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STABILITY OF EXTREMAL KÄHLER MANIFOLDS

Dedicated to Professor Shoshichi Kobayashi on his seventieth birthday

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1. Introduction

In Donaldson's study [10] of asymptotic stability for polarized algebraic manifolds (M, L), critical metrics originally defined by Zhang [39] (see also [22]) are referred to as balanced metrics and play a central role when the polarized algebraic manifolds admit Kähler metrics of constant scalar curvature. Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus in the identity component $\operatorname{Aut}^0(M)$ of the group of holomorphic automorphisms of M. In this paper, we define the concept of critical metrics relative to T, and as an application, choosing a suitable T, we shall show that a result in [26] on the asymptotic approximation of critical metrics (see [10], [39]) can be generalized to the case where (M, L) admits an extremal Kähler metric in the polarization class. Then in our forthcoming paper [27], we shall show that a slight modification of the concept of stability (see Theorem A below) allows us to obtain the asymptotic stability of extremal Kähler manifolds even when the obstruction as in [26] does not vanish. In particular, by an argument similar to [10], an extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold M will be shown to be unique^{*} up to the action of the group $\operatorname{Aut}^0(M)$.

2. Statement of results

Throughout this paper, we fix once for all an ample holomorphic line bundle L on a connected projective algebraic manifold M. Let H be the maximal connected linear algebraic subgroup of $\operatorname{Aut}^0(M)$, so that $\operatorname{Aut}^0(M)/H$ is an abelian variety. The corresponding Lie subalgebra of $H^0(M, \mathcal{O}(T^{1,0}M))$ will be denoted by \mathfrak{h} . For the complete linear system $|L^m|$, $m \gg 1$, we consider the Kodaira embedding

$$\Phi_m = \Phi_{|L^m|} \colon M \hookrightarrow \mathbb{P}^*(V_m), \qquad m \gg 1,$$

where $\mathbb{P}^*(V_m)$ denotes the set of all hyperplanes through the origin in $V_m := H^0(M, \mathcal{O}(L^m))$. Put $N_m := \dim V_m - 1$. Let *n* and *d* be respectively the dimension

^{*} For this uniquness, we choose $Z^{\mathbb{C}}$ (cf. Section 2) as the algebraic torus T.

of M and the degree of the image $M_m := \Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. Put $W_m = \{\text{Sym}^d(V_m)\}^{\otimes n+1}$. Then to the image M_m of M, we can associate a nonzero element \hat{M}_m in W_m^* such that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point associated to the irreducible reduced algebraic cycle M_m on $\mathbb{P}^*(V_m)$. Replacing L by some positive integral multiple of L if necessary, we fix an H-linearization of L, i.e., a lift to L of the H-action on M such that H acts on L as bundle isomorphisms covering the H-action on M. For an algebraic torus T in H, this naturally induces a T-action on V_m for each m. Now for each character $\chi \in \text{Hom}(T, \mathbb{C}^*)$, we set

$$V(\chi) := \{ s \in V_m; t \cdot s = \chi(t) s \text{ for all } t \in T \}.$$

Then we have mutually distinct characters $\chi_1, \chi_2, \ldots, \chi_{\nu_m} \in \text{Hom}(T, \mathbb{C}^*)$ such that the vector space $V_m = H^0(M, \mathcal{O}(L^m))$ is uniquely written as a direct sum

(2.1)
$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k).$$

Put $G_m := \prod_{k=1}^{V_m} SL(V(\chi_k))$, and the associated Lie subalgebra of $sl(V_m)$ will be denoted by \mathfrak{g}_m . More precisely, G_m and \mathfrak{g}_m possibly depend on the choice of the algebraic torus T, and if necessary, we denote these by $G_m(T)$ and $\mathfrak{g}_m(T)$, respectively. The T-action on V_m is, more precisely, a right action, while we regard the G_m -action on V_m as a left action. Since T is Abelian, this T-action on V_m can be regarded also as a left action.

The group G_m acts diagonally on V_m in such a way that, for each k, the k-th factor SL($V(\chi_k)$) of G_m acts just on the k-th factor $V(\chi_k)$ of V_m . This induces a natural G_m -action on W_m and also on W_m^* .

DEFINITION 2.2. (a) The subvariety M_m of $\mathbb{P}^*(V_m)$ is said to be *stable relative to* T or *semistable relative to* T, according as the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^* or the closure of $G_m \cdot \hat{M}_m$ in W_m^* does not contain the origin of W_m^* .

(b) Let \mathfrak{t}_c denote the Lie subalgebra of the maximal compact subgroup T_c of T, and as a real Lie subalgebra of the complex Lie algebra \mathfrak{t} , we define $\mathfrak{t}_{\mathbb{R}} := \sqrt{-1}\mathfrak{t}_c$.

Take a Hermitian metric for V_m such that $V(\chi_k) \perp V(\chi_l)$ if $k \neq l$. Put $N_m := \dim V_m - 1$ and $n_k := \dim V(\chi_k)$. We then set

$$l(k,i) := (i-1) + \sum_{j=1}^{k-1} n_j, \qquad i = 1, 2, \dots, n_k; \ k = 1, 2, \dots, \nu_m,$$

where the right-hand side denotes i - 1 in the special case k = 1. Let || || denote the Hermitian norm for V_m induced by the Hermitian metric. Take a \mathbb{C} -basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m .

DEFINITION 2.3. We say that $\{s_0, s_1, \ldots, s_{N_m}\}$ is an *admissible normal basis* for V_m if there exist positive real constants b_k , $k = 1, 2, \ldots, \nu_m$, and a \mathbb{C} -basis $\{s_{k,i}; i = 1, 2, \ldots, n_k\}$ for $V(\chi_k)$, with $\sum_{k=1}^{\nu_m} n_k b_k = N_m + 1$, such that (1) $s_{l(k,i)} = s_{k,i}$, $i = 1, 2, \ldots, n_k$; $k = 1, 2, \ldots, \nu_m$; (2) $s_l \perp s_{l'}$ if $l \neq l'$; (3) $||s_{k,i}||^2 = b_k$, $i = 1, 2, \ldots, n_k$; $k = 1, 2, \ldots, \nu_m$. Then the real vector $b := (b_1, b_2, \ldots, b_{\nu_m})$ is called the *index* of the admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m .

We now specify a Hermitian metric on V_m . For the maximal compact subgroup T_c of T above, let S be the set $(\neq \emptyset)$ of all T_c -invariant Kähler forms in the class $c_1(L)_{\mathbb{R}}$. Let $\omega \in S$, and choose a Hermitian metric h for L such that $\omega = c_1(L;h)$. Define a

(2.4) $(s, s')_{L^2} := \int_M (s, s')_{h^m} \omega^n, \qquad s, s' \in V_m,$

Hermitian metric on V_m by

where $(s, s')_{h^m}$ denotes the function on M obtained as the pointwise inner product of s, s' by the Hermitian metric h^m on L^m . Now, let us consider the situation that V_m has the Hermitian metric (2.4). Then

$$V(\chi_k) \perp V(\chi_l), \qquad k \neq l,$$

and define a maximal compact subgroup $(G_m)_c$ of G_m by $(G_m)_c := \prod_{k=1}^{\nu_m} SU(V(\chi_k))$. Again by this Hermitian metric $(,)_{L^2}$, let $\{s_0, s_1, \ldots, s_{N_m}\}$ be an admissible normal basis for V_m of a given index b. Put

(2.5)
$$E_{\omega,b} := \sum_{i=0}^{N_m} |s_i|_{h^m}^2,$$

where $|s|_{h^m} := (s, s)_{h^m}$ for all $s \in V_m$. Then $E_{\omega,b}$ depends only on ω and b. Namely, once ω and b are fixed, $E_{\omega,b}$ is independent of the choice of an admissible normal basis for $V(\chi_k)$ of index b. Fix a positive integer m such that L^m is very ample.

DEFINITION 2.6. An element ω in S is called a *critial metric relative to* T, if there exists an admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m such that the associated function $E_{\omega,b}$ on M is constant for the index b of the admissible normal basis. This generalizes a *critical metric* of Zhang [39] (see also [5]) who treated the case $T = \{1\}$. If ω is a critical metric relative to T, then by integrating the equality (2.5) over M, we see that the constant $E_{\omega,b}$ is $(N_m + 1)/c_1(L)^n[M]$.

