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## Q-HOMOLOGY PLANES WITH C\*\*-FIBRATIONS

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

MASAYOSHI MIYANISHI and TOHRU SUGIE

(Received November 13, 1989)

**Introduction.** Let  $X$  be a nonsingular algebraic surface defined over the complex number field  $\mathbf{C}$ . We call  $X$  a *homology plane* (resp. a  $\mathbf{Q}$ -*homology plane*) if the homology groups  $H_i(X; \mathbf{Z})$  (resp.  $H_i(X; \mathbf{Q})$ ) vanish for  $1 \leq i \leq 4$ . We can also define a *logarithmic homology plane*  $X$  as a normal affine surface which has only quotient singularities and  $H_i(X; \mathbf{Z})=0$  for all  $i > 0$ .

In our previous paper [7],  $\mathbf{Q}$ -homology planes with Kodaira dimension less than 2 are classified and it is shown that there are many  $\mathbf{Q}$ -homology planes which have non-trivial automorphisms of finite order. A structure theorem is given on logarithmic homology planes of Kodaira dimension  $-\infty$  and 1. In particular, it is proved that a logarithmic homology plane of Kodaira dimension  $-\infty$  is isomorphic to one of the following surfaces:

- (1)  $\mathbf{C}^2$ ;
- (2)  $\mathbf{C}^2/G$ , where  $G$  is a small finite subgroup of  $GL(2, \mathbf{C})$ ;
- (3) A surface  $X$  with an  $\mathbf{A}^1$ -fibration  $\rho: X \rightarrow \mathbf{A}^1$  such that every fiber is irreducible and that there are  $N$  multiple fibres  $H_1, \dots, H_N$  with respective multiplicities  $d_1, \dots, d_N$ , each of them carrying a cyclic quotient singular point of type  $d_i/e_i$ , where  $N$  is an arbitrary positive integer.

Similarly, logarithmic homology planes of Kodaira dimension 1 are studied by making use of  $\mathbf{C}^*$ -fibrations.

In the present paper we are interested in homology planes with  $\kappa=2$ . An example of a contractible algebraic surface with  $\kappa=2$ , which is a special case of a homology plane, was first given by C.P. Ramanujam [9] and many examples were recently found by Gurjar-Miyanishi [2], Miyanishi-Sugie [6] and Petrie-tom Dieck [11, 12]. We constructed in [6] homology planes by the blowing-up method from the configurations of two curves on the projective plane  $\mathbf{P}^2$  and Petrie-tom Dieck [11] from the line arrangements on  $\mathbf{P}^2$ . In order to construct further examples, we propose to think of algebraic surfaces with fibrations of curves. As a natural extension of the  $\mathbf{C}$ -fibrations and the  $\mathbf{C}^*$ -fibrations which are so effective in the cases of  $\kappa=-\infty$  and 1, we shall look into a surface with

a  $\mathbf{C}^{**}$ -fibration, where  $\mathbf{C}^{**}$  is the affine line with two points deleted off. We can consider a  $\mathbf{C}^{**}$ -fibration as an analogy of a fibration by curves of genus 2 in the complete case.

Besides, in the study of structures of homology planes, it is an interesting problem to verify or negate the following:

**Homology Plane Conjecture.** *Let  $X$  be a homology plane admitting a non-trivial automorphism of finite order. Then  $X$  is isomorphic to  $\mathbf{C}^2$ .*

In §1 of this paper, we classify singular fibers of  $\mathbf{C}^{**}$ -fibrations, and in §2 we classify  $\mathbf{Q}$ -homology planes which have  $\mathbf{C}^{**}$ -fibrations. In §3 we calculate the homology groups  $H_i(X; \mathbf{Z})$  and the Kodaira dimension  $\kappa = \kappa(X)$  of certain surfaces listed in §2. Thus, we obtain infinitely many homology planes and logarithmic homology planes of  $\kappa = 2$  and some of these surfaces have, indeed, nontrivial automorphisms of finite order, which negates the above homology plane conjecture. We give an explicit description of those examples in §4.

By the way, there seems to be a misunderstanding about the difference between homology planes and contractible surfaces in the case of  $\kappa = 1$ . Petrie proved that contractible surfaces of  $\kappa = 1$  have no non-trivial automorphisms. There exist, however, homology planes with  $\kappa = 1$  which have non-trivial automorphisms. We also include these examples in §5.

**NOTATIONS:** We denote by  $\mathbf{C}^{(N*)}$  a rational curve  $\mathbf{C} - \{N \text{ points}\}$ . In particular,  $\mathbf{C}^*$  is a curve  $\mathbf{C} - \{1 \text{ point}\}$  and  $\mathbf{C}^{**}$  is a curve  $\mathbf{C} - \{2 \text{ points}\}$ . A  $(-1)$  curve means an exceptional curve of the first kind. We refer to Miyanishi [5] for the definition of Kodaira dimension  $\kappa$  and relevant results. We employ also the notations and results in [7].

## 1. Singular fibers of $\mathbf{C}^{**}$ -fibrations

Let  $X$  be a normal affine surface defined over the complex number field  $\mathbf{C}$  with a  $\mathbf{C}^{(N*)}$ -fibration  $\pi: X \rightarrow B$ , where  $B$  is a smooth algebraic curve. Let  $V$  be a normal projective surface which contains  $X$  as an open subset and is smooth along  $D := V - X$ . Moreover, we assume that  $D$  is an effective divisor with simple normal crossings and that the fibration  $\pi$  is extended to a  $\mathbf{P}^1$ -fibration  $p: V \rightarrow C$ , where  $C$  is a smooth complete curve. Let  $f: W \rightarrow V$  be a minimal resolution of singularities of  $V$ . Then  $q = p \circ f: W \rightarrow C$  is a  $\mathbf{P}^1$ -fibration on a smooth projective surface  $W$  and if we set  $Y = f^{-1}(X)$ ,  $\rho := p|_Y: Y \rightarrow B$  defines a  $\mathbf{C}^{(N*)}$ -fibration on  $Y$ . We identify the divisor  $D$  on  $V$  with the divisor  $f^{-1}(D)$  on  $W$ .

The following property of a  $\mathbf{P}^1$ -fibration is well-known. We shall make use of it freely.

**Lemma 1.1.** *Let  $W$  be a smooth projective surface with a  $\mathbf{P}^1$ -fibration  $q:$*

$W \rightarrow C$  and let  $F$  be a singular fiber of  $q$ . Write  $F = \sum_{i=1}^n m_i F_i$  as a sum of irreducible components. Then the following hold:

- (1) Three irreducible components of  $F$  do not meet in one point;
- (2)  $\text{Supp}(F)$  does not contain a loop;
- (3) If  $m_k=1$  and  $n \geq 2$  there exists a  $(-1)$  curve in  $F$  other than  $E_k$ .

We consider first the case where  $D$  contains  $N+1$  different cross-sections and denote these sections by  $S_1, \dots, S_{N+1}$ . If two or more of  $S_i$ 's meet in one point, we blow up these intersection points until the proper transforms of  $S_i$ 's are disjoint from each other, and we include the resulting exceptional curves into the boundary divisor  $D$ . We may thus assume that  $S_1, S_2, \dots, S_{N+1}$  do not meet. In this case we call a  $\mathbf{C}^{(N*)}$ -fibration  $\pi: X \rightarrow C$  *untwisted*. If not all of  $S_i$ 's are cross-sections, we call a  $\mathbf{C}^{(N*)}$ -fibration *twisted*. Since singular fibers of  $\mathbf{C}^{(N*)}$ -fibrations on normal surfaces can be obtained easily from the smooth case, we consider first a smooth affine surface with a  $\mathbf{C}^{(N*)}$ -fibration. We call a fiber  $\pi^{-1}(P)$  a *singular fiber* if it is not isomorphic to  $\mathbf{C}^{(N*)}$  as a subscheme.

Let  $A$  be a singular fiber of  $\pi$ , let  $A_1, \dots, A_k$  be all connected components of  $A$  and let  $A_{ij}$  be irreducible components of  $A_i$ . Let  $T := p^{-1}(\pi(A))$  be the fiber of  $p$  containing  $A$ . We denote by  $T_i$  the connected component of  $T \cap D$  which intersects  $S_i$ . We may assume that  $T_i \neq \emptyset$  for every  $i$ . Indeed, if  $T_i = \emptyset$ , blow up the point  $T \cap S_i$  and include the exceptional curve into the boundary divisor  $D$ . Note that  $T_i$  and  $T_j$  might coincide with each other for different indices  $i$  and  $j$ . Set

$$a_i + 1 := \text{the number of points of } (\bar{A}_i - A_i) \quad \text{and} \quad a := \sum_i a_i,$$

where  $\bar{A}_i$  is the closure of  $A_i$  in  $V$ . Then we have

**Lemma 1.2.**  $a \leq N$ .

**Proof.** We have only to consider the connected components  $A_i$  for which  $a_i \geq 1$ . We assume that  $a_i \geq 1$  for  $1 \leq i \leq m$  and  $a_i = 0$  for  $m < i \leq k$ . First, there are  $a_1 + 1$  connected components  $T_1, \dots, T_{a_1+1}$  of  $T \cap D$  which intersect  $\bar{A}_1$ . Secondly, there are  $a_2 + 1$  connected components  $T_{a_1+2}, \dots, T_{a_1+a_2+2}$  of  $T \cap D$  which intersect  $\bar{A}_2$ , at most one of which can be taken from  $T_1, \dots, T_{a_1+1}$  since a fiber of a  $\mathbf{P}^1$ -fibration contains no loops. Let  $\alpha_2$  be the number of different connected components in  $\{T_1, \dots, T_{a_1+1}, T_{a_1+2}, \dots, T_{a_1+a_2+2}\}$  and let  $\beta_2 = a_1 + a_2 + 2 - \alpha_2$ . Then  $\alpha_2 \geq a_1 + a_2 + 1$  and  $2 - \beta_2$  equals to the number of connected components of the support of  $\bar{A}_1 + \bar{A}_2 + \sum_{i=1}^{a_1+a_2+2} T_i$ . In the third step, we need  $a_3 + 1$  connected components  $T_{a_1+a_2+3}, \dots, T_{a_1+a_2+a_3+3}$  which intersect  $\bar{A}_3$  and at most  $2 - \beta_2$  can be taken from  $T_1, \dots, T_{a_1+1}, T_{a_1+2}, \dots, T_{a_1+a_2+2}$ . Let  $\alpha_3$  be the number of different connected components in  $\{T_1, \dots, T_{a_1+a_2+3}\}$  and let  $\beta_3 = a_1 + a_2 + a_3 + 3 - \alpha_3$ . Then  $\alpha_3 \geq \alpha_2 + \{a_3 + 1 - (2 - \beta_2)\} = a_1 + a_2 + a_3 + 1$ . Continuing this way to the

$m$ -th step, we see  $\alpha_m \geq \sum_{i=1}^m a_i + 1$ . Since  $\alpha_m \leq N+1$ , we have the stated inequality. Q.E.D.

We have also the following lemma.

