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μ-ELEMENTS IN S^1-TRANSFER IMAGES

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1. Introduction and Results

Let $\mu_r \in \pi_{8r+1}^s(S^0)$ be a μ-element of the $(8r+1)$-dimensional stable homotopy group of the sphere. Adams [1] gave a definition of μ_r, and showed that $d_R(\mu_r) \neq 0$ and

$$\pi_{8r+1}^s(S^0) = Z/2 \langle \mu_r \rangle \oplus \mathrm{Ker}(d_R) \qquad \text{for} \quad r \geq 0\,, \tag{1.1}$$

where $d_R: \pi_{8r+1}^s(S^0) \rightarrow \mathrm{Hom}(KO^0(S^0), KO^0(S^{8r+1}))$ is the d-invariant in the KO-theory. We assume that the mod 2 Adams filtration of μ_r is equal to $4r+1$, which determines each μ_r as a uniquely defined element.

Throughout the paper, CP_n^m denotes the suspension spectrum of a Thom complex $(CP^{m-n})^{n\xi}$ for $-\infty < n \leq m \leq \infty$ and $n \neq \infty$, where ξ is the canonical complex line bundle over the complex projective space CP^{m-n}. In [5] and [10], it is shown that, for $r>0$, μ_r is not in the image of the homomorphism t_*: $\pi_{8r}^s(CP_0^\infty) \rightarrow \pi_{8r+1}^s(S^0)$ induced from a stable map t called a S^1-transfer map. On the other hand, Knapp [9] investigated S^1-transfer maps $t_n: \Sigma^{-2n+1} CP_n^\infty \rightarrow S^0$, and proved that μ_r is in the image of $(t_2)_*$. We remark that $t_0 = t$. The purpose of the present paper is to discuss whether or not μ_r is in the image of $(t_n)_*$ for other values of n.

Let $I(\mu_r)$ be an ideal of $\pi_*^s(S^0)$ generated by μ_r. Then our main result is stated as follows:

Theorem 1. *Assume that* $r \geq 0$. *If* $\binom{8r+2k+1}{4r+1} + 2\binom{8r+2k-1}{4r-1} \not\equiv 0 \bmod 4$ *for an integer* k, *then*

$$I(\mu_r) \subset \mathrm{Im}\,[(t_{2k})_*: \pi_*^s(\Sigma^{-4k+1} CP_{2k}^\infty) \rightarrow \pi_*^s(S^0)]\,.$$

In contrast with Theorem 1, it holds that $\mu_r \notin \mathrm{Im}(t_{2k+1})_*$ for any $r \geq 0$ and k. More generally, if we treat μ_r with indeterminacy $\mathrm{Ker}(d_R)$, then it is possible to give a necessary and sufficient condition for our problem. In order to state it, we need some notations. Let a_i^j be the coefficient of x^i in the power series expansion of $((e^x-1)/x)^j$, and, for $n \leq m$,

$$u(n, m) = \mathrm{Min}\,\{l>0 \mid l a_{m-j}^j \in Z \quad \text{for all } j \text{ with } n \leq j \leq m\}\,. \tag{1.2}$$

Also, for $n \leq m$, let

$$h(n, m) = |h: \pi^s_{2m}(CP^\infty_n) \to H_{2m}(CP^\infty_n; Z)| \tag{1.3}$$

be the order of the cokernel of the stable Hurewicz homomorphism. Then, we denoet by $u_2(n, m)$ and $h_2(n, m)$ the exponents of 2 in the prime power decompositions of $u(n, m)$ and $h(n, m)$ respeatively.

We denote the coset class of μ_r mod $\mathrm{Ker}(d_R)$ by

$$[\mu_r] = \{\mu_r + x \in \pi^s_{8r+1}(S^0) \mid x \in \mathrm{Ker}(d_R)\} \tag{1.4}$$

for $r \geq 0$. Then we have the following theorem.

Theorem 2. $[\mu_r] \cap \mathrm{Im}[(t_n)_*: \pi^s_{2n+8r}(CP^m_n) \to \pi^s_{8r+1}(S^0)] = \emptyset$ *if and only if one of the follswing conditions* (a)—(d) *holds*: (a) n *is odd*; (b) $m < n+4r$; (c) $m \geq n+4r$ *and* $u_2(n, n+4r) = u_2(n-1, n+4r)$; (d) $m \geq n+4r$, $u_2(n, n+4r) = u_2(n-1, n+4r)-1$ *and* $u_2(n, n+4r) < h_2(n, n+4r)$.

