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TWO CHARACTERISTIC PROPERTIES OF (ZT)-GROUPS

Dedicated to Professor K. Shoda

By

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1. Introduction. The classical linear fractional groups $L_2(q)$ over a finite field of $q=2^n$ elements are in a sense simple groups with the simplest structure. These groups $L_2(q)$ and the simple groups defined in [2] have many properties in common. Among other things they are doubly transitive permutation groups in which there is no regular normal subgroup and no non-identity element leaving three distinct elements invariant. The class of doubly transitive permutation groups of odd degree satisfying the preceding conditions is called the class of (ZT)-groups and has been studied in detail [4]. The main result of [4] says that the class of (ZT)-groups consists of simple groups $L_2(q)$ and the groups defined in [2]. There are many properties characteristic to (ZT)-groups (see [3]). The purpose of this note is to give two more characterizations of these groups.

In a (ZT)-group let H be either a Sylow 2-group or its normalizer. Then if x is an element $\neq 1$ of H, the centralizer $C_G(x)$ is a part of H. The following theorem gives a partial converse.

Theorem 1. *Let H be a proper subgroup of a finite group G satisfying the property that H contains the centralizer of any of its non-identity elements. If the order of H is even, then we have one of the following cases*:

(1) *G is a Frobenius group and H is either the Frobenius kernel or one of its complements*; *and*

(2) *G is a (ZT)-group and H is either a Sylow 2-group or the normalizer of a Sylow 2-group of G.*

The next theorem assumes a different property. In a finite group G define $\mathfrak{M}$ to be the set of maximal subgroups of G each of which contains either a Sylow 2-group of G or the centralizer of an element. In a (ZT)-group an intersection of two distinct subgroups of $\mathfrak{M}$ is cyclic

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(see the survey of subgroups given in [4]). Theorem 2 is again a partial converse to this statement.

Theorem 2. *Let G be a group of even order satisfying the property that two distinct maximal subgroups in $\mathfrak{M}$ have a cyclic intersection. Then we have one of the following cases:*

(1) *a Sylow 2-group of G is normal;*

(2) *G possesses a normal 2-complement;*

(3) *G is isomorphic to the special linear group $SL(2, 5)$ over the field of 5 elements; and*

(4) *G is a (ZT)-group.*

Both theorems characterize a (ZT)-group as a non-abelian simple group satisfying the property in question.

2. Preliminaries. Let G be a finite group. Assume that G has a subgroup U containing a Sylow 2-group S of G. We assume the following conditions to be satisfied:

(i) There exists an involution not contained in U;

(ii) $U \supseteq N_G(S)$;

(iii) If v is an inuolution in U, U contains $C_G(v)$.

Lemma 1. *Under* (i) *and* (iii), *two involutions of G are conjugate.*

Proof. Let u and t be two involutions such that $u \in U$ and $t \notin U$. Suppose by way of contradiction that u is not conjugate to t. Then the order of ut is even. Hence a power v of ut is an involution which commutes with both u and t. By (iii) applied to u we see that $v \in U$. Again by (iii) applied to v we have $t \in U$, which contradicts the definition of t. Since u and t are arbitrary Lemma 1 follows.

The set C_x of conditions is defined as the set of conditions from (i) up to (x),

Lemma 2. *If C_3 is satisfied, then U has only one conjugate class of involution.*

Proof. Let u be an involution of the center of the Sylow 2-group S of G. By definition S is a part of U. If v is an involution of S, there is an element $g \in G$ such that $v = u^g = g^{-1}ug$. Then S^g is contained in $C_G(v)$. By (iii), S^g is a Sylow 2-group of U. Hence there is an element w of U which transforms S^g into S. The element gw belongs to the normalizer $N_G(S)$ of S. By (ii) we conclude that $gw \in U$. This yields the desired conclusion $g\in U$.

