

Theorem 1. *If* $\sup_{(y,t)\in \boldsymbol{R}^{n-1}\times \boldsymbol{R}^1_+} b_0(y,t)<1$, *then* (0,1) *is* C^∞ *well-posed*[2]. *Its propagation speed is equal to*

$$v_{\max}=\sup_{(y,t)\in\Lambda}\frac{1+|b'(y,t)|^2}{-b_0(y,t)|b'(y,t)|+\sqrt{|b'(y,t)|^2-b_0(y,t)^2+1}}\ ^{3)}\ (>1)$$

if $\Lambda\equiv\{(y,t)\colon -|b'(y,t)|<b_0(y,t)\}\neq\phi$, *and equal to* 1 *if* $\Lambda=\phi$.

This theorem will be proved in §1. From Miyatake [7] it follows that (0.1) is L^2 well-posed if and only if $b_0\leqq -|b'|$ (i.e., $\Lambda=\phi$). Therefore, in the above theorem, the propagation speed is equal to one only when (0.1) is L^2 well-posed. We remark that if $n\geqq 3$ and $b_0(y_0,0)\geqq 1$ for some y_0 the problem (0.1) is not C^∞ well-posed (cf. Remark 3.1 and the proof of Theorem 2).

Next, let us consider the case where $b_j(y,t)$ is complex-valued. In general, it is expected that the condition for C^∞ well-posedness is weaker than that for L^2 well-posedness. It is true when every b_j is real. However, when $n\geqq 3$ and $b_0,\cdots,b_{n-1}$ are not all real, there is little gap between both. Set

$$\begin{cases}\alpha=-b_0(y,t)+\sum_{j=1}^{n-1}b_j(y,t)\dfrac{\eta_j}{|\eta|}\,,\\[2ex] \beta=-b_0(y,t)-\sum_{j=1}^{n-1}b_j(y,t)\dfrac{\eta_j}{|\eta|}\qquad(\eta\neq 0)\,.\end{cases}$$

Then, Miyatake [7,8] has shown that (0.1) is L^2 well-posed if and only if the following condition (0.2) is satisfied for all (y,t) and $\eta\neq 0$ (see also Agemi [1]):

$$(0.2)\qquad \begin{cases}[\mathrm{I}]\ \begin{bmatrix}2\operatorname{Re}\alpha & \operatorname{Im}(\alpha\bar\beta)\\ \operatorname{Im}(\alpha\bar\beta) & 2\operatorname{Re}\alpha\end{bmatrix}\geqq 0 \quad \text{if } |\operatorname{Re}\alpha|+|\operatorname{Re}\beta|\neq 0\,,\\[2ex] [\mathrm{II}]\ \ 1+(\operatorname{Im}\alpha)(\operatorname{Im}\beta)>0 \quad \text{if } |\operatorname{Re}\alpha|+|\operatorname{Re}\beta|=0\,.\end{cases}$$

Theorem 2. *Let* $n\geqq 3$, *and assume that* (0.2) *is violated at* $(y,t)=(y_0,0)$[4], $\eta=\eta^0$ *and that* $b_0(y_0,0),\cdots,b_{n-1}(y_0,0)$ *are not all real. Furthermore, only when* $b_0(y_0,0),\cdots,b_{n-1}(y_0,0)$ *are all purely imaginary, we assume*

$$(0.3)\qquad 1+\sum_{j=1}^{n-1}b_j(y_0,0)^2-b_0(y_0,0)^2\neq 0\,.$$

Then the problem (0.1) *is not* C^∞ *well-posed.*

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[2] Theorem 1 is valid also in the case where the initial condition is posed on $t=t_0$ (for any $t_0\in\boldsymbol{R}$).

[3] This statement implies as follows: The propagation speed is not only less than $v_{\max}$, but also for any v satisfying $0<v<v_{\max}$ there exist (x_0,y_0,t_0), δ (>0) and $u(x,y,t)$ such that $u(x_0,y_0,t_0)$ is not equal to zero although $\Box u=0$ on C_v, $Bu=0$ on $C_v|_{x=0}$ and $u=D_t u=0$ on $C_v|_{t=t_0-\delta}$ $(C_v=\{(x,y,t)\colon (t-t_0)v+|(x-x_0,y-y_0)|\leqq 0,\ x\geqq 0,\ t_0-\delta\leqq t\leqq t_0\})$.

[4] If (0.2) is violated at $(y,t)=(y_0,t_0)$, the prob'em whose initial condition is posed on $t=t_0$ is not C^∞ well-posed.

In §3, we show that the Lopatinski condition is not satisfied at $(x, y, t)=(0, y_0, 0)$, and prove Theorem 2 by applying the methods of Kajitani [6]; i.e., assuming that (0.1) is C^∞ well-posed, we construct an appropriate asymptotic solution of (0.1) violating an inequality to be satisfied. Let us note that Theorem 2 and its proof are valid also when the problem (0.1) is considered in other general domains than a half space.

In the previous note [11] we have explained only Theorem 1 together with an outline of the proof.

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1. Reduction to the equation on the boundary

In this section, to prove Theorem 1 we shall reduce the problem (0.1) to the equation on the boundary. Namely, consider the Dirichlet problem

$$\begin{cases} \Box w(x, y, t) = 0 & \text{in } \boldsymbol{R}^n_+ \times \boldsymbol{R}^1 , \\ w|_{x=0} = h(y, t) & \text{on } \boldsymbol{R}^{n-1} \times \boldsymbol{R}^1 , \end{cases}$$

and set

$$Th = Bw ;$$

then the original problem can be reduced to the equation $Th=g$. Tsuji [13] also studies the same subject as in Theorem 1 by similar methods. But it seems for the author that his discussion (corresponding to Lemma 2.3 of our paper) is not complete.

At first, let us give several comments concerning the propagation speed. For a constant $v>0$ and a point $(x_0, y_0, t_0) \in \overline{\boldsymbol{R}^n_+} \times \boldsymbol{R}^1_+$, set

$$C_v = C_v(x_0, y_0, t_0) = \{(x, y, t) : (t-t_0)v + \{(x-x_0)^2 + |y-y_0|^2\}^{1/2} < 0\} .$$

Fix (x_0, y_0, t_0) and let us have a positive constant v such that for any small constant $\delta\,(>0)$

$$u(x, y, t) = 0 \quad \text{on } C_v \cap \{0 < t_0 - t < \delta,\ x > 0\} \tag{1.1}$$

if $u \in C^\infty(\overline{\boldsymbol{R}^n_+} \times \overline{\boldsymbol{R}^1_+})$ satisfies

$$\begin{cases} \Box u = 0 & \text{on } C_v \cap \{0 < t_0 - t < \delta,\ x > 0\} , \\ Bu = 0 & \text{on } C_v \cap \{0 < t_0 - t < \delta,\ x = 0\} , \\ D_t u|_{t=t_0-\delta} = 0 & \text{on } C_v \cap \{t = t_0 - \delta,\ x > 0\} , \\ u|_{t=t_0-\delta} = 0 & \text{on } C_v \cap \{t = t_0 - \delta,\ x > 0\} . \end{cases} \tag{1.2}$$

We call the infimum of the above v the propagation speed at (x_0, y_0, t_0). Fur-

thermore, the supremum of this speed on whole $\overline{\boldsymbol{R}^n_+}\times\boldsymbol{R}^1_+$ is named the propagation speed of (0.1). Obviously, if $v\geqq$'the propagation speed of (0.1)', (1.1) follows from (1.2) for all $x_0\geqq 0$, $y_0\in\boldsymbol{R}^{n-1}$, $t_0>0$ and $0\leqq\delta<t_0$.

Consider the equation (in λ)

$$-\sqrt{1-\lambda^2}+|b'(y,t)|\lambda+b_0(y,t)=0 \tag{1.3}$$

under the assumptions of Theorem 1. Then, if $-|b'(y,t)|\leqq b_0(y,t)$, this has a positive root

$$\lambda_0(y,t)=\frac{-b_0(y,t)|b'(y,t)|+\sqrt{|b'(y,t)|^2-b_0(y,t)^2+1}}{1+|b'(y,t)|^2}\ (\leqq 1)$$

(note that $\lambda_0=1$ only if $-|b'|=b_0$), and if $-|b'(y,t)|>b_0(y,t)$, it has no real root. Set $\lambda_0(y,t)=1$ in the latter case. Then the latter half of Theorem 1 implies that the propagation speed of (0.1) is equal to $\sup_{(y,t)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+}\lambda_0(y,t)^{-1}$.