**Lemma 1.3.** *Assume  $N \geq 2$ . If  $A_{\text{red}}$  is isomorphic to  $\mathbf{C}^{(N*)}$  then  $A = A_{\text{red}}$  and  $A$  itself is isomorphic to  $\mathbf{C}^{(N*)}$ .*

Proof. In this case, the connected components  $T_1, \dots, T_{N+1}$  must be all different. If  $T$  contains a  $(-1)$  curve, we contract it. Then the images of  $T_i$ 's are all different, though one of  $T_i$ 's might become the empty set and the image of  $A_{\text{red}}$  is still isomorphic to  $\mathbf{C}^{(N*)}$ . We can thus assume from the beginning that  $T_1, \dots, T_{N+1}$  do not contain  $(-1)$  curves. Then  $\bar{A}_{\text{red}}$  must be a unique  $(-1)$  curve in  $T$ . Contract then  $\bar{A}_{\text{red}}$  and let  $\sigma$  be the contraction morphism. If  $T_i \neq \emptyset$  for every  $i$ , then more than three of  $\sigma(T_i)$ 's intersect in one point, which contradicts the property of a singular fiber of a  $\mathbf{P}^1$ -fibration. Therefore at least one component of  $T_i$ 's is the empty set, say,  $T_1 = \emptyset$ . Then  $\bar{A}_{\text{red}}$  meets the section  $S_1$  and there must be a  $(-1)$  curve in the fiber  $T$  other than  $\bar{A}$  unless  $\bar{A} = T$ . From the assumption that either  $T_i = \emptyset$  or  $T_i$  contains no  $(-1)$  curves, we conclude that  $T = \bar{A} = \mathbf{P}^1$  and  $A = \mathbf{C}^{(N*)}$  as a subscheme.

Q.E.D.

Lemma 1.3 states a property particular to the case  $N \geq 2$ . For example, a  $\mathbf{C}^*$  fibration has a singular fiber of the form  $m\mathbf{C}^*$  ( $m \geq 2$ ). The following result is easy to verify if one takes into account that  $X$  is affine.

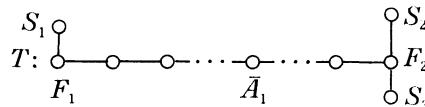
**Lemma 1.4.** *If  $a_i = 0$ , then  $A_i$  is irreducible and isomorphic to  $\mathbf{C}$ .*

From now on we restrict ourselves to the case where  $N = 2$ . Let  $\Gamma$  be the union of  $A_i$ 's for which  $a_i \geq 1$  and let  $\Delta$  be the union of  $A_i$ 's for which  $a_i = 0$ . Then  $A = \Gamma + \Delta$  and  $\Delta$  is a disjoint union of curves which are isomorphic to  $\mathbf{C}$ . With the above notations we have the following:

**Lemma 1.5.** *Let  $X$  be a smooth affine surface with an untwisted  $\mathbf{C}^{**}$ -fibration  $\rho: X \rightarrow C$  and employ the notations  $A$ ,  $T$ ,  $\Gamma$ ,  $\Delta$ ,  $T_i$ 's, etc. as above. Assume  $\Delta = \emptyset$ . Then  $\Gamma$  and the dual graph of  $T + S_1 + S_2 + S_3$  are described as one of the following:*

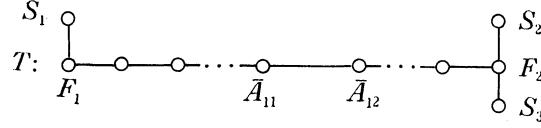
(0)  $\Gamma = \emptyset$ .

(I<sub>1</sub>)  $\Gamma = A_1 = \mathbf{C}^*$ ,  $\bar{A}_1$  is a  $(-1)$  curve and  $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_2) = 1$ :



where  $T_1$  might be empty.

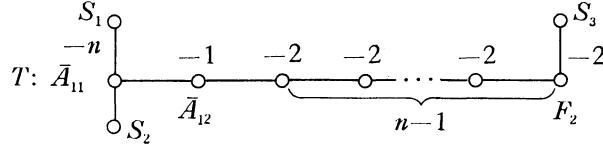
(I<sub>2</sub>)  $\Gamma = A_1 = A_{11} + A_{12}$ , where  $A_{11} \simeq A_{12} \simeq \mathbf{C}$  and either  $\bar{A}_{11}$  or  $\bar{A}_{12}$  is a  $(-1)$  curve, and  $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_2) = 1$ :



where  $T_1$  may be empty.

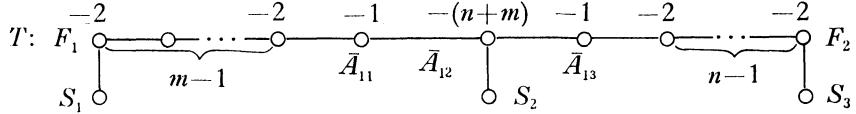
(II<sub>1</sub>)  $\Gamma = A_1 = \mathbf{C}^{**}$  (This case occurs if  $A$  is not a smooth fiber, i.e.,  $\Delta \neq \phi$ ).

(II<sub>2</sub>)  $\Gamma = A_1 = A_{11} + A_{12}$ ,  $A_{11} \simeq \mathbf{C}^*$ ,  $A_{12} \simeq \mathbf{C}$ ,  $\bar{A}_{12}$  is a  $(-1)$  curve and  $(S_1 \cdot \bar{A}_{11}) = (S_2 \cdot \bar{A}_{11}) = (S_3 \cdot F_2) = 1$ :



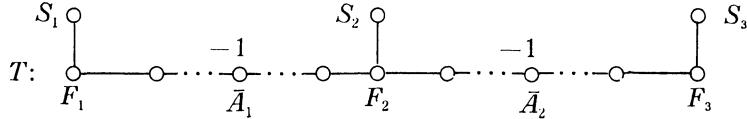
where  $T_3$  might be empty.

(II<sub>3</sub>)  $\Gamma = A_1 = A_{11} + A_{12} + A_{13}$ ,  $A_{11} \simeq A_{12} \simeq A_{13} \simeq \mathbf{C}$ ,  $\bar{A}_{11}$  and  $\bar{A}_{13}$  are  $(-1)$  curves and  $(S_1 \cdot F_1) = (S_2 \cdot \bar{A}_{12}) = (S_3 \cdot F_2) = 1$ :



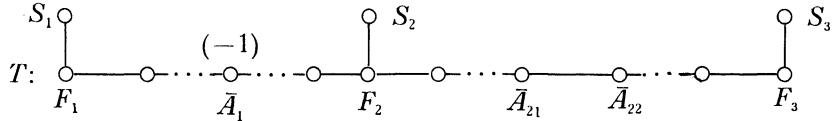
where  $T_1$  and  $T_3$  might be empty.

(III<sub>1</sub>)  $\Gamma = A_1 \sqcup A_2$ ,  $A_1 = A_2 = \mathbf{C}^*$ ,  $\bar{A}_1$  and  $\bar{A}_2$  are  $(-1)$  curves, and  $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$ :



where  $T_1$  and  $T_3$  might be empty.

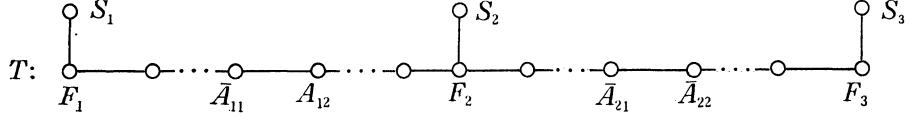
(III<sub>2</sub>)  $\Gamma = A_1 \sqcup A_2$ ,  $A_1 = \mathbf{C}^*$ ,  $A_2 = A_{21} + A_{22}$ ,  $A_{21} \simeq A_{22} \simeq \mathbf{C}$ ,  $\bar{A}_1$  is a  $(-1)$  curve, either  $\bar{A}_{21}$  or  $\bar{A}_{22}$  is a  $(-1)$  curve and  $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$ :



where  $T_1$  and  $T_3$  might be empty.

(III<sub>3</sub>)  $\Gamma = A_1 \sqcup A_2$ ,  $A_1 = A_{11} + A_{12}$ ,  $A_2 = A_{21} + A_{22}$ ,  $A_{11} \simeq A_{12} \simeq A_{21} \simeq A_{22} \simeq \mathbf{C}$ , either  $\bar{A}_{11}$  or  $\bar{A}_{12}$  is a  $(-1)$  curve, either  $\bar{A}_{21}$  or  $\bar{A}_{22}$  is a  $(-1)$  curve and  $(S_1 \cdot F_1) =$

$$(S_2 \cdot F_2) = (S_3 \cdot F_3) = 1:$$

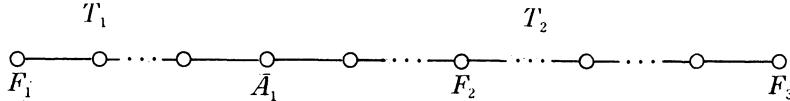


where  $T_1$  and  $T_3$  might be empty.

Proof. We use the notations set forth before. We can assume that  $T_1$ ,  $T_2$  and  $T_3$  do not contain  $(-1)$  curves other than those which intersect at least two sections.

(1) **Case  $a=1$ .** In this case  $\Gamma=A_1$ .

(1-1) Consider the case where  $A_1$  is irreducible. Then  $A_1 \simeq \mathbf{C}^*$ . We may assume that  $T_2 = T_3$  and  $\bar{A}_1$  intersects  $T_1$  and  $T_2$ . Then  $T$  must contain curves whose dual graph is given as follows:



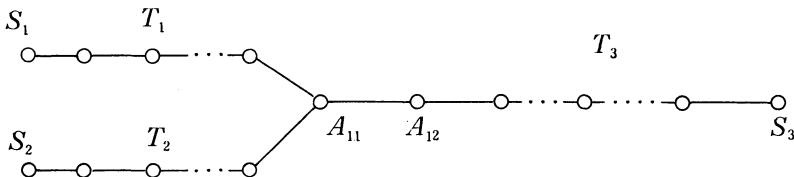
where  $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$ . From the assumption, we see that  $T$  does not contain other components than those included in the above dual graph. Since there exists a composite of blowing-downs  $\tau$  such that  $\tau(T) = \tau(F_2) \simeq \mathbf{P}^1$ , the above dual graph must contain a  $(-1)$  curve in the branch on the right hand side of  $F_2$  of the graph unless  $F_2 = F_3$ . Thus we have  $F_2 = F_3$  and  $\bar{A}_1$  must be a  $(-1)$  curve. This case corresponds to (I<sub>1</sub>) in the statement of the lemma.

(1-2) If  $\Gamma = A_1$  is reducible, we have  $A_1 = A_{11} + A_{12}$  and  $A_{11} = A_{12} = \mathbf{C}$ . We can show the statement in (I<sub>2</sub>) be a similar argument as in (1-1).

2) **Case  $a=a_1=2$ .** In this case  $\Gamma=A_1$ .

(2-1) If  $A_1$  is irreducible, Lemma 1.3 shows that  $A$  is a smooth fiber.

(2-2) If  $A_1$  is reducible and consists of two components, we have  $A_1 = A_{11} + A_{12}$  and  $A_{11} \simeq \mathbf{C}^*$  and  $A_{12} \simeq \mathbf{C}$ . We may assume here that  $T_1$  and  $T_2$  intersect  $A_{11}$  and  $T_3$  intersects with  $A_{12}$ . Then, by these assumptions we know that the dual graph of  $T + S_1 + S_2 + S_3$  is given as follows:



If  $A_{11}$  is a  $(-1)$  curve, by contracting  $A_{11}$ , we can easily show  $T_1=T_2=\phi$ . Then since  $A_{11}$  intersects the sections  $S_1$  and  $S_2$ , there exists another  $(-1)$  curve, which must be  $A_{12}$  and we have  $T_3=\phi$ . If  $A_{12}$  is a  $(-1)$  curve, first contract  $A_{12}$  and continue the contractions of the curves  $F_1, \dots, F_{n-1}$  contained in  $T_3$  and assume that after contracting  $F_{n-1}$ ,  $A_{11}$  becomes a  $(-1)$  curve. At this step, we get to a situation similar to the above. This means that  $A_{11}, A_{12}, F_1, \dots, F_{n-1}$  exhaust all curves in the fiber  $T$ . The statement in (II<sub>2</sub>) is now easy to prove. (2-3) If  $A_1$  is reducible and consists of three components,  $A_1=A_{11}+A_{12}+A_{13}$  and  $A_{11}\simeq A_{12}\simeq A_{13}\simeq C$ . Arguing as in (2,2), we can prove the statement in (II<sub>3</sub>).

