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## AFFINE FUNCTIONS ON ALEXANDROV SURFACES

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### 0. Introduction

An *Alexandrov surface*  $X$  is by definition a 2-Hausdorff dimensional, connected, locally compact and complete length space of curvature bounded from below in the sense of Alexandrov which has no boundary points. For a point  $x \in X$ ,  $\Sigma'_x$  is the set of all directions of geodesics emanating from  $x$  equipped with the angular metric  $\angle$ . Let  $\Sigma_x$  be the metric completion of  $\Sigma'_x$ . We call it the *space of directions at  $x$* . This corresponds to the unit tangent sphere in Riemannian geometry. The space of directions  $\Sigma_x$  for each  $x \in X$  is either a circle of circumference  $\leq 2\pi$  or a segment of length  $\leq \pi$  (see [1],[9]). Since, by definition,  $X$  has no boundary, we mean that  $\Sigma_x$  is a circle for every  $x \in X$ . A point  $x \in X$  is called *singular* iff the circumference of  $\Sigma_x$  is less than  $2\pi$ , and we denote by  $Sing(X)$  the set of all singular points of  $X$ . It is a well known fact in Alexandrov geometry that the pointed Gromov-Hausdorff limit  $\lim_{r \rightarrow 0} (1/rX, x)$  of the  $1/r$ -rescaling of the metric around  $x$  is the flat cone  $(C(\Sigma_x), o^*)$  over  $\Sigma_x$  with vertex  $o^*$  for every  $x \in X$ . We call  $C(\Sigma_x)$  the *tangent cone* at  $x$ , which corresponds to the tangent space in Riemannian geometry.

A real valued function  $\psi : X \rightarrow R$  on  $X$  is called *convex* iff the following inequality holds for an arbitrary geodesic  $\gamma : [a, b] \rightarrow X$  and arbitrary  $\lambda \in [0, 1]$ :

$$(*) \quad \psi \circ \gamma((1 - \lambda)a + \lambda b) \leq (1 - \lambda) \cdot \psi \circ \gamma(a) + \lambda \cdot \psi \circ \gamma(b).$$

A convex function on  $X$  is not in general continuous, because  $X$  admits the singular set  $Sing(X)$ . Nevertheless, we can introduce the notion of the  $a$ -level set of  $\psi$  for each  $a \in (\inf \psi, \infty)$  (see §0 of [6]). Every convex function on a complete Riemannian manifold  $M$  is always locally Lipschitz. Moreover,  $M$  is automatically noncompact if such a convex function is nonconstant. However, an Alexandrov surface  $X$  which admits a locally nonconstant convex function is not always noncompact (see Theorem A of [6]). The following results have been established by the author: Let  $\psi : X \rightarrow R$  be a convex function satisfying the condition

$$\text{int} \left( \bigcap_{a > \inf \psi} \overline{\{x \in X | \psi(x) \leq a\}} \right) = \emptyset.$$

Then we conclude the following:

- i.  $\sup \psi = \infty$ .
- ii. Each component of the  $a$ -level set for each  $a \in (\inf \psi, \infty)$  is either a simple closed curve or a line.
- iii. For each  $a \in (\inf \psi, \infty)$ , the  $a$ -level set has at most two components. Moreover, if the  $a$ -level set for some  $a \in (\inf \psi, \infty)$  has two components, then the same holds for all the  $b$ -level set with  $b \in (\inf \psi, \infty)$ , and in each case, the two components are both simple closed curves or both lines.
- iv.  $X$  is homeomorphic to one of three spaces,  $R^2$ ,  $S^1 \times R$  or  $(S^1 \times R)/Z_2$ .

