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EQUIVARIANT RIEMANN-ROCH THEOREMS, LOCALIZATION AND FORMAL GROUP LAW

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1. Introduction

Let G be a compact Lie group and $h_G(\ )$ be an equivariant multiplicative cohomology theory. Let M and N be closed G-manifolds. In this paper, we assume that G-actions are differentiable of class C^∞. Then for a G-map $f: M \to N$, we shall define an "equivariant Gysin homomorphism"

$$f_!: h_G(M) \to h_G(N),$$

which is an $h_G(pt)$-module homomorphism. Special cases of $f_!$ have been studied profoundly by Grothendieck, Borel-Serre [13], Hirzebruch [25], [26], Atiyah-Hirzebruch [3], [4], Dyer [21], Atiyah-Singer [9], [10], Atiyah-Segal [6], [7], tom Dieck [19] and so on.

In the present paper, we shall study the equivariant Gysin homomorphism $f_!$ systematically and conceptually.

First we shall establish a localization theory in general and shall obtain many equations between invariants of a G-manifold and invariants of its fixed point set by virtue of the uniqueness of our equivariant Gysin homomorphism. From now on, we call them briefly equations between global and local invariants.

Next we shall establish various kinds of equivariant Riemann-Roch type theorems in various categories.

As in the non-equivariant case, there are a lot of applications. For examples, we shall have integrality and divisibility theorems of equivariant characteristic numbers. On the other hand, there are particularly interesting applications peculiar to equivariant cases. Namely equivariant Riemann-Roch theorems inform us of finiteness properties of some global invariants in our localization theorem. Accordingly we can often conclude that some global invariants are independent of the actions or even zero by combining it with the localization theorem. This is in fact my motivation to establish equivariant Riemann-Roch type theorems and is essentially an idea due to Hirzebruch.

We first confine ourselves to the weakly complex category. Let $G \to EG \to BG$ be the universal principal G-bundle. By making use of the equivariant cohomology theories $K_G(M)$ and $U^*(EG \underset{G}{\times} M)$ explicitly and $K(EG \underset{G}{\times} M)$ implicitly, and the formal group law in U^*-theory, we shall have an equivariant Riemann-Roch type theorem where U^* denotes the complex cobordism theory. As mentioned above, by combining it with the localization theorem, we shall obtain an equation between an invariant of a manifold (not G-manifold!) and an invariant expressed in terms of the fixed point set and its normal bundle and the normal representation, which might be called "G-T_y-genus formula". As shown by Quillen, the formal group law in U^*-theory is universal in the sense of Lazard [39] [31]. It follows that these formulae must be the most generalized one.

Next we confine ourselves to the Spin^c-category. In this category, we also obtain an equivariant Riemann-Roch type theorem including something like an "equivariant $\hat{\mathfrak{A}}$-genus". As an application peculiar to the equivariant case, we shall obtain the vanishing theorem of $\hat{\mathfrak{A}}$-genus due to Atiyah-Hirzebruch [5] by combining it with the localization theorem. This answers the problem of I.M. Singer [40].

We now turn to the oriented differentiable category of class C^∞. Making use of the strictly multiplicative property of the L-genus [12], we shall have the G-signature theorem due to Atiyah-Bott-Singer.

In the non oriented category, we shall define equivariant Stiefel-Whitney classes and shall obtain an equivariant Wu type formula. Hence one might naturally expect to prove the following conjecture by making use of the equivariant Wu type formula: "The equivariant Stiefel-Whitney classes are invariants of the G-homotopy type". Since $EG \underset{G}{\times} M$ is not a manifold in general, the invariance does not follow from the equivariant Wu type formula. For this reason, we consider $EG^n \underset{G}{\times} M$ for all positive integers n where EG^n is an n-connected free G-manifold. Then the invariance follows from similar fomulae for $EG^n \underset{G}{\times} M$.

Last of all, we shall have an equivariant non-embedding theorem. So far equivariant non-embedding theorems have been studied for involutions [11] [32]. We here consider any compact Lie group actions.

In a future publication, we shall define equivariant characteristic numbers for G-manifolds by making use of the equivariant Gysin homomorphism. As an application we shall have, for instance, that bordism classes of oriented T^n-manifolds are characterized by their equivariant characteristic numbers modulo two torsions. Here we need the recent result of Ossa.

The intention of the present paper is to exploit general theories and various applications will appear in forthcoming papers.

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2. Equivariant Gysin homomorphism

In this section, we shall define an equivariant Gysin homomorphism in general.

Let X be a compact G-space and ξ be a G-vector bundle over X. We denote by $D(\xi)$ (resp. $S(\xi)$) the disk bundle (resp. sphere bundle) associated with ξ. Let $h_G(\ )$ be an equivariant multiplicative cohomology theory. An element $t(\xi)$ of $h_G(D(\xi), S(\xi))$ is called a Thom class (or h_G-orientation class) if for any compact G-invariant subspace Y of X, the correspondence $x \to (t(\xi) \mid Y) \cdot x$ gives an isomorphism

$$h_G(Y) \to h_G(D(\xi \mid Y), S(\xi \mid Y)) .$$

We assume that for any compact G-space Y and any G-map $f; Y \to X$, the induced element $f^*t(\xi)$ is a Thom class of the induced bundle $f^*\xi$.

Let M and N be h_G-oriented closed G-manifolds of class C^∞, that is to say, the tangent bundles of M and N are h_G-oriented. Then for a G-map $f: M \to N$, we define our equivariant Gysin homomorphism

$$f_!: h_G(M) \to h_G(N)$$

as follows. As is well-known, there is an equivariant embedding e of M in some G-vector space V. For the proof, see Palais [37]. Since f is G-homotopic to a differentiable G-map f' of class C^∞, we first define our equivariant Gysin homomorphism $f'_!$ and then define $f_!$ to be $f'_!$. The forthcoming Lemma 2.2 will assure that $f_!$ is independent of the choice of f'. Therefore we may now assume that f itself is differentiable of class C^∞. Choose a G-invariant Riemannian metric on $N \times V$ and let ν be an invariant open tubular neighborhood of

$(f\times e)(M)$ in $N\times V$. Here we need the assumption C^∞. Then ν is a G-vector bundle which may be identified with the normal bundle of $(f\times e)(M)$ in $N\times V$. Denote by $D(V)$ (resp. $S(V)$) the unit disk (resp. unit sphere) in V. Here we may assume without loss of generality that $D(\nu)$ is in $N\times \mathrm{Int}\, D(V)$.

Regarding V as a G-vector bundle over one point, we assume that V is h_G-oriented. Then the homomorphism $f_!$ is defined by the composition of the following three homomorphisms which we explain in a moment:

$$\phi_1\colon h_G(M) \to \tilde{h}_G(D(\nu)/S(\nu)),$$
$$\phi_2\colon \tilde{h}_G(D(\nu)/S(\nu)) \to \tilde{h}_G(N\times D(V)/N\times S(V)),$$
$$\phi_3\colon \tilde{h}_G(N\times D(V)/N\times S(V)) \to h_G(N).$$

EXPLANATION. Here $\tilde{h}_G$ denotes the reduced cohomology ring as usual. Let $t(M)\in\tilde{h}_G(D(TM)/S(TM))$ be the orientation class of the manifold M where TM denotes the tangent G-vector bundle. Similarly let $t(N)\in\tilde{h}(D(TN)/S(TN))$ (resp. $t(V)\in\tilde{h}_G(D(TV)/S(TV))$) be the orientation class of N (resp. V). It is easy to see that we can choose a canonical orientation class $t(\nu)$ such that

$$t(M)\times t(\nu) = (f\times e)^*(t(N)\times t(V)).$$

Then the homomorphism ϕ_1 is defined to be the Thom isomorphism by making use of the Thom class $t(\nu)$. The homomorphism ϕ_2 is the induced homomorphism by the natural collapsing map

$$N\times D(V)/N\times S(V) \to D(\nu)/S(\nu).$$

The homomorphism ϕ_3 is again defined by the Thom isomorphism in the manner of the definition of ϕ_1.

DEFINITION 2.1. Let M be a closed G-manifold and $f\colon M\to$ *point* be the constant map. Then we define an index homomorphism

$$\mathrm{Ind}\colon h_G(M) \to h_G(pt)$$

by $f_!$ where pt stands for one point.

Lemma 2.2. *The equivariant Gysin homomorphism is independent of all choices made and has the following properties*:

i) $f_!$ *depends only on the* G*-homotopy class of* f

ii) $f_!$ *is an* $h_G(pt)$*-module homomorphism*

iii) $(fg)_!=f_!\cdot g_!$

iv) $f_!(x\cdot f^*(y))=f_!(x)\cdot y$ *for* $x\in h_G(M)$, $y\in h_G(N)$

v) *if* f *is a* G*-embedding of class* C^∞ *with a normal bundle* ν, *then* $f^*f_!(x)=\chi_G(\nu)\cdot x$ *for* $x\in h_G(M)$ *where* $\chi_G(\nu)$ *denotes the equivariant Euler class of* ν.

Proof. We show that $f_!$ does not depend on the choices of V and e. Let $e': M \to V'$ be another G-embedding where V' is h_G-oriented. Then consider the following two G-embeddings:

$$\bar{e} : M \xrightarrow{e} V \subset V \oplus V' \subset (V \oplus V')^k ,$$

$$\bar{e}' : M \xrightarrow{e'} V' \subset V \oplus V' \subset (V \oplus V')^k$$

where $V \oplus V' \subset (V \oplus V')^k$ denotes the inclusion into the first factor. It follows from the transitivity of the Thom isomorphism that e and $\bar{e}$ give rise to the same equivariant Gysin homomorphism. Similarly e' and $\bar{e}'$ give rise to the same equivariant Gysin homomorphism. Denote by $\nu(f \times \bar{e})$ (resp. $\nu(f \times \bar{e}')$) the normal bundle of the G-embedding:

$$f \times \bar{e} : M \to N \times (V \oplus V')^k$$
$$(\text{resp. } f \times \bar{e}' : M \to N \times (V \oplus V')^k) .$$

Since $f \times \bar{e}$ and $f \times \bar{e}'$ are G-homotopic by a differentiable G-homotopy through differentiable G-embeddings, $\nu(f \times \bar{e})$ is G-equivalent to $\nu(f \times \bar{e}')$. If, moreover, $k \geqq 2\dim M + 4$, then two such regular G-homotopies are themselves regularly G-homotopic through regular G-homotopies [46] and the two resulting bundle equivalences are G-homotopic through G-bundle equivalences. Consequently $t(\nu(f \times \bar{e}))$ is chosen uniquely as in the non-equivariant case (cf. [21]). Namely $\bar{e}$ and $\bar{e}'$ give rise to the same equivariant Gysin homomorphism.

The rest of the proof is routine.

3. Localization

We consider the subset S of $h_G(pt)$ consisting of Euler classes of h_G-oriented G-vector spaces V such that the group G acts on V without trivial direct summand. Here we regarded a G-vector space as a G-vector bundle over one point. Then S is a multiplicative subset of $h_G(pt)$. It follows from Lemma 2.2 that we get a localization $S^{-1}h_G(M)$ and an induced homomorphism

$$S^{-1}f_! : S^{-1}h_G(M) \to S^{-1}h_G(N)$$

for a G-map $f: M \to N$ (see Bourbaki [14] for notion and notation).

Denote by F_μ each component of the fixed point set of a G-manifold M and by $i_\mu : F_\mu \to M$ the inclusion map. Denote by N_μ the normal bundle of F_μ. There exist a G-vector space V without trivial direct summand and a G-map

$$f : M - \bigcup_\mu \operatorname{Int} D(N_\mu) \to V - \operatorname{Int} D(V) .$$

This follows from the classical representation theory and so on. We assume that V is h_G-oriented.

Then we have

Lemma 3.1 (tom Dieck [18]). *The following homomorphism*

$$\sum_{\mu} S^{-1}i_{\mu}^{*}\colon S^{-1}h_G(M) \to \sum_{\mu} S^{-1}h_G(F_\mu)$$

is an isomorphism where the summation is taken over all the components F_μ of the fixed point set.

Now we assume the following

ASSUMPTION 3.2. Each N_μ is h_G-orientable so that each equivariant Euler class $\chi_G(N_\mu)$ is a unit in $S^{-1}h_G(F_\mu)$.

Fix an orientation on N_μ and we orient F_μ so that the orientation of N_μ followed by that of F_μ yields the restriction of the orientation of M.

Let x be an element of $S^{-1}h_G(M)$, then we have

$$(\sum_{\mu} S^{-1}i_{\mu}^{*})(\sum_{\mu} S^{-1}i_{\mu!})(\sum_{\mu} S^{-1}i_{\mu}^{*})x = \sum_{\mu} \chi_G(N_\mu)\cdot S^{-1}i_{\mu}^{*}x$$

by Lemma 2.2. Because of Assumption 3.2, we have that

$$(\sum_{\mu} S^{-1}i_{\mu}^{*})(\sum_{\mu} S^{-1}i_{\mu!})\left(\sum_{\mu} \frac{S^{-1}i_{\mu}^{*}x}{\chi_G(N_\mu)}\right) = \sum_{\mu} S^{-1}i_{\mu}^{*}x$$

in $\sum_{\mu} S^{-1}h_G(F_\mu)$. Since $\sum_{\mu} S^{-1}i_{\mu}^{*}$ is an isomorphism by Lemma 3.1, $\sum_{\mu} S^{-1}i_{\mu!}$ is also an isomorphism and we have

$$\sum_{\mu} \frac{S^{-1}i_{\mu}^{*}x}{\chi_G(N_\mu)} = (\sum_{\mu} S^{-1}i_{\mu!})^{-1}x\,.$$

Thus by the uniqueness and the functorial properties of our equivariant Gysin homomorphism (Lemma 2.2) we have the following localization theorem.

Theorem 3.3. *Under Assumption* 3.2, *the following diagram commutes*:

$$\begin{array}{ccc} S^{-1}h_G(M) & \xrightarrow{S^{-1}\ \mathrm{Ind}} & \\ \Big\downarrow \sum_{\mu} \frac{S^{-1}i_{\mu}^{*}}{\chi_G(N_\mu)} & & S^{-1}h_G(pt)\,. \\ \sum_{\mu} S^{-1}h_G(F_\mu) & \xrightarrow{S^{-1}\ \mathrm{Ind}} & \end{array}$$

Since we shall give several applications of Theorem 3.3 later, we now give just a few examples.

We make use of the equivariant cohomology theory $H^*(EG\underset{G}{\times}M;Q)$ now.

For an oriented G-vector bundle ξ, we use the usual Thom class $t(\xi)\in$

$H^*(EG\underset{G}{\times}(D(\xi), S(\xi)))$ as the orientation of ξ.

