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Adaptive Control for Uncertain Dynamical Systems with Performance Guarantee

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List of Publications

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1. Atsede G. Gebremedhin and Yasumasa Fujisaki.
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2. Atsede G. Gebremedhin and Yasumasa Fujisaki.
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Conference Proceedings

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2. Atsede G. Gebremedhin and Yasumasa Fujisaki.
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Abstract

There have been many experiments on adaptive control in laboratories and industries. The rapid progress in microelectronics and integrated circuit technology was a strong stimulation. Interaction between theory and experimentation resulted an active development of the field. As a result, the development of adaptive controllers is now accelerating and started to appear commercially. One of the primary reason for introducing adaptive control was to obtain controllers that could adapt to changes in process dynamics in the presence of system uncertainties. It has been found that adaptive techniques can also be used to provide automatic tuning of controllers.

Previous works have been developed for various complex systems such as interconnected and/or large-scale systems under different perspectives. However, they typically do not guarantee an optimal solution while estimating the unknown uncertainty parameters of the system dynamics. In addition, poor transient performance due to uncertainties is solved by increasing adaptive gains to rapidly suppress the uncertainties. However, large adaptive gains may lead to high frequency oscillation and system instability.

Performance guarantee related to nominal tracking is also an issue in research works for the development of distributed adaptive control architectures and realizing a desirable tracking. Therefore, it is important to evaluate the performance degradation caused by adaptation in terms of the performance index of the reference model which achieves optimal tracking and a robust performance.

This dissertation proposes four approaches for adaptive control of uncertain dynamical system for tracking problem. Each approach includes the evaluation of performance degradation of the adaptive control law. First is model reference adaptive control scheme for optimal LQ tracking in the presence of uncertainties. A new reference model selection is introduced by using linear quadratic regulator theory. Second is adaptive control proposed for H_∞ tracking of uncertain dynamical systems. A reference model which achieves a robust tracking in the presence of L_2 disturbances is introduced by using H_∞ control with transients. Third is a distributed model reference adaptive control scheme for optimal tracking of an interconnected dynamical system in the presence of system/interconnection uncertainties. Here, an adaptive control law is developed for the uncertain interconnected dynamical system, where it employs the specified reference model. The final is distributed adaptive control proposed for H_∞ tracking of interconnected uncertain dynamical systems. It is shown that the boundedness of the error dynamics behaviors as well as zero tracking error in the steady state is guaranteed by the proposed distributed adaptive control in the presence of disturbances and uncertainties. An explicit error bound of tracking is also established. Numerical examples were discussed to show applicability of the theoretical findings.

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Chapter 1

Introduction

1.1 Background

Every dynamical system is subjected to system uncertainty due to parameter variations, disturbances, and model simplifications used to describe the physical system behavior. Although the fixed gain robust controllers can deal with uncertainties, the adaptive control approaches work effectively with less information about the uncertain dynamical system to be controlled. An adaptive control system automatically compensates for variations in system dynamics by adjusting the controller characteristics so that the overall system performance is maintained at a desirable level [1].

Such controllers have been used in several applications with varying or uncertain parameters such that high-performance aerial vehicles, large scale systems, underwater vehicles, vibration suppression, flow control, drug dosing control system, and so on [2]. For example, a dissertation [3] focuses on the high performance robust control of nonlinear systems in the presence of parametric uncertainties and uncertain nonlinearities (e.g., disturbances) and its application to the control of mechanical systems.

In fact, adaptive control can achieve a good performance without precise knowledge about systems to be controlled and it can deal with uncertain systems because it can tolerate large parametric uncertainties to ensure the desired tracking performance. An adaptive control architecture was also proposed in [4, 5] to the application of longitudinal-direction motion control of a nonlinear aircraft system. The response control of a base-isolated structure equipped with magneto-rheological dampers under earthquake excitations [6] and active suppression to isolate payloads from floor vibrations and direct disturbance forces [7] is also proposed for facing the challenges of uncertainty.

The nature of fixed gain controllers is static because they cannot improve their performance based on their past and current measurements, while most systems are naturally dynamic. These fixed gain controllers are only suitable for time invariant systems, while most process dynamics in nature are inherently time-varying. For example, aircraft dynamics due to variations in atmospheric conditions, speed, and altitude. Moreover, the performance of modern linear time-invariant (LTI) robust controls like H_2, H_∞ [8], and μ-synthesis [9], and other fixed-gain controls may deteriorate for large uncertainty, caused by unexpected conditions such as failure or degradation in system components.

However, adaptive controller can improve its performance online and can be less dependent to the accuracy of the mathematical models of the system unlike the fixed gain controllers which heavily rely on mathematical models. It automatically compensates for variations in system dynamics by adjusting the controller characteristics so that the overall system performance remains the same, or rather maintained at optimum level. Adaptive controls can also be easily implemented as an addition on top of other control schemes (e.g., robust and optimal controls, which are of fixed gain nature). While robust control is a powerful method to overcome parameter variations of the system model, it also depends on the range of uncertainty domain itself [10].

Therefore, to cope with the challenging issue of tracking problems under uncertainty, modern control approaches like adaptive control need to satisfy a certain degree of robustness and adaptivity. Adaptive controllers have been widely used control schemes because of their adaptation mechanisms that mostly rely on the initial knowledge as well as previous and current measurements about the process or plant being controlled [10]. The distinctive feature of adaptive control is that the knowledge of plant parameters are not required and can still achieve boundedness of signals, system stability, and/or asymptotic output/state tracking.

Generally adaptive controllers are classified as direct and indirect adaptive controls, deterministic and stochastic systems as well as feedback and feed-forward adaptive controls. To begin with the well-known model reference adaptive control (MRAS) solved via MIT and the Lyapunov rules, it is used such that the closed loop system is matched with a reference model which has been extensively studied and its theory and design techniques have been developed for decades [11]. In MRAC, the tracking error between the plant model under uncertainty and the reference model is used as a control signal to drive the adaptation and to suppress undesired effects of uncertainty [1]. In addition to the adaptability, a stable and optimal control approach is required to achieve optimal tracking performance with some sense of robustness that follows the desired reference model as closely as possible particularly in the presence of system uncertainty and external disturbances.

In this dissertation, an adaptive control for tracking problem which is robust in handling control effectiveness uncertainty, matched plant uncertainty, and external disturbance uncertainty of a dynamical system is presented.

1.2 Related Works

One of the important research topics in large scale systems is understanding the stability and performance analysis for uncertain interconnected systems. In fact, adaptive control has been extensively studied for various complex systems such as interconnected and/or large-scale systems under different perspectives. Many adaptive control approaches have been developed for systems with uncertain dynamics to track the desired trajectory [12–16]. However, they typically do not guarantee an optimal solution while estimating the unknown uncertainty parameters of the system dynamics. Poor transient performance due to the existence of large uncertainties such as control effectiveness failure or changes in system parameters is solved by increasing the adaptive gains to rapidly suppress the uncertainties and achieve fast adaptation. However, high learning rate may introduce high frequency oscillations, which lead to system instability with poor transient performance [17, 18]. Thus, an appropriate selection of control input is needed so that the tracking error is minimized. In addition, several approaches have also been developed to adaptive control for uncertain dynamical systems with disturbances. In certain cases, due to uncertainty and/or disturbances, an adaptive control system may have poor stability margin and thus poor transient response, which requires our further investigation about performance guarantees. See, for example, standard textbooks[19, 20] and/or some papers[21–23]. In this regard, this dissertation consists on this issue in the context of linear quadratic (LQ) tracking[24].

Since several uncertainties should be treated in large-scale systems composed of several subsystems and interconnections, decentralized adaptive control was proposed in [25, 26], where bounded disturbances and nonlinear uncertainties are considered. After that, such a decentralized adaptive scheme was further investigated in [27, 28], where it is proved that the output of the system can asymptotically track the output of the reference model if all decentralized controllers share their prior information. On the other hand, the paper [13] presented a distributed control, where each controller which applies to each subsystem uses the information about her neighbors. Performance guarantee is also an issue even in decentralized adaptive control. In fact, the paper [13] provided an error bound of tracking based on a Lyapunov solution of a given reference model. This technique has further been investigated in the context of several type of uncertain systems in [14, 15, 29]. In this regard, when we introduce a *performance index* and consider an optimal tracking, it is natural to explore a *performance guarantee with respect to the index*. However, an available result does not exist. For example, the paper [12] considered an adaptive optimal control for large-scale systems, where an asymptotic optimality was investigated, while any error bound of tracking is not provided. It is important to evaluate the performance degradation caused by adaptation in terms of the performance index of the reference model which achieves optimal tracking. Then, an appropriate selection of control input which incorporates the evaluation of performance degradation is needed so that the tracking error is minimized for the nominal system and an explicit error bound of the optimal tracking is obtained. For example, an adaptive control with performance enforcement is proposed for a class of uncertain dynamical systems that consist of actuated and unactuated portions that are physically interconnected to each other [29]. A decentralized adaptive control of interconnected systems is developed for stabilization and tracking via several approaches [30, 31].