From now on, let us prove that if the assumptions of Theorem 1 are satisfied (0.1) is C^∞ well-posed and has a finite propagation speed less than $\sup_{(y,t)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+}\lambda_0(y,t)^{-1}$. We know well the following proposition:

Proposition 1.1. *For any* $(f',u'_1,u'_0)\in C^\infty(\boldsymbol{R}^n\times\overline{\boldsymbol{R}^1_+})\times C^\infty(\boldsymbol{R}^n)\times C^\infty(\boldsymbol{R}^n)$ *there exists a unique solution* $u'(x,y,t)$ *of the following Cauchy problem in* $C^\infty(\boldsymbol{R}^n\times\overline{\boldsymbol{R}^1_+})$:

$$\begin{cases}\Box u'(x,y,t)=f'(x,y,t) & \text{in } \boldsymbol{R}^n\times\boldsymbol{R}^1_+\,,\\ D_t u'|_{t=0}=u'_1(x,y) & \text{on } \boldsymbol{R}^n\,,\\ u'|_{t=0}=u'_0(x,y) & \text{on } \boldsymbol{R}^n\,.\end{cases}$$

Furthermore, this problem has a finite propagation speed, which equals one.

By this proposition it suffices to investigate (0.1) only in the case $u_1=u_0=0$ and $f=0$. Then, the compatibility condition of infinite order implies that every $D_t^j g(y,+0)$ $(j=0,1,2,\cdots)$ equals zero.

We assume that $b(z)=(b_0(z),\cdots,b_{n-1}(z))$ and $c(z)$ $(z=(y,t))$ are constant when $|z|$ is large. This assumption will be used to prove Lemma 2.3 in §2. The general case is reduced to this case. In fact, let $\{\chi_j(z)\}_{j=0,1,2,\cdots}$ be a partition of unity on $\boldsymbol{R}^n$ such that $0\leqq\chi_j\leqq 1$ and $\mathrm{supp}[\chi_j]\subset\{j-1<|z|<j+1\}$, and set $\alpha_N(z)=\sum_{j=0}^{N}\chi_j(z)$. Fix $(x_0,y_0,t_0)\in\overline{\boldsymbol{R}^n_+}\times\boldsymbol{R}^1_+$ arbitrarily, and let $u(x,y,t)$ be any C^∞ function on $\overline{\boldsymbol{R}^n_+}\times\overline{\boldsymbol{R}^1_+}$ satisfying

$$\begin{cases}\Box u=0 & \text{in } \boldsymbol{R}^n_+\times\boldsymbol{R}^1_+\,,\\ Bu=0 & \text{on } C_v(x_0,y_0,t_0)\cap\{0<t<t_0,\ x=0\}\,,\\ D_t u|_{t=0}=0 & \boldsymbol{R}^n_+\,,\\ u|_{t=0}=0 & \boldsymbol{R}^n_+\,,\end{cases} \tag{1.4}$$

where $v=\sup_{(y,t)\in \boldsymbol{R}^{n-1}\times \boldsymbol{R}^1_+}\lambda_0(y,t)^{-1}$. Then $\alpha_N u$ satisfies

$$(1.5)\qquad \begin{cases} \square(\alpha_N u)=[\square,\alpha_N]u \quad \text{in } \boldsymbol{R}^n_+\times\boldsymbol{R}^1_+\,, \\ B_{N+2}(\alpha_N u)=\alpha_N Bu+[B,\alpha_N]u \quad \text{on } \boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+\,, \\ D_t(\alpha_N u)|_{t=0}=0 \quad \text{on } \boldsymbol{R}^n_+\,, \\ \alpha_N u|_{t=0}=0 \quad \text{on } \boldsymbol{R}^n_+\,, \end{cases}$$

where [,] denotes the commutator and $B_{N+2}u=(D_x u+\alpha_{N+2}b\cdot D_z u+\alpha_{N+2}cu)|_{x=0}$. The coefficients of B_{N+2} are constant for large $|z|$, and $\lambda_0(z)$ for the equation (1.5) (defined by (1.3)) is not smaller than that for (1.4). Therefore, $\alpha_N(y,t)\cdot u(x,y,t)=0$ on $C_v\cap\{0<t<t_0\}$ for any large N, which implies that (0.1) has a finite propagation speed less than $\sup_{(y,t)\in \boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+}\lambda_0(y,t)^{-1}$. Next, consider the following problem (for $N=0, 1, \cdots$):

$$\begin{cases} \square u^{(N)}=0 \quad \text{in } \boldsymbol{R}^n_+\times\boldsymbol{R}^1_+\,, \\ B_{N+2}u^{(N)}=\chi_N\{g-\sum_{j=0}^{N-1}(B-B_{j+2})u^{(j)}\} \quad \text{on } \boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+\ (N\geqq 1)\,, \\ \qquad\qquad =\chi_0 g \quad \text{on } \boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+\ (N=0)\,, \\ D_t u^{(N)}|_{t=0}=u^{(N)}|_{t=0}=0 \quad \text{on } \boldsymbol{R}^n_+\,. \end{cases}$$

In view of finiteness of the propagation speed, we see that $u(x,y,t)=\sum_{N=0}^{\infty}u^{(N)}(x,y,t)$ belongs to $C^\infty(\overline{\boldsymbol{R}^n_+}\times\overline{\boldsymbol{R}^1_+})$ and satisfies

$$\begin{cases} \square u=0 \quad \text{in } \boldsymbol{R}^n_+\times\boldsymbol{R}^1_+\,, \\ Bu=g \quad \text{on } \boldsymbol{R}^{n-1}\times\boldsymbol{R}^n_+\,, \\ D_t u|_{t=0}=u|_{t=0}=0 \quad \text{on } \boldsymbol{R}^n_+\,. \end{cases}$$

Therefore the existence of the solution in the general case is also obtained.

Now, we denote by $C^\infty_+(M)$ the set of C^∞ functions on M ($M=\overline{\boldsymbol{R}^1_+}\times\boldsymbol{R}^n$ or $\boldsymbol{R}^n$) whose support lies in $\{t\geqq t_0\}$ for some $t_0\in\boldsymbol{R}$.

Proposition 1.2. *The Dirichlet problem*

$$(1.6)\qquad \begin{cases} \square w(x,z)=0 \quad \text{in } \boldsymbol{R}^1_+\times\boldsymbol{R}^n\,, \\ w|_{x=0}=h(z) \quad \text{on } \boldsymbol{R}^n \end{cases}$$

has a unique solution $w(x,z)$ in $C^\infty_+(\overline{\boldsymbol{R}^1_+}\times\boldsymbol{R}^n)$ for any $h(z)\in C^\infty_+(\boldsymbol{R}^n)$, and has a finite propagation speed, which equals one.

Extending $b(y,t)$ and $c(y,t)$ to $t<0$ smoothly, we set (for $h\in C^\infty_+(\boldsymbol{R}^n)$)

$$(1.7)\qquad Th=Bw\ (=(D_x+b\cdot D_z+c)w|_{x=0})\,.$$

Then T is an operator from $C^\infty_+(\boldsymbol{R}^n)$ to $C^\infty_+(\boldsymbol{R}^n)$. Furthermore we have

Theorem 1.1. *Let $b(z)$ be real-valued and* $\sup_{z\in \boldsymbol{R}^n} b_0(z)<1$. *Then, there exsists a unique solution h of the equation $Th=g$ in $C_+^\infty(\boldsymbol{R}^n)$ for any $g\in C_+^\infty(\boldsymbol{R}^n)$, and it has a finite propagation speed less than* $\sup_{z\in \boldsymbol{R}^n} \lambda_0(z)^{-1}$.

We shall prove this theorem in the next section. From this theorem and Proposition 1.2, it follows that (0.1) is C^∞ well-posed and has a finite propagation speed less than $\sup_{(y,t)\in \boldsymbol{R}^{n-1}\times \boldsymbol{R}_+^1} \lambda_0(y,t)^{-1}$. Therefore, the proof of Theorem 1 is complete if the following theorem is verified.