Let T^n be the toral group and $\boldsymbol{\rho}=(\rho_1, \cdots, \rho_n)$ be an n-tuple of integers. Ragarding T^n as $S^1_1\times\cdots\times S^1_n$, we define an irreducible representation $\boldsymbol{\rho}: T^n\to S^1=U(1)$ by

$$\boldsymbol{\rho}(s_1, \cdots, s_n) = s_1^{\rho_1}\cdots s_n^{\rho_n}$$

where $s_i\in S^1_i$. We identify $\boldsymbol{\rho}$ with the element $\rho_1 t_1+\cdots+\rho_n t_n$ of $H^2(BT^n)$ by the following translations

$$\left\{\begin{matrix}\text{complex irreducible}\\ \text{representations}\end{matrix}\right\} \leftrightarrow H^1(T^n)\xleftrightarrow{\text{transgression}} H^2(BT^n)\,.$$

Let T^n act on a closed oriented differentiable manifold M. Denote by F_μ each component of the fixed point set and by N_μ its normal bundle in M. Then we can impose a complex vector bundle structure on N_μ so that we get a decomposition

$$N_\mu = \sum_{\boldsymbol{\rho}} E_{\mu\boldsymbol{\rho}}\otimes V_{\boldsymbol{\rho}}$$

where $\boldsymbol{\rho}$ run through the complex irreducible representations of T^n and $V_{\boldsymbol{\rho}}$ denote their representation spaces and $E_{\mu\boldsymbol{\rho}}$ denote complex vector bundles.

Let $\gamma_i: T^n\to S^1=U(1)$ be the irreducible representation defined by $\gamma_i(s_1, \cdots, s_n)=s_i$ and $\Gamma_i\to BT^n$ be the γ_i extension of the universal principal T^n-bundle $ET^n\to BT^n$.

It is easy to see that

$$\begin{aligned} ET^n\underset{T^n}{\times}N_\mu &= \sum_{\boldsymbol{\rho}} ET^n\underset{T^n}{\times}(E_{\mu\boldsymbol{\rho}}\otimes V_{\boldsymbol{\rho}}) \\ &= \sum_{\boldsymbol{\rho}} (ET^n\underset{T^n}{\times}V_{\boldsymbol{\rho}})\hat{\otimes}E_{\mu\boldsymbol{\rho}} \\ &= \sum_{\boldsymbol{\rho}} (\prod \Gamma_i^{\rho_i})\hat{\otimes}E_{\mu\boldsymbol{\rho}}\,, \end{aligned}$$

where $\hat{\otimes}$ denotes the external tensor product. Hence the total Chern class $c(ET^n\underset{T^n}{\times}N_\mu)$ and the Euler class $\chi(ET^n\underset{T^n}{\times}N_\mu)$ of the bundle $ET^n\underset{T^n}{\times}N_\mu\to BT^n\times F_\mu$ are given by

$$\begin{aligned} c(ET^n\underset{T^n}{\times}N_\mu) &= \prod_{\boldsymbol{\rho},\rho_i}(1+\boldsymbol{\rho}+{}_\mu x_{\rho_i})\,, \\ \chi(ET^n\underset{T^n}{\times}N_\mu) &= \prod_{\boldsymbol{\rho},\rho_i}(\boldsymbol{\rho}+{}_\mu x_{\rho_i})\,, \end{aligned}$$

where the total chern class of $E_{\mu\boldsymbol{\rho}}$ is expressed formally as $\prod_{\rho_i}(1+{}_\mu x_{\rho_i})$. Accordingly it is easily seen that Assumption 3.2 is satisfied in this case.

Let $F(t)$ be the formal power series $F(t)=1+a_2t^2+a_4t^4+\cdots$ of t^2 and let K be the multiplicative sequence [25] belonging to the characteristic power series $F(t)$. Then we have

Corollary 3.4.

$$\text{Ind } K(ET^n \times_{T^n} TM) = \sum_G \frac{\prod_j F(z_j) \prod_{\rho,\rho_i} F(\rho+\mu x_{\rho_i})}{\prod_{\rho,\rho_i} (\rho+\mu x_{\rho_i})} / [F_\mu] .$$

Here the total Pontrjagin class of F_μ is expressed formally as $\prod(1+z_j^2)$ and $/[F_\mu]$ denotes the slant product:

$$H^*(BT^n \times F_\mu) \to H^*(BT^n) .$$

Next we consider another kind of applications. Let ω be a partition and s_ω be the characteristic class defined as in [34]. Then we have

Corollary 3.5.

$$s_\omega(M) = \sum_\mu \sum_{\omega=\omega_1\omega_2} \frac{s_{\omega_1}(\prod(1+z_j^2)) \cdot s_{\omega_2}[\prod\{1+(\rho+\mu x_{\rho_i})^2\}]}{\prod(\rho+\mu x_{\rho_i})} / [F_\mu] .$$

REMARK 3.6. Quite similar formulae hold for Stiefel-Whitney classes instead of Pontrjagin classes. Hence Corollary 3.5 gives us an explicit way to compute the bordism class $[M]$ of the oriented bordism group from the fixed point data.

In particular, we have

Corollary 3.7. *When an action is non-trivial,*

$$s_k(M) = \sum_\mu \frac{(\rho+\mu x_{\rho_i})^{2k}}{\prod(\rho+\mu x_{\rho_i})} / [F_\mu]$$

where $\dim M = 4k$.

REMARK 3.8. It is pointed out by D. Zagier that there is an interesting relation between Corollary 3.7 and a residue formula when $M = \boldsymbol{CP}^m$ and $T^n = S^1$.

4. Weakly complex G-actions

Let $MU(k)$ be the unitary spectrum (see Conner-Floyd [17]). In this section, we shall make use of the following equivariant multiplicative cohomology theory:

$$h_G(M) = \lim_{k\to\infty} [S^{2k-n} \wedge (EG \times_G M)^+, MU(k)]$$

which we denote by $U^n(EG \times_G M)$ as usual. As is well-known, there exist G-invariant, n-connected, finite CW complex EG^n such that

$$EG = \lim_{\to} EG^n .$$

It follows from Atiyah-Segal [8] that

$$\lim^1 K(EG^n \underset{G}{\times} M) = 0 .$$

On the other hand, Yosimura has shown in [47] that

$$\lim^1 U^*(EG^n \underset{G}{\times} M) = 0$$

if and only if

$$\lim^1 K(EG^n \underset{G}{\times} M) = 0 .$$

Hence we have that

$$U^*(EG \underset{G}{\times} M) \cong \lim_{\leftarrow} U^*(EG^n \underset{G}{\times} M)$$

and

$$K(EG \underset{G}{\times} M) \cong \lim_{\leftarrow} K(EG^n \underset{G}{\times} M) .$$

Let ξ be a complex G-vector bundle over M. According to Conner-Floyd [17], there exist canonical Thom classes

$$t_n(\xi) \in \tilde{U}^*(EG^n \underset{G}{\times} D(\xi)/EG^n \underset{G}{\times} S(\xi))$$

such that for the inclusion

$$i\colon EG^n \underset{G}{\times} D(\xi)/EG^n \underset{G}{\times} S(\xi) \to EG^{n+1} \underset{G}{\times} D(\xi)/EG^{n+1} \underset{G}{\times} S(\xi) ,$$

we have $i^* t_{n+1}(\xi) = t_n(\xi)$. Therefore we have a canonical Thom class

$$t(\xi) = \{t_n(\xi)\} \in U^*(EG \underset{G}{\times} D(\xi)/EG \underset{G}{\times} S(\xi)) .$$

Since the Thom class $t_n(\xi)$ is natural for a G-vector bundle map, $t(\xi)$ is also natural.

Similar arguments are valid for K-theory and we have

Lemma 4.1. *For a complex G-vector bundle ξ over M, there exist Thom classes*

$$t^U(\xi) \in U^*(EG \underset{G}{\times} D(\xi)/EG \underset{G}{\times} S(\xi))$$

and

$$t^K(\xi) \in K(EG \underset{G}{\times} D(\xi)/EG \underset{G}{\times} S(\xi))$$

which are natural for a G-vector bundle map and $\mu_1(t^U(\xi)) = t^K(\xi)$ where μ_1 is the limit of the Conner-Floyd natural transformation [17].

Let ξ be a complex G-vector bundle over a finite CW-complex M. Then we shall define a U^*-theory Chern class for the vector bundle $EG \underset{G}{\times} \xi$. Remark

that Conner-Floyd defined the U^*-theory Chern class only for complex vector bundles over finite CW-complexes. However our $\lim^1$ arguments enable us to define a U^*-theory total Chern class $c(EG\underset{G}{\times}\xi)$ for the vector bundle $EG\underset{G}{\times}\xi$.

Although there is no splitting principle in the equivariant K-theory, we shall have a "splitting principle" which answers our later purpose. Let ξ be a complex G-vector bundle over a finite CW complex M. Denote by $\pi: F(\xi)\to M$, the equivariant flag bundle. Then by the same way as in the non-equivariant case, we have

$$\pi^*\xi\cong L_1\oplus\cdots\oplus L_s$$

where L_i are complex G-line bundles. It follows that

$$(1\underset{G}{\times}\pi)^*(EG\underset{G}{\times}\xi)\cong(EG\underset{G}{\times}L_1)\oplus\cdots\oplus(EG\underset{G}{\times}L_s)\,.$$

Since $EG^n\underset{G}{\times}F(\xi)$ is canonically identified with $F(EG^n\underset{G}{\times}\xi)$ and since $EG^n\underset{G}{\times}M$ is a finite CW-complex, the homomorphism

$$(1\underset{G}{\times}\pi)^*\colon U^*(EG^n\underset{G}{\times}M)\to U^*(EG^n\underset{G}{\times}F(\xi))$$

is injective (see Dold [20] and Conner-Floyd [17]). It follows by the $\lim^1$ argument that the induced homomorphism

$$(1\underset{G}{\times}\pi)^*\colon U^*(EG\underset{G}{\times}M)\to U^*(EG\underset{G}{\times}F(\xi))$$

is injective. Repeating this argument, we have

Lemma 4.2. *Let $\xi_1,\cdots,\xi_n$ be complex G-vector bundles over a finite CW-complex M. Then there exist a compact G-space F, a G-map $\pi\colon F\to M$ and complex G-line bundles L_i^j over F such that*

$$(1\underset{G}{\times}\pi)^*\xi_i=(EG\underset{G}{\times}L_i^1)\oplus\cdots\oplus(EG\underset{G}{\times}L_i^{s_i})$$

and the induced homomorphism

$$(1\underset{G}{\times}\pi)^*\colon U^*(EG\underset{G}{\times}M)\to U^*(EG\underset{G}{\times}F)$$

is injective.

We now would like to define a multiplicative transformation

$$ch_{Gy}\colon K_G(M)\to U^{**}(EG\underset{G}{\times}M)\otimes Q[[y]]$$

which might be called an equivariant generalized Chern character in U^*-theory. For this purpose, we introduce the formal group law $F(x,y)$ in complex cobordism [1]. Generally it is known that there exists a unique formal power series $g(t)$ satisfying

$$F(x,y) = g^{-1}(g(x)+g(y)) .$$

In fact $g(t)$ is given explicitly by

$$g(t) = \sum_{n\geq 0} \frac{[\boldsymbol{CP}^n]}{n+1} t^{n+1} \in U^*(pt)\otimes Q[[t]]$$

called Mischenko series [35] [39]. Let x be an element of $K_G(M)$, then x can be written as $[\xi_1]$-$[\xi_2]$ for some complex G-vector bundles ξ_1, ξ_2 on M. Then we define:

$$ch_{Gy}(x) = \sum_j \exp\{(1+y)g(t_{1,j})\} - \sum_k \exp\{(1+y)g(t_{2,k})\}$$

where the total Chern class of $EG\times_G \xi_1$ (resp. $EG\times_G \xi_2$) is expressed formally as $\prod_j (1+t_{1,j})$ (resp. $\prod_k (1+t_{2,k})$). It is easily seen that the definition does not depend on the choices of ξ_i.

Lemma 4.3 i) $ch_{Gy}(x_1+x_2)=ch_{Gy}(x_1)+ch_{Gy}(x_2)$
ii) $ch_{Gy}(x_1\otimes x_2)=ch_{Gy}(x_1)\cdot ch_{Gy}(x_2)$.

Proof. The proof of i) is trivial. The multiplicative property (ii) follows from Lemma 4.2 and the equation

$$g(c_1(EG\times_G(\xi\otimes\eta))) = g(c_1(EG\times_G \xi))+g(c_1(EG\times_G \eta))$$

for complex G-line bundles ξ, η.

Next we define our equivariant U^*-theory generalized Todd genus as follows. Let ξ be a complex G-vector bundle over a finite CW-complex M. Then we set

$$T_{Gy}(\xi) = \prod_k \left[\frac{t_k\{\exp((1+y)g(t_k))+y\}}{\exp((1+y)g(t_k))-1}\right] \in U^*(EG\times_G M)\otimes Q[[y]] ,$$

where the total U^*-theory Chern class of $EG\times_G \xi$ is written formally as $\prod(1+t_k)$. Obviously we have

$$T_{Gy}(\xi\oplus\eta) = T_{Gy}(\xi)\cdot T_{Gy}(\eta) .$$

A weakly complex manifold M of class C^∞ is a differentiable manifold of class C^∞ together with a complex vector bundle structure on $TM\oplus\underline{\boldsymbol{R}}^\varepsilon$ where $\varepsilon=1$ or 2 according as dim $M\equiv 1$ or 0 mod 2 and TM denotes the tangent bundle.

A weakly complex G-action of class C^∞ is a G-action of class C^∞ on a weakly complex manifold of class C^∞ preserving the complex structure. Here the G-action on $\boldsymbol{R}^\varepsilon$ is given trivially.

In this category, we must modify the definitions in §2 as follows. The orientation of M is defined by making use of the complex G-vector bundle $TM\oplus\underline{\boldsymbol{R}}^{\varepsilon}$ in stead of the tangent bundle TM. In order to define our equivariant Gysin homomorphism, we choose a sufficiently large complex G-representation space V such that the normal bundle ν in §2 has a complex vector bundle structure so that we have a complex G-vector bundle isomorphism

$$TM\oplus\underline{\boldsymbol{R}}^{\varepsilon}\oplus\nu\cong f^{*}(TN\oplus\underline{\boldsymbol{R}}^{\varepsilon})\oplus\underline{V}\,,$$

where $TM\oplus\underline{\boldsymbol{R}}^{\varepsilon}$ (resp. $TN\oplus\underline{\boldsymbol{R}}^{\varepsilon}$) denotes the complex G-vector bundle representing the weakly complex G-manifold M (resp N).

We are now ready to formulate our equivariant Riemann-Roch theorem in U^{*}-theory.

