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ANALYTIC SMOOTHING EFFECT FOR SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH NONLINEARITY OF INTEGRAL TYPE[1]

Dedicated to Professor Mitsuhiro Nakao on the occasion of his sixtieth birthday

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Abstract

We study analytic smoothing effects for solutions to the Cauchy problem for the Schrödinger equation with interaction described by the integral of the intensity with respect to one direction in two space dimensions. The only assumption on the Cauchy data is the weight condition of exponential type and no regularity assumption is imposed.

1. Introduction

We study the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\triangle u = f(u), \tag{1.1}$$

where u is a complex-valued function of time and space variables denoted respectively by $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$, $\partial_t = \partial/\partial t$, $\triangle$ is the Laplacian in $\mathbb{R}^2$, and $f(u)$ is the nonlinear interaction given by

$$(f(u))(t, x, y) = \lambda \left(\int_{-\infty}^{x} \left|u(t, x', y)\right|^2 \,\mathrm{d}x' \right) u(t, x, y) \tag{1.2}$$

with $\lambda \in \mathbb{C}$. The equation (1.1) with integral type nonlinearity (1.2) appears as a model of propagation of laser beams under the influence of a steady transverse wind along the x-axis [1, 3, 37] and as a special case of the Davey-Stewartson system where the velocity potential is independent of y-variable [2, 5, 6, 9, 13, 14, 18, 25, 32].

In spite of a large literature on the nonlinear Schrödinger equations (see for instance [4] and references therein), there are not many papers on the equation (1.1) with nonlinearity of integral type [1, 3, 22, 34, 37]. The existence and uniqueness of global solutions to the Cauchy problem for (1.1) is proved in the usual Sobolev spaces

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$H^m(\mathbb{R}^2)$ with integers $m \geq 1$ [3] and in the Lebesgue spaces $L^2(\mathbb{R}^2)$ [22, 34]. The existence of modified wave operators is proved on a dense set of small and sufficiently regular asymptotic states [22]. Smoothing properties and large time asymptotics are studied in [34] (see also [10, 12, 15, 18, 21, 23–26]). The purpose of this paper is to describe analytic smoothing properties of solutions to the Cauchy problem for (1.1) in terms of the generators of Galilei and pseudo-conformal transformations. We follow the method of Hayashi and coauthors ([10–26], especially [16, 17, 23, 24]) basically, while a systematic use of Strichartz estimates and a couple of observations on the weight condition of exponential type are new ingredients in this paper.

To state our results precisely, we introduce the following notation. The generators of Galilei transformations are denoted by $J = (J_x, J_y) = (x+it\partial_x, y+it\partial_y) = (x, y)+it\nabla$. The generator of pseudo-conformal transformations is denoted by $K = x^2 + y^2 + 2it + 2itx\partial_x + 2ity\partial_y + 2it^2\partial_t = J^2 + 2t^2 L$, where $L = i\partial_t + (1/2)\triangle$. Let X be a Banach space of functions of $(t, (x, y)) \in \mathbb{R} \times \mathbb{R}^2$ or of $(x, y) \in \mathbb{R}^2$ and let A be an operator on X. Then for $a > 0$ the space $G^a(A; X)$ is defined at least formally by

$$G^a(A; X) = \left\{ f \in X;\ \left\| f; G^a(A; X) \right\| = \sum_{n \geq 0} \frac{a^n}{n!} \left\| A^n f; X \right\| < \infty \right\}.$$

Similarly, for operators $A = (A_1, A_2)$ of two components, we define

$$G^a(A; X) = \left\{ f \in X;\ \left\| f; G^a(A; X) \right\| = \sum_{\alpha \geq 0} \frac{a^{|\alpha|}}{\alpha!} \left\| A_1^{\alpha_1} A_2^{\alpha_2} f; X \right\| < \infty \right\},$$

where we have used the standard multi-index notation with $\alpha = (\alpha_1, \alpha_2)$. For simplicity, we write $G^a(A, B; X) = G^a(B; G^a(A; X))$. For $T > 0$ we define

$$\begin{aligned} X_T^a &= G^a((J_x, 0); L^\infty(0, T; L^2) \cap L^8(0, T; L_y^4 L_x^2)), \\ Y_T^a &= G^a((0, J_y); L^\infty(0, T; L^2) \cap L^8(0, T; L_y^4 L_x^2)), \\ Z_T^a &= G^a(J; L^\infty(0, T; L^2) \cap L^8(0, T; L_y^4 L_x^2)), \\ W_T^a &= G^a(J, K; L^\infty(0, T; L^2) \cap L^8(0, T; L_y^4 L_x^2)). \end{aligned}$$

We state the main result in this paper.