For the centralizer $Z_H(T)$ of T in H, let $Z_H(T)^0$ be its identity component. For m as above, the following generalization of a result in [39] is crucial to our study of

stability:

Theorem A. The subvariety M_m of $\mathbb{P}(V_m)$ is stable relative to T if and only if there exists a critical metric $\omega \in S$ relative to T. Moreover, for a fixed index b, a critical metric ω in S relative to T with constant $E_{\omega,b}$ is unique up to the action of $Z_H(T)^0$.

We now fix a maximal compact connected subgroup K of H. The corresponding Lie subalgebra of \mathfrak{h} is denoted by \mathfrak{k} . Let S_K denote the set of all Kähler forms ω in the class $c_1(L)_{\mathbb{R}}$ such that the identity component of the group of the isometries of (M, ω) coincides with K. Then $S_K \neq \emptyset$, and an extremal Kähler metric, if any, in the class $c_1(L)_{\mathbb{R}}$ is always in H-orbits of elements of S_K . For each $\omega \in S_K$, we write

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} \, dz^{\alpha} \wedge dz^{\bar{\beta}}$$

in terms of a system (z^1, \ldots, z^n) of holomorphic local coordinates on M. let \mathcal{K}_{ω} be the space of all real-valued smooth functions u on M such that $\int_M u\omega^n = 0$ and that

$$\operatorname{grad}_{\omega}^{\mathbb{C}} u \coloneqq \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial u}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^{\alpha}}$$

is a holomorphic vector field on M. Then \mathcal{K}_{ω} forms a real Lie subalgebra of \mathfrak{h} by the Poisson bracket for (M, ω) . We then have the Lie algebra isomorphism

$$\mathcal{K}_{\omega} \cong \mathfrak{k}, \qquad u \leftrightarrow \operatorname{grad}_{\omega}^{\mathbb{C}} u.$$

For the space $C^{\infty}(M)_{\mathbb{R}}$ of real-valued smooth functions on M, we consider the inner product defined by $(u_1, u_2)_{\omega} := \int_M u_1 u_2 \,\omega^n$ for $u_1, u_2 \in C^{\infty}(M)_{\mathbb{R}}$. Let pr: $C^{\infty}(M)_{\mathbb{R}} \to \mathcal{K}_{\omega}$ be the orthogonal projection. Let \mathfrak{z} be the center of \mathfrak{k} . Then the vector field

$$\mathcal{V} := \operatorname{grad}_{\omega}^{\mathbb{C}} \operatorname{pr}(\sigma_{\omega}) \in \mathfrak{z}$$

is called the *extremal Kähler vector field* of (M, ω) , where σ_{ω} denotes the scalar curvature of ω . Then \mathcal{V} is independent of the choice of ω in \mathcal{S} , and satisfies $\exp(2\pi\gamma\mathcal{V}) = 1$ for some positive integer γ (cf. [13], [32]). Next, since we have an *H*-linearization of *L*, there exists a natural inclusion $H \subset \operatorname{GL}(V_m)$. By passing to the Lie algebras, we obtain

$$\mathfrak{h} \subset \mathfrak{gl}(V_m).$$

Take a Hermitian metric h for L such that the corresponding first Chern form $c_1(L;h)$ is ω . As in [23, (1.4.1)], the infinitesimal h-action on L induces an infinitesimal

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 \mathfrak{h} -action on the complexification $\mathcal{H}_m^{\mathbb{C}}$ of the space of all Hermitian metrics \mathcal{H}_m on the line bundle L^m . The Futaki-Morita character $F: \mathfrak{h} \to \mathbb{C}$ is given by

$$F(\mathcal{Y}) \coloneqq \frac{\sqrt{-1}}{2\pi} \int_M h^{-1}(\mathcal{Y}h)\omega^n,$$

which is independent of the choice of h (see for instance [15]). For the identity component Z of the center of K, we consider its complexification $Z^{\mathbb{C}}$ in H. Then the corresponding Lie algebra is just the complexification $\mathfrak{z}^{\mathbb{C}}$ of \mathfrak{z} above. We now consider the set Δ of all algebraic tori in $Z^{\mathbb{C}}$. Let $T \in \Delta$. Put

$$q:=\frac{1}{m}.$$

For $\omega = c_1(L; h) \in S_K$, we consider the Hermitian metric (2.4) for V_m . We then choose an admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m of index $(1, 1, \ldots, 1)$. By the asymptotic expansion of Tian-Zelditch (cf. [33], [38]; see also [4]) for $m \gg 1$, there exist real-valued smooth functions $a_k(\omega)$, $k = 1, 2, \ldots$, on M such that

(2.7)
$$\frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|_{h^m}^2 = 1 + a_1(\omega)q + a_2(\omega)q^2 + \cdots$$

Then $a_1(\omega) = \sigma_{\omega}/2$ by a result of Lu [20]. Let $\mathcal{Y} \in \mathfrak{t}_{\mathbb{R}}$, and put $g := \exp^{\mathbb{C}} \mathcal{Y} \in T$, where the element $\exp(\mathcal{Y}/2)$ in T is written as $\exp^{\mathbb{C}} \mathcal{Y}$ by abuse of terminology. Recall that the T-action on V_m is a right action, though it can be viewed also as a left action. Put $h_g := h \cdot g$ for simplicity. Using the notation in Definition 2.3, we write $s_{k,i} = s_{l(k,i)}$, $k = 1, 2, \ldots, \nu_m$; $i = 1, 2, \ldots, n_k$. Then for a fixed k, $\int_M |s_{k,i}|^2_{h_g^m} g^* \omega^n = |\chi_k(\exp^{\mathbb{C}} \mathcal{Y})|^{-2}$ is independent of the choice of i. Put

$$Z(q,\omega;\mathcal{Y}) := \frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|_{h_g^m}^2 = g^* \left\{ \frac{n!}{m^n} \sum_{k=1}^{\nu_m} |\chi_k(\exp^{\mathbb{C}} \mathcal{Y})|^{-2} \sum_{i=1}^{n_k} |s_{k,i}|_{h^m}^2 \right\}, \quad \mathcal{Y} \in \mathfrak{t}_{\mathbb{R}}.$$

For extremal Kähler manifolds, the following generalization of [26] allows us to approximate arbitrarily some critical metrics relative to T:

Theorem B. Let $\omega_0 = c_1(L; h_0)$ be an extremal Kähler metric in the class $c_1(L)_{\mathbb{R}}$ with extremal Kähler vector field \mathcal{V} . Then for some $T \in \Delta$, there exist a sequence of vector fields $\mathcal{Y}_k \in \mathfrak{t}_{\mathbb{R}}$, a formal power series C_q in q with real coefficients (cf. Section 6), and smooth real-valued functions φ_k , k = 1, 2, ..., on M such that

(2.8)
$$Z(q, \omega(l); \mathcal{Y}(l)) = C_q + 0(q^{l+2}),$$

where $\mathcal{Y}(l) := (\sqrt{-1} \mathcal{V}/2) q^2 + \sum_{k=1}^{l} q^{k+2} \mathcal{Y}_k$, $h(l) := h_0 \exp(-\sum_{k=1}^{l} q^k \varphi_k)$, and $\omega(l) := c_1(L; h(l))$.

The equality (2.8) above means that there exists a positive real constant A_l independent of q such that $||Z(q, \omega(l); \mathcal{Y}(l)) - C_q||_{C^0(M)} \le A_l q^{l+2}$ for all q with $0 \le q \le 1$. By [38], for every nonnegative integer j, a choice of a larger constant $A = A_{j,l} > 0$ keeps Theorem B still valid even if the $C^0(M)$ -norm is replaced by the $C^j(M)$ -norm.

3. A stability criterion

In this section, some stability criterion will be given as a preliminary. In a forthcoming paper [27], we actually use a stronger version of Theorem 3.2 which guarantees the stability only by checking the closedness of orbits through a point for special one-parameter subgroups "perpendicular" to the isotropy subgroup. Now, for a connected reductive algebraic group G, defined over \mathbb{C} , we consider a representation of Gon an N-dimensional complex vector space W. We fix a maximal compact subgroup G_c of G. Moreover, let \mathbb{C}^* be a one-dimensional algebraic torus with the maximal compact subgroup S^1 .

DEFINITION 3.1. (a) An algebraic group homomorphism $\lambda : \mathbb{C}^* \to G$ is said to be a *special one-parameter subgroup* of G, if the image $\lambda(S^1)$ is contained in G_c . (b) A point $w \neq 0$ in W is said to be *stable*, if the orbit $G \cdot w$ is closed in W.