Theorem 4.4. *Let M and N be weakly complex G-manifolds of class C^{∞} such that* $\dim M\equiv\dim N \bmod 2$. *For a G-map $f\colon M\to N$, we have a functorial homomorphism $\tilde{f}\colon K_{G}(M)[[y]]\to K_{G}(N)[[y]]$ such that*

$$f_{!}^{U}(ch_{Gy}(x)\cdot T_{Gy}(TM\oplus\underline{\boldsymbol{R}}^{\varepsilon}))=ch_{Gy}(\tilde{f}(x))\cdot T_{Gy}(TN\oplus\underline{\boldsymbol{R}}^{\varepsilon})\,,$$

where $\varepsilon=1$ or 2 according as $\dim M\equiv 1$ *or* $0 \bmod 2$ *and $TM\oplus\underline{\boldsymbol{R}}^{\varepsilon}$* (resp. $TN\oplus\underline{\boldsymbol{R}}^{\varepsilon}$) *denotes the complex G-vector bundle representing the weakly complex G-manifold M (resp. N) and the action on $\underline{\boldsymbol{R}}^{\varepsilon}$ is given trivially. In fact $\tilde{f}(x)$ is given by*

$$\begin{aligned}(-1)^{(\dim M-\dim N)/2}f_{!}^{KG}\{&x\otimes\lambda^{(\dim M+\varepsilon)/2}(TM\oplus\underline{\boldsymbol{R}}^{\varepsilon})\otimes\\&\lambda_{y}(\overline{TM\oplus\underline{\boldsymbol{R}}^{\varepsilon}})\}\otimes\lambda^{(\dim N+\varepsilon)/2}(\overline{TN\oplus\underline{\boldsymbol{R}}^{\varepsilon}})\otimes S_{-y}(\overline{TN\oplus\underline{\boldsymbol{R}}^{\varepsilon}})\,,\end{aligned}$$

where

$$\lambda_{y}(\xi)=\sum_{i=0}y^{i}\lambda^{i}(\xi),\ \lambda^{i}(\xi)\colon \textit{exterior powers},$$

$$S_{y}(\xi)=\sum_{i=0}y^{i}S^{i}(\xi),\ S^{i}(\xi)\colon \textit{symmetric powers},$$

and $\bar{\xi}$ denotes the complex conjugate bundle of ξ.

Proof. Let ϕ_{i}^{KG}, ϕ_{i}^{K} and ϕ_{i}^{U} be the homomorphisms in §2 in the cases where $h_{G}(M)$ are $K_{G}(M)$, $K(EG\underset{G}{\times}M)$ and $U(EG\underset{G}{\times}M)$ respectively. Let $\alpha\colon K_{G}(M)\to K(EG\underset{G}{\times}M)$ be the natural transformation defined by $\alpha(x)=EG\underset{G}{\times}x$. By the choices of Thom classes [6] [17], we have that $\alpha\phi_{i}^{KG}=\phi_{i}^{K}\alpha$. It follows that $\alpha f_{!}^{KG}=f_{!}^{K}\alpha$.

In virtue of our splitting principle (Lemma 4.2), we have

Lemma 4.5. $\alpha(\lambda_{y}(x))\cdot\alpha(S_{-y}(x))=1$ *for* $x\in K_{G}(M)$, $\alpha(\lambda_{y}(\xi-\eta))=\alpha(\lambda_{y}(\xi)\cdot S_{-y}(\eta))$ *for complex G-vector bundles ξ, η.*

When we write the total U^*-theory Chern class of $EG\underset{G}{\times}\nu$ formally as $\prod(1+x_i)$, $T_{Gy}(\nu)$ is given by

$$\prod\left(\frac{x_i\{\exp((1+y)g(x_i))+y\}}{\exp((1+y)g(x_i))-1}\right).$$

As in §2 of [10], we have

$$(\phi_1^U)^{-1}ch_{Gy}\phi_1^{KG}(1)=\prod\left(\frac{1-\exp(1+y)g(x_i)}{x_i}\right).$$

Hence we have for $x\in K_G(M)$,

$$\begin{aligned}
&\phi_1^U(ch_{Gy}(x)\cdot T_{Gy}(\nu)^{-1})\\
&=\phi_1^U\left(\frac{(-1)^{\dim\nu}ch_{Gy}(x)}{\prod\{\exp((1+y)g(x_i))+y\}}\cdot\prod\left(\frac{1-\exp(1+y)g(x_i)}{x_i}\right)\right)\\
&=\frac{(-1)^{\dim\nu}ch_{Gy}(x)}{\prod\{\exp((1+y)g(x_i))+y\}}\,\phi_1^U\left(\prod\left(\frac{1-\exp(1+y)g(x_i)}{x_i}\right)\right)\\
&=\frac{(-1)^{\dim\nu}ch_{Gy}(x)}{\prod\{\exp((1+y)g(x_i))+y\}}\cdot ch_{Gy}\phi_1^{KG}(1)\,.
\end{aligned}$$

According to Lemma 4.2, we may assume that

$$\alpha(\nu)=L_1\oplus\cdots\oplus L_k \text{ and } c_1(L_i)=x_i\,,$$

where L_i are complex G-line bundles and $k=\dim\nu$. Then we have

$$\begin{aligned}
\alpha\lambda^{\dim\nu}(\nu)&=\lambda^{\dim\nu}\alpha(\nu)\\
&=L_1\otimes\cdots\otimes L_k\,.
\end{aligned}$$

It follows from the fundamental property of the formal power series $g(t)$ that

$$\begin{aligned}
&g(c_1(\alpha(\lambda^{\dim\nu}(\nu))))\\
&=g(c_1(L_1\otimes\cdots\otimes L_k))\\
&=\sum_i g(c_1(L_i))\\
&=\sum_i g(x_i)\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&g(c_1(\alpha(\lambda^{\dim\nu}(\bar{\nu}))))=g(c_1(\overline{\alpha(\lambda^{\dim\nu}(\nu))}))\\
&=-g(c_1(\alpha(\lambda^{\dim\nu}(\nu))))=-\sum g(x_i)\,.
\end{aligned}$$

Accordingly we have

$$\begin{aligned}
&ch_{Gy}(\lambda^{\dim\nu}(\bar{\nu}))=\exp(1+y)g(c_1(\alpha(\lambda^{\dim\nu}(\bar{\nu}))))\\
&=\frac{1}{\prod\exp(1+y)g(x_i)}\,.
\end{aligned}$$

On the other hand, note that

$$\begin{aligned}&\alpha\lambda^i(\bar{\nu}) = \lambda^i\alpha(\bar{\nu})\\ &= \lambda^i(\bar{L}_1\oplus\cdots\oplus\bar{L}_k)\\ &= \bigoplus_{j_1<\cdots<j_i}(\bar{L}_{j_1}\otimes\cdots\otimes\bar{L}_{j_i})\,.\end{aligned}$$

Hence we have

$$\begin{aligned}&ch_{Gy}\lambda^i(\bar{\nu}) = ch_y\alpha\lambda^i(\bar{\nu})\\ &= \sum_{j_1<\cdots<j_i}\exp(1+y)g(c_1(\bar{L}_{j_1}\otimes\cdots\otimes\bar{L}_{j_i}))\\ &= \sum_{j_1<\cdots<j_i}\exp(1+y)(g(c_1(\bar{L}_{j_1}))+\cdots+g(c_1(\bar{L}_{j_i})))\\ &= \sum_{j_1<\cdots<j_i}\exp(1+y)(-g(x_{j_1})-\cdots-g(x_{j_i}))\,.\end{aligned}$$

It follows that

$$\begin{aligned}&ch_{Gy}\lambda_y(\bar{\nu}) = ch_{Gy}(\sum y^i\lambda^i(\bar{\nu}))\\ &= \sum_i y^i\sum_{j_1<\cdots<j_i}\exp(1+y)(-g(x_{j_1})-\cdots-g(x_{j_i}))\\ &= \prod_i\{1+y\exp(1+y)(-g(x_i))\}\,.\end{aligned}$$

In view of Lemma 4.5, we have

$$\begin{aligned}&ch_{Gy}S_{-y}(\bar{\nu}) = ch_y\alpha S_{-y}(\bar{\nu})\\ &= \frac{1}{ch_y\alpha\lambda_y(\bar{\nu})} = \frac{1}{ch_{Gy}\lambda_y(\bar{\nu})}\,.\end{aligned}$$

Putting all this together, we have

$$\begin{aligned}&ch_{Gy}(\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu})) = ch_{Gy}\lambda^{\dim\nu}(\bar{\nu})\cdot ch_{Gy}S_{-y}(\bar{\nu})\\ &= \frac{ch_{Gy}\lambda^{\dim\nu}(\bar{\nu})}{ch_{Gy}\lambda_y(\bar{\nu})} = \frac{1}{\prod\{\exp((1+y)g(x_i))+y\}}\,.\end{aligned}$$

Thus we have for $x\in K_G(M)$,

$$\begin{aligned}&\phi_1^U(ch_{Gy}(x)\cdot T_{Gy}(\nu)^{-1})\\ &= ch_{Gy}(x)\cdot ch_{Gy}((-1)^{\dim\nu}\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu}))\cdot ch_{Gy}\phi_1^{K_G}(1)\\ &= ch_{Gy}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu})\cdot\phi_1^{K_G}(1))\\ &= ch_{Gy}\phi_1^{K_G}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu}))\,.\end{aligned}$$

Now we show the following formula

$$\begin{aligned}&f_!^U(ch_{Gy}(x)\cdot T_{Gy}(\nu)^{-1})\\ &= \prod\left(\frac{1-\exp(1+y)g(y_i)}{y_i}\right)ch_{Gy}f_!^{K_G}(x\cdot(-1)^{\dim\nu}\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu}))\,,\end{aligned}$$

where the total U^*-theory Chern class of the bundle $EG\underset{G}{\times}(N\times V)\to EG\underset{G}{\times}N$ is expressed formally as $\prod(1+y_i)$.

As before we have

$$\phi_3^U ch_{Gy}(\phi_3^{KG})^{-1}(1)=\prod\left(\frac{1-\exp(1+y)g(y_i)}{y_i}\right).$$

It follows that

$$\begin{aligned}
&f_!^U(ch_{Gy}(x)\cdot T_{Gy}(\nu)^{-1})\\
&=\phi_3^U\cdot\phi_2^U ch_{Gy}\phi_1^{KG}\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu))\\
&=\phi_3^U ch_{Gy}\phi_2^{KG}\cdot\phi_1^{KG}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu))\\
&=\phi_3^U ch_{Gy}(\phi_3^{KG})^{-1}f_!^{KG}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu))\\
&=\phi_3^U ch_{Gy}((\phi_3^{KG})^{-1}(1)\cdot f_!^{KG}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu)))\\
&=\phi_3^U(ch_{Gy}(\phi_3^{KG})^{-1}(1)\cdot ch_{Gy}f_!^{KG}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu)))\\
&=(\phi_3^U ch_{Gy}(\phi_3^{KG})^{-1}(1))\cdot ch_{Gy}f_!^{KG}(x\cdot(-1)^{\dim\nu}\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu))\\
&=\prod\left(\frac{(1-\exp(1+y)g(y_i)}{y_i}\right)ch_{Gy}f_!^{KG}(x\cdot(-1)^{\dim\nu}\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu)),
\end{aligned}$$

which is the required formula.

Since the genus T_{Gy} is multiplicative and invertible, we have

$$T_{Gy}(\nu)^{-1}=T_{Gy}(TM\oplus\underline{R}^{\varepsilon})\cdot f^*(T_{Gy}(\underline{V})^{-1}\cdot T_{Gy}(TN\oplus\underline{R}^{\varepsilon})^{-1}).$$

It follows from Lemma 2.2 and the above formula that

$$\begin{aligned}
&f_!^U(ch_{Gy}(x)\cdot T_{Gy}(TM\oplus\underline{R}^{\varepsilon}))\\
&=\prod\left(\frac{1-\exp(1+y)g(y_i)}{y_i}\right)ch_{Gy}f_!^{KG}(x\cdot(-1)^{\dim\nu}\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu))\\
&\quad\times T_{Gy}(\underline{V})\cdot T_{Gy}(TM\oplus\underline{R}^{\varepsilon}).
\end{aligned}$$

As before we have

$$\begin{aligned}
&\prod\left(\frac{1-\exp(1+y)g(y_i)}{y_i}\right)\cdot T_{Gy}(\underline{V})\\
&=(-1)^{\dim V}\prod\{\exp((1+y)g(y_i))+y\}\\
&=(-1)^{\dim V}\cdot ch_{Gy}(\lambda^{\dim V}(\underline{V})\cdot\lambda_y(\underline{V})).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&f_!^U(ch_{Gy}(x)\cdot T_{Gy}(TM\oplus\underline{R}^{\varepsilon})\\
&=ch_{Gy}[f_!^{KG}\{(-1)^{(\dim M-\dim N)/2}\cdot x\cdot\lambda^{\dim\nu}(\bar\nu)\cdot S_{-y}(\bar\nu)
\end{aligned}$$

$$\otimes\lambda^{\dim V}(\underline{V})\cdot\lambda_y(\overline{\underline{V}})\}]\cdot T_{Gy}(TN\oplus\underline{R}^\varepsilon)\,.$$

We now wish to have a formula without V and ν. Note that there is an isomorphism

$$\overline{TM\oplus\underline{R}^\varepsilon}\oplus\bar{\nu}\cong f^*(\overline{TN\oplus\underline{R}^\varepsilon})\oplus\overline{\underline{V}}$$

of complex G-vector bundles. By making use of this and Lemma 4.5, we have

$$\begin{aligned}
&\alpha(\lambda_y(\overline{\underline{V}})\cdot S_{-y}(\bar{\nu}))\\
&=\alpha\lambda_y(\overline{\underline{V}}-\bar{\nu})\\
&=\alpha\lambda_y(\overline{TM\oplus\underline{R}^\varepsilon}-f^*(\overline{TN\oplus\underline{R}^\varepsilon}))\\
&=\alpha(\lambda_y(\overline{TM\oplus\underline{R}^\varepsilon})\cdot S_{-y}f^*(\overline{TN\oplus\underline{R}^\varepsilon}))\,.
\end{aligned}$$

Note that

$$\alpha(\lambda^{\dim\nu}(\nu))^{-1}=\alpha\lambda^{\dim\nu}(\bar{\nu})$$

and

$$\begin{aligned}
&\alpha(\lambda^{(\dim N+\varepsilon)/2}f^*(TN\oplus\underline{R}^\varepsilon))^{-1}\\
&=\alpha(\lambda^{(\dim N+\varepsilon)/2}f^*(\overline{TN\oplus\underline{R}^\varepsilon}))\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\alpha(\lambda^{\dim\nu}(\bar{\nu})\cdot\lambda^{\dim V}(\underline{V}))\\
&=\alpha(\lambda^{(\dim M+\varepsilon)/2}(TM\oplus\underline{R}^\varepsilon)\cdot\lambda^{(\dim N+\varepsilon)/2}(f^*(\overline{TN\oplus\underline{R}^\varepsilon})))\,.
\end{aligned}$$