We remark that it is possible to prove Theorem 2 by using results of [7], although we will prove it independently.

We also consider a similar problem to the S^0-transfer maps

$$t_n: \Sigma^{-n} RP^m_n \to S^0\,, \tag{1.5}$$

which are defined on stunted real projective spaces, and to the S^3-transfer maps on stunted quaternionic (quasi-) projective spaces. As is easily seen, μ_r is not in the image of the homomorphisms induced from S^3-transfer maps (see Corollary 2.3). As for the S^0-transfer map of (1.5), we have the following:

Theorem 3. (1) *If* $n \equiv 0$ mod 4, $m \geq n+2$ (*resp.* $m \geq n+1$) *and* $r > 0$ (*resp.* $r = 0$), *then* $I(\mu_r) \subset \mathrm{Im}(t_n)_*$.

(2) *Otherwise, we have* $[\mu_r] \cap \mathrm{Im}(t_n)_* = \emptyset$.

This paper is organized as follows: In §2 we prepare some necessary properties about μ-elements and transfer maps, and we prove Theorems 1–3 in §3—5 respectively.

2. Preliminaries

First, we comment on the definition of μ_r briefly. Recall that a stable map $A: \Sigma^8 M \to M$ is called an Adams map if it induces a KO-cohomology isomorphism, where $M = S^0 \cup_2 e^1$ is the mod 2 Moore spectrum. According to [1], there exists an Adams map, and a μ-element $\mu_r \in \pi^s_{8r+1}(S^0)$ is defined to be the stable homotopy calss of the composition $\bar{\eta} \circ A^r \circ i$, where $i: S^{8r+1} \to \Sigma^{8r+1} M$ is the bottom inclusion and $\bar{\eta}: \Sigma M \to S^0$ is an extension of $\eta = \mu_0: S^1 \to S^0$. We remark that there is some choice of Adams maps and so μ_r is not necessarily unique.

But, in [8] it is proved that the highest value of the mod 2 Adams filtration of Adams maps is 4. In this paper, we define μ_r to be the μ-element defined using the Adams map of the highest filtration 4. Then μ_r is uniquely defined, and its Adams filtration is $4r+1$.

Let α be an m-dimensional vector bundle over a finite complex X, X^α its Thom space, and $i: S^m \to X^\alpha$ the bottom inclusion. Then, since $d_R(\mu_r) \neq 0$ as in (1.1), we have the following:

Lemma 2.1. *If α is a KO-orientable vector bundle, then $i_*(\mu_r') \neq 0$ in $\pi^s_{m+8r+1}(X^\alpha)$ for any $\mu_r' \in [\mu_r]$.*

Proof. The assumption says that $i^*: KO^m(X^\alpha) \to KO^m(S^m)$ is an epimorphism. The property that $d_R(\mu_r) \neq 0$ means that $(\mu_r')^*: KO^m(S^m) \to KO^m(S^{m+8r+1}) \cong Z/2$ is an epimorphism. Hence, $(i \circ \mu_r')^*$ is an epimorphism, and we have the desired result.

Let $(F, d) = (R, 1)$, $(C, 2)$ or $(H, 4)$, and FP_n^m be the suspension spectrum of a Thom space $(FP^{m-n})^{n\xi}$, where ξ is the canomical F-line bundle over the F-projective space FP^{m-n}. By [3], for $1 \leq n \leq m$, FP_n^m is the suspension spectrum of the stunted projective space FP^m/FP^{n-1}. By [9] o r[11], the cofiber of the S^{d-1}-transfer map $t_n: FP_n^m \to S^{d(n-1)+1}$ is homotopy equivalent to ΣFP_{n-1}^m. That is, we have the following:

Lemma 2.2. *For $-\infty < n \leq m \leq \infty$ and $n \neq \infty$, we have a cofiber sequence*

$$FP_{n-1}^m \xrightarrow{q} FP_n^m \xrightarrow{t_n} S^{d(n-1)+1} \xrightarrow{i} \Sigma FP_{n-1}^m ,$$

where q and i are the collapsing map and the inclusion map respectively.

Since $(4/d)\xi$ is a KO-orientable vector bundle, we have the following corollary of Lemmas 2.1 and 2.2.

