Saymeh, S.A.
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ON HASSE-SCHMIDT HIGHER DERIVATIONS

SADI ABU-SAYMEH

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Let k be a field of characteristic zero and let A be a commutative k-algebra. A higher derivation D of A over k is a sequence

$$\underline{D} = \{D_0, D_1, D_2, \cdots\}$$

of additive k-endomorphisms D_i's such that D_0 is the identity map of A and $D_n(ab) = \sum_{m=0}^{n} D_m(a) D_{n-m}(b)$ for every $a, b \in A$. This interesting notion of higher derivations was introduced by H. Hasse and F.K. Schmidt in [1].

In this paper we shall prove that a higher derivation $\underline{D}$ of A over k is represented uniquely by a certain sequence of derivations of A over k.

Let n, r be positive integers such that $n \geq r$. We shall denote by $P_{n,r}$ the set of ordered partitions of n into r-positive integers, i.e.,

$$P_{n,r} = \{(n_1, \cdots, n_r) \mid \sum_{i=1}^{r} n_i = n, n_i \in N_+\} .$$

It is easily seen that the cardinality $|P_{n,r}|$ of the set $P_{n,r}$ is given by

$$|P_{n,r}| = \binom{n-1}{r-1}.$$

Proposition 1. *Let $\underline{D} = (D_0, D_1, D_2, \cdots)$ be a higher derivation on A and let $\delta_n (n \geq 1)$ be defined by the equations*

$$\delta_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{(n_1, \cdots, n_r) \in P_{n,r}} D_{n_1} D_{n_2} \cdots D_{n_r} .$$

Then we have

(i) $\delta_n (n = 1, 2, \cdots)$ *is a k-derivation*,

(ii) $D_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1, \cdots, n_r) \in P_{n,r}} \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r}$.

Proof. (i) For $n=1$ we have $\delta_1 = D_1$ which is clearly k-derivation.

For $n\geq 2$, and $a, b\in A$ we have

$$\delta_n(ab) = D_n(ab)+\sum_{r=2}^{n}\frac{(-1)^{r+1}}{r}\sum_{(n_1,\cdots,n_r)\in P_{n,r}} D_{n_1}D_{n_2}\cdots D_{n_r}(ab)$$

$$= \sum_{m=0}^{n} D_m(a)\,D_{n-m}(b)+\sum_{r=2}^{n}\frac{(-1)^{r+1}}{r}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}\sum_{0\leq m_i\leq n_i}$$

$$\cdot\, D_{m_1}D_{m_2}\cdots\ldots D_{m_r}(a)\,D_{n_1-m_1}D_{n_2-m_2}\cdots D_{n_r-m_r}(b)$$

$$= a\delta_n(b)+\delta_n(a)\,b+\sum_{m=1}^{n-1} D_m(a)\,D_{n-m}(b)$$

$$+\sum_{r=2}^{n}\frac{(-1)^{r+1}}{r}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}\sum_{\substack{0\leq m_i\leq n_i\\ \text{and at least one } 0<m_i<n_i}}$$

$$\cdot\, D_{m_1}D_{m_2}\cdots D_{m_r}(a)\,D_{n_1-m_1}D_{n_2-m_2}\cdots D_{n_r-m_r}(b)\,.$$

Hence to prove the assertion it suffices to show that

$$\sum_{m=1}^{n-1} D_m(a)\,D_{n-m}(b)+\sum_{r=2}^{n}\frac{(-1)^{r+1}}{r}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}\sum_{\substack{0\leq m_i\leq n_i\\ \text{and at least one } 0<m_i<n_i}}$$

$$\cdot\, D_{m_1}D_{m_2}\cdots D_{m_r}(a)\,D_{n_1-m_1}\cdots D_{n_r-m_r}(b) = 0\,.$$

Let $h(e,s)$ be the coefficients of $D_{l_1}D_{k_2}\cdots D_{l_e}(a)\,D_{j_1}D_{j_2}\cdots D_{j_s}(b)$ in the reduced expression of the left hand side where l_i's and j_i's are positive integers such that $l_1+\cdots+l_e+j_1+\cdots+j_s=n$. Such a term can occur only if $r=e, e+1, \cdots, e+s$. Hence if $e\geq s\geq 1$ then it is seen without essential difficulty that we have

$$h(e, s) = \sum_{p=0}^{s}\frac{(-1)^{e+p+1}}{e+p}\binom{e+p}{e}\binom{e}{s-p}.$$

The sum correspond to the case $r=e, \cdots, r=e+s$ respectively and $\binom{e+p}{e}$ is the number of times one can select e number of m_i's to be euqal to l_i's and setting the other p-number of m_i's to be zero, while $\binom{e}{s-p}$ is the number of times one can select $s-p$ numbers of (n_i-m_i)'s to be equal to the j's.
Since

$$\frac{1}{e+p}\binom{e+p}{p} = \frac{1}{e}\binom{e+p-1}{p}.$$

Then we get

$$h(e, s) = \frac{(-1)^{e+1}}{e}\sum_{p=0}^{s}(-1)^{p}\binom{e+p-1}{p}\binom{e}{s-p}.$$

Setting $s-p=q$ we obtain

$$h(e, s) = \frac{(-1)^{e+s+1}}{e} \sum_{q=0}^{s} (-1)^q \binom{e+s-q-1}{s-q} \binom{e}{q}.$$

Hence $h(e, s)=0$ by [2, identity (35) p. 41]. Similarly $h(e, s)=0$ if $s>e\geq 1$. Hence δ_n is a k-derivation.