Theorem 1.2. *Let $b_0(y,t),\cdots,b_{n-1}(y,t)$ be real-valued and $b_0(y,t)<1$. Then the propagation speed of* (0.1) *at any* $(0,y,t)$ *is not smaller than* $\lambda_0(y,t)^{-1}$.

Note that the propagation speed is equal to one in a place distanct from the boundary $\{x=0\}$. We can prove this theorem in the same way as in the proof of Theorem 4.1 of the author [10]. Its idea is suggested by Kajitani [6] and Appendix of Ikawa [3]. Ikawa in [5] also studies the propagation speed of the same mixed problem by slightly different methods.

Let us give only a sketch of the proof of Theorem 1.2. Suppose that Theorem 1.2 is not true. Since in the case $\lambda_0(y_0,t_0)=1$ our statement is trivial, we may assume $\lambda_0(y_0,t_0)<1$ (i.e. $-|b'(y_0,t_0)|<b_0(y_0,t_0)$). Then there is a constant v $(1<v<\lambda_0(y_0,t_0)^{-1})$ such that (1.1) follows from (1.2) for any small constant $\delta\,(>0)$. Let us indicate that this is a contradiction. We construct an asymptotic solution

$$u_N(x,y,t;k)=\sum_{j=0}^{N} e^{ik\Phi(x,y,t)}v_j(x,y,t)k^{-j} \qquad (k\geqq 1) \tag{1.8}$$

such that $v_0(0,y_0,t_0)\neq 0$ and

$$\begin{cases} \Box u_N = e^{ik\Phi}\Box v_N k^{-N} \text{ in a neighborhood } U \text{ of} \\ \qquad\qquad \bar{C}_v\cap\{0\leqq t_0-t\leqq\delta,\ x\geqq 0\}\ , \\ Bu_N=0 \quad \text{on } C_v\cap\{0<t_0-t<\delta,\ x=0\}\ , \\ D_t u_N|_{t=t_0-\delta}=0 \quad \text{on } C_v\cap\{t=t_0-\delta,\ 0<x\}\ , \\ u_N|_{t=t_0-\delta}=0 \quad \text{on } C_v\cap\{t=t_0-\delta,\ 0<x\}\ . \end{cases}$$

By the former half of Theorem 1 we have a solution $w_N(x,y,t;k)$ such that

$$\begin{cases} \Box w_N = -e^{ik\Phi}\Box v_N \quad \text{in } U\ , \\ Bw_N=0 \quad \text{on } C_v\cap\{0<t_0-t<\delta,\ x=0\}\ , \\ D_t w_N|_{t=t_0-\delta}=0 \quad \text{on } C_v\cap\{t=t_0-\delta,\ 0<x\}\ , \\ w_N|_{t=t_0-\delta}=0 \quad \text{on } C_v\cap\{t=t_0-\delta,\ 0<x\} \end{cases}$$

and that the estimate

$$|w_N(0, y_0, t_0; k)| \leqq C\, k^l$$

holds for constants C and l independent of k. Set

$$u(x, y, t; k) = u_N(x, y, t; k) + w_N(x, y, t; k)k^{-N} \quad (l<N).$$

Then, u satisfies (1.2), but $u(0, y_0, t_0; k) \to v_0(0, y_0, t_0)$ $(\neq 0)$ as $k \to \infty$, which proves Theorem 1.2.

Let us show briefly the procedure to construct the asymptotic solution (1.8). We shall make a similar solution in §3. As is explained in [10], we has only to solve the eiconal equation with $B\Phi|_{x=0}=0$ and the transport equation with $B\, v_j|_{x=0}=0$. From the latter, the following equation for $\tilde{v}_j(y,t)=v_j|_{x=0}$ is obtained:

$$\begin{aligned}(1.9)\qquad &(b_0 D_x\Phi + D_t\Phi)D_t\tilde{v}_j + \sum_{k=1}^{n-1}(b_k D_x\Phi - D_{y_k}\Phi)D_{y_k}\tilde{v}_j \\ &+\left(\frac{1}{2}\Box\Phi + cD_x\Phi\right)\tilde{v}_j = -\frac{1}{2}\Box\tilde{v}_{j-1}.\end{aligned}$$

We can choose the phase function Φ so that $(\Phi_x, \Phi_y, \Phi_t) = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y_1}, \cdots, \frac{\partial\Phi}{\partial y_{n-1}}, \frac{\partial\Phi}{\partial t}\right)$ at $(0, y_0, t_0)$ satisfies $\Phi_x<0, \Phi_t>0$ and $\frac{|\Phi_y|}{\Phi_t}=\lambda_0(y_0, t_0)$. Let $(\nu, 1)$ $(\in \boldsymbol{R}^n{}_{(y,t)})$ be the direction of the characteristic curve of the equation (1.9) at (y_0, t_0). Then, it is seen that $|\nu|$ is equal to $\lambda_0(y_0, t_0)^{-1}$. By this fact we can construct the required asymptotic solution.

2. Proof of Theorem 1.1

Throughout this section we assume that $b(z)=(b_0(z), \cdots, b_{n-1}(z))$ is real-valued and $\sup_{z\in\boldsymbol{R}^n} b_0(z)<1$. These assumptions will be used to prove Lemma 2.1 and 2.2. At first we consider the equation $Th=g$ in the Sobolev space. Let $H_m(M)$ be the usual Sobolev space on M of order m. We denote by $H_{m,\gamma}(\boldsymbol{R}^1_+ \times \boldsymbol{R}^n)$ $(\gamma \in \boldsymbol{R}^n$, $m=0, 1, \cdots)$ the space $\{u(x,z)\colon e^{-\gamma z}u(x,z) \in H_m(\boldsymbol{R}^1_+ \times \boldsymbol{R}^n)\}$, and by $H_{m,\gamma}(\boldsymbol{R}^n)$ $(\gamma \in \boldsymbol{R}^n, m \in \boldsymbol{R})$ the space $\{u(z)\colon e^{-\gamma z}u(z) \in H_m(\boldsymbol{R}^n)\}$. Let us define the Laplace-Fourier transformation F_γ (in z) by

$$F_\gamma[u] = \hat{u}(\zeta) = \int e^{-i(\sigma - i\gamma)z}u(z)dz, \quad u \in C_0^\infty(\boldsymbol{R}^n)$$
$$(\zeta = \sigma - i\gamma,\ \gamma \in \boldsymbol{R}^n),$$

and denote by $\bar{F}_\gamma$ the inverse transformation, that is,

$$\bar{F}_\gamma[f](z) = (2\pi)^{-n}e^{\gamma z}\int e^{i\sigma z}f(\sigma - i\gamma)d\sigma.$$

The norm $\langle h\rangle_{m,\gamma}$ of $H_{m,\gamma}(\boldsymbol{R}^n)$ is defined by

$$\langle h\rangle^2_{m,\gamma}=(2\pi)^{-n}\int(|\sigma|^2+|\gamma|^2)^m|\hat{h}(\sigma-i\gamma)|^2d\sigma \quad (\gamma\neq 0)\,.$$

We set

$$\Gamma=\{(\xi,\gamma)=(\xi,\eta,\tau)\in \boldsymbol{R}^{n-1};\ \tau>(|\eta|^2+\xi^2)^{1/2}\}\,,$$
$$\dot{\Gamma}=\{\gamma=(\eta,\tau)\in \boldsymbol{R}^n;\ \tau>|\eta|\}\,,$$
$$\dot{\Sigma}=\{\gamma=(\eta,\tau)\in \boldsymbol{R}^n;\ \tau>(\sup_{z\in \boldsymbol{R}^n}\lambda_0(z)^{-1})|\eta|\}\,.$$

Proposition 2.1. *We have for* $(\xi,\eta,\tau)\in \boldsymbol{R}^{n+1}-i\Gamma$

$$\tau^2-\sum_{j=1}^{n-1}\eta_j^2-\xi^2\neq 0\,.$$

This proposition is obtained by the methods in §3 of Sakamoto [9].