Corollary 2.3. *Assume that $d(n-1) \equiv 0$ mod 4. Then, for the S^{d-1}-transfer map $t_n: FP_n^m \to S^{d(n-1)+1}$, we have $[\mu_r] \cap \mathrm{Im}(t_n)_* = \emptyset$.*

Let e_C be the e-invariant in the K-theory. Then Adams [1] showed that

(2.4) $\qquad e_C(\mu_r) \equiv 1/2 \bmod 1 \quad \text{and} \quad \mathrm{Ker}(d_R) = \mathrm{Ker}(e_C) \text{ in } \pi^s_{8r+1}(S^0)$.

Applying this property, we have the following lemma, where X is a finite complex and $f: X \to S^t$ is a stable map.

Lemma 2.5. *Assume $r \geq 0$. If $f^* = 0$: $\tilde{K}^t(S^t) \to \tilde{K}^t(X)$ and $H^{t+8r+1}(X; Q) = 0$, then $[\mu_r] \cap \mathrm{Im}[f_*: \pi^s_{t+8r+1}(X) \to \pi^s_{8r+1}(S^0)] = \emptyset$.*

Proof. Suppose that $\mu_r' \in \mathrm{Im}(f_*)$ for some $\mu_r' \in [\mu_r]$. Then we have maps

g and $\tilde{g}$ satisfying the following homotopy commutative diagram:

$$\begin{tikzcd} S^{t+8r+1} \arrow[r, "\mu_r'"] \arrow[d, "g"'] & S^t \arrow[r, "i"] \arrow[d, "||"'] & C(\mu_r') \arrow[d, "\tilde{g}"'] \\ X \arrow[r, "f"] & S^t \arrow[r, "i"] & C(f), \end{tikzcd}$$

where $C(\mu_r')$ and $C(f)$ are the cofibers of μ_r' and f respectively. Apply K-cohomology functor on this diagram. Then, $i^*: \tilde{K}^t(C(f)) \to \tilde{K}^t(S^t) = Z\langle\iota\rangle$ is an epimorphism by assumption, and thus there is an element $x \in \tilde{K}^t(C(f))$ with $i^*(x) = \iota$. Then, by the definition of e_C, we have $ch(\tilde{g}^*(x)) = 1_t + e_C(\mu_r')\, 1_{t+8r+2} \in H^*(C(\mu_r'); Q)$, where ch is the Chern character and 1_i denotes a generator of $H^i(C(\mu_r'); Z)$. Since $H^{t+8r+2}(C(f); Q) = 0$ by assumption, we have $ch(\tilde{g}^*(x)) = \tilde{g}^*(ch(x)) = 1_t$ and thus $e_C(\mu_r') = 0$. But this contradicts to (2.4), and thus we have the desired result.

By Lemmas 2.2 and 2.5 we have the following:

Corollary 2.6. $[\mu_r] \cap \mathrm{Im}(t_n)_* = \emptyset$ *for* $r \geq 0$, *if* t_n *is one of the following*:
(1) *the* S^0*-transfer* $t_n: RP_n^m \to S^n$ *for odd* n;
(2) *the* S^1*-transfer* $t_n: CP_n^m \to S^{2n-1}$ *for* $m < n + 4r$.

3. Proof of Theorem 1

We consider the mod 2 Adams spectral sequences

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(W; Z/2), Z/2) \Rightarrow \pi_*(W)$$

for finite CW-spectra W, and denote the r-th terms by $E_r^{s,t}(W)$. For $W = S^0$, h_i denotes the generator of $E_2^{1,2^i}(S^0) \cong Z/2$ for $0 \leq i \leq 3$. By the vanishing theorem and the periodicity theorem ([2]), $E_r^{s,t}(S^0)$ are well investigated near the vanishing line (cf. [12; Chap. 3.4]). The periodicity theorem states that there is an isomorphism $P: E_2^{s,t}(S^0) \to E_2^{s+4,t+12}(S^0)$, for $4 \leq s < t < 3s+2$, defined by $P(u) = \langle u, h_0^4, h_3 \rangle$, a Massey product, and we will apply it repeatedly to our calculation. As an example, it is known that $E_2^{s,s+8r+1}(S^0) = 0$ for $s \geq 4r+2$, and that, if we put $a_{4r+1} = P^r(h_1)$,

$$E_2^{4r+1,12r+2}(S^0) = E_\infty^{4r+1,12r+2}(S^0) = Z/2\langle a_{4r+1}\rangle, \tag{3.1}$$

where $P(h_1)$ is defined by the same way. Then, by the assumption that the mod 2 Adams filtration of μ_r is $4r+1$, μ_r just corresponds to a_{4r+1}.