Theorem 1. *Let $a > 0$. Then*:

(1) *For any $\rho > 0$ there exists $T > 0$ independent of a with the following properties*:

(a) *For any $\phi \in G^a((x, 0); L^2)$ with $\left\| \phi; G^a((x, 0); L^2) \right\| \leq \rho$*
(1.1) has a unique solution $u \in X_T^a$.

(b) *For any $\phi \in G^a((0, y); L^2)$ with $\left\| \phi; G^a((0, y); L^2) \right\| \leq \rho$*
(1.1) has a unique solution $u \in Y_T^a$.

(c) *For any* $\phi \in G^a((x, y); L^2)$ *with* $\|\phi; G^a((x, y); L^2)\| \leq \rho$
(1.1) *has a unique solution* $u \in Z_T^a$.

(2) *For any* $\rho > 0$ *there exists* $T > 0$ *depending on* a *and* ρ *such that for any* $\phi \in G^a((x, y), x^2 + y^2; L^2)$ *with* $\|\phi; G^a((x, y), x^2 + y^2; L^2)\| \leq \rho$
(1.1) *has a unique solution* $u \in W_T^a$.

REMARK 1. Theorem 1 is regarded as an infinite version of Theorem 2 of [34]. The conclusion holds when the time interval is replaced by $[-T, 0]$.

REMARK 2. Functions in the space $G^a(J; L^\infty(0, T; L^2))$ [resp. $G^a((J_x, 0); L^\infty(0, T; L^2))$, $G^a((0, J_y); L^\infty(0, T; L^2))$] are analytic in (x, y) [resp. x, y] for each $t \neq 0$ [23, 24]. Functions in the space $G^a(J, K; L^\infty(0, T; L^2))$ are analytic in $(t, (x, y))$ with $t \neq 0$ [17]. In those respects, Theorem 1 describes analytic smoothing properties of solutions. We note that no regularity assumption is imposed on the Cauchy data.

In Theorem 1, the assumptions such as $\phi \in G^a((x, 0); L^2)$ are described in terms of infinite series of weighted L^2 norms. It would be more explicit to describe the assumptions in terms of weights of exponential type.

The following proposition describes norm in the spaces $G^a((x, y); L^2)$ and $G^a((x, y), x^2 + y^2; L^2)$ in terms of weights of exponential type.

Proposition 1. *For any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that the following estimates hold*:

$$\begin{aligned} \left\|e^{a(|x|+|y|)}\phi; L^2\right\| &\leq \|\phi; G^a((x, y); L^2)\| \\ &\leq C_\varepsilon \left\|e^{(a+\varepsilon)(|x|+|y|)}\phi; L^2\right\|, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \left\|e^{a(x^2+y^2)}\phi; L^2\right\| &\leq \|\phi; G^a((x, y), x^2 + y^2; L^2)\| \\ &\leq C_\varepsilon \left\|e^{(a+\varepsilon)(x^2+y^2)}\phi; L^2\right\|. \end{aligned} \tag{1.4}$$

Moreover, the second inequality in (1.4) *is optimal in the sense that the estimate*

$$\|\phi; G^a((x, y), x^2 + y^2; L^2)\| \leq C \left\|e^{a(x^2+y^2)}\phi; L^2\right\| \tag{1.5}$$

fails to hold.

REMARK 3. The first part of Proposition 1, including (1.3) and (1.4), is a special case of Proposition 2 in Section 4 below.