(3) **Case  $a=2, a_1=a_2=1$ .** In this case  $\Gamma=A_1 \sqcup A_2$ . We divide it into the following three cases:

- (3-1)  $A_1$  and  $A_2$  are irreducible.
- (3-2)  $A_1$  is irreducible and  $A_2$  is reducible.
- (3.2)  $A_1$  and  $A_2$  are reducible.

In each case we argue as in the cases (1) and (2) and obtain accordingly the statements from (III<sub>1</sub>) to (III<sub>3</sub>). Q.E.D.

The method of obtaining a singular fiber in the case where  $\Delta \neq \phi$  from one of the above singular fibers is explained in [7]. Namely, starting from an initial point  $P_0$  in  $T \cap D$ , we obtain a component of  $\Delta$  as follows:

(a) If  $P_0$  is an intersection point  $P_0=F_i \cap F_j$  of two components  $F_i$  and  $F_j$  of  $T$ , let  $\sigma_1: Z_1 \rightarrow V$  be an oscillating sequence of blowing-ups with initial point  $P_0$ . The dual graph of the configuration of curves  $F_1+\sigma_1^{-1}(P_0)+F_j$  is given as follows:

$$L_1: \quad \begin{array}{ccccccccccccc} & & & & & & & & & & & & & & \\ & \circ & \text{---} & \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \cdots & \circ & \text{---} & \circ & \\ F_1 & & & & & & & & & & & & & & F_j \\ & & & & & & & & & & & & & & \end{array} \quad \begin{matrix} -1 \\ \text{---} \\ E_1 \end{matrix}$$

We say that this oscillating sequence of blowing-ups is of type (a).

(b) If  $P_0$  belongs to only one component  $F_i$  of  $T$ , let  $\sigma_1: Z_1 \rightarrow V$  be an oscillating sequence of blowing-ups with initial point  $P_0$ . The dual graph of curves  $F_i+\sigma_1^{-1}(P_0)$  is given by one of the following:

$$L_2: \quad \begin{array}{ccccccccccccc} & & & & & & & & & & & & & & \\ & \circ & \text{---} & \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \cdots & \circ & \text{---} & \circ & \text{---} & \circ \\ F_i & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & \end{array} \quad \begin{matrix} -1 \\ \text{---} \\ E_1 \end{matrix}$$

$$L_3: \quad \begin{array}{ccccccccccccc} & & & & & & & & & & & & & & \\ & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \cdots & \circ & \text{---} & \circ \\ F_i & & & & & & & & & & & & & & & E_1 \\ & & & & & & & & & & & & & & & \end{array} \quad \begin{matrix} -1 \\ \text{---} \\ E_1 \end{matrix}$$

We say that an oscillating sequence producing the dual graph  $L_2$  or  $L_3$  is of type

(b-1) or (b-2), respectively.

Next we choose a second initial point  $P_1$  from  $E_1$ . If  $P_1$  is an intersection point of  $E_1$  with other component of the fiber, we perform an oscillating sequence of blowing-ups of type (a). If  $P_1$  does not belong to other components of the fiber, perform an oscillating sequence of blowing-ups of type (b-1).

We proceed this way several times and at the last step we perform an oscillating sequence of blowing-ups of type (b-2). Then we obtain a surface  $\tilde{Z}$  and a  $(-1)$  curve  $\tilde{E}$  which is an end component of the dual graph of the curves contained in the fiber corresponding to  $T$ . We include all exceptional curves obtained by the above sequence of blowing-ups except for  $\tilde{E}$  into the boundary divisor  $\tilde{D}$ . Then  $\tilde{E} - \tilde{D} \simeq C$  is a component of  $\Delta$ . Every component of  $\Delta$  is given in this fashion. We denote by  $(II_2)_{(k)}$ , for example, a singular fiber which is obtained by adding  $k$  components of  $\Delta$  to  $(II_2)$ .

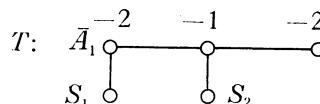
Next, we consider the case where affine surfaces have twisted  $\mathbf{C}^{**}$ -fibrations. We use here the notations similar to those employed above. Let  $\pi: X \rightarrow C$  be a twisted  $\mathbf{C}^{**}$ -fibration. There are two cases to consider, that is, the case where  $p$  has two sections  $S_1$  and  $S_2$  contained in  $D$  such that  $\deg p|_{S_1} = 1$  and  $\deg p|_{S_2} = 2$ , and the case where  $p$  has one section contained in  $D$  such that  $\deg p|_S = 3$ . We call the first the 2-section case and the second the 3-section case, respectively.

**Lemma 1.6.** *Let  $X$  be a smooth affine surface which has a twisted  $\mathbf{C}^{**}$ -fibration  $\rho: X \rightarrow C$ .*

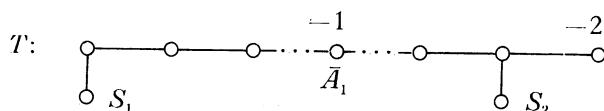
(A) **The 2-section case.** *Assume  $\Delta = \phi$ . Then  $\Gamma$  and the dual graph of  $T + S_1 + S_2$  are exhausted by one of the following graphes (IV-0) to (IV-3) which correspond to a fiber containing a branch point of  $p|_{S_2}$ :  $S_2 \rightarrow C$  and by one of the modifications of the graphes listed as (I<sub>1</sub>)-(II<sub>3</sub>) in Lemma 1.5, where  $S_2$  meets a fiber in two points and two branches of the 2-section  $S_2$  are identified suitably with two of three sections  $S_1$ ,  $S_2$  and  $S_3$ .*

(IV<sub>0</sub>)  $\Gamma = \phi$ .

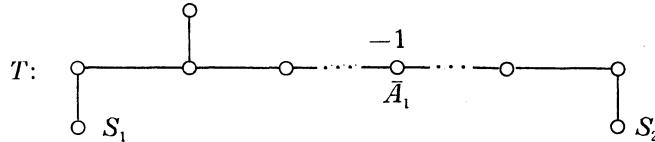
(IV<sub>1</sub>)  $\Gamma = A_1 = \mathbf{C}^*$ :



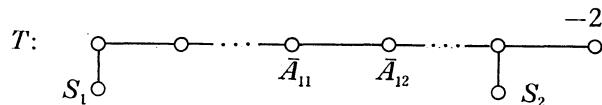
(IV<sub>2</sub>)  $\Gamma = A_1 = \mathbf{C}^*$  and  $\bar{A}_1$  is a  $(-1)$  curve:



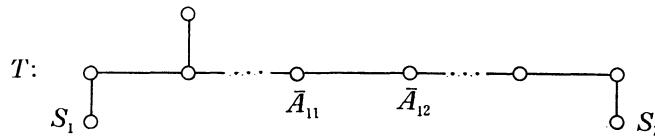
or



(IV<sub>3</sub>)  $\Gamma = A_1 = A_{11} + A_{12}$ ,  $A_{11} \simeq A_{12} \simeq C$ , either  $A_{11}$  or  $A_{12}$  is a  $(-1)$  curve:



or



(B) **The 3-section case.** Assume  $\Delta = \phi$ . Then  $\Gamma$  and the dual graph of  $T + S$  are exhausted by the modifications of the graphs (I<sub>1</sub>)-(IV<sub>3</sub>) in Lemmas 1.5 and 1.6(A), where the 3-section meets a fiber in either 3 points or 2 points and we identify accordingly branches of the 3-section  $S$  either with sections  $S_1$ ,  $S_2$  and  $S_3$  or with the section  $S_1$  and the 2-section  $S_2$ , and the following extra case:

(V)  $\Gamma = \phi$ , which corresponds to a fiber containing a totally ramified point of  $p|_S$ .

Proof. Note that we have  $a \leq 1$  in the case (A). The proof in this case is similar to the one in Lemma 1.5. We omit the details of the proof. Q.E.D.

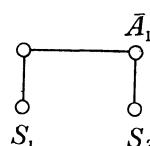
A singular fiber with  $\Delta \neq \phi$  is obtained by the same way as explained after Lemma 1.5. We use the notations like (IV<sub>1</sub>)<sub>(3)</sub>, for example, to signify a singular fiber obtained from (IV) by adding three affine lines to  $A$ .

Next we consider the case where a surface  $X$  has quotient singularities. Then we have

**Lemma 1.7.** *Let  $X$  be a logarithmic affine surface with a  $C^{**}$ -fibration  $\pi: Y \rightarrow B$ . We use the same notations as above. Then we have:*

(1)  $A = \Gamma + \Delta$ , and  $\Gamma_{\text{red}}$  together with  $T$  is given by one of (I) to (V) listed above and the following (IV<sub>4</sub>):

(IV<sub>4</sub>)  $\Gamma = A_1 = C^*$  and  $\bar{A}_1$  has one singularity of type  $A_1$ :



(2) *Each component of  $\Delta_{\text{red}}$  has at most one cyclic quotient singular point.*  
 (3) *The point (s) indicated below can be a cyclic quotient singular point:*

(I)<sub>2</sub>  $A_{11} \cap A_{12}$ ,  
 (II)<sub>2</sub>  $A_{11} \cap A_{12}$ ,  
 (II)<sub>3</sub>  $A_{11} \cap A_{12}$  and  $A_{12} \cap A_{13}$ ,  
 (III)<sub>2</sub>  $A_{21} \cap A_{22}$ ,  
 (III)<sub>3</sub>  $A_{11} \cap A_{12}$  and  $A_{21} \cap A_{22}$ ,  
 (IV)<sub>3</sub>  $A_{11} \cap A_{12}$ .

*If one of the above points is a singular point, the statement in Lemmas 1.5 and 1.6 concerning the curves  $A_{ij}$  being a  $(-1)$  curve need not be true.*

It is easy to prove the above statement. We omit the details. In order to indicate a singular fiber of a logarithmic surface as listed above, we refer to it by the same notations as in the smooth case corresponding to it.

## 2. Logarithmic $\mathbf{Q}$ -homology planes with $\mathbf{C}^{**}$ -fibrations

Let  $X$  be a logarithmic  $\mathbf{Q}$ -homology plane with a  $\mathbf{C}^{**}$ -fibration  $\pi: X \rightarrow B$ . Let  $A^{(1)}, \dots, A^{(i)}$  be all singular fibers of  $\pi$ . We define  $A_j^{(i)}$ ,  $A_{j,k}^{(i)}$  and  $a_j^{(i)}$  etc. for a singular fiber  $A = A^{(i)}$  as in §1. First we cite the following result from [MS2].