Putting all this together, we have

$$\begin{aligned}
&f_!^U(ch_{Gy}(x)\cdot T_{Gy}(TM\oplus\underline{R}^\varepsilon))\\
&=ch_y\alpha f_!^{K_G}\{(-1)^{(\dim M-\dim N)/2}\cdot x\cdot\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu})\\
&\quad\otimes\lambda^{\dim V}(\underline{V})\cdot\lambda_y(\overline{\underline{V}})\}\cdot T_{Gy}(TN\oplus\underline{R}^\varepsilon)\\
&=ch_yf_!^K\alpha\{(-1)^{(\dim M-\dim N)/2}\cdot x\cdot\lambda^{\dim\nu}(\bar{\nu})\cdot S_{-y}(\bar{\nu})\\
&\quad\otimes\lambda^{\dim V}(\underline{V})\cdot\lambda_y(\overline{\underline{V}})\}\cdot T_{Gy}(TN\oplus\underline{R}^\varepsilon)\\
&=ch_yf_!^K\alpha\{(-1)^{(\dim M-\dim N)/2}\cdot x\cdot\lambda^{(\dim M+\varepsilon)/2}(TM\oplus\underline{R}^\varepsilon)\\
&\quad\otimes\lambda^{(\dim N+\varepsilon)/2}(f^*(\overline{TN\oplus\underline{R}^\varepsilon}))\cdot\lambda_y(\overline{TM\oplus\underline{R}^\varepsilon})\cdot S_{-y}f^*(\overline{TN\oplus\underline{R}^\varepsilon})\}\\
&\quad\times T_{Gy}(TN\oplus\underline{R}^\varepsilon)\\
&=ch_{Gy}[f_!^{K_G}((-1)^{(\dim M-\dim N)/2}\cdot x\cdot\lambda^{(\dim M+\varepsilon)/2}(TM\oplus\underline{R}^\varepsilon)\\
&\quad\otimes\lambda_y(\overline{TM\oplus\underline{R}^\varepsilon}))\cdot\lambda^{(\dim N+\varepsilon)/2}(\overline{TN\oplus\underline{R}^\varepsilon})\cdot S_{-y}(\overline{TN\oplus\underline{R}^\varepsilon})]\\
&\quad\times T_{Gy}(TN\oplus\underline{R}^\varepsilon)\,.
\end{aligned}$$

This is the required formula. The proof that $\widetilde{gf}=\tilde{g}\cdot\tilde{f}$ is easy and omitted.

This makes the proof of Theorem 4.4 complete.

REMARK 4.6. If we use the conjugate Thom class $\overline{t(\xi)}$ instead of $t(\xi)$, $\tilde{f}(x)$ is given by

$$f_!^{K_G}(x\cdot\lambda_y(\overline{TM\oplus\underline{R}^s}))\otimes S_{-y}(\overline{TN\oplus\underline{R}^s})\,.$$

5. G-T_y-genus formula

In this section we shall derive a formula by combining the localization Theorem 3.3 and the equivariant Riemann-Roch Theorem 4.4. For simplicity we assume that G is a toral group T^n in this section.

We first calculate the multiplicative set S defined in §3. Let $\boldsymbol{\rho}=(\rho_1,\cdots,\rho_n)$ be an n-tuple of integers. As in §3, $\boldsymbol{\rho}$ can be regarded as an irreducible representation

$$\boldsymbol{\rho}\colon T^n\to S^1=U(1)\,.$$

Let $\gamma_i\colon T^n\to S^1=U(1)$ be the irreducible representation defined by $\gamma_{i1}(s_1,\cdots,s_n)=s_i$ and $\Gamma_i\to BT^n$ be the γ_i extension of the universal principal T^n-bundle $ET^n\to BT^n$ as in §3. Denote by t_i the U^*-theory Euler class of the bundle Γ_i. Then we have

Lemma 5.1. *The set S is generated multiplicatively by $g^{-1}(\sum\rho_i g(t_i))$ with $\rho_i\in Z$, $\sum\rho_i^2\neq 0$.*

Proof. The $\boldsymbol{\rho}$ extension of the universal bundle $ET^n\to BT^n$ is $\prod\Gamma_i^{\rho_i}$. Note that $g(c_1(\prod\Gamma_i^{\rho_i}))=\sum\rho_i g(t_i)$.
Hence we have

$$c_1(\textstyle\prod\Gamma_i^{\rho_i})=g^{-1}(\sum\rho_i g(t_i))\,.$$

Since any complex representation of T^n is a sum of such representations $\boldsymbol{\rho}$, the set S is generated multiplicatively by $g^{-1}(\sum\rho_i g(t_i))$ with $\rho_i\in Z$, $\sum\rho_i^2\neq 0$. This completes the proof of Lemma 5.1.

We next show that Assumption 3.1 is satisfied in this case. It is shown in Conner-Floyd [16] that F_μ has a canonical weakly complex structure and N_μ has a canonical complex G-vector bundle structure. Therefore it suffices to show that each Euler class $\chi_{T^n}(N_\mu)$ is a unit in $S^{-1}U^*(BT^n\times F_\mu)$. According to Atiyah-Segal [6], N_μ has the following decomposition

$$N_\mu=\sum_{\boldsymbol{\rho}}E_{\mu\boldsymbol{\rho}}\otimes V_{\boldsymbol{\rho}}\,,$$

where $\boldsymbol{\rho}$ run through the complex irreducible representations and $V_{\boldsymbol{\rho}}$ denote their representation spaces and $E_{\mu\boldsymbol{\rho}}$ denote complex vector bundles.

Lemma 5.2.

$$c(ET^n\underset{T^n}{\times}N_\mu)=\prod_{\rho,\rho_j}\{1+g^{-1}(\sum_i \rho_i g(t_i)+g({}_\mu x_{\rho_j}))\}\ ,$$
$$\chi(ET^n\underset{T^n}{\times}N_\mu)=\prod_{\rho,\rho_j} g^{-1}(\sum_i \rho_i g(t_i)+g({}_\mu x_{\rho_j}))\ .$$

Here the total U^*-theory Chern class of $E_{\mu\rho}$ is expressed formally as $\prod_{\rho_j}(1+{}_\mu x_{\rho_j})$.

Proof. As shown in §3, we have

$$ET^n\underset{T^n}{\times}N=\sum_\rho(\prod\Gamma_i^{\rho_i})\hat{\otimes}E_{\mu\rho}\ .$$

In virtue of Lemma 4.2, we may assume that $E_{\mu\rho}$ is a sum of complex line bundles $E^j_{\mu\rho}$. If we denote the U^*-theory first Chern class of $E^j_{\mu\rho}$ by ${}_\mu x_{\rho_j}$, we have that

$$c_1\{(\prod\Gamma_i^{\rho_i})\otimes E^j_{\mu\rho}\}=g^{-1}\{\sum_i\rho_i g(t_i)+g({}_\mu x_{\rho_j})\}\ .$$

This completes the proof of Lemma 5.2.

Generally the formal group law $F(x,y)$ has the property:

$$F(x,y)=x+y+\sum_{i,j\geq 1}a_{ij}x^iy^j$$

Put $t_\rho=c_1(\prod\Gamma_i^{\rho_i})$. Then $t_\rho\in S$ and

$$\chi(ET^n\underset{T^n}{\times}N_\mu)=\prod_{\rho,\rho_j}(t_\rho+{}_\mu x_{\rho_j}+\sum_{k,l\geq 1}a_{kl}t_\rho^k{}_\mu x_{\rho_j}^l)$$
$$=\prod_{\rho,\rho_j}t_\rho\left\{1+\frac{{}_\mu x_{\rho_j}(1+\sum_{k,l\geq 1}a_{kl}t_\rho^k{}_\mu x_{\rho_j}^{l-1})}{t_\rho}\right\}.$$

Consider the formal inverse of $\chi(ET^n\underset{T^n}{\times}N_\mu)$:

$$\chi(ET^n\underset{T^n}{\times}N_\mu)^{-1}$$
$$=\prod_{\rho,\rho_j}t_\rho^{-1}\left\{1+\sum_{m=1}^{\infty}(-1)^m\frac{{}_\mu x_{\rho_j}^m(1+\sum_{k,l\geq 1}a_{kl}t_\rho^k{}_\mu x_{\rho_j}^{l-1})^m}{t_\rho^m}\right\}.$$

Since the dimension of F_μ is smaller than or equal to $\dim M$,

$${}_\mu x_{\rho_j}^m=0\qquad\text{for } m>\left[\frac{\dim M}{2}\right].$$

Therefore the formal inverse of $\chi(ET^n\underset{T^n}{\times}N_\mu)$ has a meaning in the localized ring $S^{-1}U^*(BT^n\times F_\mu)$ and is given by

$$\prod_{\rho,\rho_j}t_\rho{}^{-[\dim M/2]-1}\{\sum_{m=0}^{[\dim M/2]}(-1)^m t_\rho{}^{[\dim M/2]-m}{}_\mu x_{\rho_j}^m(1+\sum_{k,l\geq 1}a_{kl}t_\rho^k{}_\mu x_{\rho_j}^{l-1})^m\}\ .$$

Thus we have shown that $\chi(ET^n\underset{T^n}{\times}N_\mu)$ is a unit in

$$S^{-1}U^*(BT^n \times F_\mu).$$

We can now apply Theorem 3.3 and get

$$S^{-1}\operatorname{Ind} x = S^{-1}\operatorname{Ind}\Big(\sum \frac{i_\mu^* x}{\chi(ET^n \underset{T^n}{\times} N_\mu)}\Big)$$

for any element x of $S^{-1}U^*(ET^n \underset{T^n}{\times} M)$.

If we take $T_{Gy}(TM \oplus \underline{\boldsymbol{R}}^s)$ as x, we have

$$\begin{aligned} &S^{-1}\operatorname{Ind} T_{Gy}(TM \oplus \underline{\boldsymbol{R}}^s) \\ &= S^{-1}\operatorname{Ind} \sum_\mu T_y(TF_\mu \oplus \underline{\boldsymbol{R}}^s) \prod_{\rho,\rho_j} \left[\frac{\exp[(1+y)\{\sum_i \rho_i g(t_i) + g(\mu x_{\rho_j})\}] + y}{\exp[(1+y)\{\sum_i \rho_i g(t_i) + g(\mu x_{\rho_j})\}] - 1}\right]. \end{aligned}$$

Here $TF_\mu \oplus \underline{\boldsymbol{R}}^s$ denotes the canonical weakly complex structure on F_μ given in [16].

The following lemma is easy to prove and we omit the proof.

Lemma 5.3. *The local index*

$$S^{-1}\operatorname{Ind}\colon S^{-1}\sum U^*(BT^n \times F_\mu) \to S^{-1}U^*(BT^n)$$

is induced by the U^*-theory slant product
$/[F_\mu]\colon U^*(BT^n \times F_\mu) \to U^*(BT^n)$ *where $[F_\mu]$ denotes the fundamental class of F_μ.*

From Theorem 4.4 we know that $\operatorname{Ind} T_{Gy}(TM \oplus \underline{\boldsymbol{R}}^s)$ has the form $ch_{Gy}(\tilde{f}(x))$ where $f\colon M \to pt$.

Let $\alpha\colon U^*(X) \to H^*(X)$ be the natural transformation. Then α induces

$$\alpha \hat{\otimes} id\colon U^{**}(X) \hat{\otimes} Q[[y]] \to H^{**}(X) \hat{\otimes} Q[[y]]$$

which is also denoted by α, where $\hat{\otimes}$ denotes the completed tensor product.

Denote by t_i' the ordinary first Chern class of the bundle Γ_i. Then $\alpha(t_i) = t_i'$. Since α is multiplicative and since $\alpha[CP^n] = 0$ for $n > 0$, we have

$$\begin{aligned} \alpha g(t_i) &= \sum_{n \geq 0} \frac{\alpha[\boldsymbol{CP}^n]}{n+1} \alpha(t_i^{n+1}) \\ &= \alpha(t_i) = t_i'. \end{aligned}$$

Note that $ch_{Gy}(\tilde{f}(x))$ has the form

$$\sum_{\boldsymbol{b}} a(\boldsymbol{b}) \exp(1+y)\,(\sum_i b_i g(t_i))$$

where $\boldsymbol{b} = (b_1, \cdots, b_n)$, $b_i \in Z$ and $a(\boldsymbol{b}) \in Z[y]$ and the summation is taken finitely. By applying α to the formula before, we have the following formula in the ordinary cohomology theory

$$(*)\qquad \sum_{\boldsymbol{b}} a(\boldsymbol{b})\exp(1+y)\,(\sum_i b_i t_i')$$
$$= \sum_{\mu} \alpha(T_y(TF\oplus \underline{R}^{\varepsilon}))\prod_{\rho,\rho_j}\left[\frac{\exp[(1+y)\{\sum_i \rho_i t_i'+\alpha(_{\mu}x_{\rho_j})\}]+y}{\exp[(1+y)\{\sum_i \rho_i t_i'+\alpha(_{\mu}x_{\rho_j})\}]-1}\right]/[F_{\mu}]\,,$$

where $/[F_\mu]$ denotes also the slant product in the ordinary cohomology theory. This equation holds even if we regard both terms as functions of t_i'. Let $\{\boldsymbol{\rho}\}$ be the set of irreducible representations which occur as normal representations of the fixed point set. Let $\boldsymbol{\tau}=(\tau_1, \cdots, \tau_n)$ be an n-tuple of integers such that $\langle \boldsymbol{\tau}, \boldsymbol{\rho}\rangle = \sum \tau_i \rho_i \neq 0$ for any $\boldsymbol{\rho}$ in $\{\boldsymbol{\rho}\}$.
Set $t_i'=t\tau_i$. Then

$$\lim_{t\to\infty} \langle t\boldsymbol{\tau}, \boldsymbol{\rho}\rangle = \operatorname{sign}\langle \boldsymbol{\tau}, \boldsymbol{\rho}\rangle \infty\,.$$

If $\operatorname{sign}\langle \boldsymbol{\tau}, \boldsymbol{\rho}\rangle = +1$ (resp. -1), then

$$\lim_{t\to\infty} \frac{\exp[(1+y)\{\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle t+\alpha(_{\mu}x_{\rho_j})\}]+y}{\exp[(1+y)\{\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle t+\alpha(_{\mu}x_{\rho_j})\}]-1} = 1 \qquad (\text{resp. } -y)\,.$$

We now define integers $d_{\tau}^{\pm}(F_\mu)$ by

$$d_{\tau}^{+}(F_\mu) = \sum_{\langle \tau,\rho\rangle>0} \dim_C E_{\mu\rho}\,.$$

and

$$d_{\tau}^{-}(F_\mu) = \sum_{\langle \tau,\rho\rangle<0} \dim_C E_{\mu\rho}\,.$$

Then we have

$$\lim_{t\to\infty} \sum_{\mu} \alpha(T_y(TF_\mu\oplus \underline{R}^{\varepsilon}))\prod_{\rho,\rho_j}\left[\frac{\exp[(1+y)\{\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle t+\alpha(_{\mu}x_{\rho_j})\}]+y}{\exp[(1+y)\{\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle t+\alpha(_{\mu}x_{\rho_j})\}]-1}\right]/[F_{\mu}]$$
$$= \sum_{\mu} (-y)^{d_{\tau}^{-}(F_\mu)}\alpha(T_y(TF_\mu\oplus \underline{R}^{\varepsilon}))\,[F_\mu]\,.$$

Namely the local index tends to a finite number. This fact is valid for any $\boldsymbol{\tau}$ with $\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle \neq 0$, $\boldsymbol{\rho}\in\{\boldsymbol{\rho}\}$.