In [17], Hayashi and Kato proved analyticity in space-time with $t \neq 0$ of solutions for the nonlinear Schrödinger equations

$$i\partial_t u + \frac{1}{2}\triangle u = \lambda |u|^{2p} u$$

in $\mathbb{R} \times \mathbb{R}^n$ with $\lambda \in \mathbb{C}$ and nonnegative integer p under the assumption on the Cauchy data ϕ such that $e^{|x|^2}\phi \in H^{[n/2]+1}$ for $n \geq 1$, where $[r]$ is the integer part of $r > 0$. The above assumption is relaxed as $e^{|x|^2}\phi \in H^m$, where $m = 0$ if $(n, p) = (1, 1)$ and $m = 1$ if $n = 2$ or $(n, p) = (3, 1)$ (see [17]). In view of (1.4) the assumption $\phi \in G^a((x, y), x^2 + y^2; L^2)$ is satisfied if $e^{(a+\varepsilon)(x^2+y^2)}\phi \in L^2$, which corresponds to $m = 0 < a < a + \varepsilon = 1$ in the last assumption in $n = 2$.

We refer the reader to [7, 8, 11, 19, 21, 29, 30, 31] for analyticity of solutions to other nonlinear evolution equations and to [33, 35, 36, 38] for analytic smoothing effects for linear dispersive equations.

We prove Theorem 1 in Section 3 by a contraction argument. Basic estimates for the proof of Theorem 1 are summarized in Section 2. We prove Proposition 1 in Section 4 in a general setting.

Throughout the paper we use the following notation without further comments. $L_y^p L_x^q = L^p(\mathbb{R}_y; L^q(\mathbb{R}_x))$ with norm

$$\left\| u; L_y^p L_x^q \right\| = \left\| \left\| u; L_x^q \right\|; L_y^p \right\|.$$

$L^p = L^p(\mathbb{R}^2) = L_y^p L_x^p$. $U(t) = \exp(i(t/2)\triangle)$ denotes the free Schrödinger group acting on functions on $\mathbb{R}^2$. $M(t)$ denotes the modulation operator realized as the multiplication by $\exp(i(x^2 + y^2)/(2t)) \cdot$ for $t \neq 0$. The operators J are represented as

$$J = U(t)(x, y)U(-t) = M(t)it\nabla M(-t),$$

while K satisfies the following useful identity

$$K - 2it = 2itM(t)PM(-t),$$

where $P = x\partial_x + y\partial_y + t\partial_t$.

2. Preliminaries

In this section we collect some basic estimates for the free Schrödinger group $U(t)$ and the nonlinearity $f(u)$ of integral type.

Lemma 1 (Hayashi-Ozawa [22]). *$U(t)$ satisfies the following estimates*:

(1) *For any (q, r) with* $0 \leq 2/q = 1/2 - 1/r \leq 1/2$

$$\left\| U(\cdot)\phi; L^q(\mathbb{R}; L_y^r L_x^2) \right\| \leq C \left\| \phi; L^2 \right\|.$$

(2) *For any* (q_j, r_j) *with* $0 \leq 2/q_j = 1/2 - 1/r_j \leq 1/2$, $j = 1, 2$, *the operator* G *defined by*

$$(Gu)(t) = \int_0^t U(t-t')u(t')\,\mathrm{d}t'$$

satisfies the estimate

$$\left\| Gu; L^{q_1}(0, T; L_y^{r_1} L_x^2) \right\| \leq C \left\| Gu; L^{q_2'}(0, T; L_y^{r_2'} L_x^2) \right\|,$$

where C *is independent of* $T > 0$ *and* p' *is the dual exponent to* p *defined by* $1/p + 1/p' = 1$.

Proof. See [22, 34]. □

Lemma 2 (Hayashi-K. Kato [17]). *Let* $1 \leq q, r \leq \infty$ *and let* $p \in \mathbb{R}$. *Then for any* a, $T > 0$ *and* $\mu \in \mathbb{C}$ *with* $amT < 1$, $m = \mathrm{Max}(|\mu|/2, 1)+1$ *the following inequality holds*:

$$\left\| f; G^a(K + i(p-2)t + \mu t; X) \right\| \leq \left(1 + \frac{a|\mu|T}{1 - amT}\right) \left\| f; G^a(K + i(p-2)t; X) \right\|,$$

where $X = L^q(0, T; L_y^r L_x^2)$ *or* $G^a(J; L^q(0, T; L_y^r L_x^2))$.