We consider two more conditions:

(iv) U contains an involution j such that the centralizer $J=C_G(j)$ satisfies the following property: if u and v are two distinct involutions and if $uv\in J$, then $u\in U$;

(v) U contains an involution j such that $J=C_G(j)$ is a normal subgroup of U.

Lemma 3. *Under C_4, U is a product of J and a group D of odd order.*

Proof. By (iv) each coset of J not contained in U contains at most one involution. In view of Lemmas 1 and 2, a counting argument proves that each coset of U contains exactly $r=[U:J]$ involutions. Let t be an involution not contained in U. Then the coset Ut contains exactly r involutions $t_0=t, t_1, \cdots, t_{r-1}$. The elements t_it $(i=1, 2, \cdots, r-1)$ are elements of U and they are incongruent modulo J by (iv). If D is defined to be the intersection $U\cap U^t$, D contains all the elements t_it. Hence $U=JD$. Since $D=D^t$, the order of D is odd by (iii).

Lemma 4. *Under C_3 the condition* (v) *implies* (iv).

Proof. C_3 implies that involutions of U are conjugate. Hence (v) implies that every involution of U is contained in the center of J. Suppose that u and v are two different involutions and that $x=uv$ belongs to J. Then u inverts x. If $x=x^{-1}$, x is an involution and (iii) yields that $u\in U$. Assume $x\neq x^{-1}$. The set of elements which transform x into x or x^{-1} form a subgroup $C_G^*(x)$ and $[C_G^*(x):C_G(x)]=2$. The set of involutions of J is contained in $C_G(x)$. We enlarge this set to a Sylow 2-group P of $C_G^*(x)$. Since u inverts x, P contains an involution w which is not contained in $C_G(x)$. This is however impossible because w commutes with some involution of $P\cap C_G(x)$, which forces w to be in U by (iii).

Lemma 5. *Under C_5 if $x\neq 1$ of U is strongly real, then $C_G(x)\subseteq U$. Hence C_5 and $U\neq J$ imply*

$$[G:U]\leqq 1+|J|.$$

Proof. By a strongly real element we mean an element which is a product of two involutions. By (iv) and Lemma 1, a strongly real element of odd order commutes with no involution. If $x^2=1$, the assertion follows from (iii). Assume that x is of odd order and that $C_G(x)\not\subseteq U$.

As is seen from the proof of Lemma 3 an element outside of U is a product of an involution t and an element y of J. If $ytx=xyt$, then

$$x^{-1}y^{-1}xy = x^{-1}txt\,.$$

This element belongs to J since J is normal in U. Hence by (iv) we have $x^{-1}tx=t$. This is impossible. Each coset of U other than U itself produces exactly $r-1$ strongly real elements of odd order in U and those elements are all distinct. Hence the inequality follows.

The last condition to be considered is the following:

(vi) The order of D in Lemma 3 is relatively prime to $|J|$.

Lemma 6. *C_6 implies that U is a Frobenius group provided $D\neq 1$.*

Proof. Let x be a non-identity element of $D=U\cap U^t$. Since $|D|$ is relatively prime to $|J|$, Lemma 1 implies that x commutes with no involution. As in the proof of Lemma 5 we have $C_G(x)\subseteq U$. Similarly $C_G(x)\subseteq U^t$. Hence $C_G(x)\subseteq D$. This is true for all element $\neq 1$ of D. The conclusion follows.

Theorem 3. *Let G be a finite group and U a subgroup of G. If the set of conditions C_6 is satisfied, then either a Sylow 2-group of G contains only one involution or G is a (ZT)-group.*

Proof. If a Sylow 2-group of G contains more than one involution, we have $U\neq J$ in the notation of Lemma 3. Hence by Lemma 5

$$[G:U]\leqq 1+|J|\,.$$

On the other hand U is by Lemma 6 a Frobenius group. Hence any element of $U-J$ is conjugate to an element of D. Since $C_G(x)\subseteq D$ for $x\in D-\{1\}$, every element of D is strongly real. Hence we have an equality $[G:U]=1+|J|$.