Similarly we have

$$\lim_{t\to-\infty} \sum_{\mu} \alpha(T_y(TF_y\oplus \underline{R}^{\varepsilon}))\prod_{\rho,\rho_j}\left[\frac{\exp[(1+y)\{\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle t+\alpha(_{\mu}x_{\rho_j})\}]+y}{\exp[(1+y)\{\langle \boldsymbol{\tau},\boldsymbol{\rho}\rangle t+\alpha(_{\mu}x_{\rho_j})\}]-1}\right]/[F_{\mu}]$$
$$= \sum_{\mu} (-y)^{d_{\tau}^{+}(F_\mu)}\alpha(T_y(TF_\mu\oplus \underline{R}^{\varepsilon}))\,[F_\mu]\,.$$

Since the global index is a finite sum of the form $a(\boldsymbol{b})\exp(1+y)\,\langle \boldsymbol{b},\boldsymbol{\tau}\rangle t$, the local property above means that the global index must be independent of t_i'. It follows that

$$a(\boldsymbol{b}) = 0 \qquad \text{for } \boldsymbol{b}\neq \boldsymbol{0}\,.$$

Thus we can conclude that the global index in U^*-theory is also a constant, which belongs to $Z[y]$. Hence we have

$$T_y(TM\oplus\underline{R}^s)[M] = \alpha(T_y(TM\oplus\underline{R}^s))[M]$$

which gives the constant above.

Consequently we have shown the following G-T_y-genus formula.

Theorem 5.4.

$$T_y(TM\oplus R^s)[M]$$
$$= \sum_\mu T_y(TF_\mu\oplus\underline{R}^s)\prod_{\rho,\rho_j}\left[\frac{\exp[(1+y)\{\sum_i \rho_i g(t_i)+g(\mu x_{\rho_j})\}]+y}{\exp[(1+y)\{\sum_i \rho_i g(t_i)+g(\mu x_{\rho_j})\}]-1}\right]/[F_\mu]\,.$$

In the proof of Theorem 5.4, we have shown the following corollary which is so called Kosniowski formula.

Corollary 5.5 ([30], [24], [27]). *We have*

$$T_y(TM\oplus\underline{R}^s)[M] = \sum_\mu (-y)^{d_\tau^+(F_\mu)}T_y(TF_\mu\oplus\underline{R}^s)[F_\mu]$$
$$= \sum_\mu (-y)^{d_\tau^-(F_\mu)}T_y(TF_\mu\oplus\underline{R}^s)[F_\mu]$$

for any τ *with* $\langle\tau,\rho\rangle\neq 0$, $\rho\in\{\rho\}$.

REMARK 5.6. Our Theorem 5.4 means that

$$\mathrm{Ind}\, T_{Gy}(TM\oplus\underline{R}^s)\in U^0(*)\otimes Q[y]\subset U^{**}(BT^n)\hat{\otimes}Q[[y]]\,.$$

Namely the evaluation of $T_{Gy}(TM\oplus\underline{R}^s)$ on $[M]$ vanishes except for the term of dim M.

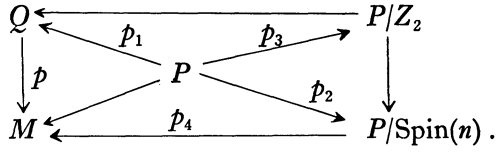
6. Spin^c-G-actions

Let π: Spin $(n)\to SO(n)$ be the standard double covering. Then Spin$^c(n)$ is the subgroup of Spin$(n+2)$ defined by $\pi^{-1}(SO(n)\times SO(2))$, which is isomorphic to $\mathrm{Spin}(n)\times_{Z_2} S^1$ as Lie groups. Let Q be a $SO(n)$-bundle over M. A Spin$^c(n)$-reduction of Q is a principal Spin$^c(n)$-bundle P over M together with an S^1-bundle p_1: $P\to Q$ such that the diagram

$$\begin{array}{ccc} P\times \mathrm{Spin}^c(n) & \longrightarrow & P \\ \downarrow{\scriptstyle p_1\times\lambda} & & \downarrow{\scriptstyle p_1} \\ Q\times SO(n) & \longrightarrow & Q \end{array} \qquad \begin{array}{l} \searrow \\ \\ \nearrow \end{array} M$$

commutes, where the horizontal arrows are the actions of Spin$^c(n)$ and $SO(n)$

from the right on P and Q respectively and λ is the composite homomorphism $\mathrm{Spin}^c(n) \xrightarrow{\pi} SO(n) \times SO(2) \to SO(n)$. The action of $\mathrm{Spin}^c(n)$ on P yields the actions of Z_2, S^1, $\mathrm{Spin}(n)$ by the natural inclusions and we have the following commutative diagram:



All maps are bundle projections and we have

Lemma 6.1. *p_4: $P/\mathrm{Spin}(n) \to M$ is the S^1-bundle characterizing the $\mathrm{Spin}^c(n)$-bundle $P \to M$ in the sense of* [38] *and P/Z_2 is nothing but the fiber product $Q \underset{M}{\times} (P/\mathrm{Spin}(n))$ where p_3 is given by $p_1 \times p_2$ in this case.*

Proof. see [38].

Suppose that the bundle $Q \to M$ admits a G-action (commuting with the action of $SO(n)$ from the right) which is compatible with the projection map. Then a lifting of the action on Q to an action on P is an action of G on P (commuting with the action of $\mathrm{Spin}^c(n)$ from the right) which is compatible with p_1.

We now study the lifting problem by using the diagram above. By the invariant integration technic [38], one can modify the results of Stewart [41] and Su [43] so that each of the followings is a sufficient condition for a lifting in the fiber bundle $P \to Q$:

"G is 1-connected",

"G is a torus, $n>2$ and $H^1(M:Z) = 0$".

But for our later use, these are not sufficient and we need a different approach. Because G-manifold Q is more complicated than M in general and we would like to express the lifting obstruction in terms of M.

Lemma 6.2 (Hattori and Yoshida [23]). *An S^1-bundle $\xi \to X$ admits a lifting of the given action of G on X (commuting with the right action of S^1) if and only if the first Chern class of the above S^1-bundle lies in the image of $j^*: H^*(EG \underset{G}{\times} X) \to H^*(X)$ where $j: X \to EG \underset{G}{\times} X$ is the inclusion given by $x \mapsto y \times x$ for some fixed $y \in EG$.*

If the left action of G on M lifts to a left action of G on $P/\mathrm{Spin}(n)$, then P/Z_2 admits a lifting of the action which commutes with the right action of $SO(n) \times S^1$. Because P/Z_2 is the fiber product $Q \underset{M}{\times} (P/\mathrm{Spin}(n))$ by Lemma 6.1. In order to lift the action on P/Z_2 to an action on P, we need the following

lemma which is a generalization of Proposition 2.1 in [5].

Lemma 6.3. *Let G be a connected topological group and M be a connected G-manifold. Let $\tilde{M}\to M$ be an n-fold covering such that $\tilde{M}$ is connected. Then there exist canonically a connected topological group $\tilde{G}$ and an m-fold covering homomorphism $h\colon \tilde{G}\to G$ such that $\tilde{G}$ acts on $\tilde{M}$ inducing via h the given action of G on M. Here m is smaller than or equal to n.*

Proof. Easy and omitted.

DEFINITION 6.4. Let $\xi\to X$ be a principal H-bundle. Then an action of G on X is called to have a pseudolifting to ξ if and only if there exist a connected topological group $\tilde{G}$ and a covering homomorphism $h\colon \tilde{G}\to G$ such that $\tilde{G}$ acts on ξ (commuting with action of H from the right) inducing *via* h the given action of G on X.

Putting all this together, we obtain easily

Proposition 6.5. *Let G be a compact connected Lie group and $p\colon Q\to M$ be a principal $SO(n)$-bundle where M is a connected manifold. Suppose that Q and M admit G-actions (commuting with the action of $SO(n)$ from the right) which is compatible with p. Then there exist a $Spin^c(n)$-reduction P of Q and a pseudolifting to P if and only if there exist a connected Lie group $\tilde{G}$ and a covering homomorphism $h\colon \tilde{G}\to G$ such that the second Stiefel-Whitney class $W_2(Q)$ lies in the image of the composition of the following homomorphisms*:

$$H^2(E\tilde{G}\underset{\tilde{G}}{\times} M)\xrightarrow{j^*} H^2(M\colon Z)\to H^2(M\colon Z_2)\,,$$

where $\tilde{G}$ acts on M via h.

We now study an equivariant Riemann-Roch type theorem in $Spin^c$-category. In this section, we make use of the equivariant cohomology theories $K_G(M)$ and $H^*(EG\underset{G}{\times} M\colon Q)$. We define a multiplicative transformation

$$ch_G\colon K_G(M)\to H^{**}(EG\underset{G}{\times} M\colon Q)$$

in the manner of the definition of ch_{Gy} in §4 by putting $y=0$ and by using the ordinary Chern classes in stead of U^*-theory Chern classes.

We define an equivariant $\hat{\mathfrak{A}}$-genus $\hat{\mathfrak{A}}_G(\xi)$ of a G-vector bundle ξ over M by

$$\hat{\mathfrak{A}}_G(\xi)=\hat{\mathfrak{A}}(EG\underset{G}{\times}\xi)\,,$$

where $\hat{\mathfrak{A}}$ is the genus belonging to the characteristics power series

$$\frac{x/2}{\sinh x/2}=\frac{x}{\exp x/2-\exp(-x/2)}\,.$$

Denote by $\hat{\mathfrak{A}}_G(M)$ the equivariant $\hat{\mathfrak{A}}$-genus of the tangent bundle TM.

Let ξ be an oriented G-vector bundle over a finite CW-complex. In this section, we make use of the usual Thom class $t_G(\xi)$ of the bundle $EG\underset{G}{\times}\xi$.

Let M and N be oriented G-manifolds of class C^∞. Then for a G-map f: $M\to N$, we get our equivariant Gysin homomorphism

$$f_!: H^*(EG\underset{G}{\times}M: Q)\to H^*(EG\underset{G}{\times}N: Q)\,.$$

A map $f: M\to N$ is called a c_1-map if we are given an element $c_1\in H^2(M: Z)$ such that $c_1=W_2(M)-f^*W_2(N)$ mod 2. We now assume that dim $M\equiv$dim N mod 2.

Let $e: M\to V$ be a G-embedding where V denotes a complex G-representation space. As in §2 we denote by ν the normal bundle of the G-embedding $f\times e: M\to N\times V$. Denote by Q the principal $SO(2k)$ bundle associated with ν. Then Q admits the induced G-action. If f is a c_1-map, then Q has the Spin$^c(2k)$-reduction P corresponding to c_1. We are now ready to state our equivariant Riemann-Roch theorem in Spin^c-category.

Theorem 6.6. *Suppose that dim $M\equiv$dim N mod 2 and that P admits a lifting of the G-action on Q. Then for given $\xi\in K_G(M)$, there exists $\eta\in K_G(N)$ such that*

$$f_!(e^{c_G/2}\cdot ch_G(\xi)\cdot\hat{\mathfrak{A}}_G(M))=ch_G(\eta)\cdot\hat{\mathfrak{A}}_G(N)$$

where c_G is the first Chern class of the complex line bundle

$$EG\underset{G}{\times}P\underset{\mathrm{Spin}^c(2k)}{\times}C\to EG\underset{G}{\times}M\,.$$

REMARK 6.7. The correspondence $\xi\to\eta$ is not functorial in general. This contrasts with the non-equivariant case (see Atiyah-Bott-Shapiro [2].). The reason is that the correspondence depends on the liftings and we can not choose a canonical lifting of the G-action in general so that the correspondence is functorial. In fact there are many examples showing that the correspondence depends on the liftings.

Proof of Theorem 6.6. Let T be the standard maximal torus of $SO(2k)\times SO(2)$ and $x_1,\cdots,x_k,y$ be the standard base of $H^1(T: Z)$. Put $T'=p^{-1}(T)$. It is a maximal torus of Spin$^c(2k)$ and covers T twofold. We regard $H^1(T: Z)$ as a subgroup of $H^1(T': Z)$ by the monomorphism $p^*: H^1(T: Z)\to H^1(T': Z)$. Then we can take $\{x_1,\cdots,x_k, (x_1,+\cdots+x_k+y)/2\}$ as a base of $H^1(T', Z)$. Denote by C_{2k} the Clifford algebra and by C^c_{2k} its complexification $C_{2k}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Previously we defined Spin$^c(2k)$ as the subgroup $\pi^{-1}(SO(2k)\times SO(2))$ of Spin$(2k+2)$. But we now regard Spin$^c(2k)$ as the multiplicative subgroup of C^c_{2k} by the following inclusions: Spin$(2k)\to C_{2k}$ and $S^1\to\boldsymbol{C}$. According to [2], there exists an ir-

reducible graded C^c_{2k}-module $\Delta_k^+ + \Delta_k^-$ of dimension 2^k over $\boldsymbol{C}$ which has the following properties:

i) Δ_k^+ and Δ_k^- are invariant if we restrict C^c_{2k} to $\mathrm{Spin}^c(2k)$ and are irreducible $\mathrm{Spin}^c(2k)$-representations,

ii) the weights of Δ_k^+ (resp. Δ_k^-) consist of

$$\{(-1)^{\delta_1}x_1 + \cdots + (-1)^{\delta_k}x_k + y\}/2\,,$$

where $\delta_i = 0$ or 1 and the number of 1 in $\{\delta_i\}$ is even (resp. odd).

Then we put

$$E = P \underset{\mathrm{Spin}^c(2k)}{\times} (\Delta_k^+ + \Delta_k^-)\,.$$

Obviously E admits a canonical G-action which makes E a G-vector bundle. Extending the notion in [2] equivariantly, we get the equivariant Clifford bundle $C(\nu)$ of ν and the equivariant $C(\nu)$-module structure on E. Hence we obtain an element $\chi_\nu^G(E)$ of $K_G(D(\nu), S(\nu))$ in the manner of [2].