Now, the condition of Theorem 1 is divided into the following two cases:

$$\binom{8r+2k+1}{4r+1} \equiv 1 \mod 2\ ; \tag{3.2}$$

$$(3.3) \qquad \binom{8r+2k+1}{4r+1}+2\binom{8r+2k-1}{4r-1}\equiv 2 \mod 4\,.$$

By Lemma 2.2, Theorem 1 is equivalent to the assertion that $i_*(\mu_r)=0$ in $\pi^s_{4k+8r-1}(CP^{2k+4r}_{2k-1})$ if (3.2) or (3.3) is satisfied, where i: $S^{4k-2}\rightarrow CP^{2k+4r}_{2k-1}$ is the bottom inclusion. Then, since $E_2^{s,s+8r+4k-1}(CP^{2k+4r}_{2k-1})=0$ for $s\geq 4r+2$ by the vanishing theorem, the following theorem implies Theorem 1 by (3.1), and the rest of this section is devoted to the proof of it.

Theorem 3.4. *If* (3.2) (*resp.* (3.3)) *is satisfied, then* $i_*(a_{4r+1})=0$ *in* $E_2^{4r+1,12r+4k}(CP^{2k+4r}_{2k-1})$(*resp.* $E_3^{4r+1,12r+4k}(CP^{2k+4r}_{2k-1})$).

We use conditions (3.2) and (3.3) to ensure the the following lemma, which is an immediate consequence of results by Crabb and Knapp in [6], where $h_2(n, m)$ denotes the integer defined below (1.3).

Lemma 3.5. *Assume that* (3.2) (*resp.* (3.3)) *is satisfied. Then,* $h_2(2k-1, 2k+4r)=4r+1$ (*resp.*$=4r$)*, and, as a generator of the free part of* $\pi^s_{4k+8r}(CP^{\infty}_{2k-1})$*, we have an element* x *whose* mod 2 *Adams filtration is* $4r+1$ (*resp.* $4r$).

Proof. In [6] and [7], the codegrees $cd_2(\alpha, X)$ and the j-theory codegrees $cd_2^j(\alpha, X)$ of vector bundles α over connected spaces X are determined under some conditions. We put $L=2k+4r+1$. Then, by [6; Cor. 1.3] and [7; Prop. 5.24], we have $cd_2^j(-L\xi, CP^{4r+1})=4r+1$ if (3.2) is satisfied. By similar calculations, we see that $cd_2^j(-L\xi, CP^{4r+1})=4r$ if (3.3) is satisfied. By [6; Th. 1.5 and 1.6] and the proofs of them, if $cd_2^j(-L\xi, CP^{4r+1})=4r+\varepsilon$ for $\varepsilon=1$ or 0, then we have $cd_2(-L\xi, CP^{4r+1})=4r+\varepsilon$, and it holds that, as a generator of the free part of the stable cohomotopy group $\pi_s^{-2L}((CP^{4r+1})^{-L\xi})$, we have an element y whose mod 2 Adams filtration is $4r+\varepsilon$. Since CP^{2k+4r}_{2k-1} is S-dual to $(CP^{4r+1})^{-L\xi}$ by [3] and $h_2(2k-1, 2k+4r)=cd_2(-L\xi, CP^{4r+1})$ by definitions, the dual element $x\in \pi^s_{4k+8r}(CP^{2k+4r}_{2k-1})$ of y is the required generator of the free part, and we have the desired result.

In order to prove Theorem 3.4, we also need the following two lemmas.

Lemma 3.6. *Assume that* $m\leq n+4r$. *Then, for the inclusion* i: $S^{2n}\rightarrow CP^m_n$, *we have* $i_*(\mu_r)\neq 0$ *in* $\pi^s_{8r+2n+1}(CP^m_n)$, *and hence* $i_*(a_{4r+1})\neq 0$ *in* $E_2^{4r+1,12r+2n+2}(CP^m_n)$.

Lemma 3.7. *For integers* $r>0$ *and* k, *we have an element* b *satisfying*

$$E_2^{4r,12r+4k-1}(CP^{2k+4r-1}_{2k-1})=Z/2\langle b\rangle \quad \text{and} \quad h_0b = i_*(a_{4r+1})\,,$$

where i: $S^{4k-2}\rightarrow CP^{2k+4r-1}_{2k-1}$ *is the bottom inclusion.*

Lemma 3.6 follows obviously from Lemma 2.2, Corollary 2.6 and the vanishing theorem. We postopone the proof of Lemma 3.7 until the end of this section, and prove Theorem 3.4 first.