Later, we apply the following stability criterion to the case where $W = W_m^*$ and $G = G_m$. Let $w \neq 0$ be a point in W.

Theorem 3.2. A point w as above is stable if and only if there exists a point w'in the orbit $G \cdot w$ of w such that $\lambda(\mathbb{C}^*) \cdot w'$ is closed in W for every special oneparameter subgroup $\lambda \colon \mathbb{C}^* \to G$ of G.

Proof. We prove this by induction on $\dim(G \cdot w)$. If $\dim(G \cdot w) = 0$, the statement of the above theorem is obviously true. Hence, fixing a positive integer k, assume that the statement is true for all $0 \neq w \in W$ such that $\dim(G \cdot w) < k$. Now, let $0 \neq w \in W$ be such that $\dim(G \cdot w) = k$, and the proof is reduced to showing the statement for such a point w. Let $\Sigma(G)$ be the set of all special one-parameter subgroups of G. Fix a G_c -invariant Hermitian metric $\| \|$ on W. The proof is divided into three steps:

STEP 1. First, we prove "only if" part of Theorem 3.2. Assume that w is stable. Since $G \cdot w$ is closed in W, the nonnegative function on $G \cdot w$ defined by

$$(3.3) G \cdot w \ni g \cdot w \mapsto ||g \cdot w|| \in \mathbb{R}, g \in G,$$

has a critical point at some point w' in $G \cdot w$. Let $\lambda \in \Sigma(G)$, and it suffices to show the closedness of $\lambda(\mathbb{C}^*) \cdot w'$ in W. We may assume that dim $\lambda(\mathbb{C}^*) \cdot w' > 0$. Then by using the coordinate system associated to an orthonormal basis for W, we can write w' as $(w'_0, \ldots, w'_r, 0, \ldots, 0)$ in such a way that $w'_{\alpha} \neq 0$ for all $0 \leq \alpha \leq r$ and that

$$\lambda(e^t) \cdot w' = (e^{t\gamma_0}w'_0, \dots, e^{t\gamma_r}w'_r, 0, \dots, 0), \qquad t \in \mathbb{C},$$

where γ_{α} , $\alpha = 0, 1, ..., r$, are integers independent of the choice of t in \mathbb{C} . Since the closed orbit $G \cdot w$ does not contain the origin of W, the inclusion $\lambda(\mathbb{C}^*) \cdot w' \subset G \cdot w$ shows that $r \geq 1$ and that the coincidence $\gamma_0 = \gamma_1 = \cdots = \gamma_r$ cannot occur. In particular,

$$f(t) := \log \|\lambda(e^t) \cdot w'\|^2 = \log \left(e^{2t\gamma_0} |w_0'|^2 + e^{2t\gamma_1} |w_1'|^2 + \dots + e^{2t\gamma_r} |w_r'|^2 \right), \quad t \in \mathbb{R},$$

satisfies f''(t) > 0 for all t. Moreover, since the function in (3.3) has a critical point at w', we have f'(0) = 0. It now follows that $\lim_{t\to+\infty} f(t) = +\infty$ and $\lim_{t\to-\infty} f(t) = +\infty$. Hence $\lambda(\mathbb{C}^*) \cdot w'$ is closed in W, as required.

STEP 2. To prove "if" part of Theorem 3.2, we may assume that w = w' without loss of generality. Hence, suppose that $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for every $\lambda \in \Sigma(G)$. It then suffices to show that $G \cdot w$ is closed in W. For contradiction, assume that $G \cdot w$ is not closed in W. Since the closure of $G \cdot w$ in W always contains a closed orbit O_1 in W, by dim $O_1 < \dim(G \cdot w) = k$, the induction hypothesis shows that there exists a point $\hat{w} \in O_1$ such that

(3.4)
$$\lambda(\mathbb{C}^*) \cdot \hat{w}$$
 is closed in W for every $\lambda \in \Sigma(G)$.

Moreover, there exist elements g_i , i = 1, 2, ..., in G such that $g_i \cdot w$ converges to \hat{w} in W. Then for each i, we can write $g_i = \kappa'_i \cdot \exp(2\pi A_i) \cdot \kappa_i$ for some κ_i , $\kappa'_i \in G_c$ and for some $A_i \in \mathfrak{a}$, where $2\pi\sqrt{-1}\mathfrak{a}$ is the Lie algebra of some maximal compact torus in G_c . Let $2\pi\sqrt{-1}\mathfrak{a}_{\mathbb{Z}}$ be the kernel of the exponential map of the Lie algebra $2\pi\sqrt{-1}\mathfrak{a}$, and put $\mathfrak{a}_{\mathbb{Q}} := \mathfrak{a}_{\mathbb{Z}} \otimes \mathbb{Q}$. Replacing $\{\kappa_i\}$ by its subsequence if necessary, we may assume that

(3.5)
$$\kappa_i \to \kappa_\infty$$
 and $\{\exp(2\pi A_i) \cdot \kappa_i\} \cdot w \to w_\infty$, as $i \to \infty$,

for some $\kappa_{\infty} \in G_c$ and $w_{\infty} \in G_c \cdot \hat{w}$. Then by (3.4), the orbit $\lambda(\mathbb{C}^*) \cdot w_{\infty}$ is also closed in W for every $\lambda \in \Sigma(G)$. Let \mathfrak{a}_{∞} denote the Lie subalgebra of a consisting of all elements in a whose associated vector fields on W vanish at $\kappa_{\infty} \cdot w$. For a Euclidean metric on a induced from a suitable bilinear form on $\mathfrak{a}_{\mathbb{Q}}$ defined over \mathbb{Q} , we write a as a direct sum $\mathfrak{a}_{\infty}^{\perp} \oplus \mathfrak{a}_{\infty}$, where $\mathfrak{a}_{\infty}^{\perp}$ is the orthogonal complement of \mathfrak{a}_{∞} in a. Let \overline{A}_i be the image of A_i under the orthogonal projection

$$\operatorname{pr}_1 : \mathfrak{a} (= \mathfrak{a}_{\infty}^{\perp} \oplus \mathfrak{a}_{\infty}) \to \mathfrak{a}_{\infty}^{\perp}, \qquad A \mapsto \overline{A} := \operatorname{pr}_1(A).$$

Note that $\{\exp(2\pi A_i) \cdot \kappa_\infty\} \cdot w = \{\exp(2\pi \bar{A}_i) \cdot \kappa_\infty\} \cdot w$. Hence,

(3.6)
$$\limsup_{i \to \infty} \| \exp \left\{ 2\pi \operatorname{Ad}(\kappa_{\infty}^{-1}) \bar{A}_i \right\} \cdot w \| = \limsup_{i \to \infty} \| \left\{ \exp(2\pi A_i) \cdot \kappa_{\infty} \right\} \cdot w \|$$

$$\leq \lim_{i \to \infty} \| \{ \exp(2\pi A_i) \cdot \kappa_i \} \cdot w \| = \| w_\infty \| < +\infty.$$

STEP 3. Since $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for every $\lambda \in \Sigma(G)$, by the boundedness in (3.6), $\{\bar{A}_i\}$ is a bounded sequence in $\mathfrak{a}_{\infty}^{\perp}$ (see Remark 3.7 below). Hence, for some element A_{∞} in $\mathfrak{a}_{\infty}^{\perp}$, replacing $\{\bar{A}_i\}$ by its subsequence if necessary, we may assume that $\bar{A}_i \to A_{\infty}$ as $i \to \infty$. Then by (3.5),

$$w_{\infty} = \lim_{i \to \infty} \{ \exp(2\pi \bar{A}_i) \cdot \kappa_i \} \cdot w = \{ \exp(2\pi \bar{A}_{\infty}) \cdot \kappa_{\infty} \} \cdot w.$$

Since we have $\exp(2\pi \bar{A}_{\infty}) \in G$, the point w_{∞} in O_1 belongs to the orbit $G \cdot w$. This contradicts $O_1 \cap (G \cdot w) = \emptyset$, as required. The proof of Lemma 3.2 is now complete.