Let $E(G, \mathrm{Spin}^c(2k)) \to B(G, \mathrm{Spin}^c(2k))$ be the universal principal $\mathrm{Spin}^c(2k)$-bundle with G-action. Obviously we can take $E(G, \mathrm{Spin}^c(2k))/\mathrm{Spin}^c(2k-1)$ as $B(G, \mathrm{Spin}^c(2k-1))$ and $E(G, \mathrm{Spin}^c(2k))/\mathrm{Spin}^c(2k-1)$ is isomorphic to $E(G, \mathrm{Spin}^c(2k)) \underset{\mathrm{Spin}^c(2k)}{\times} S^{2k-1}$ as sphere bundles over $B(G, \mathrm{Spin}^c(2k))$ with G-action. Denote by D the total space of the disk bundle $E(G, \mathrm{Spin}^c(2k)) \underset{\mathrm{Spin}^c(2k)}{\times} D^{2k}$. Then D includes $B(G, \mathrm{Spin}^c(2k-1))$ as the total space of the associated sphere bundle.

Let $h\colon M \to B(G, \mathrm{Spin}^c(2k))$ be the classifying map of P with G-action. Since h induces the G-bundle map

$$\begin{array}{ccc} D(\nu) & \xrightarrow{\bar{h}} & D \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & B(G, \mathrm{Spin}^c(2k))\,, \end{array}$$

we have the following commutative diagram:

$$\begin{array}{ccc} K_G(D(\nu), S(\nu)) & \xleftarrow{\bar{h}^*} & K_G(D, B(G, \mathrm{Spin}^c(2k-1))) \\ \downarrow ch_G & & \downarrow ch_G \\ H^{**}(EG \underset{G}{\times} (D(\nu), S(\nu)); Q) & \xleftarrow{(1 \times \bar{h})^*} & H^{**}(EG \underset{G}{\times} (D, B(G, \mathrm{Spin}^c(2k-1))); Q) \\ \uparrow \phi_1 & & \uparrow \phi \\ H^{**}(EG \underset{G}{\times} M; Q) & \xleftarrow{(1 \times h)^*} & H^{**}(EG \underset{G}{\times} B(G, \mathrm{Spin}^c(2k)); Q) \end{array}$$

where ϕ and ϕ_1 are the Thom isomorphisms.

In the manner of the construction of $\chi_\nu^G(E)$, we get an element $\chi_{\nu'}^G(E') \in K_G(D, B(G, \mathrm{Spin}^c(2k-1)))$ where $\nu' = E(G, \mathrm{Spin}^c(2k)) \underset{\mathrm{Spin}^c(2k)}{\times} \boldsymbol{R}^{2k}$ and $E' = E(G,$

$\mathrm{Spin}^c(2k)) \underset{\mathrm{Spin}^c(2k)}{\times} (\Delta^+ + \Delta^-)$. In view of the construction, we see that

$$\bar{h}^* \chi_{\nu'}^G(E') = \chi_\nu^G(E) .$$

The following diagram

$$\begin{array}{ccc} EG \underset{G}{\times} B(G, \mathrm{Spin}^c(2k-1)) & \simeq & BG \times B\ \mathrm{Spin}^c(2k-1) \\ \downarrow 1 \underset{G}{\times} j & & \downarrow 1 \times j' \\ EG \underset{G}{\times} B(G, \mathrm{Spin}^c(2k)) & \simeq & BG \times B\ \mathrm{Spin}^c(2k) \end{array}$$

is homotopy commutative where j and j' are the natural maps induced by the inclusion $\mathrm{Spin}^c(2k-1) \to \mathrm{Spin}^c(2k)$. Hence the homomorphism $(1 \underset{G}{\times} k)^*$: $H^*(EG \underset{G}{\times} (D, B(G, \mathrm{Spin}^c(2k-1))); Q) \to H^*(EG \underset{G}{\times} B(G, \mathrm{Spin}^c(2k)); Q)$ is injective where k: $B(G, \mathrm{Spin}^c(2k)) \to (D, B(G, \mathrm{Spin}^c(2k-1)))$ is the composition of the G-homotopy equivalence $B(G, \mathrm{Spin}^c(2k)) \simeq D$ and the inclusion $D \to (D, B(G, \mathrm{Spin}^c(2k-1)))$.

We are now ready to apply the Hirzebruch argument [26] equivariantly as follows.

We get the same formula for the characters of the $\mathrm{Spin}^c(2k)$-representations $\Delta_k^\pm$ in the sense of [12]:

$$ch\ \Delta_k^+ - ch\ \Delta_k^- = x_1 \cdots x_k \cdot e^{y/2} \prod \frac{\sinh x_i/2}{x_i/2} .$$

Therefore we have

$$\begin{aligned} & (1 \underset{G}{\times} k)^* ch_G \chi_{\nu'}^G(E') \\ &= ch_G k^* \chi_{\nu'}^G(E') \\ &= ch[EG \underset{G}{\times} \{E(G, \mathrm{Spin}^c(2k)) \underset{\mathrm{Spin}^c(2k)}{\times} (\Delta_k^+ - \Delta_k^-)\}] \\ &= ch[\{EG \underset{G}{\times} E(G, \mathrm{Spin}^c(2k))\} \underset{\mathrm{Spin}^c(2k)}{\times} (\Delta_k^+ - \Delta_k^-)] \\ &= W_{2k}^G \cdot e^{c'_G/2} \cdot \hat{\mathfrak{A}}_G(\nu')^{-1} , \end{aligned}$$

where W_{2k}^G is the Euler class of $\{EG \underset{G}{\times} E(G, \mathrm{Spin}^c(2k))\} \underset{\mathrm{Spin}^c(2k)}{\times} \boldsymbol{R}^{2k}$ and c'_G is the first Chern class of $\{EG \underset{G}{\times} E(G, \mathrm{Spin}^c(2k))\} \underset{\mathrm{Spin}^c(2k)}{\times} \boldsymbol{C}$. It follows from the injectivity of $(1 \underset{G}{\times} k)^*$ that

$$ch_G(\chi_\nu^G(E')) = \phi(e^{c'_G/2} \cdot \hat{\mathfrak{A}}_G(\nu')^{-1}) .$$

Thus we have

$$\begin{aligned} & ch_G \chi_\nu^G(E) \\ &= (1 \underset{G}{\times} \bar{h})^* ch_G \chi_{\nu'}^G(E') \end{aligned}$$

$$= (1 \times_G \bar{h})^* \phi(e^{c_G'/2} \cdot \hat{\mathfrak{A}}_G(\nu')^{-1})$$
$$= \phi_1(e^{c_G/2} \cdot \hat{\mathfrak{A}}_G(\nu)^{-1}) .$$

Note that there is an isomorphism

$$TM \oplus \nu = f^*(TN \oplus \underline{V})$$

of G-vector bundles.

By making use of the following properties:

1) $\phi_2 ch_G = ch_G \phi_2^{K_G}$,

2) $\phi_3 ch_G (\phi_3^{K_G})^{-1}(x) = \prod \left(\dfrac{1 - \exp y_i}{y_i} \right) ch_G(x)$

where $x \in K_G(N)$ and the total Chern class $c(EG \times_G (N \times V))$ is written formally as $\prod (1+y_i)$,

3) (IV) of Lemma 2.2,

we have that

$$f_!(e^{c_G/2} \cdot ch_G(\xi) \cdot \hat{\mathfrak{A}}_G(M)) = e^{c'/2} \cdot ch_G(\eta) \cdot \hat{\mathfrak{A}}_G(N)$$

for some $\eta \in K_G(N)$ where c' is the first Chern class of $EG \times_G N \times V \to EG \times_G N$.

Actually, we don't need $e^{c'/2}$ by the following observations. If $c' \neq 0$, we consider the conjugate representation space $\bar{V}$ of V. If we use $V \oplus \bar{V}$ in stead of V, we have $c' = 0$.

This makes the proof of Theorem 6.6 complete.

7. An answer to a problem of I.M. Singer

The purpose of this section is to give a topological proof of the following theorem, which answers the problem of I.M. Singer [40].

Theorem 7.1 [5]. *Let M be a compact connected differentiable manifold of class C^∞ with $W_2(M) = 0$. If a compact connected Lie group G acts differentiably and non-trivially on M, then $\hat{\mathfrak{A}}(M)[M] = 0$.*

Proof of Theorem 7.1. If G acts non-trivially on M, then there exists a circle subgroup S^1 of G which acts non-trivially on M. From now on we consider this S^1-action. Let $f: M \to pt$ be the constant map. By taking 0 as c_1, f becomes a c_1-map. Let P be the $\mathrm{Spin}^c(2k)$ reduction of Q corresponding to $c_1 = 0$ (see §6). According to Proposition 6.5, the S^1-action has a pseudolifting to P. We apply Theorem 6.6 to this new S^1-action and get the formula

$$f_!(\hat{\mathfrak{A}}_{S^1}(M)) = ch_{S^1}(\eta), \qquad \text{for some } \eta \in R(S^1)$$

which has the form:

$$\sum_{j, a_j \in Z} a_j \exp j\, t\,,$$

where t denotes the first Chern class of the canonical complex line bundle over BS^1 and almost all the a_j are zero.

We now wish to show that $ch_{S^1}(\eta)=0$ by making use of Theorem 3.3. Let $\rho^n: S^1 \to S^1=U(1)$ be the complex irreducible representation given by $g \mapsto g^n$. Denote by V_n its representation space. Then we can impose a complex vector bundle structure on N_μ so that we have a decomposition:

$$N_\mu = \sum E_{\mu n} \otimes V_n\,,$$

where the n are positive integers and $E_{\mu n}$ are complex vector bundles. Then we can check Assumption 3.2 as in §5. It follows from Theorem 3.3 that

$$f_!(\hat{\mathfrak{A}}_{S^1}(M)) = \sum_\mu \hat{\mathfrak{A}}(F_\mu) \prod_{n>0} \left(\frac{1}{\exp(nt+x_{n_i})/2 - \exp -(nt+x_{n_i})/2} \right) / [F_\mu]$$

in the localized ring $S^{-1}H^{**}(BS^1)$ where the total Chern class of $E_{\mu n}$ is written formally as $\prod(1+x_{n_i})$ and $/[F_\mu]$ denotes the slant product. Hence we have

$$\sum_{j, a_j \in Z} a_j \exp j\, t = \sum_\mu \hat{\mathfrak{A}}(F_\mu) \prod \left(\frac{1}{\exp(nt+x_{n_i})/2 - \exp -(nt+x_{n_i})/2} \right) / [F_\mu]\,.$$

This equation holds even if we regard both terms as functions of t. If t approaches $\pm\infty$, the local index tends to 0. But the global index with this property must be identically 0. This completes the proof of Theorem 7.1.

REMARK 7.3. In the above proof, Theorem 6.6 is essential. Because any non trivial S^1-actions on any connected oriented manifolds have the same local property as above and there are many examples (for example $\boldsymbol{CP}^{2n}$) whose $\hat{\mathfrak{A}}$-genera do not vanish. But it often happens that there exists $c \in H^2(M: Z)$ such that $\exp(c/2)\hat{\mathfrak{A}}(M)\,[M]=0$. This kind of things will be studied in a subsequent paper.

8. *G*-signature theorem

The purpose of this section is to give a topological proof of the Atiyah-Singer G-signature theorem. Manifolds which we here work with are assumed to be closed, oriented differentiable manifolds of class C^∞. Since Ossa gave a topological proof of it for periodic actions [36], we assume that G is a toral group T^n. We use the notations in §3 freely. Let $L(\cdot)$ be the Hirzebruch L-genus. Then by making use of the equivariant cohomology theory $H^*(ET^n \underset{T^n}{\times} M: Q)$, we have

Theorem 8.1.

$$\textit{Signature of } M = \sum_{\mu} L(F_\mu) \prod_{\rho,\rho_i} \left(\frac{\exp 2(\rho+\mu x_{\rho_i})+1}{\exp 2(\rho+\mu x_{\rho_i})-1}\right)/[F_\mu] .$$

Proof. First we analize the global index. It is easy to see the following

Lemma 8.2 *Regarding* $S^{2m+1}\times\cdots\times S^{2m+1}$ *as a skeleton of* $ET^n = S^\infty\times\cdots\times S^\infty$, *we have the following commutative diagram*

$$\begin{array}{ccc} H^*(ET^n\underset{T^n}{\times} M) & \xrightarrow{\text{Ind}} & H^*(BT^n) \\ \downarrow{\scriptstyle (j\underset{T^n}{\times}1)^*} & & \downarrow{\scriptstyle j^*} \\ H^*(S^{2m+1}\times\cdots\times S^{2m+1}\underset{T^n}{\times} M) & \xrightarrow{\pi_!} & H^*(\boldsymbol{CP}^m\times\cdots\times\boldsymbol{CP}^m) \end{array}$$

where j *is the natural inclusion and* $\pi_!$ *is the Poincaré dual of the homology homomorphism.*

According to Chern [15], the Gysin homomorphism $\pi_!$ is equivalent to the integration over the fiber (see Borel-Hirzebruch [12]) of the fiber bundle

$$\pi\colon S^{2m+1}\times\cdots\times S^{2m+1}\underset{T^n}{\times} M \to \boldsymbol{CP}^m\times\cdots\times\boldsymbol{CP}^m .$$

The bundle

$$S^{2m+1}\times\cdots\times S^{2m+1}\underset{T^n}{\times} TM \to S^{2m+1}\times\cdots\times S^{2m+1}\underset{T^n}{\times} M$$

is the bundle along the fiber of the fiber bundle π above. Since $\boldsymbol{CP}^m\times\cdots\times\boldsymbol{CP}^m$ is simply connected, the L-genus is strictly multiplicative in π in the sense of [12], that is,

$$\pi_! L(S^{2m+1}\times\cdots\times S^{2m+1}\underset{T^n}{\times} TM)\in H^0(\boldsymbol{CP}^m\times\cdots\times\boldsymbol{CP}^m) .$$

This holds for arbitrary m. Hence we may conclude that

$$\text{Ind } L(ET^n\underset{T^n}{\times} TM)\in H^0(BT^n) .$$

Note that the zero dimensional part of $\text{Ind}(ET^n\underset{T^n}{\times} TM)$ is equal to the signature of M. Thus we have shown the following

Proposition 8.3.

$$\textit{Signature of } M = \text{Ind } L(ET^n\underset{T^n}{\times} TM) .$$

On the other hand, one checks Assumption 3.2. It follows that we can apply Corollary 3.4.

Thus we have in the localized ring $S^{-1}H^*(BT^n)$:

Signature of M

$$= \text{Ind } L(ET^n \underset{T^n}{\times} TM)$$

$$= \sum_{\mu} \text{Ind } \frac{L(F_\mu) L(ET^n \underset{T^n}{\times} N_\mu)}{\chi(ET^n \underset{T^n}{\times} N_\mu)}$$

$$= \sum_{\mu} L(F_\mu) \prod_{\rho, \rho_i} \left(\frac{\exp 2(\rho + {}_\mu x_{\rho_i}) + 1}{\exp 2(\rho + {}_\mu x_{\rho_i}) - 1} \right) / [F] .$$

This makes the proof of Theorem 8.1 complete.