Considering a wide array of applications of interconnected systems, another major research area is the development of distributed control architectures such that the subsystems perform given tasks through local interactions. Particularly, distributed adaptive control for a class of multiagent systems with model uncertainties is proposed to track the desired trajectory [32–34]. For instance, a distributed adaptive control approach is proposed for uncertain multi-agent systems with coupled dynamics [35–37]. A distributed adaptive control law is proposed for large-scale systems with unknown interconnection parameters [38] and for large-scale modular systems interconnected physically [13] via graph theoretic approach. As explained earlier since performance guarantee is also an issue in distributed adaptive control, a work [13] provided an error bound of tracking based on a Lyapunov solution of a given reference model. This technique has further been investigated in the context of several type of uncertain systems in [35–38]. When we introduce a *performance index* for realizing a desirable tracking, it is natural to explore a *performance guarantee with respect to the index*. However, there are few available results in this context. For example, a paper [37] considers a distributed adaptive control for large-scale systems, where boundedness of the internal signals is investigated, while any performance guarantee related to nominal tracking is not provided.

In this regard, this dissertation discuses on evaluation the performance degradation *caused by adaptation* in terms of the performance index which is used for designing the reference model as LQ optimal tracking [24]. Here, an appropriate selection of control input which incorporates the evaluation of performance degradation is needed so that the tracking error is minimized for the nominal system and an explicit error bound of the optimal tracking is obtained.

This research introduces a new set of adaptive control approaches for uncertain dynamical systems. The following approaches are proposed in this dissertation:

- Adaptive control for optimal LQ tracking
- Adaptive control for H_∞ tracking
- Distributed adaptive control for optimal LQ tracking
- Distributed adaptive control for H_∞ tracking

Each proposed approach has its corresponding chapter which contains detailed explanation of its process and analyses of its properties, including stability analysis and performance evaluation. Simulations are also provided for demonstrating the theoretical results of the analyses.

1.3 Purpose and Contribution

The purpose of this dissertation is to develop novel adaptive control schemes for optimal tracking of uncertain dynamical systems which is robust to disturbances. The resulting approaches and their details are as follows:

The first approach, considers an optimal tracking problem to an uncertain dynamical system using adaptive control scheme with a reference model with performance index. The linear quadratic regulator (LQR) theory is applied to the nominal system to optimize its response, where the optimal tracking gains are calculated by solving the algebraic Riccati equation (ARE)[9]. The main contribution of this approach is to provide an adaptive control for optimal tracking of uncertain dynamical systems. The reference model parameters designed are identical with those of the optimal tracking system but without uncertainties. That is, an optimal tracking control law for the nominal system is employed and an adaptive tracking control law for the uncertain system is derived. In addition, it is shown that the performance evaluation of the proposed adaptive control law in the context of optimal tracking. The control law is based on Lyapunov stability theory and is able to tune the gains of the control input and to suppress the influence of the uncertainties. The theoretical analysis shows that the proposed algorithm guarantees closed loop stability and convergence of the uncertain dynamical system state trajectories to desired state trajectories. Finally, numerical examples demonstrate the applicability of the theoretical results.

The second, proposes an adaptive control for H_∞ tracking of uncertain dynamical systems. To this end, employing H_∞ control with transients[39–41], a reference model which achieves a robust tracking in the presence of L_2 disturbances is introduced. Then an adaptive control law for uncertain dynamical systems is developed, where the law utilizes the specified reference model. It is shown that the proposed adaptive control guarantees the boundedness of the error dynamics behaviors in the presence of disturbances and uncertainties, where it achieves zero tracking error in the steady state as well. Furthermore, an explicit error bound of tracking, which enables us to evaluate a transient performance of the control system is established. Numerical examples illustrate that the theoretical results developed in this paper are useful.

The third, considers a class of large-scale dynamical systems with uncertain interconnection between the subsystems. A distributed control, where each controller which applies to each subsystem uses the information about her neighbors is investigated. First a performance index and consider an optimal tracking problem for each nominal subsystem introduced. In addition, a reference model as optimal tracking for the nominal subsystem is constructed, where a Riccati solution of an LQ regulator determines the model. Then a distributed adaptive tracking control law is developed for the interconnected uncertain dynamical system, where the Riccati solution for optimal tracking is used in the update rule of the adaptive gain. It is shown that the proposed adaptive control law achieves the desirable optimal tracking asymptotically as well as the boundedness of all signals. It is also established that an explicit error bound with respect to the nominal optimal tracking, where a role of the learning rate of the update rule is clarified. Numerical examples illustrate that the theoretical results developed in this paper are useful. A preliminary version of this approach was presented at a conference [24], where an adaptive tracking of a single plant is considered for a step-type reference signal. Then it is extended with a distributed adaptive tracking of an interconnected system for a general reference signal, which clarifies a possible performance guarantee for a type of adaptive control law.

The fourth, follows this line of research [24]. It focuses on a class of interconnected dynamical systems that is characterized by sets of uncertain dynamics with an unknown physical interconnection between these dynamics. Specifically, a distributed adaptive control is proposed for realizing a robust tracking of uncertain dynamical systems. To this end, employing H_∞ control with transients[39–41], a reference model which achieves an H_∞ tracking in the presence of L_2 disturbances is introduced. Then a distributed adaptive control law for uncertain dynamical systems is developed, where the law utilizes the specified reference model. It is shown that the proposed distributed adaptive control guarantees the boundedness of the error dynamics behaviors in the presence of disturbances and uncertainties, where it achieves zero tracking error in the steady state as well. Furthermore, an explicit error bound of tracking, which enables us to evaluate a transient performance of the control system is established.

Numerical examples are discussed to illustrate that the results developed in this dissertation are useful. However another scenario showing the effectiveness of these theoretical findings can be further applied to electrical power systems like adaptive generator exciter control and adaptive load-frequency control, aircraft control systems, and active control of vibration suppression. In addition the results can also be used as an effective approach for mechanical robot manipulator controller design due to the presence of nonlinearity and uncertainties in robot dynamic models. More generally, the results can also be applied to distributed coordinated tracking control for a class of uncertain multiagent systems and so on.

In a very general form table 1.1 summarizes the contribution of this dissertation.

Proposed Approach	Uncertain dynamical system	Reference signal	Tracking type	Reference model selection	Adaptive controller	Stability analysis	Performance evaluation
LQ tracking	Single plant	Step type for simplicity	Optimal	LQR theory	Single	Theorem 2.2	Theorem 2.4
H_∞ tracking	Single plant	Step type for simplicity	Robust	H_∞ control with transients	Single	Theorem 3.2	Theorem 3.4
Distributed LQ tracking	Uncertain interconnected system	General type	Optimal	LQR theory	Adaptive distributed	Theorem 4.2	Theorem 4.4
Distributed H_∞ tracking	Uncertain interconnected system	General type	Robust	H_∞ control with transients	Adaptive distributed	Theorem 5.2	Theorem 5.4

TABLE 1.1: Summary of the proposed approaches

1.4 Organization

This dissertation is organized as follows. Chapter 1 contains notations and definition used in this. Related works and some important concepts related to this study are also discussed in this chapter. The succeeding chapters discuss the proposed approaches for uncertain dynamical systems. In Chapter 2, optimal tracking problem to an uncertain dynamical system using LQ adaptive control scheme is presented. Chapter 3 introduces the concept of adaptive control for H_∞ tracking of uncertain dynamical systems. This approach is applied to interconnected uncertain dynamical systems with disturbance in Chapter 4. In Chapter 5, discuss the distributed adaptive control scheme for class of interconnected dynamical systems that is characterized by sets of uncertain dynamics with an unknown physical interconnection between these dynamics. Finally, the conclusion of this dissertation is stated in Chapter 6.