The purpose of the present paper is to determine the metric structure of  $X$  admitting a non-trivial affine function. Here, a function  $\varphi : X \rightarrow R$  is by definition *affine* iff the equality in  $(*)$  always holds for arbitrary unit speed geodesic  $\gamma : [a, b] \rightarrow X$  and arbitrary  $\lambda \in [0, 1]$ . Letting  $X_a^a := \{x \in X | \varphi(x) = a\}$  define the  $a$ -level set of  $\varphi$  for convenience, we specialize the above result to the case of affine functions as follows:

**Theorem 1.** *If an Alexandrov surface  $X$  admits a non-trivial affine function  $\varphi : X \rightarrow R$ , then for every  $a \in (-\infty, \infty)$  there is an isometric map*

$$I : X_a^a \times (-\infty, \infty) \rightarrow X$$

*such that  $I(y, t) \in X_t^t$  for every  $(y, t) \in X_a^a \times (-\infty, \infty)$ . Moreover,  $X$  is isometric to either flat  $R^2$  or flat  $S^1 \times R$ .*

Note that every level set of an affine function on  $X$  is totally convex, and hence such a set is either a simple closed geodesic or a straight line. In particular  $\text{Sing}(X) = \emptyset$ , and hence  $C(\Sigma_x)$  is isometric to  $R^2$  for all  $x \in X$ . Since  $-\varphi$  is also affine, we conclude from (i) that the range of  $\varphi$  is  $(-\infty, \infty)$ .

The fundamental notion used here is the *directional derivative*  $d\varphi(v)$  of an affine function  $\varphi : X \rightarrow R$  for  $v \in \Sigma'_x$ . Set

$$d\varphi(v) := (\varphi \circ \gamma_v)'_+(0), \quad v \in \Sigma'_x, \quad x \in X,$$

where  $\gamma_v : [0, l(v)] \rightarrow X$  is a geodesic such that  $\gamma_v(0) = x$  and  $\dot{\gamma}_v(0) = v$ , and  $(\cdot)'_+$  is the right-hand derivative. Note that we do not take the limit in the above definition since  $\varphi$  is affine. We will show in Lemma 1.1 that  $d\varphi : \Sigma'_x \rightarrow R$  can be extended continuously to an affine function  $d\varphi : C(\Sigma_x) \rightarrow R$  on the whole tangent cone  $C(\Sigma_x)$ . Note that we use the same expression  $d\varphi(v)$  for  $v \in C(\Sigma_x)$ . It follows from the compactness of  $\Sigma_x$  that the function of the directional derivative  $d\varphi|_{\Sigma_x} : \Sigma_x \rightarrow R$  attains its maximum at some (unique) direction  $v_{\varphi, x} \in \Sigma_x$  (see Lemma 1.2). This allows us

to introduce the *generalized gradient*  $\nabla\varphi_x$  of  $\varphi$  at  $x \in X$ , that is,  $\nabla\varphi_x/|\nabla\varphi_x| = v_{\varphi,x}$  realizes the maximum of  $d\varphi|_{\Sigma_x}$ . Here we mean by  $|\cdot|$  the  $R^2$ -norm under identifying  $C(\Sigma_x)$  with  $R^2$ . The following lemma on the generalized gradient plays a crucial role in our investigation:

**Lemma 2.** *The following statements are true:*

(1) *We have for every  $x \in X$  and for every  $v \in C(\Sigma_x)$*

$$d\varphi(v) = |\nabla\varphi_x| |v| \cos \angle(\nabla\varphi_x, v).$$

(2) *Let  $a$  and  $b$  be arbitrary fixed numbers with  $a < b$ . Then for every  $x \in X_a^a$  and a minimal geodesic  $\sigma_x : [0, l(x)] \rightarrow X$  from  $x$  to  $X_b^b$ , we have*

$$\dot{\sigma}_x(0) = v_{\varphi,x} = \nabla\varphi_x/|\nabla\varphi_x|.$$

*Hence there is a unique minimal geodesic from  $x$  to  $X_b^b$  for every  $x \in X_a^a$ .*

(3)  $|\nabla\varphi_x|$  is constant for all  $x \in X$ .