**Lemma 2.1.** (1) *Let  $X$  be a rational logarithmic  $\mathbf{Q}$ -homology plane. Then the boundary divisor  $D$  is simply connected,  $\Gamma(X, \mathcal{O}_X)^* = \mathbf{C}^*$  and  $\text{Pic}(X)$  is a finite group.*

(2) *Assume that  $X$  is a smooth rational surface such that the boundary divisor  $D$  is connected and simply connected,  $\text{Pic}(X)$  is a finite group and  $H^2(V; \mathbf{Q}) \rightarrow H^2(D; \mathbf{Q})$  is an isomorphism. Then  $X$  is a  $\mathbf{Q}$ -homology plane and we have the following isomorphisms:*

$$\text{Pic}(X) \simeq H_1(X; \mathbf{Z}) \simeq \text{Coker}(H^2(V; \mathbf{Z}) \rightarrow H^2(D; \mathbf{Z})).$$

*Moreover, if  $H_1(X; \mathbf{Z}) = 0$  then  $X$  is a homology plane.*

(3) *Assume that  $X$  is a rational logarithmic surface such that  $D$  is connected and simply connected,  $H^2(W; \mathbf{Q}) \rightarrow H^2(D \cup \Theta; \mathbf{Q})$  is an isomorphism, where  $\Sigma$  is the singular locus of  $X$  and  $\Theta$  is  $f^{-1}(\Sigma)$  (cf. the notations in §1). Then  $X$  is a logarithmic  $\mathbf{Q}$ -homology plane and we have the following exact sequence and isomorphisms:*

$$0 \rightarrow H_1(\partial T; \mathbf{Z}) \rightarrow H_1(X^0; \mathbf{Z}) \rightarrow H_1(X; \mathbf{Z}) \rightarrow 0$$

$$H_1(X^0; \mathbf{Z}) \simeq \text{Pic}(W - D - \Theta) \simeq \text{Coker}(H_2(D \cup \Theta; \mathbf{Z}) \rightarrow H_2(W; \mathbf{Z})),$$

*where  $X^0 := X - \Sigma$  and  $\partial T$  is a disjoint union of the boundaries of closed neighbourhoods of singular points. In particular, if  $H_1(X; \mathbf{Z}) = 0$  then  $X$  is a logarithmic homology plane.*

In this section, we determine the singular fibers of a logarithmic  $\mathbf{Q}$ -homology plane with a  $\mathbf{C}^{**}$ -fibration  $\pi: X \rightarrow B$ . First, note that  $D$  can contain at most one complete fiber of  $p: V \rightarrow C$  since  $D$  is simply connected. Therefore  $B \simeq \mathbf{P}^1$  or  $\mathbf{C}$ . From Lemma 2.1. (1), we also have the following:

**Lemma 2.2.** *Assume that  $p: V \rightarrow C$  is untwisted. Then we have:*

$$\begin{cases} \sum_{i=1}^l a^{(i)} = 2(l-1) & \text{if } B \simeq \mathbf{P}^1 \\ \sum_{i=1}^l a^{(i)} = 2l & \text{if } B \simeq \mathbf{C}. \end{cases}$$

Proof. Since  $D$  is connected and simply connected, we have:

$$\sum_{i=1}^l (2 - a^{(i)}) = \begin{cases} 2 & \text{if } B \simeq \mathbf{P}^1 \\ 0 & \text{if } B \simeq \mathbf{C}. \end{cases}$$

From this follow the above inequalities. Q.E.D.

Let  $b^{(i)}$  be the number of irreducible components in  $A^{(i)}$ . Since  $\text{Pic}(X)$  is a finite group and  $\Gamma(X, \mathcal{O}_X^*) = \mathbf{C}^*$ ,  $\text{rank}(\text{Pic}(V))$  is equal to the number of irreducible components in  $D$ . Hence we have the following:

**Lemma 2.3.** *We have:*

(1) *The untwisted case:*

$$\sum_{i=1}^l (b^{(i)} - 1) = \begin{cases} 1 & \text{if } B \simeq \mathbf{P}^1 \\ 2 & \text{if } B \simeq \mathbf{C}. \end{cases}$$

(2) *The twisted case with a 2-section:*

$$\sum_{i=1}^l (b^{(i)} - 1) = \begin{cases} 0 & \text{if } B \simeq \mathbf{P}^1 \\ 1 & \text{if } B \simeq \mathbf{C}. \end{cases}$$

(3) *The twisted case with a 3-section:  $b^{(i)} = 1$  for all  $i$  if  $B \simeq \mathbf{C}$ . The case  $B \simeq \mathbf{P}^1$  does not occur.*

Now we shall determine the structure of logarithmic  $\mathbf{Q}$ -homology planes with  $\mathbf{C}^{**}$ -fibrations.

**Lemma 2.4.** *Let  $X$  be a logarithmic  $\mathbf{Q}$ -homology plane with a  $\mathbf{C}^{**}$ -fibration  $\pi: X \rightarrow B$ . Then the set of all singular fibers of  $\pi$  is given by one of the following:*

(1) *The case where  $\pi$  is untwisted and  $B \simeq \mathbf{P}^1$ ;*

(UP<sub>1</sub>)  $\pi$  has only one singular fiber;  $A^{(1)}$ : type (0)<sub>(2)</sub>;

(UP<sub>2</sub>)  $\pi$  has two singular fibers;

(UP<sub>2-1</sub>)  $A^{(1)}$ : type (I<sub>1</sub>)<sub>(1)</sub>,  $A^{(2)}$ : type (I<sub>1</sub>);

(UP<sub>2-2</sub>)  $A^{(1)}$ : type (I<sub>2</sub>),  $A^{(2)}$ : type (I<sub>1</sub>);

- (UP<sub>2-3</sub>)  $A^{(1)}$ : type (II<sub>1</sub>)<sub>(1)</sub>,  $A^{(2)}$ : type (0)<sub>(1)</sub>;
- (UP<sub>2-4</sub>)  $A^{(1)}$ : type (II<sub>2</sub>),  $A^{(2)}$ : type (0)<sub>(1)</sub>;
- (UP<sub>3</sub>)  $\pi$  has three singular fibers;
  - (UP<sub>3-1</sub>)  $A^{(1)}$ : type (III<sub>1</sub>),  $A^{(2)}$ : type (I<sub>1</sub>),  $A^{(3)}$ : type (I<sub>1</sub>);
  - (UP<sub>3-2</sub>)  $A^{(1)}$ : type (II<sub>2</sub>),  $A^{(2)}$ : type (I<sub>1</sub>),  $A^{(3)}$ : type (I<sub>1</sub>);
  - (UP<sub>3-3</sub>)  $A^{(1)}$ : type (II<sub>1</sub>)<sub>(1)</sub>,  $A^{(2)}$ : type (I<sub>1</sub>),  $A^{(3)}$ : type (I<sub>1</sub>);
- (2) The case where  $\pi$  is untwisted and  $B \simeq \mathbf{P}^1$ .
  - (UC<sub>1</sub>)  $\pi$  has one singular fiber;
    - (UC<sub>1-1</sub>)  $A^{(1)}$ : type (III<sub>1</sub>)<sub>(1)</sub>;
    - (UC<sub>1-2</sub>)  $A^{(1)}$ : type (III<sub>2</sub>);
    - (UC<sub>1-3</sub>)  $A^{(1)}$ : type (II<sub>1</sub>)<sub>(2)</sub>;
    - (UC<sub>1-4</sub>)  $A^{(1)}$ : type (II<sub>2</sub>)<sub>(1)</sub>;
    - (UC<sub>1-5</sub>)  $A^{(1)}$ : type (II<sub>3</sub>);
  - (UC<sub>2</sub>)  $\pi$  has two singular fibers;
    - (UC<sub>2-1</sub>)  $A^{(1)}$ : type (III<sub>1</sub>),  $A^{(2)}$ : type (III<sub>1</sub>);
    - (UC<sub>2-2</sub>)  $A^{(1)}$ : type (III<sub>1</sub>),  $A^{(2)}$ : type (II<sub>1</sub>)<sub>(1)</sub>;
    - (UC<sub>2-3</sub>)  $A^{(1)}$ : type (III<sub>1</sub>),  $A^{(2)}$ : type (II<sub>2</sub>);
    - (UC<sub>2-4</sub>)  $A^{(1)}$ : type (II<sub>1</sub>)<sub>(1)</sub>,  $A^{(2)}$ : type (II<sub>1</sub>)<sub>(1)</sub>;
    - (UC<sub>2-5</sub>)  $A^{(1)}$ : type (II<sub>1</sub>)<sub>(1)</sub>,  $A^{(2)}$ : type (II<sub>2</sub>);
    - (UC<sub>2-6</sub>)  $A^{(1)}$ : type (II<sub>2</sub>),  $A^{(2)}$ : type (II<sub>2</sub>);
- (3) The case where  $\pi$  is twisted with a 2-section and  $B \simeq \mathbf{P}^1$ ;
  - (TP<sub>1</sub>)  $\pi$  has two singular fibers,  $A^{(1)}$ : type (IV<sub>0</sub>)<sub>(1)</sub>,  
 $A^{(2)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>);
  - (TP<sub>2</sub>)  $\pi$  has three singular fibers,  $A^{(1)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>),  
 $A^{(2)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>),  $A^{(3)}$ : type (I<sub>1</sub>);
- (4) The case where  $\pi$  is twisted with a 2-section and  $B \simeq \mathbf{C}$ ;
  - (TC<sub>1</sub>)  $\pi$  has one singular fiber;
    - (TC<sub>1-1</sub>)  $A^{(1)}$ : type (IV<sub>1</sub>)<sub>(1)</sub> or type (IV<sub>2</sub>)<sub>(1)</sub> or type (IV<sub>4</sub>)<sub>(1)</sub>;
    - (TC<sub>1-2</sub>)  $A^{(1)}$ : type (IV<sub>3</sub>);
  - (TC<sub>2</sub>)  $\pi$  has two singular fibers;
    - (TC<sub>2-1</sub>)  $A^{(1)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>),  $A^{(2)}$ : type (III<sub>1</sub>);
    - (TC<sub>2-2</sub>)  $A^{(1)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>),  $A^{(2)}$ : type (II<sub>2</sub>);
- (5) The case where  $\pi$  is twisted with a 3-section and  $B \simeq \mathbf{C}$ .
  - (T3C<sub>1</sub>)  $\pi$  has one singular fiber,  $A^{(1)}$ : type (V<sub>0</sub>)<sub>(1)</sub>;
  - (T3C<sub>2</sub>)  $\pi$  has two singular fibers,  $A^{(1)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>),  
 $A^{(2)}$ : type (IV<sub>1</sub>) or (IV<sub>2</sub>) or (IV<sub>4</sub>);

The case  $B \simeq \mathbf{P}^1$  does not occur.

Proof. We make use of the inequalities in Lemmas 2.2 and 2.3. Consider, for example, the case where  $\pi$  is untwisted and  $B \simeq \mathbf{P}^1$ . We have

$$2(l-1) = \sum_{i=1}^l a^{(1)} \quad \text{and} \quad \sum_{i=1}^l (b^{(1)}-1) = 1,$$

where  $l$  is the number of singular fibers of  $\pi$ . If  $l=1$ , we have  $a^{(1)}=0$  and  $b^{(1)}=2$ . If  $l=2$ , we may assume  $b^{(1)}=2$ ,  $b^{(2)}=1$  and then have either  $a^{(1)}=1$ ,  $a^{(2)}=1$  or  $a^{(1)}=2$ ,  $a^{(2)}=0$ . If  $l=3$ , we may assume  $b^{(1)}=2$ ,  $b^{(2)}=1$ ,  $b^{(3)}=1$  and then have  $a^{(1)}=2$ ,  $a^{(2)}=1$ ,  $a^{(3)}=1$ . This case divides into two cases:  $a_1^{(1)}=2$  or  $a_1^{(1)}=a_2^{(1)}=1$ . If  $l \geq 4$ , we may assume  $b^{(1)}=2$  and  $b^{(i)}=1$  for  $i \geq 2$ , but  $a^{(i)}=2$  for at least two  $i$ 's with  $i \geq 2$ . This contradicts Lemma 1.3. Therefore the number of singular fibers is at most three. We argue in a similar fashion in the other cases. Q.E.D.