REMARK 3.7. The boundedness of the sequence $\{\bar{A}_i\}$ in $\mathfrak{a}_{\infty}^{\perp}$ in Step 3 above can be seen as follows: For contradiction, we assume that the sequence $\{\bar{A}_i\}$ is unbounded. Put $v := \kappa_{\infty} \cdot w$ for simplicity. Then by (3.6), we first observe that

(3.8)
$$\limsup_{i\to\infty} \|\exp(2\pi\bar{A}_i)\cdot v\| < +\infty.$$

Since $2\pi\sqrt{-1}\mathfrak{a}_{\infty}$ is the Lie algebra of the isotropy subgroup of the compact torus $\exp(2\pi\sqrt{-1}\mathfrak{a})$ at v, both \mathfrak{a}_{∞} and $\mathfrak{a}_{\infty}^{\perp}$ are defined over \mathbb{Q} in \mathfrak{a} . By choosing a complex coordinate system of W, we can write v as $(v_0, \ldots, v_r, 0, \ldots, 0)$ for some integer r with $0 \le r \le \dim W - 1$ such that $v_{\alpha} \ne 0$ for all $0 \le \alpha \le r$ and that

(3.9)
$$\exp(2\pi\bar{A}) \cdot v = (e^{2\pi\chi_0(\bar{A})}v_0, \dots, e^{2\pi\chi_r(\bar{A})}v_r, 0, \dots, 0), \qquad \bar{A} \in \mathfrak{a}_{\infty}^{\perp},$$

where $\chi_{\alpha} \colon \mathfrak{a}_{\infty}^{\perp} \to \mathbb{R}$, $\alpha = 0, 1, \ldots, r$, are additive characters defined over \mathbb{Q} . Put $n := \dim_{\mathbb{R}} \mathfrak{a}_{\infty}^{\perp}$, and let $(\mathfrak{a}_{\infty}^{\perp})_{\mathbb{Q}}$ denote the set of all rational points in $\mathfrak{a}_{\infty}^{\perp}$. Let us now identify

$$\mathfrak{a}_{\infty}^{\perp} = \mathbb{R}^n$$
 and $(\mathfrak{a}_{\infty}^{\perp})_{\mathbb{Q}} = \mathbb{Q}^n$,

as vector spaces. Since the orbit $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for all special one-parameter subgroups $\lambda \colon \mathbb{C}^* \to G$ of G, the same thing is true also for $\lambda(\mathbb{C}^*) \cdot v$. Hence,

(3.10)
$$\mathbb{Q}^n \setminus \{0\} \subset \bigcup_{\alpha,\beta=0}^r U_{\alpha\beta},$$

where $U_{\alpha\beta} := \{ A \in \mathfrak{a}; \chi_{\alpha}(A) > 0 > \chi_{\beta}(A) \}$. Note that the boundaries of the open sets $U_{\alpha\beta}, 1 \le \alpha \le r, 1 \le \beta \le r$, in \mathbb{R}^n sit in the union of \mathbb{Q} -hyperplanes

$$H_{\alpha} := \{ \chi_{\alpha} = 0 \}, \qquad \alpha = 0, 1, \ldots, r,$$

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in \mathbb{R}^r . Since an intersection of any finite number of hyperplanes H_{α} , $\alpha = 0, 1, ..., r$, has dense rational points, (3.10) above easily implies

(3.11)
$$\mathbb{R}^n \setminus \{0\} = \bigcup_{\alpha,\beta=0}^r U_{\alpha\beta}$$

Replacing $\{\bar{A}_i\}$ by its suitable subsequence if necessary, we may assume that there exists an element A_{∞} in $\mathfrak{a}_{\infty}^{\perp} (= \mathbb{R}^n)$ with $||A_{\infty}||_{\mathfrak{a}} = 1$ such that

$$\lim_{i\to\infty}\frac{\bar{A}_i}{\|\bar{A}_i\|_{\mathfrak{a}}}=A_{\infty},$$

where $\| \|_{\mathfrak{a}}$ denotes the Euclidean norm for \mathfrak{a} as in Step 2 in the proof of Theorem 3.2. By (3.11), there exist $\alpha, \beta \in \{0, 1, \ldots, r\}$ such that $A_{\infty} \in U_{\alpha\beta}$, and in particular $\chi_{\alpha}(A_{\infty}) > 0$. On the other hand, $\limsup_{i \to \infty} \|\bar{A}_i\|_{\mathfrak{a}} = +\infty$ by our assumption. Thus,

$$\limsup_{i \to \infty} \chi_{\alpha}(\bar{A}_i) = \limsup_{i \to \infty} \left\{ \|\bar{A}_i\|_{\mathfrak{a}} \cdot \chi_{\alpha}(\bar{A}_i/\|\bar{A}_i\|_{\mathfrak{a}}) \right\} = (\limsup_{i \to \infty} \|\bar{A}_i\|_{\mathfrak{a}}) \chi_{\alpha}(A_{\infty}) = +\infty,$$

in contradiction to (3.8) and (3.9), as required.

4. The Chow norm

Take an algebraic torus $T \subset \operatorname{Aut}^0(M)$, and let ι : $\operatorname{SL}(V_m) \to \operatorname{PGL}(V_m)$ be the natural projection, where we regard $\operatorname{Aut}^0(M)$ as a subgroup of $\operatorname{PGL}(V_m)$ via the Kodaira embedding $\Phi_m \colon M \hookrightarrow \mathbb{P}^*(V_m), m \gg 1$. In this section, we fix a \tilde{T}_c -invariant Hermitian metric ρ on V_m , where \tilde{T}_c is the maximal compact subgroup of $\tilde{T} := \iota^{-1}(T)$. Obviously, in terms of this metric, $V(\chi_k) \perp V(\chi_l)$ if $k \neq l$. Using Deligne's pairings (cf. [8, 8.3]), Zhang ([39, 1.5]) defined a special type of norm on W_m^* , called the *Chow norm*, as a nonnegative real-valued function

$$W_m^* \ni w \longmapsto \|w\|_{\mathrm{CH}(\rho)} \in \mathbb{R}_{>0},$$

with very significant properties described below. First, this is a norm, so that it has the only zero at the origin satisfying the homogeneity condition

$$\|c w\|_{\operatorname{CH}(\rho)} = |c| \cdot \|w\|_{\operatorname{CH}(\rho)} \quad \text{for all } (c, w) \in \mathbb{C} \times W_m^*.$$

For the group $SL(V_m)$, we consider the maximal compact subgroup $SU(V_m; \rho)$. For a special one-parameter subgroup

$$\lambda \colon \mathbb{C}^* \to \mathrm{SL}(V_m)$$

of SL(V_m), there exist integers γ_j , $j = 0, 1, ..., N_m$, and an orthonormal basis $\{s_0, s_1, ..., s_{N_m}\}$ for (V_m, ρ) such that, for all j,

(4.2)
$$\lambda_z \cdot s_j = e^{z\gamma_j} s_j, \qquad z \in \mathbb{C},$$

where $\lambda_z := \lambda(e^z)$. Recall that the subvariety M_m in $\mathbb{P}^*(V_m)$ is the image of the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ defined by

(4.3)
$$\Phi_m(p) = (s_0(p) : s_1(p) : \dots : s_{N_m}(p)), \qquad p \in M,$$

where $\mathbb{P}^*(V_m)$ is identified with $\mathbb{P}^{N_m}(\mathbb{C}) = \{(z_0 : z_1 : \cdots : z_{N_m})\}$. Put $M_{m,t} := \lambda_t(M_m)$ for each $t \in \mathbb{R}$. As in Section 2, $\hat{M}_{m,t} := \lambda_t \cdot \hat{M}_m$ is the nonzero point of W_m^* sitting over the Chow point of the irreducible reduced cycle $M_{m,t}$ on $\mathbb{P}^*(V_m)$. Then (cf. [39, 1.4, 3.4.1])

(4.4)
$$\frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)} \right) = (n+1) \int_{M} \frac{\sum_{j=0}^{N_{m}} \gamma_{j} |\lambda_{t} \cdot s_{j}|^{2}}{\sum_{j=0}^{N_{m}} |\lambda_{t} \cdot s_{j}|^{2}} \left(\Phi_{m}^{*} \lambda_{t}^{*} \omega_{\mathrm{FS}} \right)^{n} dt$$

where ω_{FS} is the Fubini-Study form $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{j=0}^{N_m}|z_j|^2)$ on $\mathbb{P}^*(V_m)$, and we regard λ_t as a linear transformation of $\mathbb{P}^*(V_m)$ induced by (4.2). Note that the term $\Phi_m^*\lambda_t^*\omega_{\text{FS}}$ above is just $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{j=0}^{N_m}|\lambda_t \cdot s_j|^2)$. Put $\Gamma := 2\pi\sqrt{-1}\mathbb{Z}$. By setting

$$\mathbb{C}/\Gamma = \{ t + \sqrt{-1} \theta ; t \in \mathbb{R}, \theta \in \mathbb{R}/(2\pi\mathbb{Z}) \},\$$

we consider the complexified situation. Let $\eta: M \times \mathbb{C}/\Gamma \to \mathbb{P}^*(V_m)$ be the map sending each $(p, t + \sqrt{-1}\theta)$ in $M \times \mathbb{C}/\Gamma$ to $\lambda_{t+\sqrt{-1}\theta} \cdot \Phi_m(p)$ in $\mathbb{P}^*(V_m)$. For simplicity, we put

$$Q \coloneqq \frac{\sum_{j=0}^{N_m} \gamma_j e^{2t\gamma_j} |s_j|^2}{\sum_{j=0}^{N_m} e^{2t\gamma_j} |s_j|^2} \left(= \frac{\sum_{j=0}^{N_m} \gamma_j |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} \right).$$