As a transitive permutation group on the cosets of U, G is doubly transitive and J is regular on cosets $\neq U$. Since D is abelian we have $D\cap D^x=\{1\}$ for $x\in G-N_G(D)$. It is easy to see that no element $\neq 1$ leaves more than 2 cosets invariant. By definition G is a (ZT)-group.

3. Proof of Theorem 1. A subgroup H of a group G is said to satisfy the condition (c) if H contains the centralizer of any of its non-identity elements. The following lemma is obvious.

Lemma 7. *If subgroups H_i $(i=1, 2, \cdots, m)$ satisfy the condition (c), then the intersection $\cap H_i$ does the same.*

Lemma 8. *If a subgroup H satisfies the condition* (c), *then H is a Hall subgroup of G.*

This is an easy consequence of a theorem of Sylow and a basic property of p-groups.

Lemma 9. *If a proper normal subgroup N of G satisfies the condition* (c), *then G is a Frobenius group with kernel N.*

Proof. Since N is a Hall normal subgroup, there is a complement H. it is easy to verify that

$$H \cap H^x = \{1\} \qquad \text{for} \quad x \notin H .$$

Suppose that G is a Frobenius group with kernel N. Let K be a complement of N. Then N is nilpotent by a theorem of Thompson [5]. A result of Zassenhaus [6] yields that if p is the smallest prime divisor of $|K|$, K contains a central element of order p. It is now easy to prove that a proper subgroup H satisfying the condition (c) is either N or a subgroup conjugate to K.

We assume in the following that G is not a Frobenius group. We distinguish two cases according as $N_G(H)=H$ or not.

Consider first the case $N_G(H) \neq H$. Then the group $U=N_G(H)$ is a Frobenius group with kernel H. Since $U \neq G$, the condition (i) of the second section is satisfied. Since H is nilpotent (ii) is also true. The condition (iii) is obvious and the nilpotency of H implies (v). If t is an involution not contained in H, $H \cap H^t$ satisfies the condition (c) by Lemma 8. Hence by the remark on Frobenius groups we have $H \cap H^t = \{1\}$. This implies the condition (vi).

Theorem 3 is applicable and yields that G is a (ZT)-group. We remark that the assumption $N_G(H) \neq H$ implies that a Sylow 2-group of G contains at least two involutions.

Suppose next that $N_G(K)=K$ for every proper subgroup K of even order which satisfies the condition (c). Let H be a subgroup which satisfies the condition (c), contains a fixed Sylow 2-group S of G and is minimal subject to these two restrictions. Then H contains $N_G(S)$. Hence C_3 of the second section is satisfied for $U=H$. We want to prove the condition (iv) for H. Suppose that u and v are involutions of G and that $x=uv \in C_G(j)$ for some involution j of H. Then a Sylow 2-group containing j is a part of H. Hence an involution w of H inverts x. Then $wu \in C_G(x) \subseteq H$, which implies that $u \in H$.

Since G is not a Frobenius group, there is an involution t such that $D=H \cap H^t$ is a proper subgroup of H. By Lemmas 7 and 9, $N_G(D)$ is a Frobenius group with kernel D. Since D satisfies condition (c), the involution t inverts every element of D. Hence D is abelian. If T is a complement of D in $N_G(D) \cap H$, an involution outside of H centralizes T. By (c) for H we have $N_G(D) \cap H = N_H(D) = D$. This means that H is a Frobenius group and D is a complement to the Frobenius kernel of H. The conditions (v) and (vi) are satisfied. Theorem 3 yields the assertion. Again the assumption $D \neq \{1\}$ implies that a Sylow 2-group of H contains at least two involutions.