We can exhibit the configurations of the divisor  $D$  on  $V$  or  $D \cup \Theta$  on  $W$  and also the configurations of the image curves of  $D$  on the relatively minimal model which is obtained from  $W$  by the contraction of curves in the fibers. We remark that there exists a contraction  $\alpha: V \rightarrow \Sigma_a$  from  $V$  to a Hirzebruch surface  $\Sigma_a$  of degree  $a$  such that the images of horizontal components of  $D$  are disjoint and smooth. This means that  $a=0$  if  $p$  is untwisted or if  $p$  is twisted and has a 3-section, that  $a=1$  if  $p$  is twisted and has a 2-section. Conversely, starting from  $\Sigma_0$  or  $\Sigma_1$ , we can construct a  $\mathbf{Q}$ -homology plane with a  $\mathbf{C}^{**}$ -fibration which has singular fibers as described in Lemma 2.4. It is rather easy to show, by means of criteria given in [5] or by looking into the configuration of the boundary curves, that  $\mathbf{Q}$ -homology planes with  $\mathbf{C}^{**}$ -fibrations have also  $\mathbf{C}$ -fibrations or  $\mathbf{C}^*$ -fibrations except for the following cases: Type( $UP_{3-1}$ ), ( $UC_{2-1}$ ), ( $UC_{2-1}'$ ) (see below), ( $TP_2$ ) and ( $TC_{2-1}$ ).

### 3. $H_i(X; \mathbf{Z})$ and $\kappa(X)$

In this section we compute the homology groups and Kodaira dimensions of  $\mathbf{Q}$ -homology planes given in Lemma 2.4. Some of these surfaces have  $\mathbf{C}$ -fibrations or  $\mathbf{C}^*$ -fibrations and the homology groups and Kodaira dimensions are computed for them in the previous paper [7]. So, we omit the computation for them and restrict ourselves to the cases listed at the end of §2.

**Type ( $UP_{3-1}$ ).** The configuration of singular fibers and sections  $S_1, S_2, S_3$  of  $p$  is given as follows:

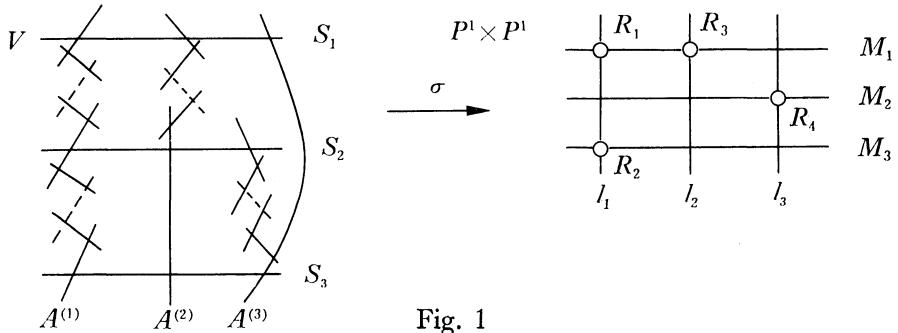


Fig. 1

We obtain the above configuration starting from a configuration as given in Fig. 1 consisting of curves on  $\mathbf{P}^1 \times \mathbf{P}^1$  by oscillating sequences of blowing-ups  $\sigma: V \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  with initial points  $R_1, R_2, R_3$  and  $R_4$ . We represent by  $l_1, l_2$  and  $l_3$  the fibers of the first projection  $p_1: \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$  and by  $M_1, M_2$  and  $M_3$  the fibers of the second projection  $p_2: \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ . The projection  $p_1$  induces a  $\mathbf{P}^1$ -fibration on  $V$ . Let  $E_i$  ( $1 \leq i \leq 4$ ) be a unique  $(-1)$  curve contained in  $\sigma^{-1}(R_i)$ . For example,  $E_1$  is  $\bar{A}_1^{(1)}$  and  $E_2$  is  $\bar{A}_2^{(1)}$  in the notations of Lemma 1.4. The total transforms of  $l_i$ 's and  $M_i$ 's are written as follows:

$$\begin{aligned}\sigma^*(l_1) &\sim u_1 E_1 + u_2 E_2 + (\text{fiber components of } D) \\ \sigma^*(l_2) &\sim u_3 E_3 + (\text{fiber components of } D) \\ \sigma^*(l_3) &\sim u_4 E_4 + (\text{fiber components of } D) \\ \sigma^*(M_1) &\sim v_1 E_1 + v_3 E_3 + (\text{other components of } D) \\ \sigma^*(M_2) &\sim v_4 E_4 + (\text{other components of } D) \\ \sigma^*(M_3) &\sim v_2 E_2 + (\text{other components of } D).\end{aligned}$$

Since  $l_1 \sim l_2 \sim l_3$  and  $M_1 \sim M_2 \sim M_3$  on  $\mathbf{P}^1 \times \mathbf{P}^1$ ,  $H_1(X; \mathbf{Z})$  has generators  $\xi_i := [E_i]$  and relations

$$\begin{aligned}u_1 \xi_1 + u_2 \xi_2 - u_3 \xi_3 &= 0 \\ u_3 \xi_3 - u_4 \xi_4 &= 0 \\ v_1 \xi_1 + v_3 \xi_3 - v_4 \xi_4 &= 0 \\ v_2 \xi_2 - v_4 \xi_4 &= 0.\end{aligned}$$

Therefore the order of  $H_1(X; \mathbf{Z})$  is equal to  $|d|$ , where

$$d = \begin{vmatrix} u_1 & u_2 & -u_3 & 0 \\ 0 & 0 & u_3 & -u_4 \\ v_1 & 0 & v_3 & -v_4 \\ 0 & v_2 & 0 & -v_4 \end{vmatrix} = u_3 u_4 v_1 v_2 + u_1 u_4 v_2 v_3 - u_2 u_3 v_1 v_4 - u_1 u_3 v_2 v_4.$$

The equation  $d = \pm 1$  has infinitely many solutions of positive integers for  $u_i$  and  $v_i$ . The following is a solution of the equation  $d = \pm 1$ :

$$u_1 = u_2 = u_3 = u_4 = 1, v_1 = m, v_2 = mn - m \pm 1, v_3 = m - n, v_4 = n - 1,$$

where  $m$  and  $n$  are positive integers such that  $n > m$  and  $n \geq 2$ . The homology planes obtained this way are isomorphic to those which we constructed in our paper [6].

Next we compute the Kodaira dimensions. Write the canonical divisor of  $V$  as  $K_V \sim \sigma^*(K_{\Sigma_0}) + G$ , where  $G$  is supported on the exceptional curves of  $\sigma$ . Starting with  $\Sigma_0$  and comparing the multiplicity of the newly obtained exceptional curve in  $G$  and  $\sigma^*(l_i)$ 's and  $\sigma^*(M_i)$ 's inductively at each step, it is easy to verify the following:

$$\begin{aligned} \sigma^*(l_1) + \sigma^*(l_2) + \sigma^*(l_3) + \sigma^*(M_1) + \sigma^*(M_2) + \sigma^*(M_3) \\ \sim l'_1 + l'_2 + l'_3 + S_1 + S_2 + S_3 + G + H, \end{aligned}$$

where  $l'_i$  is the proper transform of  $l_i$  and  $H$  is the sum of all exceptional curves of  $\sigma$  with  $H_{\text{red}}=H$ . Since  $K_{S_0} \sim -l_2 - l_3 - M_2 - M_3$ , we have the following expression for  $K_V + D$ :

$$K_V + D \sim \sigma^*(l_1) + \sigma^*(M_1) - (E_1 + E_2 + E_3 + E_4).$$

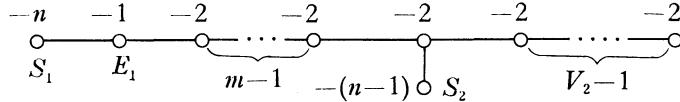
There exist many examples of homology planes of Kodaira dimension two. For example, we will show that  $\kappa=2$  for almost all values of  $u_i$ 's and  $v_i$ 's given earlier as a solution of  $d=\pm 1$ .

(a) case  $n \geq m+2$ ,  $v_2 = mn - m + 1$  or  $n \geq m+2$ ,  $m \geq 2$ ,  $v_2 = mn - m - 1$ .

In this case, we have

$$\begin{aligned} 2(K_V + D) &\sim \sigma^*(M_1) + \sigma^*(M_2) + 2\sigma^*(l_1) - 2(E_1 + E_2 + E_3 + E_4) \\ &\sim S_1 + S_2 + (v_1 E_1 + v_3 E_3) + v_4 E_4 + 2(u_1 E_1 + u_2 E_2) - 2(E_1 + E_2 + E_3 + E_4) \\ &\quad + (\text{fiber components of } D) \\ &\sim S_1 + S_2 + mE_1 + (n - m - 2)E_3 + (n - 3)E_4 + (\text{fiber components of } D). \end{aligned}$$

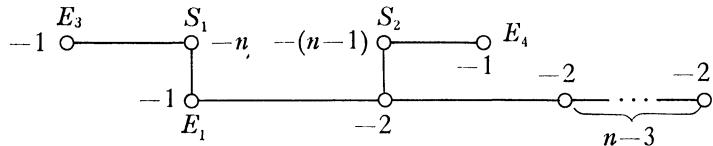
Therefore  $2(K_V + D)$  is effective and there exists an effective member  $A$  of  $|2(K_V + D)|$  whose support contains the curves, the configuration of which is given as follows:



Note that  $-n + 1 + (m - 1) + 1 + (v_2 - 1) = n(m - 1) \pm 1 \geq 1$ . Therefore if we contract all curves contained in the above graphs other than  $S_1$  and  $S_2$ , the proper transform of  $S_1$  becomes a nonsingular rational curve of positive self-intersection number. This shows that Kodaira dimension of  $X := V - D$  is equal to two.

(b) case  $m=1$ ,  $n \geq 4$  and  $v_2 = n - 2$ .

In this case also  $2(K_V + D)$  is effective and  $|2(K_V + D)|$  contains an effective member whose support contains the union of curves given by the following dual graph.



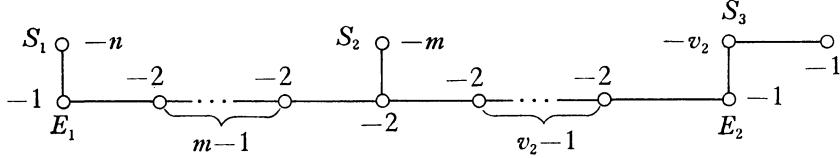
If we contract all curves contained in the above graph except  $S_1$ ,  $S_2$ ,  $E_3$  and  $E_4$ , then  $S_1$  and  $S_2$  become nonsingular rational curves whose self-intersection numbers are  $(-1)$  and the proper transform of  $S_1$  and that of  $S_2$  intersect at one point with multiplicity  $n-2$ . From this it is easy to show that  $\kappa(X)=2$ .