(ii) We use induction on n. For $n=1$ we have $\delta_1=D_1$. Since

$$D_n = \delta_n + \sum_{r=2}^{n} \frac{(-1)^r}{r} \sum_{(n_1, \cdots, n_r)\in P_{n,r}} D_{n_1} D_{n_2} \cdots D_{n_r} \text{ for } n\geq 2,$$

and the induction assumption on n implies that

$$D_{n_i} = \sum_{s=1}^{n_i} \frac{1}{s!} \sum_{(n_{i1}, \cdots, n_{is})\in P_{n_i,s}} \delta_{ni1} \delta_{ni2} \cdots \delta_{nis}$$
$$\text{for every } 1\leq n_i<n.$$

Hence after collecting similar terms we get

$$D_n = \delta_n + \sum_{r=2}^{n} \sum_{(n_1, \cdots, n_r)\in P_{n,r}} \left[\sum_{p=2}^{r} \frac{(-1)^p}{p} \sum_{(m_1, \cdots, m_p)\in P_{r,p}} \frac{1}{(m_1)!\cdots(m_p)!} \right] \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r}.$$

On the other hand the coefficient of χ^r (for every $r\geq 2$) in the Taylor's series expansion of $\chi=\ln[1+(e^\chi-1)]$ is

$$\frac{1}{r!} - \sum_{p=2}^{r} \frac{(-1)^p}{p} \sum_{(m_1, \cdots, m_p)\in P_{r,p}} \frac{1}{(m_1)!\cdots(m_p)!} = 0.$$

Hence

$$D_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1, \cdots, n_r)\in P_{n,r}} \delta_{n_1} \delta_{n_2} \cdots \delta_{n_r}.$$

A higher derivation $\underline{D}=\{D_n;\ n\geq 0\}$ is called iterative if it satisfies the condition

$$D_i D_j = \binom{i+j}{i} D_{i+j}$$

for any pair of integers (i, j). Then we have the

Corollary. *Let $\underline{D}$ be a higher derivation of A over k and $\{\delta_n, n=1, 2, \cdots\}$ be a corresponding sequence of derivations defined in Proposition 1. Then $\underline{D}$ is* *iterative if and only if* $\delta_n=0$ *for all* $n\geq 2$.

Proposition 2. *Let* $\{\delta_1, \delta_2, \cdots\}$ *be a sequence of k-derivations on A and set*

$$D_n = \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1,\cdots,n_r)\in P_{n,r}} \delta_{n_1}\delta_{n_2}\cdots\delta_{n_r}\ (n\geq 1)\,.$$

Then we have

(i) $\underline{D} = \{D_0 = id, D_1, D_2, \cdots,\}$

is a higher derivation, and

(ii) $\delta_n = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \sum_{(n_1,\cdots,n_r)\in P_{n,r}} D_{n_1} D_{n_2}\cdots D_{n_r}\,.$

Proof. (i) The assertion is clear for $n=1$.

Next we show that $D_n(ab)=\sum_{m=0}^{n} D_m(a)\, D_{n-m}(b)$ for every $a,b\in A$ and $n\geq 2$. For convenience let δ_m^0 satnds for the identity mapping and δ_m^1 sands for δ_m. Then we have

$$\begin{aligned} D_n(ab) &= \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1,\cdots,n_r)\in P_{n,r}} \delta_{n_1}\delta_{n_2}\cdots\delta_{n_r}(ab) \\ &= \sum_{r=1}^{n} \frac{1}{r!} \sum_{(n_1,\cdots,n_r)\in P_{n,r}} \sum_{0\leq e_i\leq 1} \\ &\quad \delta_{n_1}^{e_1}\delta_{n_2}^{e_2}\cdots\delta_{n_r}^{e_r}(a)\, \delta_{n_1}^{1-e_1}\delta_{n_2}^{1-e_2}\cdots\delta_{n_r}^{1-e_r}(b) \\ &= aD_n(b)+D_n(a)\cdot b+\sum_{r=2}^{n} \frac{1}{r!} \sum_{(n_1,\cdots,n_r)\in P_{n,r}} \sum_{\substack{0\leq e_i\leq 1 \\ \text{such that} \\ 1\leq e_1+\cdots+e_r\leq r-1}} \\ &\quad \delta_{n_1}^{e_1}\cdots\delta_{e_r}^{e_r}(a)\cdot \delta_{n_1}^{1-e_1}\cdots\delta_{n_r}^{1-e_r}(b) \\ &= aD_n(b)+D_n(a)\cdot b+\sum_{r=2}^{n} \sum_{(t,s)\in P_{r,2}} \frac{1}{(t+s)!}\binom{t+s}{t} \\ &\quad \sum_{(m^1,\cdots,m_t,l_1,\cdots,l_s)\in P_{n,t+s}} \\ &\quad \delta_{m_1}\delta_{m_2}\cdots\delta_{m_t}(a)\cdot\delta_{l_1}\delta_{l_2}\cdots\delta_{l_s}(b)\,. \end{aligned}$$