Proof. We argue as in [17]. By the commutation relation

$$[\mu t, K + i(p-2)t] = -\frac{2i}{\mu}(\mu t)^2,$$

we have

$$\begin{aligned}(K + i(p-2)t + \mu t)^l &\\ = \sum_{k=1}^{l} \binom{l}{k} \left[\prod_{j=0}^{k-1} (\mu + 2ij)\right] t^k (K + i(p-2)t)^{l-k} &\\ + (K + i(p-2t))^l. &\qquad (2.1)\end{aligned}$$

We note that $\|tf; X\| \leq T\,\|f; X\|$. By (2.1), we obtain

$$\begin{aligned}&\left\| f; G^a(K + i(p-2)t + \mu t; X) \right\| \\ &= \sum_{l=0}^{\infty} \frac{a^l}{l!} \left\| (K + i(p-2)t + \mu t)^l f; X \right\|\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{l=0}^{\infty} \sum_{k=1}^{l} \frac{a^l}{l!} \binom{l}{k} \left[\prod_{l=0}^{k-1} (|\mu| + 2j) \right] T^k \left\| (K + i(p-2)t)^{l-k} f; X \right\| \\
&\quad + \left\| f; G^a(K + i(p-2)t; X) \right\| \\
&\le \sum_{l=0}^{\infty} \sum_{k=1}^{l} \frac{a^{l-k}}{(l-k)!} \left[\frac{1}{k!} \prod_{j=0}^{k-1} (|\mu| + 2j) \right] (aT)^k \left\| (K + i(p-2)t)^{l-k} f; X \right\| \\
&\quad + \left\| f; G^a(K + i(p-2)t; X) \right\| \\
&\le \left[\sum_{k=1}^{\infty} \frac{(aT)^k}{k!} \prod_{j=0}^{k-1} (|\mu| + 2j) + 1 \right] \left\| f; G^a(K + i(p-2)t; X) \right\| .
\end{aligned}
$$

The lemma then follows by the following inequalities:

$$
\begin{aligned}
\frac{1}{k!} \prod_{j=0}^{k-1} (|\mu| + 2j) &= |\mu| \prod_{j=1}^{k-1} \left(2 + \frac{|\mu| - 2}{j+1} \right) \\
&\le |\mu| \prod_{j=1}^{k-1} \left(2 + \mathrm{Max} \left(\frac{|\mu| - 2}{2}, 0 \right) \right) = |\mu| m^{k-1}. \qquad \square
\end{aligned}
$$

Lemma 3. *Let* (q_j, r_j), $j = 0, 1, 2, 3$, *satisfy* $1 \le q_j, r_j \le \infty$, $1/q_0 = \sum_{j=1}^{3} 1/q_j$, $1/r_0 = \sum_{j=1}^{3} 1/r_j$. *Then*:

(1) *For any* $a, T > 0$

$$
\begin{aligned}
&\left\| \psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}(t, x', y)\, \mathrm{d}x'; G^a((J_x, 0); L^{q_0}(I; L_y^{r_0} L_x^2)) \right\| \\
&\le \prod_{j=1}^{3} \left\| \psi_j; G^a((J_x, 0); L^{q_j}(I; L_y^{r_j} L_x^2)) \right\| , \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&\left\| \psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}(t, x', y)\, \mathrm{d}x'; G^a((0, J_y); L^{q_0}(I; L_y^{r_0} L_x^2)) \right\| \\
&\le \prod_{j=1}^{3} \left\| \psi_j; G^a((0, J_y); L^{q_j}(I; L_y^{r_j} L_x^2)) \right\| , \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
&\left\| \psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}(t, x', y)\, \mathrm{d}x'; G^a(J; L^{q_0}(I; L_y^{r_0} L_x^2)) \right\| \\
&\le \prod_{j=1}^{3} \left\| \psi_j; G^a(J; L^{q_j}(I; L_y^{r_j} L_x^2)) \right\| , \qquad (2.4)
\end{aligned}
$$

where $I = [0, T]$.

(2) *For any* $a, T > 0$ *with* $2aT < 1$

$$\left\| \psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}(t, x', y)\, \mathrm{d}x'; G^a(J, K - 2it; L^{q_0}(I; L_y^{r_0} L_x^2)) \right\| \tag{2.5}$$
$$\leq \frac{e^a}{(1-2aT)^3} \prod_{j=1}^{3} \left\| \psi_j; G^a(J, K; L^{q_j}(I; L_y^{r_j} L_x^2)) \right\|,$$

where $I = [0, T]$.