Corollary 8.4 ([10] [24] [27] [28] [29]). *Let M be a closed oriented T^n-manifold of class C^∞. Then we can orient each component F_μ of the fixed point set so that we have*

$$\textit{Signature of } M = \sum_{\mu} \textit{Signature of } F_\mu .$$

9. G-homotopy type invariance of equivariant Stiefel-Whitney classes

In this section we don't assume orientability of manifolds and make use of the equivariant cohomology theory $H^*(EG \underset{G}{\times} M; Z_2)$. Denote by Sq^i the Steenrod *i-th* squaring operation. Put $Sq = \sum_{i=0}^{\infty} Sq^i$ and $Sq^{-1} = \dfrac{1}{1+(Sq-1)}$ as formal power series. Then both Sq and Sq^{-1} are multiplicative and $Sq \cdot Sq^{-1} = Sq^{-1} \cdot Sq = 1$. For a G-vector bundle ξ over M, we put

$$W_G(\xi) = \phi^{-1} Sq \phi(1), \qquad V_G(\xi) = Sq^{-1} \phi^{-1} Sq \phi(1) ,$$
$$\bar{W}_G(\xi) = Sq \phi^{-1} Sq^{-1} \phi(1), \quad \bar{V}_G(\xi) = \phi^{-1} Sq^{-1} \phi(1) ,$$

where $\phi: H^*(EG \underset{G}{\times} M; Z_2) \to H^*(EG \underset{G}{\times} (D(\xi), S(\xi)); Z_2)$ is the Thom isomorphism. Then $W_G(\xi)$ is nothing but the total Stiefel-Whitney class of the vector bundle

$$EG \underset{G}{\times} \xi \to EG \underset{G}{\times} M ,$$

and is called the equivariant total Stiefel-Whitney class of ξ. Obviously we have $Sq V_G(\xi) = W_G(\xi)$ and $Sq \bar{V}_G(\xi) = \bar{W}_G(\xi)$. It is easy to see that

$$W_G(\xi) \cdot \bar{W}_G(\xi) = 1 ,$$
$$V_G(\xi) \cdot \bar{V}_G(\xi) = 1 .$$

We call $V_G(\xi)$, $\bar{W}_G(\xi)$ and $\bar{V}_G(\xi)$ equivariant Wu class, the equivariant dual total Stiefel-Whitney class and the equivariant dual Wu class respectively. When ξ is the tangent G-vector bundle TM of M, $W_G(TM)$, $V_G(TM)$, $\bar{W}_G(TM)$

and $\bar{V}_G(TM)$ are simply denoted by $W_G(M)$, $V_G(M)$, $\bar{W}_G(M)$ and $\bar{V}_G(M)$ respectively.

Then we have

Theorem 9.1. *Let M and N be closed (non-oriented) differentiable G-manifolds of class C^∞. For a G-map $f: M \to N$, we have*

$$f_!(x \cdot V_G(M)) = (Sq^{-1} f_!(Sqx)) \cdot V_G(N),$$
$$f_!(x \cdot \bar{W}_G(M)) = (Sq f_!(Sq^{-1}x)) \cdot \bar{W}_G(N),$$

for $x \in H^*(EG \underset{G}{\times} M: Z_2)$.

Proof. The idea of the proof is similar to those of the previous theorems and we have only to remark the following:

Lemma 9.2. *For any G-vector bundles ξ and η over M, we have that*

i) *$W_G(\xi)$, $V_G(\xi)$, $\bar{W}_G(\xi)$ and $\bar{V}_G(\xi)$ are all invertible and $W_G(\xi) \cdot \bar{W}_G(\xi) = 1$, $V_G(\xi) \cdot \bar{V}_G(\xi) = 1$;*

ii) *$W_G(\xi \oplus \eta) = W_G(\xi) \cdot W_G(\eta)$, $V_G(\xi \oplus \eta) = V_G(\xi) \cdot V_G(\eta)$, $\bar{W}_G(\xi \oplus \eta) = \bar{W}_G(\xi) \cdot \bar{W}_G(\eta)$, $\bar{V}_G(\xi \oplus \eta) = \bar{V}_G(\xi) \cdot \bar{V}_G(\eta)$.*

REMARK 9.3. The following statement does not hold in the equivariant case:

"If ξ and η are G-vector bundles such that $\xi \oplus \eta$ are trivial, then $\bar{W}_G(\xi) = W_G(\eta)$, $\bar{V}_G(\xi) = V_G(\eta)$."

As is well-known, EG can be written as $EG = \varinjlim EG^n$ where EG^n is an n-connected free G-manifold. By making use of EG^n in stead of EG, we can define

$$f_!: H^*(EG^n \underset{G}{\times} M: Z_2) \to H^*(EG^n \underset{G}{\times} N: Z_2)$$

and $W_G^n(\xi)$, $V_G^n(\xi)$, $\bar{W}_G^n(\xi)$ and $\bar{V}_G^n(\xi)$ similarly.

Then quite a similar argument proves:

Theorem 9.4. *Let M and N be closed differentiable G-manifolds of class C^∞. For a G-map $f: M \to N$, we have*

$$f_!^n(x \cdot V_G^n(M)) = (Sq^{-1} f_!^n(Sqx)) \cdot V_G^n(N),$$
$$f_!^n(x \cdot \bar{W}_G^n(M)) = (Sq f_!^n(Sq^{-1}x)) \cdot \bar{W}_G^n(N),$$

for $x \in H^*(EG^n \underset{G}{\times} M: Z_2)$.

Theorem 9.4 is used to prove the following

Theorem 9.5. *$W_G(M)$, $V_G(M)$, $\bar{W}_G(M)$ and $\bar{V}_G(M)$ are all invariants of the G-homotopy type.*

Proof. Note that $EG^n \underset{G}{\times} M$ and $EG^n \underset{G}{\times} N$ are closed manifolds. *A* G-map $f: M \to N$ induces a map

$$1 \underset{G}{\times} f: EG^n \underset{G}{\times} M \to EG^n \underset{G}{\times} N .$$

For a closed manifold X, we denote by $[X]$ the fundamental homology class mod 2. It is easy to prove the following.

Lemma 9.6. *The homomorphism $f_!^n$ is equal to the Poincaré dual of the homology homomorphism of $1 \underset{G}{\times} f$. Namely for $x \in H^*(EG^n \underset{G}{\times} M; Z_2)$ we have*

$$f_!^n(x) = (1 \underset{G}{\times} f)_!(x) = (\cap [EG^n \underset{G}{\times} N])^{-1} \cdot (1 \underset{G}{\times} f)_*(x \cap [EG^n \underset{G}{\times} M]) .$$

Suppose now that $f: M \to N$ is a G-homotopy equivalence. Then

$$1 \underset{G}{\times} f: EG^n \underset{G}{\times} M \to EG^n \underset{G}{\times} N$$

is a homotopy equivalence. Moreover we have

$$(1 \underset{G}{\times} f)_! \cdot (1 \underset{G}{\times} f)^* = \text{identity}.$$

Because for $x \in H^*(EG^n \underset{G}{\times} N; Z_2)$

$$\begin{aligned}
&(1 \underset{G}{\times} f)_! \cdot (1 \underset{G}{\times} f)^*(x) \\
&= (\cap [EG^n \underset{G}{\times} N])^{-1} \cdot (1 \underset{G}{\times} f)_*((1 \underset{G}{\times} f)^*(x) \cap [EG^n \underset{G}{\times} M]) \\
&= (\cap [EG^n \underset{G}{\times} N])^{-1} \cdot (x \cap (1 \underset{G}{\times} f)_* [EG^n \underset{G}{\times} M]) \\
&= (\cap [EG^n \underset{G}{\times} N])^{-1} \cdot (x \cap [EG^n \underset{G}{\times} N]) \\
&= x .
\end{aligned}$$

Now consider the following commutative diagram:

$$\begin{array}{ccc}
EG^n \underset{G}{\times} M & \xrightarrow{1 \underset{G}{\times} P} & \\
{\scriptstyle 1 \underset{G}{\times} f} \downarrow & & EG^n \underset{G}{\times} * = BG^n \\
EG^n \underset{G}{\times} N & \xrightarrow[1 \underset{G}{\times} P']{} &
\end{array}$$

where $P: M \to *$ and $P': N \to *$ are the maps to one point $*$. Let $\alpha_1, \cdots, \alpha_k$ be a basis of $H^*(EG^n \underset{G}{\times} N; Z_2)$. Then we have

$$\begin{aligned}
&\langle \alpha_i \cdot V_G^n(N), [EG^n \underset{G}{\times} N] \rangle \\
&= \langle (1 \underset{G}{\times} P')_!(\alpha_i \cdot V_G^n(N)), [BG^n] \rangle \\
&= \langle P'^n_!(\alpha_i \cdot V_G^n(N)), [BG^n] \rangle \\
&= \langle Sq^{-1} P'^n_! (Sq \alpha_i), [BG^n] \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle Sq^{-1}P'^{n}_{!} \cdot (1 \underset{G}{\times} f)_{!} \cdot (1 \underset{G}{\times} f)^{*}(Sq\alpha_i), [BG^n] \rangle \\
&= \langle Sq^{-1}P'^{n}_{!} \cdot f^{n}_{!} Sq(1 \underset{G}{\times} f)^{*}(\alpha_i), [BG^n] \rangle \\
&= \langle Sq^{-1}P^{n}_{!} Sq(1 \underset{G}{\times} f)^{*}(\alpha_i), [BG^n] \rangle \\
&= \langle P^{n}_{!}((1 \underset{G}{\times} f)^{*}(\alpha_i) \cdot V^{n}_{G}(M)), [BG^n] \rangle \\
&= \langle (1 \underset{G}{\times} P)_{!}((1 \underset{G}{\times} f)^{*}(\alpha_i) \cdot V^{n}_{G}(M)), [BG^n] \rangle \\
&= \langle (1 \underset{G}{\times} f)^{*}(\alpha_i) \cdot V^{n}_{G}(M), [EG^n \underset{G}{\times} M] \rangle .
\end{aligned}
$$

Since $1 \underset{G}{\times} f$ is a homotopy equivalence, we have

$$
\begin{aligned}
&\langle \alpha_i \cdot V^{n}_{G}(N), [EG^n \underset{G}{\times} N] \rangle \\
&= \langle (1 \underset{G}{\times} f)^{*}(\alpha_i \cdot V^{n}_{G}(N)), [EG^n \underset{G}{\times} M] \rangle \\
&= \langle (1 \underset{G}{\times} f)^{*}(\alpha_i) \cdot (1 \underset{G}{\times} f)^{*}(V^{n}_{G}(N)), [EG^n \underset{G}{\times} M] \rangle .
\end{aligned}
$$

By combining the above equations, we have

$$
\begin{aligned}
&\langle (1 \underset{G}{\times} f)^{*}(\alpha_i) \cdot V^{n}_{G}(M), [EG^n \underset{G}{\times} M] \rangle \\
&= \langle (1 \underset{G}{\times} f)^{*}(\alpha_i) \cdot (1 \underset{G}{\times} f)^{*}(V_{G}(N)), [EG^n \underset{G}{\times} M] \rangle .
\end{aligned}
$$

Notice that $\{(1 \underset{G}{\times} f)^{*}\alpha_i\}$ is a basis of $H^{*}(EG^n \underset{G}{\times} M; Z_2)$. According to the Poincaré duality theorem, we can conclude that

$$
V^{n}_{G}(M) = (1 \underset{G}{\times} f)^{*}(V^{n}_{G}(N)) .
$$

This equation holds for an arbitrary integer n. It follows that

$$
\begin{aligned}
V_G(M) &= \varprojlim V^{n}_{G}(M) = \varprojlim (1 \underset{G}{\times} f)^{*}(V^{n}_{G}(N)) \\
&= (1 \underset{G}{\times} f)^{*}(\varprojlim V^{n}_{G}(N)) = (1 \underset{G}{\times} f)^{*}(V_G(N)) .
\end{aligned}
$$

Namely $V_G(M)$ is an invariant of the G-homotopy type. As a consequence,

$$
\begin{aligned}
W_G(M) &= SqV_G(M) \\
&= Sq(1 \underset{G}{\times} f)^{*}(V_G(N)) \\
&= (1 \underset{G}{\times} f)^{*}(SqV_G(N)) \\
&= (1 \underset{G}{\times} f)^{*}(W_G(N)) .
\end{aligned}
$$

Namely $W_G(M)$ is also an invariant of the G-homotopy type.

Similarly we prove that $\bar{V}_G(M)$ and $\bar{W}_G(M)$ are invariants of the G-homotopy type.

This makes the proof of Theorem 9.5 complete.

Corollary 9.7. *$(Z_2)^k$-homotopy equivalent manifolds are $(Z_2)^k$-bordant.*

Proof. It suffices to remark that $(Z_2)^k$-bordism classes are characterized by equivariant Stiefel-Whitney numbers [19].

10. Equivariant non-embedding theorem

This section is devoted to a study of an equivariant non-embedding theorem and its application. We define an equivariant $\mathfrak{A}$-genus $\mathfrak{A}_G(\xi)$ of a G-vector bundle ξ over M by

$$\mathfrak{A}_G(\xi) = \mathfrak{A}(EG \underset{G}{\times} \xi),$$

where $\mathfrak{A}$ is the genus belonging to the characteristic power series

$$\frac{x}{\sinh x} = \frac{2x}{\exp x - \exp(-x)}.$$

Denote by $\mathfrak{A}_G(M)$ the equivariant $\mathfrak{A}$-genus of the tangent bundle TM. We make use of the equivariant cohomology theory $H^*(EG \underset{G}{\times} M : Q)$. For an oriented G-vector bundle ξ, we employ the ordinary Thom class $t(\xi) \in H^*(EG \underset{G}{\times} (D(\xi), S(\xi)), Q)$ as the orientation of ξ.