(c) case  $n=m+1$ ,  $m \geq 3$  and  $v_2=m^2 \pm 1$ .

In this case we have the following expression for  $3(K_V+D)$ .

$$\begin{aligned} 3(K_V+D) &\sim \sigma^*(M_1) + \sigma^*(M_2) + \sigma^*(M_3) + \sigma^*(l_1) + 2\sigma^*(l_2) - 3(E_1 + E_2 + E_3 + E_4) \\ &\sim S_1 + S_2 + S_3 + (v_1 E_1 + v_3 E_3) + v_4 E_4 + v_2 E_2 + (u_1 E_1 + u_2 E_2) + 2u_3 E_3 \\ &\quad + (\text{fiber components of } D) - 3(E_1 + E_2 + E_3 + E_4) \\ &\sim S_1 + S_2 + S_3 + (m-2)E_1 + (v_2-2)E_2 + (m-3)E_4 + (\text{fiber components of } D) \end{aligned}$$

Therefore  $3(K_V+D)$  is effective and there exists an effective member of  $|3(K_V+D)|$  whose support contains the union of curves given by the following dual graph.



The similar argument shows that  $\kappa(X)=2$ .

**Type (UC<sub>2-1</sub>).** The configuration of singular fibers and sections  $S_1$ ,  $S_2$ ,  $S_3$  of  $p$  is given as follows:

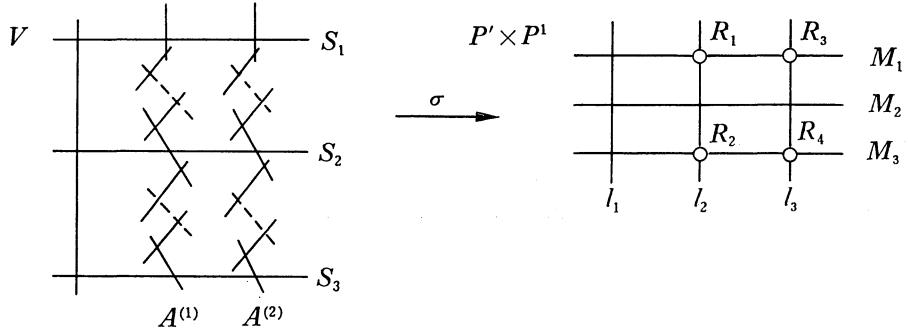


Fig. 2

We obtain the above configuration starting from a configuration of curves as given in Fig. 2 on  $P^1 \times P^1$  by oscillating sequences of blowing-ups  $\sigma: V \rightarrow P^1 \times P^1$  with initial points  $R_1$ ,  $R_2$ ,  $R_3$  and  $R_4$ . We use the notations  $l_i$ ,  $M_i$  and  $E_i$  in the same way as in the previous case. Write the total transforms of  $l_i$ 's and  $M_i$ 's as follows:

$$\begin{aligned}
\sigma^*(l_2) &\sim u_1 E_1 + u_2 E_2 + (\text{fiber components of } D) \\
\sigma^*(l_3) &\sim u_3 E_3 + u_4 E_4 + (\text{fiber components of } D) \\
\sigma^*(M_1) &\sim v_1 E_1 + v_3 E_3 + (\text{other components of } D) \\
\sigma^*(M_3) &\sim v_2 E_2 + v_4 E_4 + (\text{other components of } D).
\end{aligned}$$

$H_1(X; \mathbf{Z})$  has generators  $\xi_i := [E_i]$  and relations:

$$\begin{aligned}
u_1 \xi_1 + u_2 \xi_3 &= 0 \\
u_3 \xi_3 + u_4 \xi_4 &= 0 \\
v_1 \xi_1 + v_3 \xi_3 &= 0 \\
v_2 \xi_2 + v_4 \xi_4 &= 0.
\end{aligned}$$

The order of  $H_1(X; \mathbf{Z})$  is equal to  $|d|$ , where  $d = u_2 u_3 v_1 v_4 - u_1 u_4 v_2 v_3$  and the equation  $d = \pm 1$  should take positive integer solutions for  $u_i$ 's and  $v_i$ 's.

On the other hand, a computation shows that we have the same expression as above for  $K_V + D$  and there are many examples of homology planes of  $\kappa=2$  of this type.

**Type  $(UC_{2-1})'$ .** The configuration of singular fibers and sections  $S_1, S_2, S_3$  of  $p$  is given as follows:

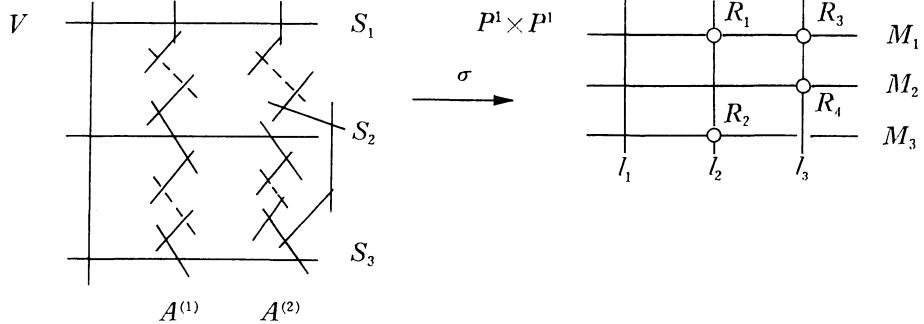


Fig. 3

We obtain the above configuration starting from a configuration of curves on  $\mathbf{P}^1 \times \mathbf{P}^1$  as given in Fig. 3 by oscillating sequences of blowing-ups  $\sigma: V \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  with initial points  $R_1, R_2, R_3$  and  $R_4$ . We use the notations  $l_i, M_i$  and  $E_i$  in the same way as in the previous case. Write the total transforms of  $l_i$ 's and  $M_i$ 's as follows.

$$\begin{aligned}
\sigma^*(l_2) &\sim u_1 E_1 + u_2 E_2 + (\text{fiber components of } D) \\
\sigma^*(l_3) &\sim u_3 E_3 + u_4 E_4 + (\text{fiber components of } D) \\
\sigma^*(M_1) &\sim v_1 E_1 + v_3 E_3 + (\text{other components of } D) \\
\sigma^*(M_2) &\sim v_4 E_4 + (\text{other components of } D) \\
\sigma^*(M_3) &\sim v_2 E_2 + (\text{other components of } D).
\end{aligned}$$

$H_1(X; \mathbf{Z})$  has generators  $\xi_i := [E_i]$  and relations:

$$\begin{aligned} u_1\xi_1 + u_2\xi_2 &= 0 \\ u_3\xi_3 + u_4\xi_4 &= 0 \\ v_1\xi_1 + v_3\xi_3 - v_4\xi_4 &= 0 \\ -v_2\xi_2 + v_4\xi_4 &= 0. \end{aligned}$$

The order of  $H_1(X; \mathbf{Z})$  is equal to  $d = u_1u_3v_2v_4 + u_1u_4v_2v_3 + u_2u_3v_1v_4$ . Since  $u_i$ 's and  $v_j$ 's are positive integers, there are no solutions for  $d = \pm 1$ . Hence there exist no homology planes of this type. The divisor  $K_V + D$  is also given by the same expression as in the previous cases and there are many examples of  $\mathbf{Q}$ -homology planes of  $\kappa = 2$  of this type. We remark that the surface corresponding to the values  $u_1 = u_2 = u_3 = u_4 = v_1 = v_3 = 1$  and  $v_2 = v_4$  has  $\kappa = 0$  and isomorphic to  $Y\{2, 4, 4\}$  according to Fujita [F].

**Type ( $\mathbf{TP}_2$ ).** The configuration of singular fibers and sections  $S_1, S_2$  of  $p$  is given as follows:

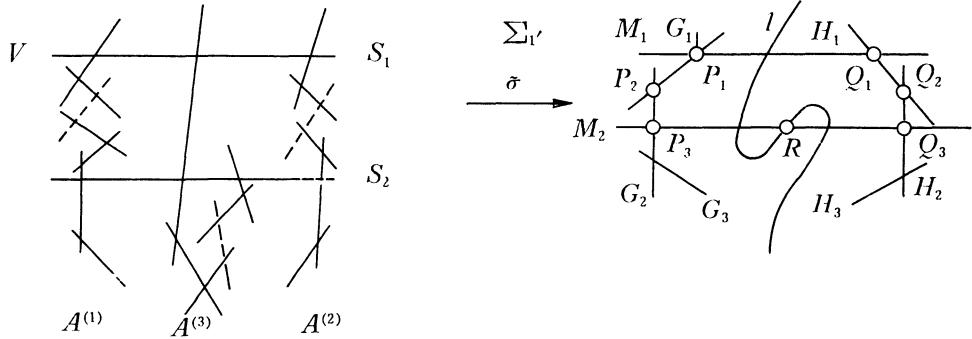


Fig. 4

We obtain this configuration starting from a configuration of curves  $\Sigma_1$  as given in Fig. 4. Let  $M_1$  be the minimal section of  $\Sigma_1$  and let  $p_1$  be the morphism from  $\Sigma_1$  to  $\mathbf{P}^1$  which gives the natural  $\mathbf{P}^1$ -bundle structure on  $\Sigma_1$ . Let  $C$  be a 2-section of  $\Sigma_1$  disjoint from  $M_1$  and let  $l_1$  and  $l_2$  be fibers of  $p_1$  containing ramification points of  $p_1|_C$  and let  $l_3$  be a third fiber of  $p_1$ . Consider a divisor  $D_0 = M_1 + C + l_1 + l_2 + l_3$  on  $\Sigma_1$ . First we blow up  $l_1 \cap C$ ,  $l_2 \cap C$  and their infinitely near points over  $C$  in order to get a simple normal crossing divisor. We call this surface  $\Sigma'_1$  and let  $\tau: \Sigma'_1 \rightarrow \Sigma_1$  be the composition of these four blowing-ups. The configuration of curves on  $\Sigma'_1$  is given in Fig. 4, where  $M_2$  is the proper transform of  $C$ . Next we perform oscillating sequences of blowing-ups  $\sigma: V \rightarrow \Sigma'_1$ . We choose an initial point  $R_1$  on  $\tau^{-1}(l_1)$  to be one of  $P_1, P_2$  and  $P_3$  if  $A^{(1)}$  has type  $(\text{IV}_2)$ . Similarly, we choose an initial point  $R_2$  on  $\tau^{-1}(l_2)$ . Let  $R_3$  be an initial point on  $\tau^{-1}(l_3)$ . Let  $\sigma = \tau \cdot \sigma$  and let  $E_i (1 \leq i \leq 3)$  be a unique  $(-1)$  curve con-

tained in  $\sigma^{-1}(R_i)$ .  $E_1$  is  $\bar{A}^{(1)}$  and  $E_2$  is  $\bar{A}^{(2)}$  in the notations of Lemma 1.4. We choose  $G_1$  as  $E_1$  (resp.  $G_2$  as  $E_1$ ) if  $A^{(1)}$  has type  $(IV_1)$  (resp.  $(IV_4)$ ). The same remark applies also to  $A^{(2)}$ . Write the total transforms of  $l_i$ 's and  $M_i$ 's as follows:

$$\begin{aligned}\sigma^*(l_1) &\sim u_1 E_1 + (\text{fiber components of } D) \\ \sigma^*(l_2) &\sim u_2 E_2 + (\text{fiber components of } D) \\ \sigma^*(l_3) &\sim u_3 E_3 + (\text{fiber components of } D) \\ \sigma^*(M_1) &\sim v_1 E_1 + v_2 E_2 + (\text{other boundary components}) \\ \sigma^*(C) &\sim w_1 E_1 + w_2 E_2 + w_3 E_3 + (\text{other boundary components}).\end{aligned}$$