Proof. We define $\widetilde{\psi}_j = M_x^{-1} \psi_j$, $M_x = M_x(t) = \exp(ix^2/(2t))$. By the relation $J_x = M_x\,(it\partial_x)\,M_x^{-1}$, we obtain

$$J_x^l \left(\psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}\, \mathrm{d}x' \right) = M_x\,(it\partial_x)^l \left(\widetilde{\psi}_1 \int_{-\infty}^{x} \widetilde{\psi}_2 \overline{\widetilde{\psi}_3}\, \mathrm{d}x' \right)$$
$$= M_x(it)^l \left(\partial_x^l \widetilde{\psi}_1 \cdot \int_{-\infty}^{x} \widetilde{\psi}_2 \overline{\widetilde{\psi}_3}\, \mathrm{d}x' + \sum_{\substack{l_1+l_2=l \\ l_2 \geq 1}} \frac{l!}{l_1!\, l_2!} \partial_x^{l_1} \widetilde{\psi}_1 \cdot \int_{-\infty}^{x} \partial_x^{l_2} \left(\widetilde{\psi}_2 \overline{\widetilde{\psi}_3} \right) \mathrm{d}x' \right)$$
$$= M_x(it)^l \sum_{l_1+l_2+l_3=l} \frac{l!}{k_1!\, k_2!\, k_3!} \partial_x^{k_1} \widetilde{\psi}_1 \int_{-\infty}^{x} \partial_x^{k_2} \widetilde{\psi}_2 \cdot \overline{\partial_x^{k_3} \widetilde{\psi}}\, \mathrm{d}x'$$
$$= \sum_{l_1+l_2+l_3=l} \frac{l!(-1)^{k_3}}{k_1!\, k_2!\, k_3!} J_x^{k_1} \psi_1 \int_{-\infty}^{x} J_x^{k_2} \psi_2 \overline{J_x^{k_3} \psi_3}\, \mathrm{d}x'. \tag{2.6}$$

We estimate (2.6) by the Hölder inequalities in space-time as

$$\left\| J_x^l \left(\psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}\, \mathrm{d}x' \right); L^{q_0}(I; L_y^{r_0} L_x^2) \right\|$$
$$\leq l! \sum_{k_1+k_2+k_3=l} \prod_{j=1}^{3} \frac{1}{k_j!} \left\| J_x^{k_j} \psi_j; L^{q_j}(I; L_y^{r_j} L_x^2) \right\|.$$

This implies

$$\left\| \psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3}\, \mathrm{d}x'; G^a((J_x, 0); L^{q_0}(I; L_y^{r_0} L_x^2)) \right\|$$
$$\leq \sum_{l=0}^{\infty} \sum_{k_1+k_2+k_3=l} \prod_{j=1}^{3} \frac{1}{k_j!} \left\| J^{k_j} \psi_j; L^{q_j}(I; L_y^{r_0} L_x^2) \right\|$$
$$\leq \prod_{j=1}^{3} \left\| \psi_j; G^a((J_x, 0); L^{q_0}(I; L_y^{r_0} L_x^2)) \right\|.$$

This proves (2.2). By a similar calculation, we obtain (2.3). Similarly, (2.4) follows from a two dimensional generalization of (2.6).

To prove (2.5), by using

$$P \int_{-\infty}^{x} f \, \mathrm{d}x' = \int_{-\infty}^{x} (P+1) f \, \mathrm{d}x',$$
$$(P+1)^k (fg) = \sum_{j=0}^{k} \binom{k}{j} P^{k-j} f \cdot (P+1)^j g,$$