Theorem 10.1. *Let M^{2n} be a closed oriented differentiable G-manifold of class C^∞ and V^{n+k} a complex G-representation space of complex dimension $n+k$. Suppose that a G-manifold N which is G-bordant to M is G-embeddable in V^{n+k}. Then for any $\xi \in Im(K_G(W) \to K_G(M))$ (where W is a G-bordism between M and N), there exists $\eta \in R(G)$ such that $f_!(2^{k-1} \cdot ch_G(\xi) \cdot \mathfrak{A}_G(M)) = ch_G(\eta) \cdot \prod_{i=1}^{n+k} \left(\frac{2 \exp y_i}{1 + \exp y_i} \right)$ where $f : M \to pt$ and the total Chern class $c(EG \underset{G}{\times} V^{n+k})$ is expressed formally as $\prod_{i=1}^{n+k} (1 + y_i)$.*

Proof. We first show that we are able to reduce the proof of Theorem 10.1 to that in the case where M itself is G-embeddable in V^{n+k}. Let $i_1 : M \to W$ and $i_2 : N \to W$ be the inclusion maps. By assumption, there exists $\xi_0 \in K_G(W)$ such that $i_1^* \xi_0 = \xi$. Put $\xi' = i_2^* \xi_0$. Let $f' : N \to pt$. Then it is easy to prove that

$$f_!(2^{k-1} \cdot ch_G(\xi) \cdot \mathfrak{A}_G(M)) = f'_!(2^{k-1} \cdot ch_G(\xi') \cdot \mathfrak{A}_G(N)).$$

Hence it suffices to check the case where M itself is G-embeddable in V^{n+k}.

The idea of the rest of the verification is similar to that of Theorem 6.6. In stead of the representation $\Delta^+ + \Delta^-$, we use the following virtual representation

$$\mu_+ = \sum_{i=0}^{k-1} (-1)^i \lambda^i + (-1)^k \lambda_+^k \in R(SO(2k)),$$

due to Atiyah-Hirzebruch [4]. It is proved in a way similar to the proof of Theorem 6.6 that there exists $\tau \in K_G(D(\nu), S(\nu))$ such that

$$ch_G(\tau) = \phi_1\Big\{\frac{1}{2}(-1)^k \prod_{i=1}^{k} (\exp z_i - \exp(-z_i))/z_i + \frac{1}{2}\prod_{i=1}^{k}(\exp z_i - 1)(\exp(-z_i)-1)/z_i\Big\},$$

where ν is the normal G-bundle of M^{2n} in V^{n+k} and the total Pontrjagin class $P(EG\underset{G}{\times}\nu)$ is expressed formally as $\prod(1+z_i^2)$ and the Euler class $\chi(EG\underset{G}{\times}\nu)$ is expressed as $z_1\cdots z_k$.

We now show that the term $\frac{1}{2}\prod_{i=1}^{k}(\exp z_i - 1)(\exp(-z_i)-1)/z_i$ vanishes in the equivariant case too. Let $i: M \to D(\nu) \subset V^{n+k}$ and $j: M \to D^{2n+2k} \subset V^{n+k}$ be the inclusion maps. Since $H^{2k}(EG\underset{G}{\times}(D^{2n+2k}, S^{2n+2k-1}))=0$ for $n>0$, the following commutative diagram

$$\begin{array}{ccc} H^*(EG\underset{G}{\times}(D(\nu), S(\nu)); Q) & \xrightarrow{\phi_2} & H^*(EG\underset{G}{\times}(D^{2n+2k}, S^{2n+2k-1}); Q) \\ \Big\downarrow{\scriptstyle (1\underset{G}{\times} i)^*} & \swarrow{\scriptstyle (1\underset{G}{\times} j)^*} & \\ H^*(EG\underset{G}{\times} M; Q) & & \end{array}$$

implies that the Euler class $z_1\cdots z_k=0$. Since $\prod(\exp z_i - 1)(\exp(-z_i)-1)/z_i$ is divisible by $z_1\cdots z_k$, we may assert its vanishing.

Denote by $\prod_{i=1}^{n}(1+x_i^2)$ the total Pontrjagin class of the bundle $EG\underset{G}{\times}TM$. Notice the following isomorphisms of G-vector bundles:

$$TM \oplus \nu \cong M \times V^{n+k} \cong f^*V^{n+k}.$$

Accordingly we have

$$\prod_{i=1}^{n}(1+x_i^2)\cdot\prod_{i=1}^{k}(1+z_i^2) = f^*\prod(1+y_i^2).$$

It follows that

$$\prod_{i=1}^{k}\Big(\frac{\exp z_i - \exp(-z_i)}{2z_i}\Big) = \prod_{i=1}^{n}\Big(\frac{2x_i}{\exp x_i - \exp(-x_i)}\Big)\cdot f^*\prod_{i=1}^{n+k}\Big(\frac{\exp y_i - \exp(-y_i)}{2y_i}\Big).$$

By our routine arguments, we have

$$f_!\Big(2^{k-1}\cdot ch_G(\xi)\cdot\prod_{i=1}^{n}\Big(\frac{2x_i}{\exp x_i - \exp(-x_i)}\Big)\Big)\prod_{i=1}^{n+k}\Big(\frac{\exp y_i - \exp(-y_i)}{2y_i}\Big)$$

$$= f_!\Big[2^{k-1}\cdot ch_G(\xi)\cdot \prod_{i=1}^{n}\Big(\frac{2x_i}{\exp x_i-\exp(-x_i)}\Big)\cdot f^*\prod_{i=1}^{n+k}\Big(\frac{\exp(y_i)-\exp(-y_i)}{2y_i}\Big)\Big]$$

$$= f_!\Big(2^{k-1}\cdot ch_G(\xi)\cdot \prod_{i=1}^{k}\Big(\frac{\exp z_i-\exp(-z_i)}{2z_i}\Big)\Big)$$

$$= f_!\Big[ch_G(\xi)\cdot(-1)^k\Big\{\frac{1}{2}(-1)^k\prod_{i=1}^{k}\Big(\frac{\exp z_i-\exp(-z_i)}{z_i}\Big)$$
$$+\frac{1}{2}\prod_{i=1}^{k}\frac{(\exp z_i-1)(\exp(-z_i)-1)}{z_i}\Big\}\Big]$$

$$= \phi_3\cdot\phi_2\Big[ch_G((-1)^k\xi)\phi_1\Big\{\frac{1}{2}(-1)^k\prod_{i=1}^{k}\Big(\frac{\exp z_i-\exp(-z_i)}{z_i}\Big)$$
$$+\frac{1}{2}\prod_{i=1}^{k}\frac{(\exp z_i-1)(\exp(-z_i)-1)}{z_i}\Big\}\Big]$$

$$= \phi_3\phi_2 ch_G((-1)^k\xi\otimes\tau)$$

$$= \phi_3 ch_G\phi_2^{K_G}((-1)^k\xi\otimes\tau)$$

$$= \phi_3\cdot ch_G\{(\phi_3^{K_G})^{-1}\cdot\phi_3^{K_G}\cdot\phi_2^{K_G}((-1)^k\xi\otimes\tau)\}$$

$$= \phi_3\cdot[ch_G(\phi_3^{K_G})^{-1}(1)\cdot ch_G\phi_3^{K_G}\cdot\phi_2^{K_G}((-1)^k\xi\otimes\tau)]$$

$$= \{\phi_3\cdot ch_G(\phi_3^{K_G})^{-1}(1)\}\cdot ch_G\phi_3^{K_G}\cdot\phi_2^{K_G}((-1)^k\xi\otimes\tau)$$

$$= \prod_{i=1}^{n+k}\Big(\frac{1-\exp y_i}{y_i}\Big)\cdot ch_G\phi_3^{K_G}\cdot\phi_2^{K_G}((-1)^k\xi\otimes\tau)\,.$$

Set $\eta=\phi_3^{K_G}\cdot\phi_2^{K_G}((-1)^n\xi\otimes\tau)$. Then we obtained

$$f_!\Big(2^{k-1}\cdot ch_G(\xi)\cdot\prod_{i=1}^{n}\Big(\frac{2x_i}{\exp x_i-\exp(-x_i)}\Big)\Big)$$

$$= ch_G(\eta)\cdot(-1)^{n+k}\cdot\prod_{i=1}^{n+k}\Big(\frac{1-\exp y_i}{y_i}\Big)\cdot\prod_{i=1}^{n+k}\Big(\frac{2y_i}{\exp y_i-\exp(-y_i)}\Big)$$

$$= ch_G(\eta)\cdot\prod_{i=1}^{n+k}\Big(\frac{2\exp y_i}{1+\exp y_i}\Big)\,.$$

This completes the proof of Theorem 10.1.

REMARK 10.2. By making use of the localization theory, we can calculate the left hand side of the equation in Theorem 10.1

For example, we have

Corollary 10.3. *Let $(\boldsymbol{CP}^4, \varphi, S^1)$ be the semi-free S^1-action given by*

$$(z_0,z_1,z_2,z_3,z_4)\xmapsto{g\in S^1}(z_0,z_1,z_2,gz_3,gz_4)\,.$$

Then any S^1-manifold M which is S^1-bordant to $(\boldsymbol{CP}^4, \varphi, S^1)$ is not equivariantly embeddable in $3\rho\oplus\boldsymbol{C}^k$ for any k where ρ is the representation

$$\rho = id\colon S^1 \to S^1 = U(1)$$

and $\boldsymbol{C}^k$ denotes the trivial S^1-representation space.

Proof. We calculate our index homomorphism by making use of the localization Theorem 3.3. The fixed point set is the disjoint union $\boldsymbol{CP}^2 \cup S^2$. Denote by $i_1\colon \boldsymbol{CP}^2 \to \boldsymbol{CP}^4$, $i_2\colon S^2 \to \boldsymbol{CP}^4$ the inclusion maps. Since

$$\frac{2x}{\exp x - \exp(-x)} = 1 + \text{higher terms},$$

we can express $\mathfrak{A}_{S^1}(\boldsymbol{CP}^4)$ as $\prod_{i=1}^{5} \frac{2x_i}{\exp x_i - \exp(-x_i)}$ instead of $\prod_{i=1}^{4} \frac{2x_i}{\exp x_i - \exp(-x_i)}$ as usual. Denote by x the first Chen class $c_1(L)$ of the canonical complex line bundle L over $\boldsymbol{CP}^4$. Similarly denote by t the first Chern class of the bundle $ES^1 \to BS^1$. Then the equivariant Euler class χ_1 (resp. χ_2) of the normal bundle of $\boldsymbol{CP}^2$ (resp. S^2) in $\boldsymbol{CP}^4$ is given by $(x+t)^2$ (resp. $(x-t)^3$). Then we compute

$$\begin{aligned}
&f_!\left(2^{k-1} \cdot \prod_{i=1}^{5} \left(\frac{2x_i}{\exp x_i - \exp(-x_i)}\right)\right) \\
&= 2^{k-1}\left\{\frac{1}{\chi_1} i_1^* \prod_{i=1}^{5} \left(\frac{2x_i}{\exp x_i - \exp(-x_i)}\right)/[\boldsymbol{CP}^2]\right. \\
&\quad \left. + \frac{1}{\chi_2} i_2^* \prod_{i=1}^{5} \left(\frac{2x_i}{\exp x_i - \exp(-x_i)}\right)/[S^2]\right\} \\
&= 2^{k-1}\left\{\frac{1}{(x+t)^2}\left(\frac{2x}{\exp x - \exp(-x)}\right)^3\left(\frac{2(x+t)}{\exp(x+t) - \exp(-x-t)}\right)^2/[\boldsymbol{CP}^2]\right. \\
&\quad \left. + \frac{1}{(x-t)^3}\left(\frac{2x}{\exp x - \exp(-x)}\right)^2\left(\frac{2(x-t)}{\exp(x-t) - \exp(-x+t)}\right)^3/[S^2]\right\} \\
&= 2^{k-1}\left\{\left(\frac{2x}{\exp x - \exp(-x)}\right)^3\left(\frac{2}{\exp(x+t) - \exp(-x-t)}\right)^2/[\boldsymbol{CP}^2]\right. \\
&\quad \left. + \left(\frac{2x}{\exp x - \exp(-x)}\right)^2\left(\frac{2}{\exp(x-t) - \exp(-x+t)}\right)^3/[S^2]\right\}.
\end{aligned}$$

For a fixed $t \neq 0$, the functions above are analytic on x. We compute the coefficients of x^2 and x as follows. Notice that

$$\left(\frac{2x}{\exp x - \exp(-x)}\right)^3 = 1 - \frac{x^2}{2} + \text{higher terms}.$$

Set

$$f(x) = \left(\frac{2}{\exp(x+t) - \exp(-x-t)}\right)^2.$$

Then we have

$$f''(x) = 24(\exp(x+t)-\exp(-x-t))^{-4}(\exp(x+t)+\exp(-x-t))^2 \\ -8(\exp(x+t)-\exp(-x-t))^{-2}.$$

Hence the coefficient of x^2 in $f(x)$ is given by

$$\frac{1}{2}f''(0) = 12(\exp t-\exp(-t))^{-4}(\exp t+\exp(-t))^2 \\ -4(\exp t-\exp(-t))^{-2}.$$

The constant term of $f(x)$ is given by

$$4(\exp t-\exp(-t))^{-2}.$$

Accordingly the coefficient of x^2 in

$$\left(\frac{2x}{\exp x-\exp(-x)}\right)^3\left(\frac{2}{\exp(x+t)-\exp(-x-t)}\right)^2$$

is given by

$$1\times\{12(\exp t-\exp(-t))^{-4}(\exp t+\exp(-t))^2-4(\exp t-\exp(-t))^{-2}\} \\ -\frac{1}{2}\times 4(\exp t-\exp(-t))^{-2}.$$

Next we compute the local value at S^2. Set

$$f(x) = 8(\exp(x-t)-\exp(-x+t))^{-3}.$$

Then

$$f'(0) = -24(\exp(-t)-\exp t)^{-4}(\exp(-t)+\exp t).$$

Since $2x/(\exp x-\exp(-x))$ has no x term, the coefficient of x in

$$\left(\frac{2x}{\exp x-\exp(-x)}\right)^2\left(\frac{2}{\exp(x-t)-\exp(-x+t)}\right)^3$$

is given by $f'(0)$. By combining the above computations, we have

$$\begin{aligned} &f_!(2^{k-1}\cdot\mathfrak{A}_{S^1}(\boldsymbol{CP}^4)) \\ &= 2^k(\exp t-\exp(-t))^{-4}\{6(\exp t+\exp(-t))^2-3(\exp t-\exp(-t))^2 \\ &\quad -12(\exp t+\exp(-t))\} \\ &= 3\cdot 2^{k-4}\cdot\exp(-2t)\cdot\left(\frac{2\exp t}{1+\exp t}\right)^4. \end{aligned}$$

Hence we need at least $4\rho\oplus\boldsymbol{C}^k$ ($k\geqq 4$).

This completes the proof of Corollary 10.3.

REMARK 10.4. If we look at the tangential representation at a point $(0, 0, 0, *, *)$, we see trivially that at least 3ρ is necessary. On the other hand, by the equivariant embedding theorem, $(\boldsymbol{CP}^4, \varphi, S^1)$ is equivariantly embeddable in $j\rho \oplus \boldsymbol{C}^k$ for some j and k.

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Added in proof. Recently Mr. I. Yasui succeeded in computing the homomorphism in Theorem 10.1 for semi-free S^1-actions on cohomology complex projective spaces. As a consequence he obtained numerical conditions for equivariant embeddings.

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