Note that if the initial point  $R_1$  is  $P_1$ ,  $w_1$  is 0 and if  $R_1$  is  $P_2$  or  $P_3$ ,  $v_1$  is 0. If  $A^{(1)}$  has type  $(IV_4)$ ,  $u_1=w_1=2$  and  $v_1=0$ . The same remark applies also to  $A^{(2)}$ . Since  $l_1 \sim l_2 \sim l_3$  and  $2M_1 + 2l_3 \sim C$ ,  $H_1(X; \mathbf{Z})$  has generators  $\xi_i := [E_i]$  and relations:

$$\begin{aligned}u_1 \xi_1 - u_2 \xi_2 &= 0 \\ u_2 \xi_2 - u_3 \xi_3 &= 0 \\ (2v_1 - w_1) \xi_1 + (2v_2 - w_2) \xi_2 + (2u_3 - w_3) \xi_3 &= 0,\end{aligned}$$

and the order of  $H_1(X; \mathbf{Z})$  is equal to  $|d|$ , where

$$d = u_1 u_2 (2u_3 - w_3) + u_2 u_3 (2v_1 - w_1) + u_1 u_3 (2v_2 - w_2).$$

The equation  $d = \pm 1$  has infinitely many solutions of positive integers for  $u_i$ ,  $v_i$  and  $w_i$ . The following is a solution of the equation  $d = \pm 1$ .

$$w_1 = w_2 = 0, u_1 = u_2 = 1, u_3 = m, v_1 = v_2 = n, w_3 = 2m + 4mn \pm 1,$$

where  $m, n$  are positive integers. In the next section we prove that homology planes corresponding to the above values have involutions.

We compute the Kodaira dimensions of the above examples. Write the canonical divisor of  $V$  as follows:

$$K_V \sim \sigma^*(K_{\Sigma_1}) + 2G'_2 + G'_3 + 2H'_2 + H'_3 + Z_1 + Z_2$$

where  $\text{Supp}(Z_1) = \tau^{-1}(P_1) \cup \tau^{-1}(Q_1)$ ,  $\text{Supp}(Z_2) = \tau^{-1}(R)$  and we denote the proper transform of the curve using'. Starting with  $\Sigma_1$  and checking inductively at each step, it is easy to prove the following:

$$\begin{aligned}\sigma^*(l_1) + \sigma^*(l_2) + \sigma^*(l_3) + \sigma^*(M_1) + \sigma^*(C) \\ \sim l'_1 + l'_2 + l'_3 + S_1 + S_2 + 4G'_2 + 2G'_3 + 4H'_2 + 2H'_3 + Z_1 + U_1 + Z_2 + U_2\end{aligned}$$

where  $U_1$  is the sum of all exceptional curves contained in  $\tau^{-1}(P_1) \cup \tau^{-1}(Q_1)$  with  $(U_1)_{\text{red}} = U_1$  and  $U_2$  is the sum of all exceptional curves contained in  $\tau^{-1}(R)$  with

$(U_2)_{\text{red}} = U_2$ . Since  $K_{\mathbf{z}_1} \sim -C - l_3$ , we have the following expression for  $K_V + D$ :

$$\begin{aligned} K_V + D &\sim K_V + l'_1 + l'_2 + l'_3 + S_1 + S_2 + G'_2 + G'_3 + H'_2 + H'_3 + U_1 + U_2 \\ &\quad - (E_1 + E_2 + E_3) \\ &\sim -\sigma^*(C) - \sigma^*(l_3) + Z_1 + Z_2 + 2G'_2 + G'_3 + 2H'_2 + H'_3 \\ &\quad + \sigma^*(l_1) + \sigma^*(l_2) + \sigma^*(l_3) + \sigma^*(M_1) + \sigma^*(C) - 3G'_2 - G'_3 - 3H'_2 - H'_3 \\ &\quad - Z_1 - Z_2 - (E_1 + E_2 + E_3) \\ &\sim \sigma^*(l_1) + \sigma^*(l_2) + \sigma^*(M_1) - G'_2 - H'_2 - (E_1 + E_2 + E_3). \end{aligned}$$

Since

$$4\{\sigma^*(l_1) + \sigma^*(l_2) + \sigma^*(M_1)\} \sim \sigma^*(C) + 2\sigma^*(M_1) + 3\sigma^*(l_1) + 3\sigma^*(l_2),$$

we have

$$\begin{aligned} 4(K_V + D) &\sim 2(v_1 E_1 + v_2 E_2) + (w_3 E_3 + 2G'_2 + 2H'_2) + 3(E_1 + 2G'_2) + 3(E_2 + 2H'_2) \\ &\quad - 4(E_1 + E_2 + E_3 + G'_2 + H'_2) + 2S_1 + S_2 + (\text{fiber components of } D) \\ &\sim (2v_1 - 1)E_1 + (2v_2 - 1)E_2 + (W_3 - 4)E_3 + 4G'_2 + 4H'_2 \\ &\quad + 2S_1 + S_2 + (\text{fiber components of } D) \end{aligned}$$

and  $4(K_V + D)$  is effective. From this expression, it is easy to see that  $\kappa(X) = 2$ .

If  $A^{(1)}$  has type  $(IV_4)$ , the above number  $|d|$  is the order of  $H_1(X - \Sigma; \mathbf{Z})$ . The surface obtained this way has a unique  $A_1$  singular point. We have  $d = \pm 2$  for the following values:

$$u_1 = 2, v_1 = 0, w_1 = 2, u_2 = 1, v_2 = m, w_2 = 0, u_3 = n, w_3 = 2mn + n \pm 1.$$

Thus we have examples of logarithmic homology planes, each of which has a unique  $A_1$  singular point.

**Type  $(TC_{2-1})$ .** We use the same notations as in the previous cases. We obtain a surface  $V$  starting from  $\Sigma'_1$  by oscillating sequences of blowing-ups with initial points  $R_1, R_2$  and  $R_3$ .

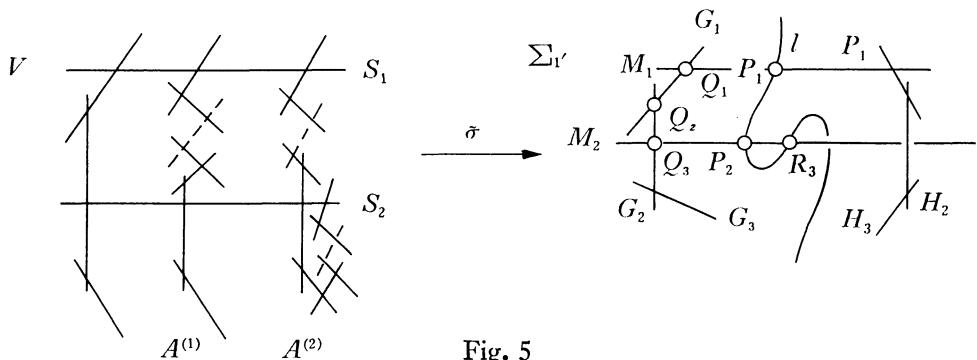


Fig. 5

We choose an initial point  $R_1$  from  $\mathbf{Q}_1$ ,  $\mathbf{Q}_2$  and  $\mathbf{Q}_3$  if  $A^{(1)}$  has type (IV<sub>2</sub>) and we choose an initial point  $R_2$  from  $P_1$  and  $P_2$ . Let  $E_i$  be the unique exceptional curve contained in  $\sigma^{-1}(R_i)$ .  $E_1$  is the proper transform of  $G_1$  (resp.  $G_2$ ) if  $A^{(1)}$  has type (IV<sub>1</sub>) (resp. (IV<sub>4</sub>)).

Write the total transform of  $l_i$ 's and  $M_i$ 's as follows:

$$\begin{aligned}\sigma^*(l_1) &\sim u_1 E_1 + (\text{fiber components of } D) \\ \sigma^*(l_2) &\sim u_2 E_2 + u_3 E_3 + (\text{fiber components of } D) \\ \sigma^*(M_1) &\sim v_1 E_1 + v_2 E_2 + (\text{other boundary components}) \\ \sigma^*(C) &\sim w_1 E_1 + w_2 E_2 + w_3 E_3 + (\text{other boundary components}).\end{aligned}$$

The order of  $H_1(X; \mathbf{Z})$  (or  $H_1(X - \Sigma; \mathbf{Z})$  if  $A^{(1)}$  has type (IV<sub>4</sub>)) is equal to  $|d|$  where  $d = 2u_1u_3v_2 + u_1u_2w_3 - u_1u_3w_2$ . We note that  $v_2w_2 = 0$ . Therefore we have homology planes when  $v_2 = 0$ ,  $u_1 = 1$  and  $u_2w_3 - u_3v_2 = \pm 1$  and logarithmic homology planes when  $v_2 = 0$ ,  $u_2 = 2$  and  $u_2w_3 - u_3v_2 = \pm 1$ . The computation of the Kodaira dimension is similar to the previous case.

**Type (T3C<sub>2</sub>).** The configuration of singular fibers and sections of  $p$  is given as follows:

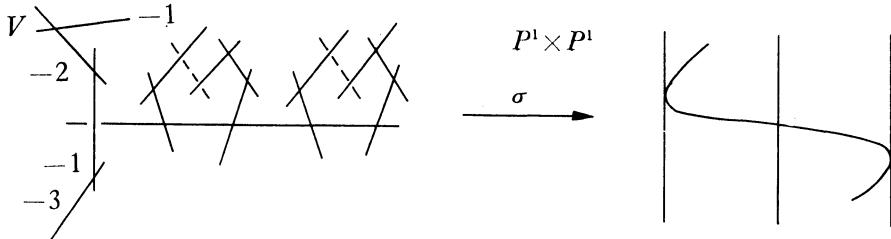


Fig. 6

We start from a nonsingular member  $C$  of  $|3M+l|$  on  $P^1 \times P^1$  which totally ramifies at one point and has two other ramification points when it is considered a covering of  $P^1$  through the first projection  $p_1$ . Here  $M$  is a section and  $l$  is a fiber of  $p_1$ . We can prove that such a curve exists. The calculation of  $H_1(X; \mathbf{Z})$  is same as before. In this case, there exist no homology planes. Also we can show that there exist  $\mathbf{Q}$ -homology planes of  $\kappa=2$  of this type. We omit the details.

Summarizing the results of this section we obtain the following theorem:

**Theorem 1.** *There exist infinitely many homology planes of  $\kappa=2$  with  $\mathbf{C}^{**}$ -fibrations of type (UP<sub>3-1</sub>), (UC<sub>2-1</sub>), (TP<sub>2</sub>) and (TC<sub>2-1</sub>) and there exist infinitely many logarithmic homology planes of  $\kappa=2$  with  $\mathbf{C}^{**}$ -fibrations of type (TP<sub>2</sub>) and (TC<sub>2-1</sub>). Conversely, if  $X$  is a homology plane or a logarithmic homology plane of  $\kappa=2$  with a  $\mathbf{C}^{**}$ -fibration,  $X$  belongs to one of the above classes.*

#### 4. Homology planes which have automorphisms

In this section we give examples of homology planes which admit non-trivial automorphisms of finite order. First we show that homology planes of type (TP<sub>2</sub>) with values of  $u_i$ ,  $v_i$  and  $w_i$  assigned in the previous section have involutions. These examples are constructed by tom Dieck and Petrie. We found out them by different approach.