We further put $z := t + \sqrt{-1} \theta$. For the time being, on the total complex manifold $M \times \mathbb{C}/\Gamma$, the ∂ -operator and the $\bar{\partial}$ -operator will be written simply as ∂ and $\bar{\partial}$ respectively, while on M, they will be denoted by ∂_M and $\bar{\partial}_M$ respectively. Then

$$\eta^* \omega_{\rm FS} = \Phi_m^* \lambda_t^* \omega_{\rm FS} + \frac{\sqrt{-1}}{2\pi} \left(\partial_M Q \wedge d\bar{z} + dz \wedge \bar{\partial}_M Q \right) + \frac{\sqrt{-1}}{4\pi} \frac{\partial Q}{\partial t} dz \wedge d\bar{z}.$$

For $0 \neq r \in \mathbb{R}$, we consider the 1-chain $I_r := [0, r]$, where [0, r] means the 1-chain -[r, 0] if r < 0. Let pr: $\mathbb{C}/\Gamma \to \mathbb{R}$ be the mapping sending each $t + \sqrt{-1}\theta$ to t. We now put $B_r := \operatorname{pr}^* I_r$. Then $\int_{M \times B_r} \eta^* \omega_{\mathrm{FS}}^{n+1}$ is nothing but

$$(n+1)\int_0^r dt \int_M \left(\frac{\partial Q}{\partial t} \Phi_m^* \lambda_t^* \omega_{\rm FS}^n + \frac{\sqrt{-1}}{\pi} \bar{\partial}_M Q \wedge \partial_M Q \wedge n \Phi_m^* \lambda_t^* \omega_{\rm FS}^{n-1}\right)$$

$$= \int_0^r \frac{d^2}{dt^2} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)} \right) dt = \frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)} \right) \Big|_{t=0}^{t=r},$$

and by assuming $r \ge 0$, we obtain the following convexity formula:

Theorem 4.5.
$$\frac{d}{dt} \left(\log \| \hat{M}_{m,t} \|_{\operatorname{CH}(\rho)} \right) \Big|_{t=0}^{t=r} = \int_{M \times B_r} \eta^* \omega_{\operatorname{FS}}^{n+1} \ge 0.$$

REMARK 4.6. Besides special one-parameter subgroups of $SL(V_m)$, we also consider a little more general smooth path λ_t , $t \in \mathbb{R}$, in $GL(V_m)$ written explicitly by

$$\lambda_t \cdot s_j = e^{t\gamma_j + \delta_j} s_j, \qquad j = 0, 1, \dots, N_m,$$

where γ_j , $\delta_j \in \mathbb{R}$ are not necessarily rational. In this case also, we easily see that the formula (4.4) and Theorem 4.5 are still valid.

5. Proof of Theorem A

The statement of Theorem A is divided into "if" part, "only if" part, and the uniqueness part. We shall prove these three parts separately.

Proof of "*if*" part. Let $\omega \in S$ be a critical metric relative to T. Then by Definition 2.6, in terms of the Hermitian metric defined in (2.4), there exists an admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m of index b such that the associated function $E_{\omega,b}$ has a constant value C on M. By operating $(\sqrt{-1/2\pi})\partial\bar{\partial}\log$ on the identity $E_{\omega,b} = C$, we have

(5.1)
$$\Phi_m^* \omega_{\rm FS} = m \,\omega.$$

Besides the Hermitian metric defined in (2.4), we shall now define another Hermitian metric on V_m . By the identification $V_m \cong \mathbb{C}^{N_m}$ via the basis $\{s_0, s_1, \ldots, s_{N_m}\}$, the standard Hermitian metric on \mathbb{C}^{N_m} induces a Hermitian metric ρ on V_m . As a maximal compact subgroup of G_m , we choose $(G_m)_c$ as in Section 2 by using the metric defined in (2.4). Then the Hermitian metric ρ is also preserved by the $(G_m)_c$ -action on V_m . Let

$$\lambda\colon \mathbb{C}^*\to G_m$$

be a special one-parameter subgroup of G_m . By the notation l(k, i) as in Definition 2.3, we put $s_{k,i} := s_{l(k,i)}$. If necessary, replacing $\{s_0, s_1, \ldots, s_{N_m}\}$ by another admissible normal basis for V_m of the same index b, we may assume without loss of generality that there exist integers $\gamma_{k,i}$, $i = 1, 2, \ldots, n_k$, satisfying

(5.2)
$$\lambda_t \cdot s_{k,i} = e^{l\gamma_{k,i}} s_{k,i}, \qquad t \in \mathbb{C},$$

where $\lambda_t := \lambda(e^t)$ is as in (4.2), and the equality $\sum_{i=1}^{n_k} \gamma_{k,i} = 0$ is required to hold for every k. Put $\gamma_{k,i} = \gamma_{l(k,i)}$ for simplicity. Then by (4.4) and (5.1),

$$\frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)} \right)_{|t=0} = (n+1) \int_{M} \frac{\sum_{j=0}^{N_{m}} \gamma_{j} |s_{j}|^{2}}{\sum_{j=0}^{N_{m}} |s_{j}|^{2}} (\Phi_{m}^{*}\omega_{\mathrm{FS}})^{n}$$

$$= (n+1) m^{n} \int_{M} \frac{\sum_{j=0}^{N_{m}} \gamma_{j} |s_{j}|^{2}_{h^{m}}}{\sum_{j=0}^{N_{m}} |s_{j}|^{2}_{h^{m}}} \omega^{n} = (n+1) m^{n} \int_{M} \frac{\sum_{k=1}^{\nu_{m}} (\sum_{i=1}^{n_{k}} \gamma_{k,i} |s_{i}|^{2}_{h^{m}})}{E_{\omega,b}} \omega^{n}$$

$$= \frac{(n+1) m^{n}}{C} \int_{M} \sum_{k=1}^{\nu_{m}} \left(\sum_{i=1}^{n_{k}} \gamma_{k,i} |s_{i}|^{2}_{h^{m}} \right) \omega^{n} = \frac{(n+1) m^{n}}{C} \sum_{k=1}^{\nu_{m}} b_{k} \left(\sum_{i=1}^{n_{k}} \gamma_{k,i} \right) = 0.$$

Note also that, by Theorem 4.5, we have $c := (d^2/dt^2)(\log \|\hat{M}_{m,t}\|_{CH(\rho)})|_{t=0} \ge 0.$

CASE 1. If *c* is positive, then $\lim_{t\to\infty} \|\hat{M}_{m,t}\|_{CH(\rho)} = +\infty = \lim_{t\to\infty} \|\hat{M}_{m,t}\|_{CH(\rho)}$, and in particular $\lambda(\mathbb{C}^*) \cdot \hat{M}_m$ is closed.

CASE 2. If c is zero, then by applying Theorem 4.5 infinitesimally, we see that $\lambda(\mathbb{C}^*)$ preserves the subvariety M_m in $\mathbb{P}^*(V_m)$, and moreover by

$$\frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\operatorname{CH}(\rho)} \right)_{|t=0} = 0,$$

the isotropy representation of $\lambda(\mathbb{C}^*)$ on the complex line $\mathbb{C}\hat{M}_m$ is trivial. Hence, $\lambda(\mathbb{C}^*) \cdot \hat{M}_m$ is a single point, and in particular closed.

Thus, these two cases together with Theorem 3.2 show that the subvariety M_m of $\mathbb{P}^*(V_m)$ is stable relative to T, as required.