1.5 Preliminaries

In this dissertation, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of n-dimensional real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. In addition, P^T is written for the transpose of a real matrix P, Q^T for the transpose of a real matrix Q, R^{-1} for the inverse of a matrix R, rank S for the rank of a matrix S, and $\mathrm{tr}(U)$ for the trace of a square matrix U. Furthermore, it is defined

$$\|f\| = \left(\int_0^\infty f^T(t) f(t) dt \right)^{1/2},$$

that is, $\|f\|$ denotes the L_2 norm of a function f.

Chapter 2

Adaptive control for optimal LQ tracking of uncertain dynamical systems

2.1 Overview

As mentioned in the previous chapter, optimal tracking problem to an uncertain dynamical system using adaptive control scheme with a reference model and performance index will be discussed. This approach addressed the use of LQR theory which is applied to the nominal system in order to optimize the response of the uncertain dynamical system.

This chapter describes theoretical analysis that shows the performance evaluation of the proposed adaptive control law in the context of optimal tracking. The control law is based on Lyapunov stability theory which is able to tune the gains of the control input and to suppress the influence of the uncertainties. Theoretical analysis that shows that the proposed algorithm guarantees closed loop stability and convergence of the uncertain dynamical system state trajectories to the desired state trajectory will also be discussed in this chapter. Finally, numerical examples demonstrate the applicability of the theoretical results.

2.2 Problem formulation

Consider an uncertain dynamical system given by

$$\begin{aligned}\dot{x}_p(t) &= Ax_p(t) + B[\Lambda u_p(t) + \Delta(x_p(t))],\\ y_p(t) &= Cx_p(t),\end{aligned} \tag{2.1}$$

where $x_p(t) \in \mathbb{R}^n$ is the state vector which is assumed to be available for control, $u_p(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, and $y_p(t) \in \mathbb{R}^m$ is the controlled output. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ represent the nominal *known* part of this system, where the pair (A, B) is controllable and the pair (C, A) is observable. On the other hand, the matrix $\Lambda \in \mathbb{R}^{m\times m}$ and the vector-valued function $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ represent the uncertain *unknown* part of the system.

When $\Lambda = I$ and $\Delta(x_p) \equiv 0$, we have the nominal system

$$\begin{aligned}\dot{x}_n(t) &= Ax_n(t) + Bu_n(t),\\ y_n(t) &= Cx_n(t),\end{aligned} \tag{2.2}$$

where $x_n(t) \in \mathbb{R}^n$, $u_n(t) \in \mathbb{R}^m$, and $y_n(t) \in \mathbb{R}^m$ correspond to $x_p(t)$, $u_p(t)$, and $y_p(t)$ for the nominal system of (2.1), respectively.

In this regard, we introduce the following assumption for Λ and $\Delta(x_p)$.

Assumption 2.1. *The control effectiveness Λ is an unknown symmetric and positive definite matrix. The state dependent matched uncertainty $\Delta(x_p)$ is linearly parameterized as*

$$\Delta(x_p) = F\alpha(x_p),$$

where $F \in \mathbb{R}^{m\times s}$ is an unknown weight matrix and $\alpha : \mathbb{R}^n \to \mathbb{R}^s$ is the corresponding basis function.

For this system (2.1), we consider a reference signal $r(t) \in \mathbb{R}^m$ of a step type function

$$r(t) = \begin{cases} r_-, & t < 0,\\ r_+, & 0 \le t,\end{cases} \tag{2.3}$$

where r_- and r_+ are constant vectors. It is known that the controlled output $y_n(t)$ of (2.2) can achieve any r_+ in the steady state if and only if

$$\text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + m. \tag{2.4}$$

We assume this condition for (2.1) as well.

The objective of this paper is to construct an adaptive control law for $u_p(t)$ such that the output $y_p(t)$ of the given system (2.1) asymptotically tracks the reference signal $r(t)$ of (2.3) in the presence of the system uncertainty described by Λ and $\Delta(x_p)$ satisfying Assumption 2.1. In particular, we employ an optimal tracking control law for the nominal system (2.2) and derive an adaptive tracking control law for the uncertain system (2.1) with a performance guarantee related to the nominal optimality.

2.3 Reference model selection

We first consider the nominal system (2.2) and revisit a standard optimal tracking for the reference signal (2.3)[42].

It should be noted that there exist a unique state x_∞ and a unique control input u_∞ for which $y_n(t) = r_+$ in the steady state. They are calculated as

$$\begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ r_+ \end{bmatrix} \tag{2.5}$$

under the condition (2.4). We denote the variations of $x_n(t)$ and $u_n(t)$ from x_∞ and u_∞ by

$$\tilde{x}_n(t) = x_n(t) - x_\infty, \qquad \tilde{u}_n(t) = u_n(t) - u_\infty, \tag{2.6}$$

and the tracking error of the controlled output $y_n(t)$ by

$$e_n(t) = r(t) - y_n(t).$$

Using these notations, the variation system is defined by

$$\begin{aligned} \dot{\tilde{x}}_n(t) &= A\tilde{x}_n(t) + B\tilde{u}_n(t), \\ e_n(t) &= -C\tilde{x}_n(t). \end{aligned} \tag{2.7}$$

To obtain a good transient behavior of tracking to the reference signal $r(t)$, we apply linear quadratic regulator theory to the variation system (2.7) with the performance index

$$J_n = \int_0^\infty \{e_n^T(t)Qe_n(t) + \tilde{u}_n^T(t)R\tilde{u}_n(t)\}dt, \tag{2.8}$$

where $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric and positive definite matrices. Then the optimal control law which minimize J_n with respect to (2.7) is given by

$$\tilde{u}_n(t) = -R^{-1}B^TP\tilde{x}_n(t), \tag{2.9}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite solution of the Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + C^TQC = 0. \tag{2.10}$$

When the control law (2.9) is applied to the variation system (2.7), the resultant closed loop system is stable, and $\tilde{x}_n(t) \to 0$, $e_n(t) \to 0$ as $t \to \infty$, where the minimum value of J_n is given by

$$\min_{\tilde{u}_n} J_n = \tilde{x}_n^T(0)P\tilde{x}_n(0). \tag{2.11}$$

Using (2.5) and (2.6), we rewrite the control law (2.9) as

$$u_n(t) = Kx_n(t) + Hr(t) \tag{2.12}$$

for the nominal system (2.2), where

$$K = -R^{-1}B^TP \tag{2.13}$$

$$\begin{aligned} H &= \begin{bmatrix} -K & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} A+BK & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \left\{-C(A+BK)^{-1}B\right\}^{-1}. \end{aligned} \tag{2.14}$$

That is, the optimal tracking control law for the nominal system (2.2) is composed of a feedback from the state $x_n(t)$ and a feedforward from the reference signal $r(t)$. The resultant control system with (2.2) and (2.12) is described by

$$\begin{aligned} \dot{x}_n(t) &= (A + BK)x_n(t) + BHr(t), \\ y_n(t) &= Cx_n(t). \end{aligned} \tag{2.15}$$

When there is no uncertainty in the system (2.1), the optimal tracking (2.15) represents the best achievable behavior. We therefore consider an adaptive control framework for (2.1) which asymptotically realizes the optimal tracking (2.15). To this end, we employ the optimal tracking system (2.15) as the reference model. Then we develop a model reference adaptive control and derive its performance guarantee regarding the index (2.8).

2.4 Adaptive control scheme

Let us go back to the uncertain dynamical system (2.1). Since we have selected (2.15) as the reference model, we rewrite (2.1) as

$$\begin{aligned} \dot{x}_p(t) &= (A + BK)x_p(t) + BHr(t) \\ &\qquad + B\Lambda[u_p(t) + \delta(t)], \\ y_p(t) &= Cx_p(t), \end{aligned} \tag{2.16}$$

where we define

$$\begin{aligned} \delta(t) &= -\Lambda^{-1}[Kx_p(t) + Hr(t) - \Delta(x_p(t))] \\ &= -W\sigma(x_p(t), r(t)). \end{aligned}$$

In fact, from Assumption 2.1, the signal $\delta(t)$ must be parameterized by using an unknown weight $W \in \mathbb{R}^{m\times q}$ and the corresponding basis function $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^q$ which contains $x_p(t)$, $r(t)$, and $\alpha(x_p(t))$, where $q \leq n + m + s$.