We can start from the following configuration of curves on  $\mathbf{P}^2$  to obtain the homology plane of type (TP<sub>2</sub>):

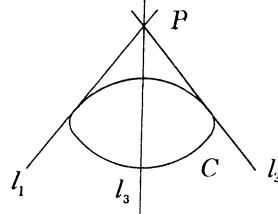


Fig. 7

Let  $(X_0: X_1: X_2)$  be the homogeneous coordinates of  $\mathbf{P}^2$  and  $l_1$ ,  $l_2$  and  $l_3$  be lines on  $\mathbf{P}^2$  and let  $C$  be a conic on  $\mathbf{P}^2$  whose equations are given respectively as follows:

$$\begin{aligned} l_1: X_1 + X_0 &= 0, & l_2: X_1 - X_0 &= 0, \\ l_3: X_1 &= 0, & C: X_1^2 + X_2^2 &= X_0^2. \end{aligned}$$

Let  $i: \mathbf{P}^2 \rightarrow \mathbf{P}^2$  be an involution defined by  $(X_0: X_1: X_2) \mapsto (X_0: -X_1: X_2)$ . Then we have  $i(C) = C$ ,  $i(l_1) = l_2$  and  $l_3$  is pointwise fixed. Moreover,  $i$  has an isolated fixed point  $(0, 1, 0)$ . To obtain a homology plane, we first perform the blowing-ups with centers  $P$ ,  $l_1 \cap C$  plus its infinitely near point over  $C$ , and  $l_2 \cap C$  plus its infinitely near point over  $C$ . Let  $\sigma: \Sigma'_1 \rightarrow \mathbf{P}^1$  be the composition of these blowing-ups. Then we have the following configuration of curves on  $\Sigma'_1$ :

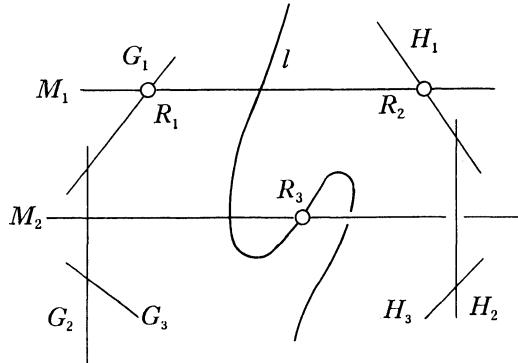


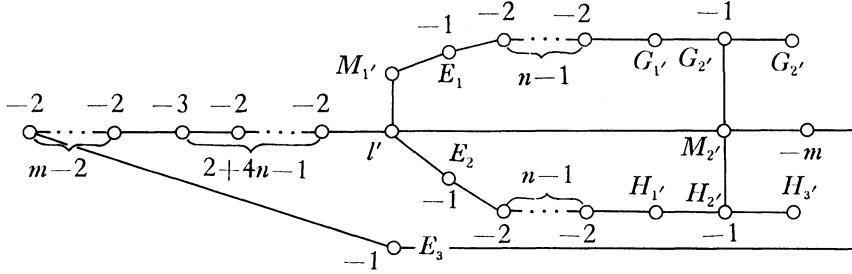
Fig. 8

Next we perform the blowing-ups with centers the point  $R_1$ , its  $(n-1)$  infinitely

near points consecutively lying on  $M_1$ , the point  $R_2$  and its  $(n-1)$  infinitely near points consecutively lying on  $M_1$ , and perform an oscillating sequence of blowing-ups with initial point  $R_3$  such that, if we set  $\sigma: V \rightarrow \Sigma'_1$  the composition of the above blowing-ups, we have

$$\begin{aligned}\sigma^*(M_2) &\sim (2m+4mn\pm 1)E_3 + (\text{other boundary components}) \quad \text{and} \\ \sigma^*(l) &\sim mE_3 + (\text{other boundary components}),\end{aligned}$$

where  $E_i$  represents a unique  $(-1)$  curve contained in  $\sigma^{-1}(R_i)$ . Set  $D = \sigma^{-1}(M_1 + M_2 + G_1 + G_2 + G_3 + H_1 + H_2 + H_3 + l)_{\text{red}} - (E_1 + E_2 + E_3)$ . We give the dual graph of divisor  $D + E_1 + E_2 + E_3$  on  $V$  in the case where  $w_3 = 2m+4mn-1$  ( $m \geq 2$ ):

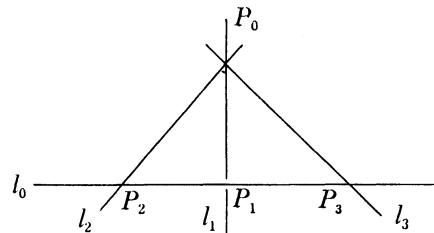


It is shown in §3 that the affine surface  $X := V - D$  is a homology plane. Since we perform the blowing-ups at the intersection points of pairwise stable curves or at the fixed points of the involution  $i$  or the involutions induced by  $i$ , the involution  $i$  is liftable to an involution  $\tilde{i}$  on  $V$  such that  $\tilde{i}(D) = D$ . Hence  $\tilde{i}$  induces an involution  $\tilde{i}$  on  $X$ , which has a unique fixed point inside  $X$ . Thus we obtain homology planes which have involutions. The quotient surface  $X/\tilde{i}$  is a logarithmic homology plane with a cyclic quotient singular point of Dynkin type  $A_1$  and belongs to the class (TP<sub>2</sub>).

Next we show that there exist homology planes of  $\kappa=1$  which have non-trivial automorphisms. A method of constructing homology planes with  $\kappa=1$  is given in [2]. Let  $(X_0: X_1: X_2)$  be the homogeneous coordinates of  $\mathbf{P}^2$ . We take four lines on  $\mathbf{P}^2$  as follows:

$$l_0: X_2 = 0, \quad l_1: X_1 = 0, \quad l_2: X_1 = -X_2, \quad l_3: X_1 = X_2.$$

The configuration of four lines is as follows:



First blow-up  $P_0=(0:0:1)$  and then perform the oscillating sequences of blowing-ups of type (a) (cf. §1) with initial points  $P_2$  and  $P_3$ . Finally we perform an oscillating sequence of blowing-ups with initial point  $P_1$  to produce a singular fiber isomorphic to  $\mathbf{C}$  on  $X:=V-D$ . Let  $\sigma: V \rightarrow \mathbf{P}^2$  be the composition of all the above blowing-ups. Let  $E_i$  be a unique  $(-1)$  curve contained in  $\sigma^{-1}(P_i)$ . Write the total transform of  $l_i$  as follows:

$$\begin{aligned}\sigma^*(l_i) &\sim E_0 + u_i E_i + (\text{other boundary components}) \quad \text{for } 1 \leq i \leq 3 \\ \sigma^*(l_0) &\sim \sum_{i=1}^3 v_i E_i + (\text{other boundary components}).\end{aligned}$$

Set  $D = \sigma^*(l_0 + l_1 + l_2 + l_3)_{\text{red}} - (E_1 + E_2 + E_3)$  and  $X = V - D$ . Then the order of  $H_1(X; \mathbf{Z})$  is  $|d|$ , where  $d = u_1 u_2 u_3 - u_1 u_2 v_3 - u_1 u_3 v_2 - u_2 u_3 v_1$  (cf. [2]). The equation  $d = \pm 1$  has the following solutions for any prime number  $p \neq 2$ :

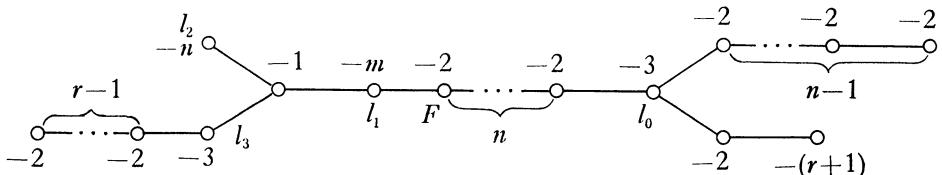
- $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$  is a permutation of  $\{(2, 1), (4r+1, 1), (2r+1, r)\}$  if  $p = 4r+1$ ,
- $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$  is a permutation of  $\{(2, 1), (4r+3, 1), (2r+1, r)\}$  if  $p = 4r+3$ .

Since the role of  $l_2$  and  $l_3$  is symmetric, there are three homology planes according to which one of three pairs is assigned to  $l_i$ . Among them, we consider only those homology planes with  $v_1=1$  and therefore we may assume that  $(u_3, v_3)=(2r+1, r)$ . For a different prime number  $p$ , we denote by  $X_m^{(p)}$  a homology plane with  $u_1=m$ ,  $m$  being 2 or  $4r+1$  if  $p=4r+1$  (resp. 2 or  $4r+3$  if  $p=4r+3$ ). We shall show that  $X_m^{(p)}$  admits an automorphism of order  $m$ .

Let  $\eta = \exp(2\pi\sqrt{-1}/m)$ . Define an action of  $\mathbb{Z}/m$  on  $P^2$  by  $(X_0: X_1: X_2) \rightarrow (\eta^{-1}X_0: \eta^{-1}X_1: X_2)$ . Then  $P_0 = (0: 0: 1)$  is a fixed point under this action and  $l_0$  is pointwise fixed. We state explicitly the process of blowing-ups  $\sigma: V \rightarrow P^2$ ;

- (i) blow up  $P_0$ ,
- (ii) blow up  $P_2$  and its infinitely near points over  $l_2$  altogether  $n$  times,
- (iii) blow up  $P_3$  and its infinitely near points over  $l_3$  altogether  $s$  times, in such a way that  $\sigma^*(l_3) \sim u_3 E_3 + \dots$  and  $\sigma^*(l_0) \sim v_3 E_3 + \dots$ ,
- (iv) blow up  $P_1$  and its infinitely near points over  $l_1$  altogether  $m$  times and blow up an arbitrary point on the last exceptional curve which is not an intersection point,

where  $(m, n, s) = (u_1, u_2, u_3)$ . Define  $D$  on  $V$  as stated before. Then the dual graph of  $D$  looks like:



We note that the last blowing-up on the line  $l_1$  is performed on  $F$ . Since the first  $1+n+s+m$  blowing-ups are performed at the intersection points of two components of fibers, it is easy to see that the action of  $\mathbb{Z}/m$  is liftable to the blown-up surface. Then at the last step,  $F$  becomes pointwise fixed under the induced action. Thus the action of  $\mathbb{Z}/m$  is liftable onto  $V$  and induces an action on  $X_m^{(p)}$ . It is clear that this action on  $X$  has a unique isolated fixed point which lies on the unique  $(-1)$  curve contained in  $\sigma^{-1}(P_1)$ , i.e., on  $E_1$ .

Summarizing the results in this section we obtain the following theorem:

**Theorem 2.** (1) *There exist infinitely many homology planes with  $\kappa=2$ , each of which admits an involution  $\iota$ . This involution  $\iota$  has a unique isolated fixed point.*

(2) *For every prime number  $p \neq 2$ , there exist homology planes  $X_m^{(p)}$  of  $\kappa=1$  on which a cyclic group  $\mathbb{Z}/m$  acts, where  $m=2$ , or  $4r+1$  if  $p=4r+1$  and  $m=2$  or  $4r+3$  if  $p=4r+3$ . This action has a unique isolated fixed point.*

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