we compute

$$\begin{aligned}
&J^{\alpha} (K - 2it)^l \left(\psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3} \, \mathrm{d}x' \right) \\
&= (2it)^{|\alpha|+l} M \partial^{\alpha} P^l \left(\widetilde{\psi}_1 \int_{-\infty}^{x} \widetilde{\psi}_2 \overline{\widetilde{\psi}_3} \, \mathrm{d}x' \right) \\
&= (2it)^{|\alpha|+l} M \sum_{\substack{\beta'+\beta''+\beta'''=\alpha \\ j_1+j_2+j_3=l}} \frac{\alpha! \, l! \, (-1)^{|\beta'''|}}{j_1! \, j_2! \, j_3! \, \beta'! \, \beta''! \, \beta'''!} \\
&\qquad \cdot \partial^{\beta'} P^{j_1} \widetilde{\psi}_1 \int_{-\infty}^{x} \partial^{\beta''} P^{j_2} \widetilde{\psi}_2 \cdot \overline{\partial^{\beta'''} (P+1)^{j_3} \widetilde{\psi}_3} \, \mathrm{d}x' \\
&= \sum_{\substack{\beta'+\beta''+\beta'''=\alpha \\ j_1+j_2+j_3=l}} \frac{\alpha! \, l! (-1)^{|\beta'''|}}{j_1! \, j_2! \, j_3! \, \beta'! \, \beta''! \, \beta'''!} \\
&\qquad \cdot J^{\beta'} (K - 2it)^{j_1} \psi_1 \int_{-\infty}^{x} J^{\beta''} (K - 2it)^{j_2} \psi_2 \cdot \overline{J^{\beta'''} (K - 2it + 1)^{j_3} \psi_3} \, \mathrm{d}x'.
\end{aligned}$$

In the same way as above, we estimate

$$\begin{aligned}
&\left\| \psi_1 \int_{-\infty}^{x} \psi_2 \overline{\psi_3} \, \mathrm{d}x'; G^a(J, K - 2it; L^{q_0}(I; L_y^{r_0} L_x^2)) \right\| \\
&\leq \left\| \psi_3; G^a(J, K - 2it + 1; L^{q_3}(I; L_y^{r_3} L_x^2)) \right\| \prod_{j=1}^{2} \left\| \psi_j; G^a(J, K - 2it; L^{q_j}(I; L_y^{r_j} L_x^2)) \right\| \\
&\leq e^a \prod_{j=1}^{3} \left\| \psi_j; G^a(J, K - 2it; L^{q_j}(I; L_y^{r_j} L_x^2)) \right\| \\
&\leq \frac{e^a}{(1 - 2aT)^3} \prod_{j=1}^{3} \left\| \psi_j; G^a(J, K; L^{q_j}(I; L_y^{r_j} L_x^2)) \right\|,
\end{aligned}$$

where we have used Lemma 2 and the inequality

$$\left\| f; G^a(A+1; X) \right\| \leq e^a \left\| f; G^a(A; X) \right\|,$$

which holds for any operator A with $[A, 1] = 0$. □

3. Proof of Theorem 1

We solve the integral equation

$$u(t) = U(t)\phi - i\int_0^t U(t-t')f(u(t'))\,\mathrm{d}t' \tag{3.1}$$

by a contraction argument on X_T^a, Y_T^a, Z_T^a, and W_T^a. Let $\phi \in G^a((x,0);L^2)$ with $\|\phi; G^a((x,0);L^2)\| \le \rho$. For $R > 0$ we define

$$X_T^a(R) = \left\{u \in X_T^a;\ \|u; X_T^a\| \le R\right\},$$

where

$$\|u; X_T^a\| = \mathrm{Max}\left(\|u; G^a((J_x,0); L^\infty(0,T;L^2))\|, \|u; G^a((J_x,0); L^8(0,T;L_y^4L_x^2))\|\right).$$

For $u \in X_T^a$ we define $(\Phi(u))(t)$ as the RHS of (3.1). We have

$$\left(J_x^l\Phi(u)\right)(t) = U(t)x^l\phi - i\int_0^t U(t-t')\left(J_x^l f(u)\right)(t')\,\mathrm{d}t'. \tag{3.2}$$

Applying Lemma 1 to the RHS of (3.2) and using the Hölder inequality in time, we obtain

$$\begin{aligned}
&\mathrm{Max}\left(\|J_x^l\Phi(u); L^\infty(0,T;L^2)\|, \|J_x^l\Phi(u); L^8(0,T;L_y^4L_x^2)\|\right)\\
&\le C\|x^l\phi; L^2\| + C\|J_x^l(f(u)); L^{4/3}(0,T;L_y^1L_x^2)\|\\
&\le C\|x^l\phi; L^2\| + CT^{1/2}\|J_x^l(f(u)); L^4(0,T;L_y^1L_x^2)\|.
\end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $a^l/l!$, making a summation on l and applying Lemma 3, we obtain

$$\begin{aligned}
\|\Phi(u); X_T^a\| &\le \|\phi; G^a((x,0);L^2)\|\\
&\quad + CT^{1/2}\|u; G^a((J_x,0); L^8(0,T;L_y^4L_x^2))\|^2\,\|u; G^a((J_x,0); L^\infty(0,T;L^2))\|.
\end{aligned}$$