4. Proof of Theorem 2. Let G be a group satisfying the assumption of Theorem 2. We assume that a Sylow 2-group S of G is not normal and that G does not have a normal 2-complement. This implies in particular that S is not cyclic. Hence S is contained in a unique maximal subgroup of G. Let U be the maximal subgroup containing S.

We assume furthermore that G contains no normal subgroup of prime order. We want to prove that U satisfies C_6 of the section 2.

Since U is the unique maximal subgroup of G containing S, U contains $N_G(S)$. Hence by a theorem of Sylow U coincides with its normalizer.

If the condition (i) is not satisfied, the set of involutions of U generates a normal subgroups I of G. Since $N_G(U) = U$, I is cylic. Hence G contains a central involuton contrary to the assumption.

The condition (ii) has been verified. If an involution u is in the center of S, $C_G(u)$ is a proper subgroup containing S. Hence $C_G(u)$ is a part of U. If j is any other involution of S, $C_G(j) \cap U$ contains a non-cyclic subgroup of order 4. Hence by the basic assumption $C_G(j) \subseteq U$. This proves (iii).

In order to prove the condition (iv) for U let u and v be involutions of G such that $x = uv \in C_G(j)$ for some involution j of U. By the same argument as in the corresponding part of the proof of Theorem 1, the interesection of U and the group $G_G^*(x)$ which consists of the totality of elements transforming x into x or x^{-1} is not cyclic. Hence U contains $C_G^*(x)$ and in particular $u \in U$.

By Lemma 3, U is a product of $J = C_G(j)$ and D, where $D = U \cap U^t$ for an involution t not contained in U. If $J = U$, S is a (generalized) quaternion group. Hence by a theorem of Brauer-Suzuki [0], $G = JN$ where N is a normal subgroup of maximal odd order. Since SN is not contained in J, we have $G = SN$. This is not the case. Hence $U \neq J$. Put $r = [U : J]$. Then $J \cap D$ is a subgroup of index r in D. If $J \cap D \neq \{1\}$, we find an element $x \neq 1$ of $J \cap D$ so that $C_G^*(x)$ contains j, t and D. This is implies that $\{j, D\}$ is cyclic. This is clearly not the case. Hence $J \cap D = \{1\}$. This implies in particular that the involution t inverts every element of D.

If $x \neq 1$ is in D, $C_G^*(x)$ contains t and D. Hence $C_U(x)$ is cyclic. This implies that $|D|$ is relatively prime to $|J|$. At the same time we see that for each prime divisor of $|D|$ the transfer theorem of Burnside is applicable. Hence U contains a normal complement to D. This normal complement must coincide with J. Thus the conditions (v) and (vi) are satisfied. Theorem 2 is an easy consequence of Theorem 3.

It remains to treat the case when G contains a normal subgroup of prime order. Suppose that G contains a normal subgroup N of prime order p but G/N contains no normal subgroup of prime order. Since G/N satisfies the same assumption as G, we may assume that G/N is a (ZT)-group. If $|N|$ is an odd prime p, then a Sylow p-group of G is abelian because a Sylow p-group of G/N is cyclic. Hence a theorem of Zassenhaus [6] yields the existence of a normal subgroup of index p. Assume $p=2$. If a Sylow 2-group of G splits over N, then G splits over N. Hence the non-trivial extension is possible only when a Sylow 2-group of G/N is of order 4. In this case G/N is isomorphic to $L_2(5)$ and a classical theorem of Schur [1] says that G is isomorphic with $SL(2, 5)$. Theorem 2 follows by induction.

We remark that the following theorem is true.

Theorem 4. *Let N be a subgroup of the center of G. If G/N is a (ZT)-group, then the extension of G over N splits unless $G=SL(2, 5)$.*

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After completing this work the author learned that J. G. Thompson has used some of the lemmas in the section 2 in his unpublished work. The same idea appeared also in the recent work of the author to appear elsewhere. The last half of section 2 is closely related to the idea of W. Feit in his paper appeared in Amer. J. of Math (1960).

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