Proof of Theorem 3.4. Put $X=CP_{2k-1}^{2k+4r}$ and $X'=CP_{2k-1}^{2k+4r-1}$. Then, the cofiber sequence $X'\xrightarrow{i'}X\xrightarrow{p}S^{4k+8r}$ induces the following exact sequence of E_2-terms:

$$E_2^{s,s+8r+4k}(X)\xrightarrow{p_*}E_2^{s,s}(S^0)\xrightarrow{\partial}E_2^{s+1,s+8r+4k}(X')\xrightarrow{i'_*}E_2^{s+1,s+8r+4k}(X).$$

By Lemma 3.5, as a generator of the free part of $\pi_{8r+4k}^s(X)$, we have an element x whose mod 2 Adams filtration is $4r+1$ (resp. $4r$), if (3.2) (resp. (3.3)) is satisfied. Also, by Lemma 3.7, we have $E_2^{4r,12r+4k-1}(X')=Z/2\langle b\rangle$ and $h_0b=i_*(a_{4r+1})$.

First, we assume that (3.2) is satisfied. Then x represents an element $y\in E_\infty^{4r+1,12r+4k+1}(X)$ which satisfies $p_*(y)=h_0^{4r+1}$. It holds that $\partial(h_0^{4r})=i_*(a_{4r+1})$ in $E_2^{4r+1,12r+4k}(X')$, because otherwise we have $h_2(2k-1,2k+4r)\leq 4r$ which does not occur by Lemma 3.5. Hence, $i_*(a_{4r+1})=0$ in $E_2^{4r+1,12r+4k}(X)$, and thus we have the desired result in this case.

Next, we assume that (3.3) is satisfied. Then x represents an element $y\in E_\infty^{4r,12r+4k}(X)$, and then we have $p_*(y)=h_0^{4r}$. In this case, we have $h_0^{4r-1}=(p_*)(z)$ for some $z\in E_2^{4r-1,12r+4k-1}(X)$. In fact, if $h_0^{4r-1}\notin\mathrm{Im}(p_*)$, then we have $\partial h_0^{4r-1}=b$ and $\partial h_0^{4r}=h_0b=i_*(a_{4r+1})\neq 0$ by Lemmas 3.6 and 3.7, which contradicts to the equality $p_*(y)=h_0^{4r}$. Hence $E_2^{4r+1,12+4k}(X)=Z/2\langle i_*(a_{4r+1})\rangle$, and $E_2^{s+1,s+8r+4k}(X)=0$ foı $s\geq 4r+1$. But we have $z=0\in E_\infty^{4r-1,12r+4k-1}(X)$, because otherwise we have $h_2(2k-1,2k+4r)\leq 4r-1$ which does not occur by Lemma 3.5. Therefore we have $d_2(z)=i_*(a_{4r+1})$, and thus $i_*(a_{4r+1})=0$ in $E_3^{4r+1,12r+4k}(X)$, which establishes the required result.

Proof of Lemma 3.7. As in (3.1), we have $P(a_{4r+1})=a_{4r+5}$ for $r\geq 1$, where P is the isomorphism in the periodicity theorem. We put $a_{4r}=P^{r-1}(h_0^3h_3)\in E_2^{4r,12r-1}(S^0)$. Then, it is known that a_{4r} is represented by an element α_{4r} which is in $\mathrm{Im}(J)\subset\pi_{8r-1}^s(S^0)$ and of order 2, and we have $E_2^{4r,12r-1}(S^0)=E_\infty^{4r,12r-1}(S^0)=Z/2\langle a_{4r}\rangle$ and $E_2^{s,s+8r-1}(S^0)=0$ for $s\geq 4r+1$ by the periodicity theorem.

Consider a cofiber sequence $S^1\xrightarrow{\eta}S^0\xrightarrow{i}M_\eta\xrightarrow{q}S^2$. Then, $\Sigma^{4k-2}M_\eta=CP_{2k-1}^{2k}$, and we have an isomorphism $i'_*\colon E_2^{s,s+8r+4k-1}(CP_{2k-1}^{2k})\to E_2^{s,s+8r+4k-1}(CP_{2k-1}^{2k+4r-1})$ for $s\geq 4r$, where i' is the usual inclusion. Thus, to obtain the required result, it is sufficient to prove that there is an element $w\in E_2^{4r,12r+1}(M_\eta)$ with $h_0w=i_*(a_{2r+1})$, because then $b=i'_*(w)$ is a desired element.