Corollary. *If* $(\eta,\tau)\in \boldsymbol{R}^n-i\dot{\Gamma}$, *the equation* $\tau^2-\sum_{j=1}^{n-1}\eta_j^2-\xi^2=0$ *in* ξ *has a root* $\xi_+(\eta,\tau)$ *with a positive imaginary part and a root with a negative one.*

We know that the Dirichlet problem (1.6) is solvable also in the Sobolev space:

Proposition 2.2. *For any* $h(z)\in H_{m,\gamma}(\boldsymbol{R}^n)$ $(\gamma\in\dot{\Gamma},\ m=1,2,\cdots)$ *there exists a unique solution* $w(x,z)$ *of* (1.6) *in* $H_{m,\gamma}(\boldsymbol{R}^1_+\times \boldsymbol{R}^n)$, *and the solution is represented by the form*

$$w(x,z)=\bar{F}_\gamma[e^{ix\xi_+(\zeta)}\hat{h}(\zeta)] \quad (\zeta=\sigma-i\gamma)\,.$$

Furthermore, for any compact set S *in* $\dot{\Gamma}$ *there is a constant* $\gamma_0(m,S)$ *such that* $w\in\bigcap_{\substack{|\gamma|\geqq\gamma_0(m,S)\\ \gamma\in K_S}}H_{m,\gamma}(\boldsymbol{R}^1_+\times \boldsymbol{R}^n)$ *follows from* $h\in\bigcap_{\substack{|\gamma|\geqq\gamma_0(m,S)\\ \gamma\in K_S}}H_{m,\gamma}(\boldsymbol{R}^n)$ *(where* $K_S=\{\zeta=\mu\gamma;\ \gamma\in S,\ \mu>0\}$*).*

Define Th for $h\in H_{m,\gamma}(\boldsymbol{R}^n)$ $(\gamma\in\dot{\Gamma})$ by (1.7). Then, by Proposition 2.2 we have

$$Th=R_\gamma h\equiv\bar{F}_\gamma[(\xi_+(\sigma-i\gamma)+b(\sigma-i\gamma)+c)\hat{h}(\sigma-i\gamma)]$$
$$(\gamma\in\dot{\Gamma}\text{ and }h\in H_{m,\gamma}(\boldsymbol{R}^n))\,.$$

Let us note that if $h\in H_{m,\gamma}(\boldsymbol{R}^n)\cap H_{m,\gamma'}(\boldsymbol{R}^n)$ then $R_\gamma h=R_{\gamma'}h$ when $\gamma,\gamma'\in K_S$ and $|\gamma|,\ |\gamma'|\geqq\gamma_0(m,S)$ (K_S and $\gamma_0(m,S)$ are defined in Proposition 2.2). Set

$$R^*_{-\gamma}h=\bar{F}_{-\gamma}[\overline{(\xi_+(\sigma-i\gamma)}+b\cdot(\sigma+i\gamma)+D_z\cdot b+\bar{c})\hat{h}(\sigma+i\gamma)] \quad (\gamma\in\dot{\Gamma})\,.$$

Then it follows that

$$(R_\gamma h,g)_{L^2}=(h,R^*_{-\gamma}g)_{L^2} \quad \text{for } h,g\in C_0^\infty(\boldsymbol{R}^n) \quad (\gamma\in\dot{\Gamma})\,.$$

The following lemma plays a basic role.

Lemma 2.1. *Let $m\in\boldsymbol{R}$ and S be any compact set of $\dot{\Sigma}$. Then, there is a constant $\gamma_0(m, S)$ such that if $|\gamma|\geqq\gamma_0(m, S)$ and $\gamma\in K_S=\{\gamma=\mu\zeta: \zeta\in S, \mu>0\}$ the following estimates hold for a constant C independent of γ:*

(i) $|\gamma|\langle h\rangle_{m,\gamma}\leqq C\langle R_\gamma h\rangle_{m,\gamma}, \quad h\in C_0^\infty(\boldsymbol{R}^n)$,

(ii) $|\gamma|\langle h\rangle_{-m,-\gamma}\leqq C\langle R^*_{-\gamma}h\rangle_{-m,-\gamma}, \quad h\in C_0^\infty(\boldsymbol{R}^n)$.

Proof. Noting that $b(z)$ is real-valued, by an easy calculation we have

$$\begin{aligned}\operatorname{Im}\langle R_\gamma h, h\rangle_{0,\gamma} &= \operatorname{Im}(e^{-\gamma z}R_\gamma h(z), e^{-\gamma z}h(z))_{L^2}\\ &\geqq(2\pi)^{-n}\int\{\operatorname{Im}\xi_+(\sigma-i\gamma)-\sup_{z\in\boldsymbol{R}^n}(b\cdot\gamma)\}\,|\hat{h}(\sigma-i\gamma)|^2d\sigma\\ &\quad -C_1\langle h\rangle_{0,\gamma}^2 \quad (C_1=\sup_{z\in\boldsymbol{R}^n}|D_z\cdot b(z)|+\sup_{z\in\boldsymbol{R}^n}|c(z)|)\,.\end{aligned}$$

From Lemma 2.2 below, it follows that

$$\operatorname{Im}\langle R_\gamma h, h\rangle_{0,\gamma}\geqq(\delta|\gamma|-C_1)\langle h\rangle_{0,\gamma}^2\,.$$

Therefore we obtain (i) in the case $m=0$. Define Λ^m ($m\in\boldsymbol{R}$) by

$$\Lambda^m h=\bar{F}_\gamma[(|\sigma|^2+|\gamma|^2)^{m/2}\hat{h}(\sigma-i\gamma)]\,.$$

Then, we have

$$\begin{aligned}\langle R_\gamma h\rangle_{m,\gamma} &= \langle\Lambda^m R_\gamma h\rangle_{0,\gamma}\\ &\geqq\langle R_\gamma\Lambda^m h\rangle_{0,\gamma}-\langle[\Lambda^m, b]\cdot D_z h\rangle_{0,\gamma}-\langle[\Lambda^m, c]h\rangle_{0,\gamma}\\ &\geqq\delta_1|\gamma|\langle h\rangle_{m,\gamma}-C_2\langle h\rangle_{m,\gamma}\,,\end{aligned}$$

which yields (i). In the same way we can get (ii). The proof is complete.

Lemma 2.2. *Let S be a compact set in $\dot{\Sigma}$. Then there is a constant $\delta\,(>0)$ such that*

$$\operatorname{Im}\xi_+(\zeta)+b(z)\cdot\operatorname{Im}\zeta\geqq\delta|\zeta|,\ \zeta\in\boldsymbol{R}^n-iK_S,\ z\in\boldsymbol{R}^n\,.$$

Proof. In view of the corollary of Proposition 2.1, we have $(-\operatorname{Im}\xi_+(\zeta), -\operatorname{Im}\zeta)\notin\Gamma$ if $\zeta\in\boldsymbol{R}^n-i\dot{\Gamma}$. On the other hand, noting that $b(z)$ is real-valued and $\sup_{z\in\boldsymbol{R}^n}b_0(z)<1$, we see that if $\gamma\in K_S$, $\xi<0$ and $(\xi, \gamma)\notin\Gamma$ then there is a small constant $\delta\,(>0)$ such that $\xi\leqq-(b+\delta\omega)\cdot\gamma$ for any ω ($\omega\in\boldsymbol{R}^n$, $|\omega|=1$). Therefore we have

$$\operatorname{Im}\xi_+(\zeta)+(b-\delta\operatorname{Im}\zeta/|\operatorname{Im}\zeta|)\cdot\operatorname{Im}\zeta\geqq0,\ \zeta\in\boldsymbol{R}^n-iK_S,\ z\in\boldsymbol{R}^n\,.$$

The proof is complete.

Now, let us prove Theorem 1.1. We set

$$\dot{\Sigma}'=\{\gamma'\in\boldsymbol{R}^n;\ \gamma'\cdot\gamma\geqq0 \quad \text{for any } \gamma\in\dot{\Sigma}\}$$

$$(=\{(\eta,\tau)\in \boldsymbol{R}^n;\ |\eta|<(\sup_{z\in \boldsymbol{R}^n}\lambda_0(z)^{-1})\tau\})\,.$$

It suffices to verify

Lemma 2.3. *Let* $g(z)\in \bigcap_{\substack{\gamma\in\dot{\Sigma}\\ |\gamma|\geqq 1}} H_{m,\gamma}(\boldsymbol{R}^n)$ $(m\geqq 0)$ *and* $\operatorname{supp}[g]\subset\dot{\Sigma}'+z_1$ *for some* $z_1\in\boldsymbol{R}^n$. *Then there exists a unique solution* $h(z)$ *of* $T\,h=g$ *in* $\bigcap_{\substack{\gamma\in\dot{\Sigma}\\ |\gamma|\geqq 1}} H_{m,\gamma}(\boldsymbol{R}^n)$ *such that* $\operatorname{supp}[h]\subset\dot{\Sigma}'+z_1$.