To show Theorem 1, the flatness of every geodesic triangle in  $X$  is required. Therefore we prove the similarities of geodesic triangles as follows. Let  $a$  and  $b$  be as in Lemma 2(2), and let  $\gamma : [0, l] \rightarrow X$  be a geodesic from a point on  $X_a^a$  to a point on  $X_b^b$ . For every  $s \in (0, l]$ , let  $\sigma_s : [0, l(s)] \rightarrow X$  be the (unique) minimal geodesic from  $\gamma(s)$  to  $X_a^a$ . Then it follows from Lemma 2 (1) and (3) that the angle between  $\sigma_s$  and  $\gamma$  is constant for all  $s \in (0, l]$ . This is true for the angle between  $\sigma_s$  and  $X_a^a$ . Let  $\Delta(t)$  for  $t \in (0, l]$  be a geodesic triangle spanned by geodesics  $\{\sigma_s | 0 < s \leq t\}$ . Using this and the first variation formula, we conclude the following:

**Proposition 3.** *With the above notation,  $\Delta(t_1)$  and  $\Delta(t_2)$  for all  $t_1, t_2 \in (0, l]$  are similar triangles, i.e., all ratio of the lengths of corresponding edges are same.*

In §1 we prove assertions (1)-(3) of Lemma 2, and in §2 we construct the isometric map  $I$  indicated in Theorem 1.

## 1. Proof of Lemma 2

From this point let  $X$  be an Alexandrov surface admitting an affine function  $\varphi : X \rightarrow R$ . We denote by  $|x, y|$  the distance between  $x$  and  $y$  for  $x, y \in X$ . We use the following fact through this paper:

FACT 1.0. The pointed Gromov-Hausdorff limit  $\lim_{t \rightarrow 0} (1/tX, x)$  of the  $(1/t)$ -rescaling of the metric around  $x$  is the flat cone  $(C(\Sigma_x), o^*)$  over  $\Sigma_x$  with vertex  $o^*$  for every  $x \in X$ .

Since  $X$  admits the affine function  $\varphi$ ,  $\Sigma_x$  is the circle of length  $2\pi$  for all  $x \in X$ . Thus  $C(\Sigma_x)$  is identified with  $R^2$ , and  $\Sigma_x$  is identified with the unit circle centered at origin of  $R^2$ . Hence we can denote an arbitrary element of  $C(\Sigma_x)$  by  $\lambda u$  for some  $\lambda \in [0, \infty)$  and some  $u \in \Sigma_x$ .

We first discuss the directions in  $\Sigma'_x$  for arbitrary fixed  $x \in X$ . Let  $u, v$  be fixed directions in  $\Sigma'_x$  with  $0 < \angle(u, v) < \pi$ . Then we choose the direction  $w_\lambda \in \Sigma'_x$  for some  $\lambda \in (0, 1)$  such that (by identifying  $C(\Sigma_x)$  with  $R^2$ )

$$w_\lambda = \frac{(1 - \lambda)u + \lambda v}{|(1 - \lambda)u + \lambda v|},$$

where  $|\cdot|$  denotes the standard norm in  $R^2$ . Using this notation, the following holds:

**Lemma 1.1.** *We have*

$$d\varphi(w_\lambda) = [(1 - \lambda) \cdot d\varphi(u) + \lambda \cdot d\varphi(v)] \cdot \frac{\sin \angle(u, w_\lambda)}{\lambda \sin \angle(u, v)}.$$

Moreover,  $d\varphi : \Sigma'_x \rightarrow R$  has the continuous extension  $d\varphi : \Sigma_x \rightarrow R$ , and  $d\varphi : C(\Sigma_x) \rightarrow R$  becomes an affine function again.

Proof. Since the directional derivatives are defined locally, we discuss only in (sufficiently small) disk neighborhood  $U_x$  of  $x$ . The bracket part in the above equation follows from the definition of affine functions, and the other part follows from Euclidean geometry on  $C(\Sigma_x)$ , the sine formula and from Fact 1.0.

With the equation established, the second assertion easily follows. The third assertion follows from the property that  $d\varphi(\lambda v) = \lambda d\varphi(v)$  for all  $\lambda \in (0, \infty)$  and  $v \in \Sigma_x$ .  $\square$

For every  $x \in X$ , we denote by  $O_x$  the directions in  $\Sigma_x$  tangent to  $X_{\varphi(x)}^{\varphi(x)}$ . Clearly,  $O_x$  consists of exactly two elements,  $O_{1,x}$  and  $O_{2,x}$  such that  $\angle(O_{1,x}, O_{2,x}) = \pi$  and  $d\varphi(O_{1,x}) = d\varphi(O_{2,x}) = 0$ . Put

$$M_\varphi^x := \{v \in \Sigma_x \mid d\varphi(v) = \max_{w \in \Sigma_x} d\varphi(w)\} \text{ and } m_\varphi^x := \{v \in \Sigma_x \mid d\varphi(v) = \min_{w \in \Sigma_x} d\varphi(w)\}.$$

Then the configuration of  $O_x$ ,  $M_\varphi^x$  and  $m_\varphi^x$  is determined as follows.