By using the exact sequence of E_2-terms induced from the above cofiber sequence, we see that $E_2^{s,s+8r+1}(M_\eta)=0$, $Z/2\langle i_*(a_{4r+1})\rangle$ and $Z/2\langle w\rangle$ according as $s\geq 4r+2$, $s=4r+1$ and $s=4r$, and we have $q_*(w)=a_{4r}$ since $h_1a_{4r}=0$. Also we have $E_2^{s,s+8r}(M_\eta)=0$ for $s\geq 4r+1$. Thus $E_2^{4r,12r+1}(M_\eta)=E_\infty^{4r,12r+1}(M_\eta)=Z/2\langle w\rangle$. We show that w is the desired element. To do it, we consider an exact sequence

$\pi_{8r}^s(S^0) \xrightarrow{\eta_*} \pi_{8r+1}^s(S^0) \xrightarrow{i_*} \pi_{8r+1}^s(M_\eta) \xrightarrow{q_*} \pi_{8r-1}^s(S^0)$. Let $v \in \pi_{8r+1}^s(M_\eta)$ represents w. Then $q_* v = \alpha_{4r}$, and we have $2v \in i_* \langle \eta, \alpha_{4r}, 2\rangle$ by [15; Prop. 1.8], where $\langle\ ,\ ,\ \rangle$ denotes a Toda bracket. But, $\mu_r \in \langle \eta, 2, \alpha_{4r}\rangle$ by definition, and thus by [15; (3.9)] we have $2v = i_*(\mu_r + 2u)$ for some $u \in \pi_{8r+1}^s(S^0)$. This yields that $h_0 w = i_*(a_{4r+1})$, and we complete the proof.

4. Proof of Theorem 2

Theorem 2 gives a necessary and sufficient condition for

$$(4.1) \qquad [\mu_r] \cap \operatorname{Im}[(t_n)_*\colon \pi_{2n+8r}^s(CP_n^m) \to \pi_{8r+1}^s(S^0)] = \emptyset\,,$$

where $r \geq 0$ and $m \geq n$. By Corollaries 2.3 and 2.6, (4.1) holds if the condition (a) or (b) of Theorem 2 is satisfied. In this section, we prove that, for $m \geq n+4r$, (4.1) holds if and only if the condition (c) or (d) of Theorem 2 is satisfied. Then it establishes Theorem 2.

When we treat μ_r with indeterminacy $\operatorname{Ker}(d_R)$, the Adams-Novikov spectral sequence

$$E_{s,t}^2(X) = \operatorname{Ext}_{MU_*MU}^{s,t}(MU_*, MU_*(X)) \Rightarrow \pi_*^s(X)$$

is an efficient tool, where MU is the Thom spectrum of the complex cobordism theory. The cofiber sequence in Lemma 2.2 induces an exact sequence

$$(4.2) \qquad 0 \to E_{0,2n+8r}^2(CP_{n-1}^\infty) \xrightarrow{q_*} E_{0,2n+8r}^2(CP_n^\infty) \xrightarrow{\delta} E_{1,8r+2}^2(S^0)\,.$$

Let $PMU_k(\ )$ be the group of all primitive elements in $MU_k(\ )$ with respect to the MU_*MU-comodule structure of $MU_*(\ )$. Then, it is a fundamental fact that $E_{0,k}^2(\ ) = PMU_k(\ )$, and for $k \geq h$ we have

$$(4.3) \qquad E_{0,2k}^2(CP_h^\infty) \cong PMU_{2k}(CP_h^\infty) \cong Z\langle g_{h,k}\rangle \quad \text{for some generator } g_{h,k}\,.$$

Also we need the following well known fact.

Lemma 4.4. (1) *$E_{1,8r+2}^2(S^0) = E_{1,8r+2}^\infty(S^0) = Z/2\langle\alpha_{4r+1}\rangle$ and α_{4r+1} is represented by μ_r.*

(2) *δ induces a homomorphism $E_{0,2n+8r}^\infty(CP_n^\infty) \to E_{1,8r+2}^\infty(S^0)$, which is associated with $(t_n)_*\colon \pi_{2n+8r}^s(CP_n^\infty) \to \pi_{8r+1}^s(S^0)$.*

By (4.2–4), we have the following:

Corollary 4.5. *Assume that $m \geq n+4r$. Then, (4.1) holds if and only if one of the following holds:*

(1) $q_*(g_{n-1,n+4r}) = \pm g_{n,n+4r}$;

(2) $q_*(g_{n-1,n+4r}) = \pm 2g_{n,n+4r}$ *and* $g_{n,n+4r} \notin E_{0,2n+8r}^\infty(CP_n^\infty)$.