In fact, take $g(z)\in C_+^\infty(\boldsymbol{R}^n)$ arbitrarily, and let $\{\chi_j(z)\}_{j=0,1,\dots}$ be the partition of unity used in §1. By Lemma 2.3 we have a solution $h_j\in C^\infty(\boldsymbol{R}^n)$ of $T\,h_j=\chi_j\,g$ for $j=0,1,\cdots$ whose support lies in $\bigcup_{z\in\operatorname{supp}(\chi_j g)}(\dot{\Sigma}'+z)$. As is easily seen, $h(z)=\sum_{j=0}^{\infty}h_j(z)$ belongs to $C_+^\infty(\boldsymbol{R}^n)$ and $\operatorname{supp}[h]\subset\bigcap_{z\in\operatorname{supp}(g)}(\dot{\Sigma}'+z)$. Therefore Theorem 1.1 is obtained except the uniqueness in $C_+^\infty(\boldsymbol{R}^n)$. Obviously Lemma 2.1 guarantees the uniqueness in $H_{m,\gamma}(\boldsymbol{R}^n)$ for some $\gamma\in\dot{\Sigma}$. Hence, if the the data (f,g,u_1,u_0) and the solution u of (0.1) have compact support, (0.1) has a finite propagation speed less than $\sup_{(y,t)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}_+^1}\lambda_0(y,t)^{-1}$. Assume that $T\,h=0$, $h\in C_+^\infty(\boldsymbol{R}^n)$ and $\operatorname{supp}[h(y,t)]\subset\{t_0\leqq t\}$. Let $w(x,z)\in C_+^\infty(\overline{\boldsymbol{R}_+^1}\times\boldsymbol{R}^n)$ be the solution of (1.6) for the h. Then $w(x,z)$ satisfies

$$\begin{cases}\Box w(x,z)=0 & \text{in } \boldsymbol{R}_+^1\times\boldsymbol{R}^n\,,\\ Bw(z)=0 & \text{on } \boldsymbol{R}^n\,.\end{cases}$$

Let $\psi_N(x)\in C^\infty(\boldsymbol{R}^1)$ be equal to 1 if $x<N$ and to 0 if $N+1<x$ $(N=0,1,\cdots)$, and set

$$\beta_N(x,y,t)=\psi_N(x)\sum_{j=0}^{N}\chi_j(y,t)\,.$$

Then, it follows that

$$\begin{cases}\Box(\beta_N w)(x,y,t)=[\Box,\beta_N]w & \text{in } \boldsymbol{R}_+^n\times(t_0,\infty)\,,\\ B(\beta_N w)(y,t)=[B,\beta_N]w & \text{on } \boldsymbol{R}^{n-1}\times(t_0,\infty)\,,\\ D_t(\beta_N w)|_{t=t_0}=0 & \text{on } \boldsymbol{R}_+^n\,,\\ (\beta_N w)|_{t=t_0}=0 & \text{on } \boldsymbol{R}_+^n\,.\end{cases}$$

Since $\operatorname{supp}([\Box,\beta_N]w)$ and $\operatorname{supp}([B,\beta_N]w)$ lie in $\{N-1\leqq|z|\leqq N+1,\ N\leqq x\leqq N+1\}$ and $\{N-1\leqq|z|\leqq N+1\}$ respectively, we have

$$\beta_N w(x,z)=0 \quad\text{on}\quad \{|z|\leqq M(N,v_{\max}),\ 0\leqq x\leqq M(N,v_{\max})\}\,,$$

where the constant $M(N,v_{\max})\to\infty$ as $N\to\infty$. This implies $w=0$, hence $h=0$. Thus we obtain the uniqueness.

Proof of Lemma 2.3. Obviously the uniqueness follows from Lemma 2.1. Let $b(z)=\tilde{b}$ and $c(z)=\tilde{c}$ when $|z|$ is large, and set

$$\tilde{T}\,h = (D_x+\tilde{b}\cdot D_z+\tilde{c})w|_{x=0}\,,$$

where $w(x,z)$ is the solution of (1.6) for $h(z)$. Then, in the same way as T, $\tilde{T}$ is expressed by the form

$$\tilde{T}\,h = \bar{F}_\gamma[(\xi_+(\zeta)+\tilde{b}\cdot\zeta+\tilde{c})\hat{h}(\zeta)],\quad h\in H_{m,\gamma}(\boldsymbol{R}^n)$$
$$(m\geqq 0,\ \zeta=\sigma-i\gamma,\ \gamma\in\dot{\Gamma})\,.$$

Noting that $\tilde{b}$ and $\tilde{c}$ are constant, we see that the statement of Lemma 2.3 is true for $\tilde{T}$ (cf. Sakamoto [9]).

Let $g(z)\in\bigcap_{\substack{\gamma\in\dot{\Sigma}\\|\gamma|\geqq 1}} H_{m,\gamma}(\boldsymbol{R}^n)$ and $\mathrm{supp}[g]\subset\dot{\Sigma}'+z_1$. By Lemma 2.1 there is a solution $h_{\tilde{\gamma}}\in H_{m,\tilde{\gamma}}(\boldsymbol{R}^n)$ of $T\,h_{\tilde{\gamma}}=g$ for some $\tilde{\gamma}\in\dot{\Sigma}$ (let $|\tilde{\gamma}|$ be large enough). Then we can write

$$\begin{aligned}\tilde{T}\,h_{\tilde{\gamma}} &= (\tilde{T}-T)h_{\tilde{\gamma}}+T\,h_{\tilde{\gamma}}\\ &= (\tilde{b}-b(z))\cdot D_z h_{\tilde{\gamma}}+(\tilde{c}-c(z))h_{\tilde{\gamma}}+g\,.\end{aligned}$$

Since the support of the right side lies in $\dot{\Sigma}'+\tilde{z}$ (for some $\tilde{z}\in\boldsymbol{R}^n$), we have $h_{\tilde{\gamma}}\in\bigcap_{\substack{\gamma\in\dot{\Sigma}\\|\gamma|\geqq 1}} H_{m,\gamma}(\boldsymbol{R}^n)$. Fix $\omega\in\dot{\Sigma}$ ($|\omega|=1$) arbitrarily. If $\dfrac{\gamma}{|\gamma|}=\omega$ and $|\gamma|$ is large enough, it follows from Lemma 2.1 that

$$|\gamma|\langle h_{\tilde{\gamma}}\rangle_{m,\gamma}\leqq C\langle R_\gamma h_{\tilde{\gamma}}\rangle_{m,\gamma}=C\langle g\rangle_{m,\gamma}\,.$$

Noting that the above constant C does not depend on γ, we see that $\mathrm{supp}[h_{\tilde{\gamma}}]\subset\dot{\Sigma}'+z_1$. The proof is complete.

3. Proof of Theorem 2

We denote by $\xi_+(\eta,\tau)$ the root of the equation $\tau^2-\xi^2-|\eta|^2=0$ in ξ ($\eta\in\boldsymbol{R}^{n-1}$, $\tau=\sigma-i\gamma$ ($\gamma>0$, $\sigma\in\boldsymbol{R}^1$)) whose imaginary part is positive (cf. Corollary of Proposition 2.1). Set

$$R(\eta,\tau)=\xi_+(\eta,\tau)+\sum_{j=1}^{n-1}b_j(y_0,0)\eta_j+b_0(y_0,0)\tau\,,$$

which is homogeneous of order one in (η,τ).

To begin with, we shall show that if the assumptions of Theorem 2 are fulfilled the Lopatinski condition is not satisfied:

Lemma 3.1. *Let $b_0(y_0,0)\neq 1$, and assume that $b_0(y_0,0),\cdots,b_{n-1}(y_0,0)$ ($n\geqq 3$) are not all real. Furthermore, let* (0.3) *be satisfied when every $b_j(y_0,0)$ is purely*

imaginary. Then, if the condition (0.2) *is violated, there exist* τ_0 (Im $\tau_0<0$) *and* η^0 ($\in \boldsymbol{R}^{n-1}$) *such that* $R(\eta^0, \tau_0)=0$.