REMARK 5.3. About the one-parameter subgroup $\{\lambda_t : t \in \mathbb{R}\}$ of G_m , we consider a more general situation that $\gamma_{k,i}$ in (5.2) are just real numbers which are not necessarily rational. The above computation together with Remark 4.6 shows that, even in this case, $(d/dt)_{t=0}(\log || \hat{M}_{m,t} ||_{CH(\rho)})$ vanishes.

Proof of "only if" part. Assume that the subvariety M_m in $\mathbb{P}^*(V_m)$ is stable relative to T. Take a Hermitian metric ρ for V_m such that $V(\chi_k) \perp V(\chi_l)$ for $k \neq l$. For this ρ , we consider the associated Chow norm. Since the orbit $G_m \cdot \hat{M}_m$ is closed in W_m , the Chow norm restricted to this orbit attains an abosolute minimum. Hence, for some $g_0 \in G_m$,

$$0 \neq \|g_0 \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)} \le \|g \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)}, \quad \text{for all } g \in G_m.$$

By choosing an admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for $(V_m; \rho)$ of index $(1, 1, \ldots, 1)$, we identify V_m with $\mathbb{C}^{N_m} = \{(z_0, z_1, \ldots, z_{N_m})\}$. Then $SL(V_m)$ is identi-

fied with $SL(N_m+1; \mathbb{C})$. Let \mathfrak{g}_m be the Lie subalgebra of $\mathfrak{sl}(N_m+1; \mathbb{C})$ associated to the Lie subgroup G_m of $SL(N_m+1; \mathbb{C})$. We can now write $g_0 = \kappa' \cdot \exp\{\operatorname{Ad}(\kappa)D\}$ for some $\kappa, \kappa' \in (G_m)_c$ and a real diagonal matrix D in \mathfrak{g}_m . By $\|\exp\{\operatorname{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)} = \|g_0 \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)}$, we have

(5.4)
$$\|\exp\{\operatorname{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)} \le \|\exp\{t\operatorname{Ad}(\kappa)A\} \cdot \exp\{\operatorname{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\operatorname{CH}(\rho)}, t \in \mathbb{R},$$

for every real diagonal matrix A in \mathfrak{g}_m . For $j = 0, 1, \ldots, N_m$, we write the j-th diagonal element of A and D above as a_j and d_j , respectively. Put $c_j := \exp d_j$ and $s'_j := \kappa^{-1} \cdot s_j$. Then $\{s'_0, s'_1, \ldots, s'_{N_m}\}$ is again an admissible normal basis for (V_m, ρ) of index $(1, 1, \ldots, 1)$. By the notation in Definition 2.3, we rewrite s'_j , a_j , c_j , z_j as $s'_{k,i}$, $a_{k,i}$, $c_{k,i}$, $z_{k,i}$ by

$$s'_{k,i} := s'_{l(k,i)}, \quad a_{k,i} := a_{l(k,i)}, \quad c_{k,i} := c_{l(k,i)}, \quad z_{k,i} := z_{l(k,i)},$$

where $k = 1, 2, ..., \nu_m$ and $i = 1, 2, ..., n_k$. By (5.4), the derivative at t = 0 of the right-hand side of (5.4) vanishes. Hence by (4.4) together with Remark 4.6, fixing an arbitrary real diagonal matrix A in g_m , we have

(5.5)
$$\int_{M} \frac{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} a_{k,i} c_{k,i}^{2} |s_{k,i}'|^{2}}{\sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} c_{k,i}^{2} |s_{k,i}'|^{2}} \Phi_{m}^{*}(\Theta^{n}) = 0,$$

where we set $\Theta := (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}c_{k,i}^2|z_{k,i}|^2)$. Let $k_0 \in \{1, 2, ..., \nu_m\}$ and let $i_1, i_2 \in \{1, 2, ..., n_k\}$ with $i_1 \neq i_2$. Using Kronecker's delta, we specify the real diagonal matrix A by setting

$$a_{k,i} = \delta_{kk_0}(\delta_{ii_1} - \delta_{ii_2}), \qquad k = 1, 2, \dots, \nu_m; \ i = 1, 2, \dots, n_k.$$

Apply (5.5) to this A, and let (i_1, i_2) run through the set of all pairs of two distinct elements in $\{1, 2, ..., n_k\}$. Then there exists a positive constant $b_k > 0$ independent of the choice of i in $\{1, 2, ..., n_k\}$ such that

(5.6)
$$\frac{N_m + 1}{m^n c_1(L)^n[M]} \int_M \frac{c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = b_k, \qquad k = 1, 2, \dots, \nu_m.$$

The following identity (5.7) allows us to define (cf. [39]) a Hermitian metric $h_{\rm FS}$ on L^m by

(5.7)
$$|s|_{h_{\mathrm{FS}}}^2 \coloneqq \frac{(N_m+1)}{c_1(L)^n[M]} \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |(s,s'_{k,i})_\rho|^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2}, \qquad s \in V_m.$$

Then for this Hermitian metric, it is easily seen that

(5.8)
$$\sum_{j=0}^{N_m} |c_j s'_j|^2_{h_{\rm FS}} = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2_{h_{\rm FS}} = \frac{N_m + 1}{c_1(L)^n[M]},$$

By operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on both sides of (5.8), we obtain $\Phi_m^*\Theta = c_1(L^m; h_{\rm FS})$. We now set $h := (h_{\rm FS})^{1/m}$ and $\omega := c_1(L; h)$. Then

$$\omega = \frac{1}{m} \Phi_m^* \Theta.$$

Put $s_{k,i}'' := c_{k,i}s_{k,i}'$, and as in Definition 2.3, we write $s_{k,i}''$ as $s_{l(k,i)}''$. Then by (5.8), we have the equality $\sum_{j=0}^{N_m} |s_j''|_{h^m}^2 = (N_m + 1)/c_1(L)^n [M]$. Moreover, in terms of the Hermitian metric defined in (2.4), the equality (5.6) is interpreted as

$$\|s_{k,i}''\|_{L^2}^2 = b_k, \qquad k = 1, 2, \dots, \nu_m; \ i = 1, 2, \dots, n_k,$$

while by this together with (5.8) above, we obtain $\sum_{k=1}^{\nu_m} n_k b_k = N_m + 1$, as required.

Proof of *uniqueness*. Let $\omega = c_1(L;h)$ and $\omega' = c_1(L;h')$ be critical metrics relative to *T*, and let $\{s_j; j = 0, 1, ..., N_m\}$ and $\{s'_j; j = 0, 1, ..., N_m\}$ be respectively the associated admissible normal bases for V_m of index *b*. We use the notation in Definition 2.3. Then

$$E_{\omega,b} := \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}|_{h^m}^2$$
 and $E_{\omega',b} := \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}'|_{h'^m}^2$

take the same constant value $C := (N_m + 1)/c_1(L)^n[M]$ on M. Note here that, by operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on both of these identities, we obtain

$$m\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}|^2 \right) \quad \text{and} \quad m\omega' = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}'|^2 \right).$$

If necessary, we replace each $s_{k,i}$ by $\zeta_k s_{k,i}$ for a suitable complex number ζ_k , independent of *i*, of absolute value 1. Then for each $k = 1, 2, ..., \nu_m$, we may assume that there exists a matrix $g^{(k)} = (g_{i\,i}^{(k)}) \in \text{GL}(n_k; \mathbb{C})$ satisfying

$$s'_{k,\hat{\imath}} = \sum_{i=1}^{n_k} s_{k,i} g^{(k)}_{i\,\hat{\imath}},$$

where *i* and $\hat{\imath}$ always run through the integers in $\{1, 2, ..., n_k\}$. Then the matrix $g^{(k)}$ above is written as $\kappa^{(k)} \cdot (\exp A^{(k)}) \cdot (\kappa'^{(k)})^{-1}$ for some real diagonal matrix $A^{(k)}$ and

$$\kappa^{(k)} = (\kappa_{i\,\hat{i}\,\hat{i}\,}^{(k)}) \text{ and } \kappa^{\prime(k)} = (\kappa^{\prime(k)}_{i\,\hat{i}\,\hat{i}\,})$$

in SU(n_k). Let $a_i^{(k)}$ be the *i*-th diagonal element of $A^{(k)}$. For each $\hat{\imath}$, we put $\tilde{s}_{k,\hat{\imath}} := \sum_{i=1}^{n_k} s_{k,i} \kappa_{i\hat{\imath}}^{(k)}$ and $\tilde{s}'_{k,\hat{\imath}} := \sum_{i=1}^{n_k} s'_{k,i} \kappa'_{i\hat{\imath}}^{(k)}$. If necessary, we replace the bases