Then we introduce an adaptive feedback control law

$$u_p(t) = \hat{W}(t)\sigma(x_p(t), r(t)), \tag{2.17}$$

where we define the update rule of the adaptive control gain $\hat{W}(t) \in \mathbb{R}^{m\times q}$ as

$$\dot{\hat{W}}(t) = -\eta B^T P(x_p(t) - x_n(t))\sigma^T(x_p(t), r(t)). \tag{2.18}$$

It should be noted that $x_n(t)$ of (2.18) is generated by (2.15). The learning rate η is a positive real number and P is the symmetric and positive definite solution of the Riccati equation (2.10).

Now, let us define the errors from the ideal case as

$$x_e(t) = x_p(t) - x_n(t), \qquad y_e(t) = y_p(t) - y_n(t),$$
$$W_e(t) = \hat{W}(t) - W,$$

where $x_n(t)$ and $y_n(t)$ are from (2.15). With (2.15), (2.16), (2.17), and (2.18), we have

$$\dot{x}_e(t) = (A + BK)x_e(t) + B\Lambda W_e(t)\sigma(x_p(t), r(t)),$$
$$y_e(t) = Cx_e(t), \tag{2.19}$$
$$\dot{W}_e(t) = -\eta B^T P x_e(t)\sigma^T(x_p(t), r(t)) \tag{2.20}$$

which describes the error dynamics from the reference model (2.15).

The next theorem presents the result of this section.

Theorem 2.2. *Consider the uncertain dynamical system described by* (2.1) *subject to Assumption 2.1. Consider, in addition, the reference model given by* (2.15) *and the adaptive feedback controller given by* (2.17) *and* (2.18)*. Then, the solution* $(x_e(t), W_e(t))$ *given by* (2.19) *and* (2.20) *is Lyapunov stable and*

$$\lim_{t\to\infty} y_e(t) = 0$$

for all $(x_e(0), W_e(0))$*.*

Proof. Consider a candidate of Lyapunov function

$$V(x_e, W_e) = x_e^T P x_e + \frac{1}{\eta}\mathrm{tr} W_e^T \Lambda W_e,$$

where η and P are taken from (2.18) and Λ of (2.1) satisfies Assumption 2.1, which means that $P = P^T > 0$ and $\Lambda = \Lambda^T > 0$. Thus the function $V(x_e, W_e)$ is in fact a continuously differentiable function such that $V(0, 0) = 0$ and $V(x_e, W_e) > 0$ for all $(x_e, W_e) \neq (0, 0)$.

Differentiating this candidate along the trajectories of (2.19) and (2.20), we have

$$
\begin{aligned}
&\dot{V}(x_e(t), W_e(t)) \\
&= \dot{x}_e^T(t) P x_e(t) + x_e^T(t) P \dot{x}_e(t) + \frac{2}{\eta} \mathrm{tr} \dot{W}_e^T(t) \Lambda W_e(t) \\
&= x_e^T(t) \left((A+BK)^T P + P(A+BK) \right) x_e(t) \\
&\quad + 2 x_e^T(t) P B \Lambda W_e(t) \sigma(x_p(t), r(t)) \\
&\quad - 2 \mathrm{tr} \left(\sigma(x_p(t), r(t)) x_e^T(t) P B \Lambda W_e(t) \right) \\
&= x_e^T(t) \left((A+BK)^T P + P(A+BK) \right) x_e(t) \\
&= -x_e^T(t) \left(C^T Q C + K^T R K \right) x_e(t), \qquad (2.21)
\end{aligned}
$$

where we use the fact that the Riccati equation (2.10) can be rewritten as

$$
(A+BK)^T P + P(A+BK) = -(C^T Q C + K^T R K)
$$

with K of (2.13). Since $Q = Q^T > 0$ and $R = R^T > 0$, we see that

$$
\dot{V}(x_e(t), W_e(t)) \leq 0,
$$

which means that

$$
V(x_e(t), W_e(t)) \leq V(x_e(0), W_e(0))
$$

holds $\forall t \geq 0$. Hence the solution $(x_e(t), W_e(t))$ given by (2.19) and (2.20) is Lyapunov stable.

Now, let us recall a standard fact

$$
\begin{aligned}
&\int_0^t \dot{V}(x_e(\tau), W_e(\tau)) d\tau \\
&= V(x_e(t), W_e(t)) - V(x_e(0), W_e(0)),
\end{aligned}
$$

which holds for all $t \geq 0$. With (2.21), we have

$$
\begin{aligned}
-\dot{V}(x_e(t), W_e(t)) &= x_e^T(t) \left(C^T Q C + K^T R K \right) x_e(t) \\
&\geq y_e^T(t) Q y_e(t).
\end{aligned}
$$

Note also that $V(x_e(t), W_e(t)) \geq 0$.

Thus we obtain

$$\begin{aligned}
&\int_0^t y_e^T(\tau)Qy_e(\tau)d\tau \\
&\leq -\int_0^t \dot{V}(x_e(\tau), W_e(\tau))d\tau \\
&\leq -\int_0^t \dot{V}(x_e(\tau), W_e(\tau))d\tau + V(x_e(t), W_e(t)) \\
&= V(x_e(0), W_e(0))
\end{aligned} \tag{2.22}$$

for all $t \geq 0$.

We therefore see that

$$\int_0^\infty y_e^T(t)Qy_e(t)dt \leq V(x_e(0), W_e(0)), \tag{2.23}$$

Furthermore, since $\dot{y}_e(t) = C\dot{x}_e(t)$ and $\dot{x}_e(t)$ of (2.19) is represented by the signals $x_e(t)$, $W_e(t)$, and $r(t)$, we can see that

$$\sup_{t\geq 0} \left(\dot{y}_e^T(t)Q\dot{y}_e(t)\right) < \infty. \tag{2.24}$$

In fact, $x_e(t)$ and $W_e(t)$ are bounded as we have proved above, while $r(t)$ of (2.3) is bounded as we have assumed.

The boundedness of $x_p(t)$ follows the boundedness of $x_e(t)$ and $x_n(t)$ of (2.15). Using (2.23), (2.24), and A.1 we conclude that the tracking error $y_e(t)$ satisfy

$$\lim_{t\to\infty} y_e(t) = 0$$

with $Q = Q^T > 0$. for any $(x_e(0), W_e(0))$. □

Remark 2.3. Theorem 2.2 establishes Lyapunov stability for the error dynamics of the proposed adaptive control in the presence of uncertainties. The theorem also shows that zero steady state tracking error is achieved by this control.

2.5 Performance evaluation

Since the proposed adaptive control given by (2.17) and (2.18) employs the *optimal* tracking system (2.15) as the reference model, one of our interests should be to evaluate the performance degradation from the optimal response.

To this end, we first rewrite the minimum value (2.11) of the performance index (2.8) as

$$\min_{\tilde{u}_n} J_n = \int_0^\infty \tilde{x}_n^T(t) \left(C^T QC + K^T RK\right) \tilde{x}_n(t)dt, \tag{2.25}$$

where K is the optimal gain (2.13). Referring the above, we define a performance index for the adaptive control as

$$J_e = \int_0^\infty x_e^T(t) \left(C^T QC + K^T RK\right) x_e(t)dt. \tag{2.26}$$

This index (2.26) is in fact suitable for evaluating the degradation caused by adaptation since its weight matrix coincides with that of (2.25) and J_e becomes 0 if the perfect tracking $x_p(t) = x_n(t)$ is achieved.

For this index (2.26), we have the following result.

Theorem 2.4. *Consider the uncertain dynamical system described by* (2.1) *subject to Assumption 2.1. Consider, in addition, the reference model given by* (2.15) *and the adaptive feedback controller given by* (2.17) *and* (2.18)*. Then, the index* (2.26) *is bounded as*

$$J_e \leq x_e^T(0)Px_e(0) + \frac{1}{\eta}\mathrm{tr}W_e^T(0)\Lambda W_e(0). \tag{2.27}$$

Proof. In the proof of Theorem 2.2, we have established (2.21) and (2.22), which implies that

$$\begin{aligned}\int_0^t x_e^T(\tau) \left(C^T QC + K^T RK\right) x_e(\tau)d\tau \\ \leq V(x_e(0), W_e(0))\end{aligned}$$

holds for all $t \geq 0$. Thus it turns out that

$$\begin{aligned}\int_0^\infty x_e^T(t) \left(C^T QC + K^T RK\right) x_e(t)dt \\ \leq V(x_e(0), W_e(0)),\end{aligned}$$

which establishes the bound (2.27). □

Remark 2.5. The upper bound given by Theorem 2.4 guarantees the transient performance of the proposed adaptive control. It shows that the performance of the adaptive controller applied to the uncertain dynamical system captured by $x_e(t)$ cannot be more than the right hand side of (2.27). In addition, if we make the learning rate η large, the transient performance will be better which will be confirmed in the numerical example below.