REMARK 3.1. If $b_0(y_0, 0)=1$, the Lopatinski condition is not satisfied; in fact, $R(0, -i\gamma)=0$ for $\gamma>0$. Moreover, also if $b_0(y_0, 0)>1$ and $b_1(y_0, 0), \cdots, b_{n-1}(y_0, 0)$ ($n\geqq 3$) are all real, it is violated; because $R(\eta^0, -i(b_0^2-1)^{-1/2})=0$ for η^0 ($|\eta^0|=1$) orthogonal to $(b_1(y_0, 0), \cdots, b_{n-1}(y_0, 0))$.

Proof of Lemma 3.1. Let us abbreviate $b_j(y_0, 0)$ $(j=0, \cdots, n-1)$ to b_j. For $\eta'\in S=\{\eta'\colon |\eta'|=1\}$ and μ (Im $\mu<0$) we set

$$\varphi_-(\mu)=R(\eta', \mu)\,.$$

This has an analytic continuation in $\{\mu\in \boldsymbol{C};\ \mu\notin[-1, 1]\}$, which is of the form

$$\varphi(\mu)=-\sqrt{\mu^2-1}+b'\cdot\eta'+b_0\mu\,.$$

Here b' denotes $(b_1, \cdots, b_{n-1})$ and $\sqrt{\mu^2-1}$ is the branch which is positive for $\mu>1$ (note that $\sqrt{\mu^2-1}$ is single-valued in $\{\mu\in \boldsymbol{C};\ \mu\notin[-1, 1]\}$). It is easily seen that

$$\varphi(\mu)=\begin{cases} R(\eta', \mu) & \text{for } \operatorname{Im}\mu<0\,, \\ -R(-\eta', -\mu) & \text{for } \operatorname{Im}\mu>0\,. \end{cases}$$

We employ the following transformation $\mu\mapsto z$, introduced by Miyatake [7] (cf. §3 of Chapter I in [7]):

$$z^2=\frac{1-\mu}{\mu+1}\quad (\operatorname{Im} z<0)\,.$$

Then, as Miyatake [7] shows, it follows that

(i) $\{\mu\colon \mu\notin[-1, 1]\}$ is mapped to $\{z\colon \operatorname{Im} z<0, z\neq -i\}$,

(ii) the lines $\{\mu\colon \mu\in(-1, 1)\}$, $\{\mu\colon \mu\in(-\infty, -1)\}$, $\{\mu\colon \mu\in(1, +\infty)\}$ are mapped to $\{z\colon z\in(-\infty, +\infty), z\neq 0\}$, $\{z\colon -iz\in(-\infty, -1)\}$, $\{z\colon -iz\in(-1, 0)\}$ respectively; $\pm\infty$, 1 and -1 in the μ-plane to $-i$, 0 and $\pm\infty$, $-i\infty$ in the z-plane respectively,

(iii) $\varphi(\mu)$ is transformed to

$$\varphi^*(z)=\frac{\alpha z^2-2iz-\beta}{z^2+1}\left(\equiv\frac{f_\eta(z)}{z^2+1}\right).$$

From these facts, it suffices to prove that the equation $f_{\eta^0}(z)=0$ has a root in $\{z\colon \operatorname{Im} z<0, \operatorname{Re} z\neq 0\}$ for some $\eta^0\in S$.

At first, we show that if (0.2) is violated for $\eta'=\eta^0$ ($\in S$) there is a (connected) neighborhood $V(\subset S)$ such that for any $\eta'\in V$ $f_{\eta'}(z)=0$ has a root in $\{z\colon \operatorname{Im} z<0\}$. If (0.2) is not satisfied, the following three cases can be considered:

(i) [I] of (0.2) is violated;
(ii) [II] of (0.2) is violated and Re $b' \neq 0$;
(iii) [II] of (0.2) is violated and Re $b'=0$.

The case (i): Since $|\mathrm{Re}\,\alpha|+|\mathrm{Re}\,\beta|\neq 0$ for $\eta'=\eta^0$, $f_{\eta'}(z)=0$ has no complex conjugate pair of roots near $\eta'=\eta^0$. Therefore we can apply the following proposition (due to Miyatake [7]) to the polynomial $f_{\eta'}(z)$.

Proposition 3.1. *Assume that a polynomial $f(z)$ of degree m has not any complex conjugate pair of non-real roots. Then all the roots of $f(z)=0$ lie in $\{z\colon \mathrm{Im}\, z\geqq 0\}$ if and only if the Bézout form for $\{f(z), -i\bar{f}(z)\}$ is non-negative.*

This is proved in [7] (see Corollary of the Hermite theorem in [7]). Here the Bézout form for $\{f(z), g(z)\}$ (f and g are polynomials of degree m) means the quadratic form defined by the symmetric matrix $A=(a_{ij})_{i,j=0,\cdots,m-1}$ whose components a_{ij} are given by

$$G[f,g]=\frac{f(x)g(y)-g(x)f(y)}{x-y}=\sum_{i,j=0}^{m-1}a_{ij}x^iy^j\,.$$

Since the matrix defining the Bézout form for $\{f_{\eta'}(z), -i\bar{f}_{\eta'}(z)\}$ is of the form $2\begin{bmatrix}2\,\mathrm{Re}\,\alpha & \mathrm{Im}(\alpha\bar{\beta})\\ \mathrm{Im}(\alpha\bar{\beta}) & 2\,\mathrm{Re}\,\beta\end{bmatrix}$, Proposition 3.1 yields the requirement in the case (i).

The case (ii): Since $|\mathrm{Re}\,\alpha|+|\mathrm{Re}\,\beta|=0$ (for $\eta'=\eta^0$), we have Re $b_0=0$. Therefore, it follows that

$$(1,\,0)\begin{bmatrix}2\,\mathrm{Re}\,\alpha & \mathrm{Im}(\alpha\bar{\beta})\\ \mathrm{Im}(\alpha\bar{\beta}) & 2\,\mathrm{Re}\,\beta\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=2\,\mathrm{Re}\,\alpha=2(\mathrm{Re}\,b')\cdot\eta'\,.$$

Noting Re $b'\neq 0$, we see that the matrix $\begin{bmatrix}2\,\mathrm{Re}\,\alpha & \mathrm{Im}(\alpha\bar{\beta})\\ \mathrm{Im}(\alpha\bar{\beta}) & 2\,\mathrm{Re}\,\beta\end{bmatrix}$ is not non-negative for $\eta'\in V$ (V is a neighborhood in S). Hencc, by Proposition 3.1 $f_{\eta'}(z)=0$ has a root in Im $z<0$ for $\eta'\in V$.

The case (iii): Then $f_{\eta'}(z)$ is of the form

$$f_{\eta'}(z)=i\{(\mathrm{Im}\,\alpha)z^2-2z-(\mathrm{Im}\,\beta)\}\,.$$

From this form, $f_{\eta'}(z)=0$ has a root in Im $z<0$ when

$$D(\eta')=1+(\mathrm{Im}\,\alpha)\,(\mathrm{Im}\,\beta)$$

is negative. Therefore, if '$D(\eta^0)<0$' or '$D(\eta^0)=0$ and $\eta^0\neq\pm|\mathrm{Im}\,b'|^{-1}\cdot\mathrm{Im}\,b''$', the requirement is obtained. When $D(\eta^0)=0$ and '$\eta^0=|\mathrm{Im}\,b'|^{-1}\cdot\mathrm{Im}\,b'$ or $-|\mathrm{Im}\,b'|^{-1}\cdot\mathrm{Im}\,b''$, it follows that

$$1+\sum_{j=1}^{n-1}b_j^2-b_0^2=0\,,$$

which is contrary to (0.3). Therefore we get the requirement in the case (iii).