 $\{s_{k,1}, s_{k,2}, \ldots, s_{k,n_k}\}$ and $\{s'_{k,1}, s'_{k,2}, \ldots, s'_{k,n_k}\}$ for $V(\chi_k)$ by the bases $\{\tilde{s}_{k,1}, \tilde{s}_{k,2}, \ldots, \tilde{s}_{k,n_k}\}$ and $\{\tilde{s}'_{k,1}, \tilde{s}'_{k,2}, \ldots, \tilde{s}'_{k,n_k}\}$, respectively. Then we may assume, from the beginning, that

$$s'_{k,i} = \{ \exp a_i^{(k)} \} s_{k,i}, \qquad i = 1, 2, \dots, n_k.$$

We now set $\tau_{k,i} := s_{k,i}/\sqrt{b_k}$, and the Hermitian metric for V_m defined in (2.4) will be denoted by ρ . Then $\{\tau_{k,i}; k = 1, 2, ..., \nu_m, i = 1, 2, ..., n_k\}$ is an admissible normal basis of index (1, 1, ..., 1) for (V_m, ρ) . Let $\{\lambda_t; t \in \mathbb{C}\}$ be the smooth one-parameter family of elements in $GL(V_m)$ defined by

$$\lambda_t \cdot \tau_{k,i} = \{ \exp(t \, a_i^{(k)}) \} \sqrt{b_k} \, \tau_{k,i}, \qquad k = 1, 2, \dots, \nu_m; \ i = 1, 2, \dots, n_k.$$

Put $\hat{M}_{m,t} := \lambda_t \cdot \hat{M}_m$, $0 \le t \le 1$. Then by Remark 4.6 applied to the formula (4.4), the derivative $\mathfrak{d}(t) := (d/dt)(\log \|\hat{M}_{m,t}\|_{CH(\rho)})/(n+1)$ at $t \in [0, 1]$ is expressible as

$$\int_{M} \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_i^{(k)} |\lambda_t \cdot \tau_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2} \left\{ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2 \right) \right\}^n.$$

Hence at t = 0, we see that

$$\mathfrak{d}(0) = \int_{M} \sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_{k}} \left\{ \frac{a_{i}^{(k)} |s_{k,i}|_{h^{m}}^{2}}{C} \right\} (m\omega)^{n} = \frac{m^{n}}{C} \sum_{k=1}^{\nu_{m}} \left\{ b_{k} \sum_{i=1}^{n_{k}} a_{i}^{(k)} \right\},$$

while at t = 1 also, we obtain

$$\mathfrak{d}(1) = \int_M \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \left\{ \frac{a_i^{(k)} |S'_{k,i}|_{h'^m}}{C} \right\} (m\omega')^n = \frac{m^n}{C} \sum_{k=1}^{\nu_m} \left\{ b_k \sum_{i=1}^{n_k} a_i^{(k)} \right\}.$$

Thus, $\mathfrak{d}(0)$ coincides with $\mathfrak{d}(1)$, while by Remark 4.6, we see from Theorem 4.5 that $(d^2/dt^2)\{\log \|\hat{M}_{m,t}\|_{CH(\rho)}\} \ge 0$ on [0, 1]. Hence, for all $t \in [0, 1]$,

$$\frac{d^2}{dt^2} \{ \log \| \hat{M}_{m,t} \|_{\operatorname{CH}(\rho)} \} = 0, \qquad \text{on } M.$$

By Remark 4.6, the formula in Theorem 4.5 shows that λ_t , $t \in [0, 1]$, belong to H up to a positive scalar multiple. Since λ_1 commutes with T, the uniqueness follows, as required.

6. Proof of Theorem B

Throughout this section, we assume that the first Chern class $c_1(L)_{\mathbb{R}}$ admits an extremal Kähler metric $\omega_0 = c_1(L; h_0)$. Then by a theorem of Calabi [3], the identity

component K of the group of isometries of (M, ω_0) is a maximal compact connected subgroup of H, and we obtain $\omega_0 \in S_K$ by the notation in the introduction.

DEFINITION 6.1. For a K-invariant Kähler metric $\omega \in S_K$ on M in the class $c_1(L)_{\mathbb{R}}$, we choose a Hermitian metric h on L such that $\omega = c_1(L;h)$. Then the power series in q given by the right-hand side of (2.8) will be denoted by $\Psi(\omega, q)$. Given ω and q, the power series $\Psi(\omega, q)$ is independent of the choice of h.

Let \mathcal{D}_0 be the Lichnérowicz operator as defined in [3], (2.1), for the extremal Kähler manifold (M, ω_0) . Then by $\mathcal{V} \in \mathfrak{k}$, the operator \mathcal{D}_0 preserves the space \mathcal{F} of all real-valued smooth K-invariant functions φ such that $\int_M \varphi \omega_0^n = 0$. Hence, we regard \mathcal{D}_0 just as an operator $\mathcal{D}_0: \mathcal{F} \to \mathcal{F}$, and the kernel in \mathcal{F} of this restricted operator will be denoted simply by Ker \mathcal{D}_0 . Then Ker \mathcal{D}_0 is a subspace of \mathcal{K}_{ω_0} , and we have an isomorphism

(6.2)
$$e_0: \operatorname{Ker} \mathcal{D}_0 \cong \mathfrak{z}, \qquad \varphi \leftrightarrow e_0(\varphi) := \operatorname{grad}_{\omega_0}^{\mathbb{C}} \varphi.$$

By the inner product $(,)_{\omega_0}$ defined in the introduction, we write \mathcal{F} as an orthogonal direct sum Ker $\mathcal{D}_0 \oplus \text{Ker } \mathcal{D}_0^{\perp}$. We then consider the orthogonal projection

$$P: \mathcal{F} (= \operatorname{Ker} \mathcal{D}_0 \oplus \operatorname{Ker} \mathcal{D}_0^{\perp}) \to \operatorname{Ker} \mathcal{D}_0.$$

Now, starting from $\omega(0) := \omega_0$, we inductively define a Hermitian metric h(k), a Kähler metric $\omega(k) := c_1(L; h(k)) \in S_K$, and a vector field $\mathcal{Y}(k) \in \sqrt{-1} \mathfrak{z}, k = 1, 2, ...$, by

(6.3)
$$\begin{cases} h(k) \coloneqq h(k-1) \exp(-q^{k}\varphi_{k}), \\ \omega(k) \equiv \omega(k-1) + \frac{\sqrt{-1}}{2\pi} q^{k} \partial \bar{\partial} \varphi_{k}, \\ \mathcal{Y}(k) \equiv \mathcal{Y}(k-1) + \sqrt{-1} q^{k+2} e_{0}(\zeta_{k}), \end{cases}$$

for appropriate $\varphi_k \in \text{Ker } \mathcal{D}_0^{\perp}$ and $\zeta_k \in \text{Ker } \mathcal{D}_0$, where $\omega(k)$ and $\mathcal{Y}(k)$ are required to satisfy the condition (2.8) with *l* replaced by *k*. We now set $g(k) := \exp^{\mathbb{C}} \mathcal{Y}(k)$. Then

$$\{h(k) \cdot g(k)\}^{-m} h(k)^{m} \{Z(q, \omega(k); \mathcal{Y}(k)) - C_{q}\}$$

$$= \frac{n!}{m^{n}} \left\{ \sum_{j=0}^{N_{m}} |s_{j}|_{h(k)^{m}} \right\} - C_{q} \{g(k) \cdot h(k)^{-m}\} h(k)^{m}$$

$$= \Psi(\omega(k), q) - C_{q} h(k)^{m} \{(\exp^{\mathbb{C}} \mathcal{Y}(k)) \cdot h(k)^{-m}\}$$

$$= \Psi(\omega(k), q) - C_{q} \left\{ 1 + h(k) \frac{\mathcal{Y}(k)}{q} \cdot h(k)^{-1} + R(\mathcal{Y}(k); h(k)) \right\},$$

where $C_q = 1 + \sum_{k=0}^{\infty} \alpha_k q^{k+1}$ is a power series in q with real coefficients α_k spec-

ified later, and the last term $R(\mathcal{Y}(k); h(k)) := h(k)^m \sum_{j=2}^{\infty} \{\mathcal{Y}(k)^j / j!\} \cdot h(k)^{-m}$ will be taken care of as a higher order term in q. Consider the truncated term $C_{q,l} = 1 + \sum_{k=0}^{l} \alpha_k q^{k+1}$. Put

$$\Xi(\omega(k), \mathcal{Y}(k), C_{q,k}) \coloneqq \Psi(\omega(k), q) - C_{q,k} \left\{ 1 - \frac{\mathcal{Y}(k)}{q} \cdot \log h(k) + R(\mathcal{Y}(k); h(k)) \right\}$$

for each k. Then, in terms of $\omega(k)$, $\mathcal{Y}(k)$ and $C_{q,k}$, the condition (2.8) with l replaced by k is just the equivalence