Regarding the conservativeness of the performance evaluation if perfect tracking is achieved ($x_e(t) = x_p(t) - x_n(t) = 0$) and there is no estimation error ($W_e(t) = \hat{W}(t) - W = 0$), then the performance index $J_e \leq 0$.

2.6 Numerical example

Consider an uncertain second order mass-spring-damper system shown in Fig. 2.1, where a mass is hanging on a spring and damper. The equation of motion is

$$M_p\ddot{x}(t) + C_p\dot{x}(t) + K_p x(t) = u(t), \tag{2.28}$$

where M_p, C_p, and K_p refers to mass, damping constant and spring constant of the system respectively, which are unknown and uncertain parameters.

Let us introduce the state $x_p(t)$ as

$$x_p(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}.$$

Then the mass-spring-damper system (2.28) can be represented as the form of (2.1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$\Lambda = \frac{1}{M_p}, \quad F = \begin{bmatrix} -\frac{K_p}{M_p} & -\frac{C_p}{M_p} \end{bmatrix}, \quad \alpha(x_p(t)) = x_p(t),$$

where the uncertain parameters are represented as Λ and F which satisfy Assumption 2.1.

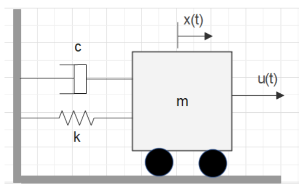

FIGURE 2.1: Mass-spring-damper system

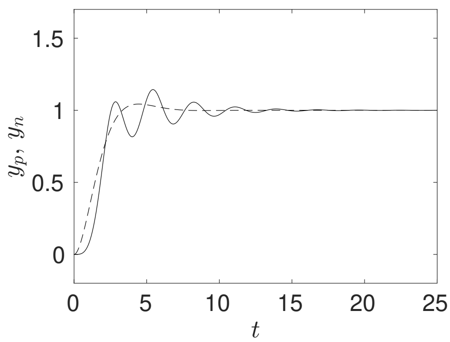

FIGURE 2.2: Output with Riccati P ($\eta = 5$)

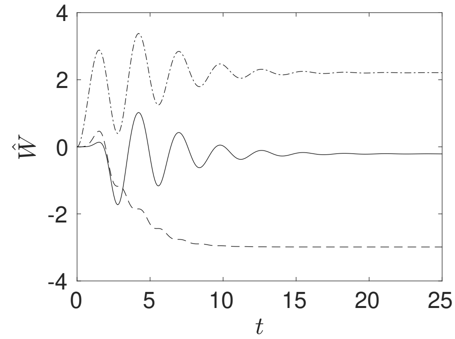

FIGURE 2.3: Adaptive gains with Riccati P ($\eta = 5$)

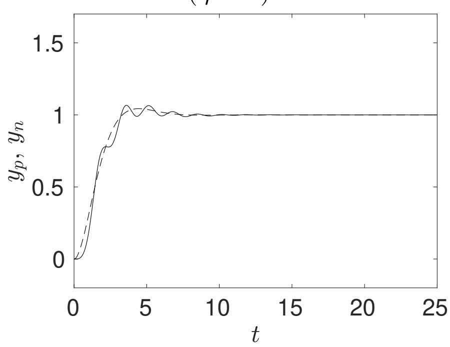

FIGURE 2.4: Output with Riccati P ($\eta = 15$)

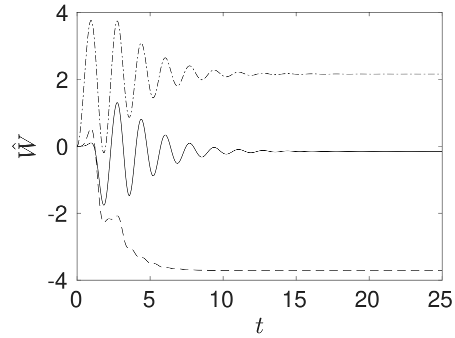

FIGURE 2.5: Adaptive gains with Riccati P ($\eta = 15$)

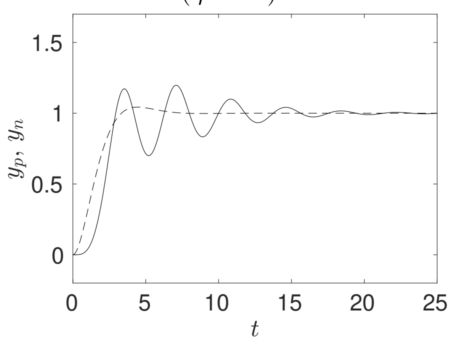

FIGURE 2.6: Output with Lyapunov P ($\eta = 5$)

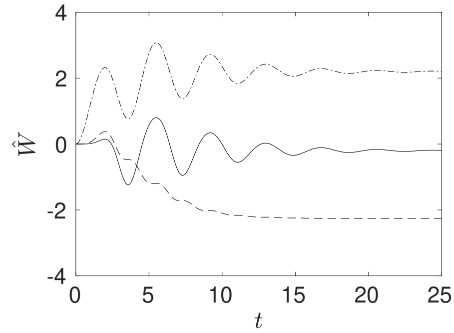

FIGURE 2.7: Adaptive gains with Lyapunov P ($\eta = 5$)

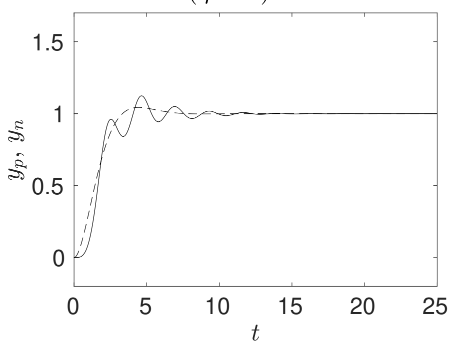

FIGURE 2.8: Output with Lyapunov P ($\eta = 15$)

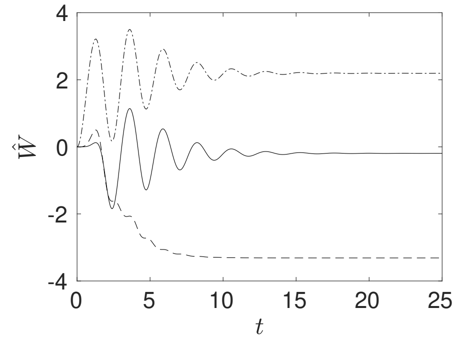

FIGURE 2.9: Adaptive gains with Lyapunov P ($\eta = 15$)

Since (A, B) is controllable, (C, A) is observable, and the condition (2.4) is satisfied, we can obtain an optimal tracking control law (2.12) for the nominal system (2.2). In fact, we have a symmetric and positive definite solution P of (2.10) as

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \tag{2.29}$$

when we choose $Q = 1$ and $R = 1$ in (2.8). Then the optimal tracking gains are

$$K = \begin{bmatrix} -1 & -\sqrt{2} \end{bmatrix}, \qquad H = 1. \tag{2.30}$$

In this case, the proposed adaptive control gain (2.18) employs the Riccati solution P of (2.29), where

$$\sigma(x_p(t), r(t)) = \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}.$$

We remark that the standard adaptive control gain [1] (see B.5) is also updated according to the same form (2.18), but P is chosen as a Lyapunov solution of

$$P(A + BK) + (A + BK)^T P = -Q.$$

For the reference model (2.15) with the gains (2.30), an example is

$$P = \frac{1}{2} \begin{bmatrix} 2\sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \tag{2.31}$$

with $Q = I$.

Generally, Theorem 2.2 and 2.4 shows the boundedness and convergence of all signals including $\hat{W}(t)$. However, W can be calculated as:

$$W = \Lambda^{-1} \begin{bmatrix} K & -F & H \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 - 3\sqrt{2} \end{bmatrix}$$

Let us compare the proposed adaptive control based on the Riccati solution (2.29) with the standard adaptive control [1] based on the Lyapunov solution (2.31) via a numerical example. We set the unknown and uncertain parameters of (2.28) as $M_p = 3$, $C_p = 1$, and $K_p = 2$. We set all of the initial states and gains as zero.