**Lemma 1.2.** *For every  $v \in M_\varphi^x$  and  $u \in m_\varphi^x$ , we have*

$$\angle(O_x, v) = \angle(O_x, u) = \frac{\pi}{2}.$$

Hence each of the sets  $M_\varphi^x$  and  $m_\varphi^x$  consists of only one element.

Proof. Suppose that  $\angle(O_x, v) \neq \pi/2$  for some  $v \in M_\varphi^x$ . Since  $\angle(O_{1,x}, O_{2,x}) = \pi$ , we can choose a direction  $w \in \Sigma_x$  such that  $d\varphi(w) > 0$  and  $\angle(O_x, w) = \pi/2$ . Here we assume that  $\angle(O_{1,x}, v) > \pi/2$ . Then, under identifying  $C(\Sigma_x)$  with  $R^2$ , we have  $w = [(1 - \lambda)O_{1,x} + \lambda v]/|(1 - \lambda)O_{1,x} + \lambda v|$  for some  $\lambda \in (0, 1)$ . Therefore, from the equation in Lemma 1.1, we have

$$\begin{aligned} d\varphi(w) &= [(1 - \lambda)d\varphi(O_{1,x}) + \lambda \cdot d\varphi(v)] \cdot \frac{\sin \angle(O_{1,x}, w)}{\lambda \sin \angle(O_{1,x}, v)} \\ &= \frac{1}{\sin \angle(O_{1,x}, v)} \cdot d\varphi(v) > d\varphi(v). \end{aligned}$$

This contradicts the choice of  $v \in M_\varphi^x$ .

Since  $-\varphi$  is also affine,  $\angle(O_x, u) = \pi/2$  follows for every  $u \in m_\varphi^x$ .  $\square$

Proof of Lemma 2 (1). We see from Lemma 1.1 that  $d\varphi$  is an affine function on  $C(\Sigma_x)$  isometric to  $R^2$ . We can easily see a fact that every affine function on  $R^2$  satisfies the equation in Lemma 2 (1).  $\square$

Proof of Lemma 2 (2). Suppose that  $\dot{\sigma}_x(0) \neq \nabla\varphi_x$  for some minimal geodesic  $\sigma_x : [0, l(x)] \rightarrow X$  from  $x$  to  $X_b^b$ . Then we construct a broken geodesic segment

$$\xi = \bigcup_i \gamma_i : [0, l(\xi)] \rightarrow X$$

such that  $(\varphi \circ \xi)'_+(s) > d\varphi(\dot{\sigma}_x(0))$  for every  $s \in [0, l(\xi))$  and  $\xi(0) = x$ ,  $\xi(l(\xi)) \in X_b^b$ . The construction of  $\xi$  is achieved by inductive steps as follows. First of all, we note that  $d\varphi(\dot{\gamma}(s))$  is constant in  $s$  on each geodesic  $\gamma : [0, l(\gamma)] \rightarrow X$ . By the continuity of  $d\varphi : \Sigma_x \rightarrow R$ , we can find a direction  $v_1 \in \Sigma'_x$  such that  $d\varphi(v_1) > d\varphi(\dot{\sigma}_x(0))$ . Let  $\gamma_1 : [0, l_1] \rightarrow X$  be a maximal geodesic tangent to  $v_1$ . If  $\gamma_i(l_i)$  does not reach  $X_b^b$  for the  $i$ -th maximal geodesic  $\gamma_i : [0, l_i] \rightarrow X$  tangent to  $v_i \in \Sigma_{\gamma_i(0)}$ , then using the continuity of  $d\varphi : \Sigma_x \rightarrow R$  and Lemma 2(1), we can find a direction  $v_{i+1} \in \Sigma_{\gamma_i(l_i)}$  such that  $d\varphi(v_{i+1}) > d\varphi(\dot{\sigma}_x(0))$ , and we denote the maximal geodesic tangent to  $v_{i+1}$  by  $\gamma_{i+1} : [0, l_{i+1}] \rightarrow X$ . Then, put a broken geodesic segment  $\xi := \bigcup_i \gamma_i : [0, \sum_i l_i] \rightarrow X$ , and  $x_1 := \xi(\sum_i l_i)$ ,  $l(\xi) := \sum_i l_i$ .