In the same way as above, for $u, v \in X_T^a(R)$, $\Phi(u)$ satisfies

$$\|\Phi(u); X_T^a\| \le C\rho + CT^{1/2}R^3,$$
$$\|\Phi(u)-\Phi(v); X_T^a\| \le CT^{1/2}R^2\|u-v; X_T^a\|.$$

For $\rho > 0$ let R and T satisfy $R \ge 2C\rho$, $T \le 1/(4C^2R^4)$. Then $\Phi(u)$ has a unique fixed point in $X_T^a(R)$. This proves Part (1). Parts (2) and (3) follow in the same way.

For Part (4) we write, with $u \in W_T^a$

$$\left(J^\alpha K^l\Phi(u)\right)(t) = U(t)x^{\alpha_1}y^{\alpha_2}(x^2+y^2)^l\phi$$

$$- i \int_0^t U(t-t') \left(J^{\alpha}(K+4it')^l f(u) \right)(t')\, \mathrm{d}t',$$

where we have used the relation

$$K(t)U(t-t') = U(t-t') \left(K(t') + 4it' \right),$$

which follows from the commutation relation $[K, L] = -4itL$, see [17, 20]. In the same way as above, we have

$$\begin{aligned} &\left\| \Phi(u); W_T^a \right\| \\ &\leq C \left\| \phi; G^a((x,y), x^2+y^2; L^2) \right\| \\ &\quad + CT^{1/2} \left\| f(u); G^a(J, K+4it; L^4(0,T; L_y^1 L_x^2)) \right\| . \end{aligned} \tag{3.4}$$

By Lemmas 2 and 3, the last norm on the RHS of (3.4) is estimated as

$$\begin{aligned} &\left\| f(u); G^a(J, K+4it; L^4(0,T; L_y^1 L_x^2)) \right\| \\ &\leq \frac{1}{1-(5/2)aT} \left\| f(u); G^a(J, K-2it; L^4(0,T; L_y^1 L_x^2)) \right\| \\ &\leq \frac{1}{1-(5/2)aT} \frac{e^a}{(1-2aT)^3} \left\| u; G^a(J, K; L^8(0,T; L_y^4 L_x^2)) \right\|^2 \\ &\quad \cdot \left\| u; G^a(J, K; L^{\infty}(0,T; L^2)) \right\| . \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we have for $u \in W_T^a(R)$

$$\left\| \Phi(u); W_T^a \right\| \leq C\rho + \frac{CT^{1/2} R^3 e^a}{(1-(5/2)aT)(1-2aT)^3}.$$

Similarly, for $u, v \in W_T^a(R)$

$$\left\| \Phi(u) - \Phi(v); W_T^a \right\| \leq \frac{CT^{1/2} R^2 e^a}{(1-(5/2)aT)(1-2aT)^3} \left\| u - v; W_T^a \right\| .$$

For $\rho > 0$ let R and T satisfy $R \geq 2C\rho$, $T \leq \mathrm{Min}(1/(5a), 1/(20CR^2e^a)^2)$. Then $\Phi(u)$ has a unique fixed point in $W_T^a(R)$.

4. Proof of Proposition 1

First, we prove the last part of the proposition. The LHS of (1.5) is estimated as

$$\begin{aligned} &\left\| \phi; G^a((x,y), x^2+y^2; L^2) \right\| \\ &= \sum_{j,k,l \geq 0} \frac{a^{j+k+l}}{j!\,k!\,l!} \left\| |x|^j |y|^k (x^2+y^2)^l \phi; L^2 \right\| \end{aligned}$$

$$\geq \left\| \sum_{j,k,l\geq 0} \frac{a^{j+k+l}}{j!\,k!\,l!} |x|^j |y|^k (x^2+y^2)^l \phi; L^2 \right\|$$
$$= \left\| e^{a(|x|+|y|+x^2+y^2)} \phi; L^2 \right\|.$$

Therefore (1.5) fails to hold for $\phi = e^{-a(|x|+|y|+x^2+y^2)}$.