Figs. 2.2-2.5 show the step responses and the gain behaviors of the proposed adaptive control based on the Riccati solution (2.29), where Figs. 2.2-2.3 are with $\eta = 5$, while Figs. 2.4-2.5 are with $\eta = 15$. On the other hand, Figs. 2.6-2.9 show the step responses and the gain behaviors of the standard adaptive control based on the Lyapunov solution (2.31), where Figs. 2.6-2.7 are with $\eta = 5$, while Figs. 2.8-2.9 are with $\eta = 15$. Note that $y_p(t)$ and $y_n(t)$ are indicated as solid and dashed lines, respectively, in Figs. 2.2, 2.4, 2.6, and 2.8. The elements $\hat{W}_1(t), \hat{W}_2(t)$ and $\hat{W}_3(t)$ of the adaptive gain $\hat{W}(t) = \begin{bmatrix} \hat{W}_1(t) & \hat{W}_2(t) & \hat{W}_3(t) \end{bmatrix}$ are indicated as solid, dashed, and dash-dotted lines, respectively, in Figs 3, 5, 7 and 9. In Figs. 2.2-2.5, all signals are bounded and $y_p(t)$ tends to $y_n(t)$ as t tends to infinity, which is consistent with Theorem 2.2. Furthermore, comparing Fig. 2.2 and Fig. 2.4, we see that a larger η gives a better performance, which is consistent with Theorem 2.4. Regarding optimality, comparing Fig. 2.2 and Fig. 2.6 or Fig. 2.4 and Fig. 2.8, we see that the proposed adaptive control shows a better transient response. This justifies employing the Riccati solution in the update rule of the adaptive control gain.

2.7 Summary

In this chapter, model reference adaptive control scheme is proposed for optimal *LQ* tracking in the presence of uncertainties. A new reference model selection is introduced by using linear quadratic regulator theory. That is, we employ an optimal tracking control law for the nominal system and derive an adaptive tracking control law for the uncertain system with a performance guarantee related to the nominal optimality. Then the adaptive control law achieves the desired behavior such that the output of the uncertain dynamical system asymptotically tracks a given reference signal in the presence of the system uncertainty. The optimal and stable performance of the uncertain dynamical system is guaranteed by the LQR controller introduced to the nominal system and the adaptive tuning law developed based on Lyapunov stability theory to compensate the uncertainties. Numerical examples show that the proposed control algorithm is able to adaptively track the optimal response of the reference model and to suppress the influence of the uncertainties.

Chapter 3

Adaptive control for H_∞ tracking of uncertain dynamical systems

3.1 Overview

The previous chapter introduced optimal tracking for uncertain dynamical systems using LQR control theory approach for stability analysis and evaluation of performance degradation. This chapter proposes an adaptive H_∞ control for robust tracking of uncertain dynamical systems by employing H_∞ control with transients[39–41] in the presence of L_2 disturbances. A reference model which achieves a robust tracking in the presence of L_2 disturbances is introduced by using H_∞ control with transients. Then an adaptive control law is developed for uncertain dynamical systems, where it employs the specified reference model.

This chapter shows that the proposed adaptive control guarantees the boundedness of the error dynamics as well as zero tracking error in the steady state is guaranteed by the proposed adaptive control in the presence of disturbances and uncertainties. Furthermore, an explicit error bound of tracking, which enables us to evaluate a transient performance of the control system is also established. Finally, numerical examples illustrate that the theoretical results developed in this paper are useful.

3.2 Problem formulation

Consider an uncertain dynamical system given by

$$\begin{aligned}\dot{x}_p(t) &= Ax_p(t) + B[\Lambda u_p(t) + \Delta(x_p(t))] + Ed(t),\\ x_p(0) &= x_{p0},\\ y_p(t) &= Cx_p(t),\end{aligned} \tag{3.1}$$

where $x_p(t) \in \mathbb{R}^n$ is the state vector which is assumed to be available for control, $u_p(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, $y_p(t) \in \mathbb{R}^m$ is the controlled output, and $d(t) \in \mathbb{R}^p$ is an *unknown* disturbance. Here we assume that $d \in L_\infty \cap L_2$ and $\dot{d} \in L_\infty$. In other words, we assume that $d(t)$ satisfies

$$\sup_{t\geq 0} d^T(t)d(t) < \infty, \qquad \int_0^\infty d^T(t)d(t)dt < \infty,$$
$$\sup_{t\geq 0} \dot{d}^T(t)\dot{d}(t) < \infty.$$

Notice here that we have $\lim_{t\to\infty} d(t) = 0$ if $d \in L_2$ and $\dot{d} \in L_\infty$ [43]. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $E \in \mathbb{R}^{n\times p}$ represent the nominal *known* part of this system, where the pair (A, B) is controllable and the pair (C, A) is observable. On the other hand, the matrix $\Lambda \in \mathbb{R}^{m\times m}$ and the vector-valued function $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ represent the uncertain *unknown* part of the system. Here we introduce the following assumption.

Assumption 3.1. *The control effectiveness Λ is an unknown symmetric and positive definite matrix. The state dependent matched uncertainty $\Delta(x_p)$ is linearly parameterized as*

$$\Delta(x_p) = F\alpha(x_p), \tag{3.2}$$

where $F \in \mathbb{R}^{m\times s}$ is an unknown weight matrix and $\alpha : \mathbb{R}^n \to \mathbb{R}^s$ is the corresponding given basis function.

For this system (3.1), we define its nominal system as

$$\begin{aligned}\dot{x}_n(t) &= Ax_n(t) + Bu_n(t) + Ed(t), \qquad x_n(0) = x_{p0}\\ y_n(t) &= Cx_n(t),\end{aligned} \tag{3.3}$$

where $x_n(t) \in \mathbb{R}^n$, $u_n(t) \in \mathbb{R}^m$, and $y_n(t) \in \mathbb{R}^m$ correspond to $x_p(t)$, $u_p(t)$, and $y_p(t)$ for the nominal system of (3.1), respectively. That is, the system (3.1) takes its nominal behavior when $\Lambda = I$ and $\Delta(x_p) \equiv 0$.

In this chapter, we consider a reference signal $r(t) \in \mathbb{R}^m$ of a step type function

$$r(t) = \begin{cases} r_-, & t < 0, \\ r_+, & 0 \leq t, \end{cases} \tag{3.4}$$

where r_- and r_+ are constant vectors. It is known that the controlled output $y_n(t)$ of (3.3) can achieve any r_+ in the steady state if and only if

$$\text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + m. \tag{3.5}$$

We assume this condition for (3.1) as well.

The objective of this chapter is to construct a robust adaptive control law for $u_p(t)$ such that the output $y_p(t)$ of the given system (3.1) asymptotically tracks the reference signal $r(t)$ of (3.4) in the presence of the L_2 disturbance $d(t)$ and the system uncertainty described by Λ and $\Delta(x_p)$ satisfying Assumption 3.1. In particular, we provide a reference model via an H_∞ type robust tracking control for the nominal system (3.3) and derive a robust adaptive tracking control law for the uncertain system (3.1) with a performance guarantee related to an H_∞ type measure. Here, the performance measure is the induced norm of the tracking error over all possible disturbances and initial states.

3.3 Reference model selection

In this section, we select a suitable reference model for our adaptive control. To this end, we employ an H_∞ control design [39–41] in order to design a robust tracking control law for the nominal system (3.3) to the reference signal (3.4).