Now next, let us show that if $f_{\eta'}(z)=0$ has a root $z_-(\eta')$ in $\operatorname{Im} z<0$ for $\eta'\in V$ $\operatorname{Re} z_-(\eta')$ does not always vanish. Proving this, we use the assumption $n\geqq 3$. Suppose that $z_-(\eta')=i\lambda(\eta')$ for $\eta'\in V$ (where $\lambda(\eta')=\operatorname{Im} z_-(\eta')<0$). Then it follows that

$$\begin{cases}-(\operatorname{Re}\alpha)\lambda^2+2\lambda-(\operatorname{Re}\beta)=0\,,\\ -(\operatorname{Im}\alpha)\lambda^2-(\operatorname{Im}\beta)=0\,.\end{cases}\tag{3.1}$$

Eliminating λ from these, we have

$$\psi_2^2-(\operatorname{Im} b_0)^2=(\psi_1\operatorname{Im} b_0-\psi_2\operatorname{Re} b_0)^2\qquad\text{for }\eta'\in V\,,\tag{3.2}$$

where $\psi_1=\operatorname{Re}(b'\cdot\eta')$ and $\psi_2=\operatorname{Im}(b'\cdot\eta')$. $\operatorname{Im} b'\neq 0$ follows from the assumption that $\operatorname{Im} b_0,\cdots,\operatorname{Im} b_{n-1}$ are not all equal to 0; because, if $\operatorname{Im} b'=0$, $(1+\psi_1(\eta'))^2\cdot(\operatorname{Im} b_0)^2=0$ holds for any $\eta'\in V$ and so $\operatorname{Im} b_0=0$. When $\operatorname{Re} b'$ and $\operatorname{Im} b'$ are linearly dependent, we can write $\psi_1(\eta')=\nu\psi_2(\eta')$ for a constant ν. Putting it into (3.2), we have

$$\{(\nu\operatorname{Im} b_0-\operatorname{Re} b_0)^2-1\}\psi_2(\eta')^2+(\operatorname{Im} b_0)^2=0\qquad\text{for }\eta'\in V\,.$$

This holds if and only if $\operatorname{Im} b_0=0$ and $(\operatorname{Re} b_0)^2=1$. Therefore, $b_0=-1$ ($b_0\neq 1$ is assumed). $b_0=-1$ yields $\operatorname{Re}\alpha+\operatorname{Re}\beta=2$ and $\operatorname{Im}\alpha+\operatorname{Im}\beta=0$, which is incompatible with (3.1). When $\operatorname{Re} b'$ and $\operatorname{Im} b'$ are linearly independent, we take an orthogonal base $\{e_j\}_{j=1,\cdots,n-1}$ in $\boldsymbol{R}^{n-1}$ such that e_1, e_2 are contained by the plane expanded by $\operatorname{Re} b'$ and $\operatorname{Im} b'$ and that $e_3,\cdots,e_{n-1}$ are orthogonal to $\operatorname{Re} b'$ and $\operatorname{Im} b'$. Then, there exist constants $\tilde{\eta}_3,\cdots,\tilde{\eta}_{n-1}$, $r(\neq 0)$ and an interval $[\theta_1,\theta_2]$ such that $\eta(\theta)\equiv(r\cos\theta)e_1+(r\sin\theta)e_2+\tilde{\eta}_3e_3+\cdots+\tilde{\eta}_{n-1}e_{n-1}$ belongs to V for any $\theta\in[\theta_1,\theta_2]$. When $\eta'=\eta(\theta)$, ψ_1 and ψ_2 are written by the form

$$\begin{aligned}\begin{bmatrix}\psi_1(\theta)\\ \psi_2(\theta)\end{bmatrix}&=r\begin{bmatrix}\langle\operatorname{Re} b',e_1\rangle & \langle\operatorname{Re} b',e_2\rangle\\ \langle\operatorname{Im} b',e_1\rangle & \langle\operatorname{Im} b',e_2\rangle\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\\ &\equiv {}^tA\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\quad(\det A\neq 0)\,.\end{aligned}$$

Therefore, from (3.2) it follows that

$$\begin{aligned}(\cos\theta\ \sin\theta)\,{}^tA\begin{bmatrix}(\operatorname{Im} b_0)^2 & -(\operatorname{Re} b_0)(\operatorname{Im} b_0)\\ -(\operatorname{Re} b_0)(\operatorname{Im} b_0) & -1+(\operatorname{Re} b_0)^2\end{bmatrix}A\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\\ =-(\operatorname{Im} b_0)^2\qquad\text{for }\theta\in[\theta_1,\theta_2]\,.\end{aligned}$$

It holds if and only if ${}^tA\begin{bmatrix}(\operatorname{Im} b_0)^2 & -(\operatorname{Re} b_0)(\operatorname{Im} b_0)\\ -(\operatorname{Re} b_0)(\operatorname{Im} b_0) & -1+(\operatorname{Re} b_0)^2\end{bmatrix}A=-(\operatorname{Im} b_0)^2I$. This implies that $b_0=\pm 1$. We have seen earlier that $b_0=\pm 1$ cannot be admitted. Thus (3.2) does not always hold for $\eta'\in V$, which proves Lemma 3.1.

Proof of Theorem 2. For simplicity, let $y_0=0$. Suppose that (0.1) is C^∞ well-posed, and let the propagation speed be less than v. We set

$$\Omega_0 = \{(x, y, t): 0 \leqq t < 2,\ (x^2+|y|^2)^{1/2} < 2v,\ 0 \leqq x\}\ .$$

Then there exist a domain $\Omega_0'(\supset \Omega_0)$ and an integer l such that

$$|u|_{0,\Omega_0} \leqq C(|\Box u|_{l,\Omega_0'} + |Bu|_{l,D_0'} + |D_t u|_{l,G_0'} + |u|_{l,G_0'})\ , \qquad u(x, y, t) \in C^\infty(\overline{\boldsymbol{R}_+^n} \times \overline{\boldsymbol{R}_+^1})\ , \tag{3.3}$$

where $D_0' = \Omega_0'|_{x=0}$ and $G_0' = \Omega_0'|_{t=0}$. Choose a function $\theta(s) \in C^\infty([0, \infty))$ satisfying (i) $0 < -\dfrac{d\theta}{ds}(s) \leqq \dfrac{1}{v}$ for $s>0$ (ii) $\dfrac{d^j\theta}{ds^j}(0)=0$ (for $j=0, 1, \cdots$), and set for a parameter k ($\geqq 1$)

$$\Omega_k = \{(x, y, t): 0 \leqq kt \leqq 1,\ kt-1 \leqq \theta(k(x^2+|y|^2)^{1/2}),\ 0 \leqq x\}\ ,$$
$$D_k = \Omega_k|_{x=0},\ G_k = \Omega_k|_{t=0}\ .$$

Then we have the following lemma (corresponding to Lemma 2.1 of Kajitani [6]):

Lemma 3.2. *There exist a constant C and integers l_0, l_1 independent of k such that the estimate*

$$|u|_{0,\Omega_k} \leqq C\, k^{l_0}\{|\Box u|_{l_1,\Omega_k} + |Bu|_{l_1,D_k} + |D_t u|_{l_1,G_k} + |u|_{l_1,G_k}\}\ , \qquad u(x, y, t) \in C^\infty(\Omega_k) \tag{3.4}$$

holds for $k \geqq 1$.

Proof. We have the following extension operator E: For data $F=(f, g, u_1, u_0) \in \boldsymbol{C}^l(\Omega_1)$ $(\equiv C^l(\Omega_1) \times C^l(D_1) \times C^l(G_1) \times C^l(G_1))$ with the compatibility condition of order $l-1$, $EF=(\tilde{f}, \tilde{g}, \tilde{u}_1, \tilde{u}_0)$ belongs to $\boldsymbol{C}_0^{\tilde{l}}(\Omega_0)$ (i.e., $EF \in \boldsymbol{C}^{\tilde{l}}(\Omega_0)$ and supp $EF \subset \Omega_0$) and satisfies the compatibility condition of order $\tilde{l}-1$; furthermore E: $\boldsymbol{C}^l(\Omega_1) \to \boldsymbol{C}_0^{\tilde{l}}(\Omega_0)$ is continuous. For data $F(X) \in \boldsymbol{C}^l(\Omega_k)$ $(X=(x,y,t))$ with the compatibility condition of order $l-1$, we transform them as follows:

$$F(X) \xrightarrow[(\tilde{X}=kX)]{} F\left(\frac{\tilde{X}}{k}\right) \equiv \tilde{F}(\tilde{X}) \xrightarrow{E} E\tilde{F}(\tilde{X}) \xrightarrow[(X=k^{-1}\tilde{X})]{} E\tilde{F}(kX)\ .$$

Then, $E\tilde{F}(kX)=(f', g', u_1', u_0')$ belongs to $\boldsymbol{C}_0^{\tilde{l}}(\Omega_0)$ and satisfies the compatibility condition of order $\tilde{l}-1$. Furthermore, there are constants C and l_0 independent of k ($\geqq 1$) such that

$$\begin{aligned} &|f'|_{\tilde{l},\Omega_0} + |g'|_{\tilde{l},D_0} + |u_1'|_{\tilde{l},G_0} + |u_0'|_{\tilde{l},G_0} \\ &\quad \leqq C\, k^{l_0}(|f|_{l,\Omega_k} + |g|_{l,D_k} + |u_1|_{l,G_k} + |u_0|_{l,G_k}). \end{aligned} \tag{3.5}$$

In view of finiteness of the propagation speed and the shape of the domain Ω_k, we see that the solution for (f', g', u_1', u_0') coincides with that for (f, g, u_1, u_0) on Ω_k. Therefore, from (3.3) and (3.5) the estimate (3.4) is derived. The proof is complete.