From now on we consider functions in $\mathbb{R}^n$. We use the standard multi-index notation.

Proposition 2. *Let n and l be positive integers and let $a > 0$. Let $w = (w_1, \dots, w_l)$ be functions of $x \in \mathbb{R}^n$. Then the following inequalities hold.*

$$\left\| \left(\prod_{j=1}^{l} e^{a|w_j|} \right) \phi; L^2(\mathbb{R}^n) \right\|$$
$$\leq \sum_{\substack{\alpha\in\mathbb{Z}^l \\ \alpha\geq 0}} \frac{a^{|\alpha|}}{\alpha!} \left\| w^\alpha \phi; L^2(\mathbb{R}^n) \right\|$$
$$\leq C_0^l \left\| \left(\prod_{j=1}^{l} (1+a|w_j|)^{1/2} e^{a|w_j|} \right) \phi; L^2(\mathbb{R}^n) \right\|, \tag{4.1}$$

where $C_0 = (1 + 2\log 2)^{1/2}$.

REMARK 4. To derive (1.3) from (4.1), we put $l = 2$, $w_1(x, y) = x$, $w_2(x, y) = y$. To derive (1.4) from (4.1), we put $l = 3$, $w_1(x, y) = x$, $w_2(x, y) = y$, $w_3(x, y) = x^2 + y^2$.

Proof of Proposition 2. The first inequality of (4.1) is proved by the Maclaurin expansion of e^x and the triangle inequality.

The second inequality of (4.1) is proved as follows. By the Wallis formula

$$\frac{(2k)!}{2^{2k}(k!)^2}\frac{1}{k} \sim \frac{1}{\sqrt{\pi}}\frac{1}{k^{1+1/2}},$$

we see that the series

$$\sum_{k\geq 0} \frac{(2k)!}{2^{2k}(k!)^2}\frac{1}{k_+}$$

converges to a finite value C_0^2 $(= 1 + 2\log 2)$, where $k_+ = \mathrm{Max}(k, 1)$. By Schwarz' inequality, this yields the following estimate:

$$
\begin{aligned}
&\sum_{\alpha} \frac{a^{|\alpha|}}{\alpha!} \left\| w^{\alpha}\phi; L^2(\mathbb{R}^n) \right\| \\
&\le \left(\sum_{\alpha} \frac{(2\alpha)!}{2^{2|\alpha|}((\alpha_1)!)^2 (\alpha_1)_+ \cdots (\alpha_l)_+} \right)^{1/2} \\
&\quad \cdot \left(\sum_{\alpha} \frac{(\alpha_1)_+ \cdots (\alpha_l)_+ 2^{2|\alpha|}}{(2\alpha)!} \cdot a^{2|\alpha|} \left\| w^{\alpha}\phi; L^2(\mathbb{R}^n) \right\|^2 \right)^{1/2} \\
&= C_0^l \left\langle \sum_{\alpha} \frac{(\alpha_1)_+ \cdots (\alpha_l)_+}{(2\alpha)!} (2aw)^{2\alpha}\phi, \phi \right\rangle^{1/2}, \qquad (4.2)
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^n)$. Since

$$
\sum_{k=0}^{\infty} \frac{k_+}{(2k)!} \xi^{2k} = 1 + \frac{\xi}{2} \sinh \xi,
$$

the RHS of (4.2) is dominated by

$$
\begin{aligned}
&C_0^l \left\langle \prod_{j=1}^{l} \left(1 + \frac{2aw_j}{2} \sinh 2aw_j \right) \phi, \phi \right\rangle^{1/2} \\
&= C_0^l \left\| \prod_{j=1}^{l} \left(1 + \frac{2aw_j}{2} \sinh 2aw_j \right)^{1/2} \phi; L^2(\mathbb{R}^n) \right\|.
\end{aligned}
$$

Using $1 \le (1 + \xi \sinh 2\xi) \le (1 + |\xi|) e^{2|\xi|}$, we have the second inequality of (4.1). □

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