It may happen that the endpoint  $x_1$  of  $\xi$  does not reach to  $X_b^b$ . We then join  $x$  to  $x_1$  by a minimal geodesic  $\alpha : [0, |x, x_1|] \rightarrow X$ . By the minimizing property of  $\alpha$ , we see that  $d\varphi(\dot{\alpha}(|x, x_1|)) \geq (\varphi \circ \xi)'_+(s)$  for all  $s \in [0, l(\xi)]$ . Since  $d\varphi(\dot{\alpha}(|x, x_1|)) > d\varphi(\dot{\sigma}_x(0))$ , using the continuity of  $d\varphi : \Sigma_x \rightarrow R$ , we can find a direction  $w_1 \in \Sigma'_x$  with  $d\varphi(w_1) > d\varphi(\dot{\sigma}_x(0))$ , and hence we proceed with inductive steps to construct  $\xi$ .

From the above reason, we may assume that  $x_1 \in X_b^b$ . Clearly, we have

$$\int_0^{l(\xi)} (\varphi \circ \xi)'_+(s) ds > \int_0^{l(\xi)} d\varphi(\dot{\sigma}_x(0)) ds.$$

Moreover, we conclude that  $l(x) > l(\xi)$  since  $\varphi \circ \xi$  is almost everywhere differentiable. This contradicts the minimizing property of  $\sigma_x$ .  $\square$

**Proof.** Proof of Lemma 2 (3) We prove that  $|\nabla \varphi_{x_1}| = |\nabla \varphi_{x_2}|$  for every  $x_1, x_2 \in X$ . The first step of the proof is to show that  $|\nabla \varphi_x|$  is constant for all  $x \in X_a^a$  and for arbitrary fixed  $a \in (-\infty, \infty)$ . Choose  $x_1, x_2 \in X_a^a$  and let  $\tau : [0, |x_1, x_2|] \rightarrow X$  be a minimal geodesic from  $x_1$  to  $x_2$ . Necessarily,  $\tau \subset X_a^a$ . Set  $\sigma_s : [0, l(s)] \rightarrow X$  for the minimal geodesic from  $\tau(s)$  to  $X_b^b$ . Then it follows from (1) and (2) of Lemma 2 and the first variation formula that the function  $g = g(s) := l(s)$  is differentiable in  $s \in (0, |x_1, x_2|)$ , and  $\frac{dg}{ds} = 0$  for all  $s \in (0, |x_1, x_2|)$ . This therefore implies that  $|\nabla \varphi_{x_1}| = (b - a)/l(0) = (b - a)/l(|x_1, x_2|) = |\nabla \varphi_{x_2}|$ .

The second step of the proof is to show that  $|\nabla \varphi_{x_1}| = |\nabla \varphi_{x_2}|$  when  $x_1 \in X_a^a$  and  $x_2 \in X_b^b$  for distinct numbers  $a, b \in (-\infty, \infty)$ . Here we assume  $a < b$ . Set  $\sigma_{x_1} : [0, l(x_1)] \rightarrow X$  for the minimal geodesic from  $x_1$  to  $X_b^b$  and  $z := \sigma_{x_1}(l(x_1))$ . Then it follows from (1) and (2) of Lemma 2 that  $|\nabla \varphi_{x_1}| = |\nabla \varphi_z|$ . From the first step of the proof, we see that  $|\nabla \varphi_z| = |\nabla \varphi_{x_2}|$ , and hence  $|\nabla \varphi_{x_1}| = |\nabla \varphi_{x_2}|$ .  $\square$