We first introduce a variation system for tracking control. Since $\lim_{t\to\infty} d(t) = 0$, there exist a state x_∞ and a control input u_∞ which achieve $y_n(t) = r_+$ in the steady state, independently of $d(t)$. They are uniquely determined as

$$\begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ r_+ \end{bmatrix} \tag{3.6}$$

under the condition (3.5). We write the variations of $x_n(t)$ and $u_n(t)$ from x_∞ and u_∞ as

$$\begin{aligned} \tilde{x}_n(t) &= x_n(t) - x_\infty, \qquad\qquad \tilde{x}_{n0} = x_{p0} - x_\infty \\ \tilde{u}_n(t) &= u_n(t) - u_\infty, \end{aligned} \tag{3.7}$$

and the tracking error of the controlled output $y_n(t)$ as

$$e_n(t) = r(t) - y_n(t).$$

Using these notations, the variation system is defined by

$$\begin{aligned} \dot{\tilde{x}}_n(t) &= A\tilde{x}_n(t) + B\tilde{u}_n(t) + Ed(t), \qquad\qquad \tilde{x}_n(0) = \tilde{x}_{n0}, \\ e_n(t) &= -C\tilde{x}_n(t). \end{aligned} \tag{3.8}$$

In this way, we can recast the original *tracking* problem as a *stabilization* problem of the variation system. That is, if a feedback control law

$$\tilde{u}_n(t) = K\tilde{x}_n(t) \tag{3.9}$$

stabilizes the variation system given by (3.8), it turns out that $\lim_{t\to\infty} \tilde{x}_n(t) = 0$, i.e., $\lim_{t\to\infty} y_n(t) = r_+$.

In order to select a suitable control law (3.9) for tracking in the presence of L_2 disturbance $d(t)$, we employ a robust control design which is called *H_∞ control with transients* [39]. That is, in this chapter, we utilize a performance specification

$$\sup \left\{ \frac{\|e_n\|^2}{\|d\|^2 + \tilde{x}_{n0}^T R \tilde{x}_{n0}} \right\}^{1/2} < \gamma, \tag{3.10}$$

where $\gamma \in \mathbb{R}$ is a specified positive number and $R > 0$ is a positive definite matrix which is weight for the initial uncertainty. The supremum is taken over all $\tilde{x}_{n0} \in \mathbb{R}^n$ and $d \in L_2$ which satisfy $\|d\|^2 + \tilde{x}_{n0}^T R \tilde{x}_{n0} \neq 0$. Then, we see that there exists a state feedback (3.9) which stabilizes (3.8) and achieves (3.10) if and only if there exist $X = X^T \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{m\times n}$ which satisfy the linear matrix inequalities

$$\begin{aligned} &\begin{bmatrix} AX + XA^T + BG + G^T B^T & E & XC^T \\ E^T & -\gamma^2 I & 0 \\ CX & 0 & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} X & I \\ I & \gamma^2 R \end{bmatrix} > 0, \end{aligned} \tag{3.11}$$

where such a state feedback gain K of (3.9) is obtained by

$$K = GX^{-1}. \tag{3.12}$$

This is a direct consequence of the existing work [41]. Throughout this chapter, we assume that such a X exists for a given $\gamma > 0$.

As a matter of fact, the performance (3.10) is guaranteed by the feedback gain (3.12) as follows. When we define

$$P = X^{-1} \tag{3.13}$$

and use (3.12), we can rewrite (3.11) as

$$\begin{aligned} & P(A+BK) + (A+BK)^T P + \frac{1}{\gamma^2} PEE^T P \\ & \quad + C^T C < 0, \end{aligned} \tag{3.14}$$

$$\gamma^2 R > P = P^T > 0, \tag{3.15}$$

where we see that $A + BK$ is Hurwitz, i.e., the resultant closed-loop system is stable. The inequality (3.14) together with (3.8) and (3.9) implies that

$$\begin{aligned} & \frac{d}{dt}\left(\tilde{x}_n^T(t) P \tilde{x}_n(t)\right) \\ & = \tilde{x}_n^T(t)\left(P(A+BK) + (A+BK)^T P\right)\tilde{x}_n(t) \\ & \quad + \tilde{x}_n^T(t) PEd(t) + d^T(t) E^T P \tilde{x}_n(t) \\ & < -\frac{1}{\gamma^2}\tilde{x}_n^T(t) PEE^T P \tilde{x}_n(t) - \tilde{x}_n^T(t) C^T C \tilde{x}_n(t) \\ & \quad + \tilde{x}_n^T(t) PEd(t) + d^T(t) E^T P \tilde{x}_n(t) \\ & = -\left(\gamma d(t) - \frac{1}{\gamma} E^T P \tilde{x}_n(t)\right)^T \left(\gamma d(t) - \frac{1}{\gamma} E^T P \tilde{x}_n(t)\right) \\ & \quad - \tilde{x}_n^T(t) C^T C \tilde{x}_n(t) + \gamma^2 d^T(t) d(t) \\ & \leq -e_n^T(t) e_n(t) + \gamma^2 d^T(t) d(t) \end{aligned} \tag{3.16}$$

for any $\tilde{x}_n(t) \neq 0$. Integrating this inequality from 0 to ∞, with (3.15), we have

$$\|e_n\|^2 - \gamma^2 \|d\|^2 < \tilde{x}_{n0}^T P \tilde{x}_{n0} < \gamma^2 \tilde{x}_{n0}^T R \tilde{x}_{n0},$$

where we use $\lim_{t\to\infty} \tilde{x}_n^T(t) P \tilde{x}_n(t) = 0$ which is guaranteed by the closed-loop stability. The above inequality shows that the performance specification (3.10) holds.

Using (3.6) and (3.7), we rewrite the control law (3.9) as

$$\begin{aligned} u_n(t) &= Kx_n(t) - Kx_\infty + u_\infty \\ &= Kx_n(t) + Hr(t) \end{aligned} \tag{3.17}$$

for the nominal system (3.3), where K is given by (3.12) and H is represented as

$$\begin{aligned} H &= \begin{bmatrix} -K & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \left\{ -C(A+BK)^{-1}B \right\}^{-1}. \end{aligned} \tag{3.18}$$

The resultant control system with (3.3) and (3.17) is described by

$$\begin{aligned} \dot{x}_n(t) &= (A+BK)x_n(t) + BHr(t) + Ed(t), \qquad x_n(0) = x_{p0} \\ y_n(t) &= Cx_n(t). \end{aligned} \tag{3.19}$$

Since it achieves the performance specification (3.10), it is suitable as a reference model for adaptive control. However, the *unknown* $d(t)$ should be excluded in the reference model. Also, the initial states of the given system and the reference model may be different. Thus, in this chapter, we use the system

$$\begin{aligned} \dot{x}_m(t) &= (A+BK)x_m(t) + BHr(t), \qquad x_m(0) = x_{m0} \\ y_m(t) &= Cx_m(t) \end{aligned} \tag{3.20}$$

as the reference model for adaptive control, where $x_m(t) \in \mathbb{R}^n$ is the state, K is given by (3.12) based on the LMIs (3.11), and H is given by (3.18) with this K.

3.4 Adaptive control scheme

Let us go back to the uncertain dynamical system (3.1). According to the selected reference model (3.20), we rewrite (3.1) as

$$\begin{aligned} \dot{x}_p(t) &= (A+BK)x_p(t) + BHr(t) + Ed(t) \\ &\qquad + B\Lambda[u_p(t) + \delta(t)], \qquad x_p(0) = x_{p0} \\ y_p(t) &= Cx_p(t), \end{aligned} \tag{3.21}$$

where we define

$$\begin{aligned}\delta(t) &= -\Lambda^{-1}[Kx_p(t) + Hr(t) - \Delta(x_p(t))] \\ &= -W\sigma(x_p(t), r(t)).\end{aligned}$$

In fact, from Assumption 3.1, the signal $\delta(t)$ must be parameterized by using an unknown weight $W \in \mathbb{R}^{m\times q}$ and the corresponding basis function $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^q$ which contains $x_p(t)$, $r(t)$, and $\alpha(x_p(t))$, where $q \leq n + m + s$.

Then we introduce an adaptive feedback control law

$$u_p(t) = \hat{W}(t)\sigma(x_p(t), r(t)), \tag{3.22}$$

where we define the update rule of the adaptive control gain $\hat{W}(t) \in \mathbb{R}^{m\times q}$ as

$$\begin{aligned}\dot{\hat{W}}(t) &= -\frac{\eta}{\gamma^2}B^T P(x_p(t) - x_m(t))\sigma^T(x_p(t), r(t)) \\ \hat{W}(0) &= \hat{W}_0.\end{aligned} \tag{3.23}$$

Notice that $x_m(t)$ of (3.23) is generated by (3.20). The learning rate η is any positive real number, the specification γ for the nominal system is the one in (3.10), and P is a symmetric and positive definite matrix defined by (3.13), where X is a solution of the linear matrix inequalities (3.11).