Note that $\binom{t+s}{t}$ is the number of ways of selecting t number of e_i's equal to one in the expression

$$\delta_{n_1}^{e_1}\delta_{n_2}^{e_2}\cdots\delta_{n_r}^{e_r}(a)\cdot\delta_{n_1}^{1-e_1}\delta_{n_2}^{1-e_2}\cdots\delta_{n_r}^{1-e_r}(b) \quad \text{where } r=t+s\,.$$

On the other hand we have

$$\sum_{m=0}^{n} D_m(a)\, D_{n-m}(b) = aD_n(b)+D_n(a)b$$

$$+\sum_{m=1}^{n-1}\left[\left(\sum_{t=1}^{m}\frac{1}{t!}\sum_{(m_1,\cdots,m_t)\in P_{m,t}}\delta_{m_1}\delta_{m_2}\cdots\delta_{m_t}(a)\right)\right.$$

$$\left.\left(\sum_{s=1}^{n-m}\frac{1}{s!}\sum_{(l_1,\cdots,l_s)\in P_{n-m,s}}\delta_{l_1}\delta_{l_2}\cdots\delta_{l_s}(b)\right)\right]$$

$$= aD_n(b)+D_n(a)\cdot b+\sum_{r=2}^{n}\sum_{(t,s)\in P_{r,2}}\frac{1}{t!s!}$$

$$\sum_{(m_1,\cdots,m_t,l_1,\cdots,l_s)\in P_{n,t+s}}$$

$$\delta_{m_1}\delta_{m_2}\cdots\delta_{m_t}(a)\cdot\delta_{l_1}\delta_{l_2}\cdots\delta_{l_s}(b)\,.$$

Since $\frac{1}{(t+s)!}\binom{t+s}{t}=\frac{1}{t!s!}$ we have $D_n(ab)=\sum_{m=0}^{n} D_m(a)\, D_{n-m}(b)$

(ii) Since $\underline{D}=\{D_0, D_1, D_2, \cdots,\}$ is a higher derivation we can associate to $\underline{D}$ a sequence of derivations $\{\delta_1', \delta_2', \cdots\}$ by Proposition 1(i). From Proposition 1(ii) it follows that

$$D_n = \delta_n'+\sum_{r=2}^{n}\frac{1}{r!}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}\delta_{n_1}'\delta_{n_2}'\cdots\delta_{n_r}'\,.$$

On the other hand

$$D_n = \delta_n+\sum_{r=2}^{n}\frac{1}{r!}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}\delta_{n_1}\delta_{n_2}\cdots\delta_{n_r}$$

by definition of D_n. Since $\delta_1=D_1=\delta_1'$ we get easily $\delta_n=\delta_n'$ by induction on n.

The following theorem follows from Propositions 1 and 2.

Theorem. *There is a one to one correspondence between the set of ordered sequences of k-derivations on A and the set of higher derivations on A in such a way if $\{\delta_n\colon n\geq 0,\ \delta_0$ identity, δ_n is a k-derivation$\}$ and the higher derivation $\underline{D}=\{D_n\colon n\geq 0\}$ correspond, then*

$$D_n = \sum_{p=1}^{n}\frac{1}{r!}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}\delta_{n_1}\delta_{n_2}\cdots\delta_{n_r}$$

and

$$\delta_n = \sum_{r=1}^{n}\frac{(-1)^{r+1}}{r}\sum_{(n_1,\cdots,n_r)\in P_{n,r}}D_{n_1}D_{n_2}\cdots D_{n_r}$$

for every $n\geq 1$ and $D_0=\delta_0$.

Corollary. *Let $D_{A/k}$ be the set of all higher derivations on A. Let $\underline{D}, \underline{E}\in$*

$D_{A/k}$ correspond respectively to the sequences $\{\delta_{1,n}: n\geq 0\}$ and $\{\delta_{2,n}: n\geq 0\}$ of k-derivations on A. Then $D_{A/k}$ is a Lie algebra with respect to the operations $\alpha\underline{D}+\underline{E}=\underline{L}$ and $[\underline{D}, \underline{E}]=\underline{G}$ where $\alpha\in k$ and $\underline{L}$ is the higher derivation corresponding to the sequence $\{\delta_0, \alpha\delta_{1,n}+\delta_{2,n}: n\geq 1\}$ and $\underline{G}$ is the higher derivation corresponding to the sequence $\{\delta_0, [\delta_{1,n}, \delta_{2,n}]=\delta_{1,n}\,\delta_{2,n}-\delta_{2,n}\,\delta_{1,n}: n\geq 1\}$ respectively.

Proof. It follows easily from the fact that the set of k-derivations on A is a Lie algebra.

References

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Department of Mathematics
Yarmouk University
Irbid, Jordan