## 2. Proof of Theorem 1

In this section, we construct a isometric map  $I$  in Theorem 1. Lemma 2 (2) guarantees that for an arbitrary fixed  $a \in (-\infty, \infty)$  there exist the gradient flow  $\phi_x : (-\infty, \infty) \rightarrow X$  passing through  $x \in X_a^a$  such that  $\phi_x(t) \in X_t^t$  for every  $t \in (-\infty, \infty)$ . Then the required bijective map  $I : X_a^a \times (-\infty, \infty) \rightarrow X$  is obtained by  $I(x, t) := \phi_x(t)$  for  $(x, t) \in X_a^a \times (-\infty, \infty)$ . We will verify that the map  $I : X_a^a \times (-\infty, \infty) \rightarrow X$  satisfies the following:

$$|I(x_1, t_1), I(x_2, t_2)|^2 = |x_1, x_2|^2 + |t_1 - t_2|^2$$

for every  $(x_1, t_1), (x_2, t_2) \in X_a^a \times (-\infty, \infty)$ .

It follows from Lemma 2 and the first variation formula that this flow  $\phi_x$  satisfies the following:

$$(2.1) \quad \phi_x \text{ is perpendicular to } X_t^t \text{ for every } t \in (-\infty, \infty).$$

$$(2.2) \quad |\phi_{x_1}(t), \phi_{x_2}(t)| \text{ is constant for all } t \in (-\infty, \infty).$$

We first normalize  $\varphi$  so that  $|\nabla\varphi_x| = 1$  for all  $x \in X$ . From (2.2), we may assume without loss of generality that the geodesic  $\gamma : [0, l] \rightarrow X$  in Proposition 3 is a minimal geodesic from  $I(x_2, t_2) \in X_b^b$  to  $I(x_1, t_1) \in X_a^a$ . Put  $\theta := \angle(\gamma, X_b^b) \in [0, \pi/2]$ . Then it suffices to prove the distance-preserving property of  $I$  in the case that  $\theta \neq 0, \pi/2$ . With the same notation as in Proposition 3, if we denote by  $\bar{\Delta}$  the  $1/t$ -rescaling limit triangle of  $\Delta(t)$  for  $t \rightarrow 0+$ , fixing the vertex  $\gamma(0)$  of  $\Delta(t)$ , it follows from Proposition 3 that  $\bar{\Delta}$  and  $\Delta(l)$  are similar triangles. Moreover, it follows from Fact 1.0 that  $\bar{\Delta}$  is a flat right triangle with an inner angle  $\theta$ . Together with this and the similarity of  $\bar{\Delta}$  and  $\Delta(l)$ , we observe that

$$\begin{aligned} |I(x_1, t_1), I(x_2, t_2)|^2 &= \frac{1}{\cos^2 \theta} |x_1, x_2|^2 \\ &= |x_1, x_2|^2 + \tan^2 \theta |x_1, x_2|^2. \end{aligned}$$

Using again the similarity of  $\bar{\Delta}$  and  $\Delta(l)$ , we have  $\tan \theta = |t_1 - t_2|/|x_1, x_2|$ . Hence the proof is complete.

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### References

- [1] Yu.D.Burago, M.Gromov and G.Perelman: *A. D. Alexandrov's spaces with curvatures bounded below* (english version), Russian Math. Surveys, **47**:2 (1992), 1-58
- [2] J.Cheeger and D.Gromoll: *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413-443
- [3] J.Cheeger and D.Gromoll: *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **61** (1971), 119-128
- [4] R.E.Greene and K.Shihama: *Convex functions on complete noncompact manifolds: Topological structure*, Invent. Math. **63** (1981), 129-157
- [5] R.E.Greene and K.Shihama: *The isometry groups of manifolds admitting nonconstant convex functions*, J. Math. Soc. Japan, **39** (no.1) (1987), 1-16
- [6] Y.Mashiko: *Convex functions on Alexandrov surfaces*, to appear in Trans. AMS
- [7] Y.Otsu and T.Shioya: *The Riemannian structure of Alexandrov spaces*, J. Diff. Geom. **39** (1994), 629-658
- [8] V.A.Sharafutdinov: *The Pogorelov-Klingenberg theorem for manifolds homeomorphic to  $R^n$* , Transl. from Sib. Mat. Zhur. **18** (4) (1977), 649-657
- [9] K.Shihama: An Introduction to the Geometry of Alexandrov Spaces, Seoul National univ. Lecture notes series 8, 1992

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