In order to combine Corollary 4.5 with the required result, we need to represent $g_{h,k}$ in an explicit form, and we use a method given by [13] and [4]. In order to describe it, we prepare the following notations to denote generators of E-homology groups of CP_n^∞ and MU for $E=H$, MU or K.

$$E_*(CP_n^\infty)=E_*\langle \beta_n^E, \beta_{n+1}^E, \cdots\rangle \quad\text{and}\quad E_*(MU)=E_*[b_1^E, b_2^E, \cdots],$$

where $|\beta_i^E|=|b_i^E|=2$. Also, $(b)_j^i$ denotes the $2j$-dimensional part of $(1+b_1^H+\cdots+b_j^H)^i$, and we identify it with an element of $\pi_{2j}(MU)\otimes Q$. Then $g_{h,k}$ is represented by the following formula, in which $u(h,k)$ is the integer gievn in (1.2).

Proposition 4.6. *Assume that $h\leq k$. Then we have*

$$g_{h,k}=u(h,k)\sum_{i=h}^{k}(b)_{k-i}^i\,\beta_i^{MU} \quad \textit{up to sign.}$$

Proof. We put $B=\sum_{i=h}^k(b)_{k-i}^i\,\beta_i^{MU}$. By the similar reason as in [13], it is easy to see that $B\in PMU_{2k}(CP_h^\infty)\otimes Q\cong Q$. By Baker ([4]) it is shown that the Todd genus τ: $MU\to K$ induces an isomorphism τ_*: $PMU_*(CP_h^\infty)\to PK_*(CP_h^\infty)$ by the Hattori-Stong theorem, and it is known that $\tau_*(\beta_i^{MU})=\beta_i^K$ and $\tau_*(b_i^H)=(1/(i+1)\,!)\,t^i\in\pi_{2i}(K)\otimes Q$ (cf. [14]), where $t\in\pi_2(K)\cong Z$ is a generator. Thus we have

$$\tau_*(B)=\sum_{i=h}^{k}a_{k-i}^i\,t^{k-i}\,\beta_i^K,$$

where a_j^i is the coefficient of x^j in the power series expansion of $((e^x-1)/x)^i$. Hence, by (1.2), $u(h,k)$ is the minimum positive integer such that $u(h,k)B$ is an integral class, and thus we have the desired result.

Now, we can complete the proof of Theorem 2. The condition that $q_*(g_{n-1,n+4r})=\pm g_{n,n+4r}$ (resp. $q_*(g_{n-1,n+4r})=\pm 2g_{n,n+4r}$) in Corollary 4.5 is equivalent to that $u_2(n,n+4r)=u_2(n-1,n+4r)$ (resp. $u_2(n,n+4r)=u_2(n-1,n+4r)-1$), by Proposition 4.6. Also, we have $u_2(n,n+4r)<h_2(n,n+4r)$ if and only if $g_{n,n+4r}\notin E_{0,2n+8r}^\infty(CP_n^\infty)$, by (4.3), Proposition 4.6 and the definition of $h(n,m)$. Therefore, for $m\geq n+4r$, the conditions (c) and (d) in Theorem 2 are equivalent to those of (1) and (2) in Corollary 4.5 respectively. Thus we have the desired result.

5. Proof of Theorem 3

Let $M_t=S^t\cup_2 CS^t$ be the mod 2 Moore spectrum. Then the stable cell structure of RP_{4k-1}^{4k+2} is given by

$$RP_{4k-1}^{4k+2}\simeq S^{4k-1}\cup_{2\vee\bar{\eta}}C(S^{4k-1}\vee M_{4k}), \tag{5.1}$$

where $\bar{\eta}$ is an extension of $\eta=\mu_0$, and thus the following lemma is easily seen, where $[X, Y]$ denotes the abelian group of all stable homotopy classes of stable maps from X to Y.

Lemma 5.2. (1) $\pi^s_{4k}(RP^{4k+1}_{4k-1})=0$.
(2) $(2\vee\bar{\eta})_*$: $[M_{4k+1}, RP^{4k+2}_{4k}]\to[M_{4k+1}, S^{4k}]$ *is an epimorphism.*

Now, Theorem 3 for $n\equiv 1 \bmod 2$ has already shown in Corollary 2.6. By Lemma 2.2, we have a cofiber sequence

$$(5.3) \qquad RP^m_n \xrightarrow{t_n} S^n \xrightarrow{i} \Sigma RP^m_{n-1}\,,$$

where t_n is the S^0-transfer map and i is the bottom inclusion. Then we have Theorem 3 for $n\equiv 2 \bmod 4$ as follows:

Lemma 5.4. *If* $n\equiv 2 \bmod 4$, *then* $[\mu_r]\cap \mathrm{Im}(t_n)_*=\emptyset$ *for* $r\geq 0$ *and* $m\geq n$.