(6.4)
$$\Xi(\omega(k), \mathcal{Y}(k), C_{q,k}) \equiv 0, \qquad \text{modulo } q^{k+2}.$$

We shall now define $\omega(k)$, $\mathcal{Y}(k)$ and $C_{q,k}$ inductively in such a way that the condition (6.4) is satisfied. If k = 0, then we set $\omega(0) = \omega_0$, $\mathcal{Y}(0) = \sqrt{-1} q^2 \mathcal{V}/2$ and $C_{q,0} = 1 + \alpha_0 q$, where we put $\alpha_0 := \{2c_1(L)^n[M]\}^{-1}\{\int_M \sigma_\omega \omega^n + 2\pi F(\mathcal{V})\}\$ for $\omega \in \mathcal{S}_K$. This α_0 is obviously independent of the choice of ω in \mathcal{S}_K . Then, modulo q^2 ,

$$\begin{split} \Psi(\omega(k), q) &- C_{q,0} \left\{ 1 - \frac{\mathcal{Y}(0)}{q} \cdot \log h(0) + R(\mathcal{Y}(0); h(0)) \right\} \\ &\equiv \left(1 + \frac{\sigma_{\omega_0}}{2} q \right) - (1 + \alpha_0 q) \left\{ 1 - q h_0^{-1} \sqrt{-1} \frac{\mathcal{V}}{2} \cdot h_0 \right\} \\ &\equiv \left(1 + \frac{\sigma_{\omega_0}}{2} q \right) - (1 + \alpha_0 q) \left\{ 1 + \left(\frac{\sigma_{\omega_0}}{2} - \alpha_0 \right) q \right\} \equiv 0, \end{split}$$

and we see that (6.4) is true for k = 0. Here, the equality $h_0^{-1}\sqrt{-1}(\mathcal{V}/2) \cdot h_0 = \alpha_0 - (\sigma_{\omega_0}/2)$ follows from a routine computation (see for instance [23]).

Hence, let $l \ge 1$ and assume (6.4) for k = l - 1. It then suffices to find φ_l , ζ_l and α_l satisfying (6.4) for k = l. Put $\mathcal{Y}_l := \sqrt{-1} e_0(\zeta_l)$. For each $(\varphi_l, \zeta_l, \alpha_l) \in \text{Ker } \mathcal{D}_0^{\perp} \times \text{Ker } \mathcal{D}_0 \times \mathbb{R}$, we consider

$$\begin{split} \Phi(q;\varphi_l,\zeta_l,\alpha_l) &:= \Psi\left(\omega(l-1) + \frac{\sqrt{-1}}{2\pi}q^l\partial\bar{\partial}\varphi_l,q\right) - \left(C_{q,l-1} + \alpha_l q^{l+1}\right) \\ &\times \left\{1 - \left(\frac{\mathcal{Y}(l-1)}{q} + q^{l+1}\mathcal{Y}_l\right) \cdot \log\left\{h(l-1)\exp(-q^l\varphi_l)\right\} \\ &+ R\left(\frac{\mathcal{Y}(l-1)}{q} + q^{l+1}\mathcal{Y}_l;\ h(l-1)\exp(-q^l\varphi_l)\right)\right\}. \end{split}$$

By the induction hypothesis, $\Xi(\omega(l-1), \mathcal{Y}(l-1), C_{q,l-1}) \equiv 0 \mod q^{l+1}$. Since $\Phi(q; 0, 0, 0) = \Xi(\omega(l-1), \mathcal{Y}(l-1), C_{q,l-1})$, we have

$$\Phi(q;0,0,0) \equiv u_l q^{l+1}, \qquad \text{modulo } q^{l+2},$$

for some real-valued *K*-invariant smooth function u_l on *M*. Let $(\varphi_l, \zeta_l, \alpha_k) \in \text{Ker } \mathcal{D}_0^{\perp} \times \text{Ker } \mathcal{D}_0 \times \mathbb{R}$. Since φ_k is *K*-invariant, by $\mathcal{V} \in \mathfrak{k}$, we see that $\sqrt{-1} \mathcal{V} \varphi_k$ is a real-valued

function on *M*. Note also that $\mathcal{Y}(0) = (\sqrt{-1} \mathcal{V}/2) q^2$. Then the variation formula for the scalar curvature (see for instance [3, (2.5)]) shows that, modulo q^{l+2} ,

$$\begin{split} \Phi(q;\varphi_l,\zeta_l,\alpha_l) \\ &\equiv \Phi(q;0,0,0) + \frac{q^{l+1}}{2} \left(-\mathcal{D}_0 + \sqrt{-1} \mathcal{V} \right) \varphi_l - \alpha_l q^{l+1} + q^{l+1} h_0^{-1} (\mathcal{Y}_l \cdot h_0) - \frac{\sqrt{-1}}{2} \mathcal{V} \varphi_l q^{l+1} \\ &\equiv \left\{ u_l - \mathcal{D}_0 \frac{\varphi_l}{2} - \alpha_l - \hat{F}_m(\mathcal{Y}_l) + e_0^{-1} (\sqrt{-1} \mathcal{Y}_l) \right\} q^{l+1}, \end{split}$$

where we put $\hat{F}(\mathcal{Y}) := \{c_1(L)^n[M]\}^{-1} 2\pi F(\sqrt{-1}\mathcal{Y})$ for each $\mathcal{Y} \in \sqrt{-1}\mathfrak{z}$. By setting $\mu_l := \{c_1(L)^n[M]\}^{-1}(\int_M u_l \omega_0^n)$, we write u_l as a sum

$$u_l = \mu_l + u_l' + u_l'',$$

where $u'_l := (1 - P)(u_l - \mu_l) \in \text{Ker } \mathcal{D}_0^{\perp}$ and $u''_l := P(u_l - \mu_l) \in \text{Ker } \mathcal{D}_0$. Now, let φ_l be the unique element of $\text{Ker } \mathcal{D}_0^{\perp}$ such that $\mathcal{D}_0(\varphi_l/2) = u'_l$. Moreover, we put

$$\zeta_l := u_l''$$
 and $\alpha_l := \mu_l - \hat{F}(\mathcal{Y}_l).$

Then by $\mathcal{Y}_l = \sqrt{-1} e_0(\zeta_l) = \sqrt{-1} e_0(u_l'')$, we obtain

$$\begin{split} \Phi(q;\varphi_l,\zeta_l,\alpha_l) &\equiv \left\{ \mu_l + u_l' + u_l'' - \mathcal{D}_0 \frac{\varphi_l}{2} - \alpha_l - \hat{F}_m(\mathcal{Y}_l) + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_l) \right\} q^{l+1} \\ &\equiv \left\{ u_l'' + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_l) \right\} q^{l+1} \equiv 0, \qquad \text{mod } q^{l+2}, \end{split}$$

as required. Write $\sqrt{-1} \mathcal{V}/2$ as \mathcal{Y}_0 for simplicity. Now, for the real Lie subalgebra \mathfrak{b} of \mathfrak{z} generated by \mathcal{Y}_k , k = 0, 1, 2, ..., its complexification $\mathfrak{b}^{\mathbb{C}}$ in $\mathfrak{z}^{\mathbb{C}}$ generates a complex Lie subgroup $B^{\mathbb{C}}$ of $Z^{\mathbb{C}}$. Then it is easy to check that the algebraic subtorus T of $Z^{\mathbb{C}}$ obtained as the closure of $B^{\mathbb{C}}$ in $Z^{\mathbb{C}}$ has the required properties.

REMARK 6.5. In Theorem B, assume that ω_0 is a Kähler metric of constant scalar curvature, and moreover that the actions $\rho_{m(\nu)}$, $\nu = 1, 2, ...$, coincide (cf. [26, (2.3)]) for all sufficiently large ν . Then by [26], the trivial group {1} can be chosen as the algebraic subtorus T above of $Z^{\mathbb{C}}$.

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