Let us set

$$B(k) = k\{D_x+\sum_{j=1}^{n-1} b_j(k^{-1}y, k^{-1}t)D_{y_j}+b_0(k^{-1}y, k^{-1}t)D_t\}+c(k^{-1}y, k^{-1}t)\,.$$

By change of the variable $x'=k^{-1}x$, $y'=k^{-1}y$, $t'=k^{-1}t$ and Lemma 3.2, we obtain the following lemma (corresponding to Lemma 2.2 of Kajitani [6]):

Lemma 3.3. *There are integers l_1, l_2 independent of k, ν $(\geqq 1)$ such that*

$$|u|_{0,\Omega_\nu}\leqq C_\nu k^{l_2}\{|\Box u|_{l_1,\Omega_\nu}+|B(k)u|_{l_1,D_\nu}+|D_t u|_{l_1,G_\nu}+|u|_{l_1,G_\nu}\}\,, \qquad u(x, y, t)\in C^\infty(\Omega_\nu)\,, \tag{3.6}$$

where the constant C_ν does not depend on k.

From now on, we shall construct an asymptotic solution violating (3.6) in the same way as in Kajitani [6]. Set

$$B^{(0)} = D_x+\sum_{j=1}^{n-1} b_j(0)D_{y_j}+b_0(0)D_t\,,$$

$$B^{(l)} = \sum_{j=1}^{n-1}\sum_{|\alpha|+i=l}\frac{y^\alpha t^i}{\alpha!i!}\left(\frac{\partial}{\partial y}\right)^\alpha\frac{\partial^i}{\partial t^i}b_j(0)D_{y_j}+\sum_{i+|\alpha|=l-1}\frac{y^\alpha t^i}{\alpha!i!}\left(\frac{\partial}{\partial y}\right)^\alpha\frac{\partial^i}{\partial t^i}c(0) \qquad (l=1, 2, \cdots)\,.$$

Then $B(k)$ is written by the form

$$B(k) = kB^{(0)}+B^{(1)}+\cdots+k^{-N+1}B^{(N)}+k^{-N}r^{(N+1)}\,.$$

Here $r^{(N+1)}$ is a first order operator and the norms $(|\cdot|_{l,\Omega_\nu})$ of its coefficients are bounded as $k\to\infty$. If the assumptions of Theorem 2 are satisfied, by Lemma 3.1 and Remark 3.1 we have constants ξ_0, η^0, τ_0 ($\mathrm{Im}\,\xi_0>0$, $\eta^0\in\boldsymbol{R}^{n-1}$, $\mathrm{Im}\,\tau_0<0$) such that

$$\begin{cases}\tau_0^2-\xi_0^2-|\eta^0|^2=0\,,\\ B^{(0)}(\xi_0, \eta^0, \tau_0)\equiv\xi_0+\sum_{j=1}^{n-1}b_j(0)\eta_j^0+b_0(0)\tau_0=0\,.\end{cases} \tag{3.7}$$

We set $\Phi(x, y, t)=\xi_0 x+\eta^0\cdot y+\tau_0 t$. Let us make

$$u_N(x, y, t; k) = \sum_{l=0}^{N} e^{ik\Phi(x,y,t)}v_l(x, y, t)k^{-l}$$

satisfy both $\Box u_N=0$ on Ω_ν and $B(k)u_N=0$ on D_ν (asymptotically). Noting (3.7), we have only to solve the following transport equation with a boundary condition:

$$\begin{cases} 2(\tau_0 D_t - \xi_0 D_x - \eta^0 \cdot D_y) v_l + \Box v_{l-1} = 0, \ x \geqq 0 \ (v_{-1} = 0), \\ \{B^{(0)} + (B^{(1)}\Phi)\} v_l + \Delta_{l-1} = 0, \ x = 0, \end{cases}$$

where $\Delta_{l-1} = \{(B_0^{(2)}\Phi) v_{l-1} + \cdots + (B_0^{(l+1)}\Phi) v_0\} + \{B^{(1)} v_{l-1} + \cdots + B^{(l)} v_0\}$ ($B_0^{(l)}$ is the principal part of $B^{(l)}$). Combining these equations on $\{x=0\}$, we have the equation for $\tilde{v}_l(y, t) = v_l|_{x=0}$:

$$(\tau_0 + \xi_0 b_0(0)) D_t \tilde{v}_l + \sum_{j=1}^{n-1} (\xi_0 b_j(0) - \eta_j^0) D_{y_j} \tilde{v}_l + (B_0^{(1)}\Phi) \tilde{v}_l + \tilde{\Delta}_{l-1} = 0,$$

where $\tilde{\Delta}_{l-1} = \left(\frac{1}{2} \Box v_{l-1} + \xi_0 \Delta_{l-1}\right)|_{x=0}$. Here, we can choose (τ_0, η^0) so that the above coefficients $\tau_0 + \xi_0 b_0(0)$, $\xi_0 b_1(0) - \eta_1^0, \cdots, \xi_0 b_{n-1}(0) - \eta_{n-1}^0$ do not all vinish ($\xi_0 = \xi_+(\tau_0, \eta^0)$)[1]. Therefore, by the Cauchy-Kowalewski theorem we get the solution $\tilde{v}_l$ with $\tilde{v}_0(0) \neq 0$, and so v_l with $v_0(0) \neq 0$. Take the integer N satisfying $N > l_1 + l_2 + 1$ (l_1 and l_2 are the integers in Lemma 3.3), and fix ν so largely that $v_0, \cdots, v_N$ are all defined on Ω_ν and $v_0(x, y, t) \neq 0$ on Ω_ν. Then it follows that

$$|u_N|_{0,\Omega_\nu} \geqq \left\{ \left| v_0\left(0, 0, \frac{1}{\nu}\right) \right| - C_1 k^{-1} \right\} e^{(-\mathrm{Im}\,\tau_0)k/\nu},$$

$$|\Box u_N|_{l_1,\Omega_\nu} + |B(k) u_N|_{l_1,D_\nu} \leqq C_2 k^{l_1+1-N} e^{(-\mathrm{Im}\,\tau_0)k/\nu},$$

$$|D_t u_N|_{l_1,G_\nu} + |u_N|_{l_1,G_\nu} \leqq C_3 k^{l_1+1}.$$

Therefore, by Lemma 3.3 we have

$$\left\{ \left| v_0\left(0, 0, \frac{1}{\nu}\right) \right| - C_1 k^{-1} \right\} e^{(-\mathrm{Im}\,\tau_0)k/\nu} \leqq C_\nu C_2 k^{l_1+l_2+1-N} e^{(-\mathrm{Im}\,\tau_0)k/\nu} + C_\nu C_3 k^{l_1+l_2+1}.$$

This cannot hold when $k \to \infty$, which proves Theorem 2.

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[1] In the case $b_0(0) \neq 1$, as is seen from the proof of Lemma 3.1, we have infinitely numerous $\eta_0 (\in \{|\eta'| = 1\})$ and $\tau_0(\eta^0)$ ($\mathrm{Im}\,\tau_0(\eta^0) < 0$) such that $R(\eta^0, \tau_0(\eta^0)) = 0$. However, if $\eta_j^0 - \xi_0 b_j(0) = 0$ for $j = 1, 2, \cdots, n-1$ ($\xi_0 = \xi_+(\eta^0, \tau_0(\eta^0))$), then $\xi_0 = \pm(\sum b_j(0)^2)^{-1/2}$ and $\eta_j^0 = \pm(\sum b_j(0)^2)^{-1/2} \cdot b_j(0)$. In the case $b_0(0) = 1$, we see it easily.

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