Proof. We prove that $i_*(\mu'_r)\neq 0$ in $\pi^s_{n+8r}(RP^m_{n-1})$ for any $\mu'_r\in[\mu_r]$. Then it yields the desired result by (5.3). Let ∂: $\Sigma RP^m_{n-1}\to\Sigma^2 RP^{n-2}_{n-3}$ be the attaching map, and i': $S^{n-1}\to\Sigma^2 RP^{n-2}_{n-3}$ the bottom inclusion. Since the attaching map of the top cell of $\Sigma^2 RP^{n-1}_{n-3}$ is homotopic to $i'\circ\eta$ by (5.1), and, since it is also homotopic to $\partial\circ i$, we ha have

$$\partial_* i_*(\mu'_r) = i'_*(\eta\mu'_r)\,.$$

But $\Sigma^2 RP^{n-2}_{n-3}\simeq M_{n-1}$, and $2\not|\,\eta\mu'_r$ in $\pi^s_{8r+2}(S^0)$ by [1; Th.1.4]. Hence we have $i'_*(\eta\mu'_r)\neq 0$, and thus the desired result that $i_*(\mu'_r)\neq 0$.

Now, we begin the proof of theorem 3 for $n\equiv 0 \bmod 4$, and so we put $n=4k$. Let $i_{t,m}$: $RP^t_{4k-1}\to RP^m_{4k-1}$ be the inclusion for $4k\leq t\leq m$. For $m\geq 4k+1$, $i_*(\mu_0)\in\pi^s_{4k}(RP^m_{4k-1})$ is in the image of $(i_{4k+1,m})_*$, and thus we have $i_*(\mu_0)=0$ by Lemma 5.2(1). When $m=4k$, $i_*(\mu_0)\neq 0$, since t_{4k}: $S^{4k}\to S^{4k}$ is of degree 2 and $2\not|\,\mu_0$. Thus we have the following lemma, which is the assertion of Theorem 3 for μ_0 and $n=4k$.

Lemma 5.5. $\mu_0\in\mathrm{Im}(t_{4k})_*$ *if and only if* $m\geq 4k+1$.

Lastly, the following lemma completes the proof of Theorem 3.

Lemma 5.6. *Assume that* $r>0$. *Then,* $\mu_r\in\mathrm{Im}(t_{4k})_*$ *if* $m\geq 4k+2$, *and* $[\mu_r]\cap\mathrm{Im}(t_{4k})_*=\emptyset$ *if* $m=4k$ *or* $4k+1$.

Proof. First we assume that $m\geq 4k+2$, and prove that $i_*(\mu_r)=0$ for the bottom inclusion i. Then it yields the desired result in this case by (5.3). By Lemma 5.2(2), we have $i_*(\bar{\eta})=0$ in $[M_{4k}, RP^{4k+2}_{4k-1}]$. But, since $\mu_r=\bar{\eta}\circ A^r\circ i$ by definition, we have $i_*(\mu_r)=0$, and the desired result. For the case of $m=4k$, the assertion is obvious, because $t_{4k}: S^{4k} \to S^{4k}$ is of degree 2 and μ'_r is not devided by 2 for any $\mu'_r \in [\mu_r]$. Next, we assume $m=4k+1$. Then, by (5.1) and (5.3), $t_{4k} \simeq 2 \vee \eta: S^{4k} \vee S^{4k+1} \to S^{4k}$. Suppose that $\mu'_r \in \mathrm{Im}(t_{4k})_*$ for some $\mu'_r \in [\mu_r]$. Then we have $\mu'_r = \eta a_0 + 2a_1$ for some $a_i \in \pi^s_{8r+i}(S^0)$, and $\eta\mu'_r = \eta^2 a_0$. But this is impossible, because the Adams-Novikov filtration of $\eta\mu'_r$ is 2 (cf. [12; Chap. 5.4]) and that of $\eta^2 a_0$ is greater than 2. Thus we have $\mu'_r \notin \mathrm{Im}(t_{4k})_*$, and complete the proof.

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