Now, let us define the errors from the ideal case as

$$\begin{aligned}x_e(t) &= x_p(t) - x_m(t), & x_{e0} &= x_{p0} - x_{m0} \\ y_e(t) &= y_p(t) - y_m(t), & & \\ W_e(t) &= \hat{W}(t) - W, & W_{e0} &= \hat{W}_0 - W\end{aligned}$$

where $x_m(t)$ and $y_m(t)$ are defined in (3.20). With (3.20), (3.21), (3.22), and (3.23), we have

$$\begin{aligned}\dot{x}_e(t) &= (A + BK)x_e(t) + Ed(t) \\ &\qquad + B\Lambda W_e(t)\sigma(x_p(t), r(t)), \\ x_e(0) &= x_{e0}, \\ y_e(t) &= Cx_e(t), \qquad (3.24)\\ \dot{W}_e(t) &= -\frac{\eta}{\gamma^2}B^T Px_e(t)\sigma^T(x_p(t), r(t)), \\ W_e(0) &= W_{e0}, \qquad (3.25)\end{aligned}$$

which describes the error dynamics from the reference model (3.20).

The next theorem presents the result of this section.

Theorem 3.2. *Consider the uncertain dynamical system described by* (3.1) *subject to Assumption 3.1. Consider, in addition, the reference model given by* (3.20) *and the adaptive feedback controller given by* (3.22) *and* (3.23). *Then, the solution* $(x_e(t), W_e(t))$ *given by* (3.24) *and* (3.25) *is bounded and*

$$\lim_{t\to\infty} y_e(t) = 0$$

for all (x_{e0}, W_{e0}).

Proof. Consider a candidate of Lyapunov function

$$V(x_e, W_e) = x_e^T P x_e + \frac{\gamma^2}{\eta}\mathrm{tr} W_e^T \Lambda W_e,$$

where η and P are taken from (3.23) and Λ of (3.1) satisfies Assumption 3.1, which means that $P = P^T > 0$ and $\Lambda = \Lambda^T > 0$. Thus the function $V(x_e, W_e)$ is in fact a continuously differentiable function such that $V(0,0) = 0$ and $V(x_e, W_e) > 0$ for all $(x_e, W_e) \neq (0,0)$.

Differentiating this candidate along the trajectories of (3.24) and (3.25), we have

$$\begin{aligned}
&\frac{d}{dt}V(x_e(t), W_e(t)) \\
&= \dot{x}_e^T(t)Px_e(t) + x_e^T(t)P\dot{x}_e(t) + \frac{2\gamma^2}{\eta}\mathrm{tr}\dot{W}_e^T(t)\Lambda W_e(t) \\
&= x_e^T(t)\left(P(A+BK) + (A+BK)^T P\right)x_e(t) \\
&\qquad + x_e^T(t)PEd(t) + d^T(t)E^T P x_e(t) \\
&\qquad + 2x_e^T(t)PB\Lambda W_e(t)\sigma(x_p(t), r(t)) \\
&\qquad - 2\mathrm{tr}\left(\sigma(x_p(t), r(t))x_e^T(t)PB\Lambda W_e(t)\right) \\
&= x_e^T(t)\left((A+BK)^T P + P(A+BK)\right)x_e(t) \\
&\qquad + x_e^T(t)PEd(t) + d^T(t)E^T P x_e(t) \\
&< -y_e^T(t)y_e(t) + \gamma^2 d^T(t)d(t)
\end{aligned}$$

for any $\tilde{x}_e(t) \neq 0$, which partially follows completing the square used in (3.16). Integrating this inequality from 0 to any $T > 0$, we have

$$\int_0^T y_e^T(t)y_e(t)dt + V(x_e(T), W_e(T)) \leq \int_0^T \gamma^2 d^T(t)d(t)dt + V(x_{e0}, W_{e0}). \tag{3.26}$$

Since $d \in L_2$, the right-hand side of the inequality is bounded. Thus, $V(x_e(T), W_e(T))$ is bounded for any $T > 0$, which implies that the solution $(x_e(t), W_e(t))$ is bounded. Also, the boundedness of the right-hand side of (3.26) implies that

$$\lim_{T\to\infty} \int_0^T y_e^T(t)y_e(t)dt < \infty. \tag{3.27}$$

Furthermore, since $\dot{y}_e(t) = C\dot{x}_e(t)$ and $\dot{x}_e(t)$ of (3.24) is represented by the signals $x_e(t)$, $d(t)$, $W_e(t)$, $x_p(t)$, and $r(t)$, we can see that

$$\sup_{t\geq 0} \dot{y}_e^T(t)\dot{y}_e(t) < \infty. \tag{3.28}$$

In fact, $x_e(t)$ and $W_e(t)$ are bounded as we have proved above, while $d \in L_\infty$ is bounded and a step type function $r(t)$ of (3.4) is bounded. The boundedness of $x_p(t)$ follows the boundedness of $x_e(t)$ and $x_m(t)$ of (3.20). With (3.27), (3.28), and [43], we obtain

$$\lim_{t\to\infty} y_e(t) = 0.$$

This completes the proof of the theorem. □

Remark 3.3. Theorem 3.2 establishes boundedness of the error dynamics behaviors via the proposed adaptive control in the presence of disturbances and uncertainties. The theorem also shows that zero steady state tracking error is achieved by this control.

3.5 Performance evaluation

Since the proposed adaptive control given by (3.22) and (3.23) employs an H_∞ control (3.20) for the nominal system (3.3) as the reference model, one of our interests should be to evaluate the performance degradation due to introduction of adaptive mechanism.

In this regard, let us recall the performance specification (3.10) for the nominal system. It says that

$$\begin{aligned}&\|r - y_n\| \\ &\leq \gamma \left(\|d\|^2 + (x_{p0} - x_\infty)^T R (x_{p0} - x_\infty) \right)^{1/2}\end{aligned} \tag{3.29}$$

holds for any $d \in L_2$, $x_{p0} \in \mathbb{R}^n$, and $x_\infty \in \mathbb{R}^n$. On the other hand, when we use the adaptive control of (3.20), (3.22), and (3.23), we can evaluate the tracking error $\|r - y_p\|$ similarly as follows.

Theorem 3.4. *Consider the uncertain dynamical system described by* (3.1) *subject to Assumption 3.1. Consider, in addition, the reference model given by* (3.20) *and the adaptive feedback controller given by* (3.22) *and* (3.23). *Then, the tracking error is bounded as*

$$\begin{aligned}&\|r - y_p\| \\ &\leq \gamma \Big\{ \big(\|d\|^2 + (x_{p0} - x_{m0})^T R (x_{p0} - x_{m0}) \\ &\qquad + \frac{1}{\eta} \mathrm{tr}(\hat{W}_0 - W)^T \Lambda (\hat{W}_0 - W) \big)^{1/2} \\ &\qquad + \big((x_{m0} - x_\infty)^T R (x_{m0} - x_\infty) \big)^{1/2} \Big\}\end{aligned} \tag{3.30}$$

for any $d \in L_2$, $x_{p0} \in \mathbb{R}^n$, $x_{m0} \in \mathbb{R}^n$, $x_\infty \in \mathbb{R}^n$, *and* $\hat{W}_0 \in \mathbb{R}^{m \times q}$.

Proof. Since (3.29) holds for (3.19), we have

$$\begin{aligned}&\|r - y_m\| \\ &\leq \gamma \left((x_{m0} - x_\infty)^T R (x_{p0} - x_\infty) \right)^{1/2}\end{aligned} \tag{3.31}$$

for the reference model (3.20) which does not contain $d \in L_2$. Regarding the adaptive control, we have established (3.26), which means that

$$\|y_e\|^2 \leq \gamma^2 \|d\|^2 + V(x_{e0}, W_{e0}).$$

Thus we have

$$\begin{aligned}&\|y_p - y_m\| \\ &\leq \gamma \big(\|d\|^2 + (x_{p0} - x_{m0})^T R (x_{p0} - x_{m0}) \\ &\qquad + \frac{1}{\eta} \mathrm{tr}(\hat{W}_0 - W)^T \Lambda (\hat{W}_0 - W) \big)^{1/2}.\end{aligned} \tag{3.32}$$

These evaluations (3.31) and (3.32) together with the triangle inequality

$$\|r - y_p\| \le \|r - y_m\| + \|y_m - y_p\| \tag{3.33}$$

gives the tracking error bound of the theorem. □

Remark 3.5. This bound given by Theorem 3.4 shows a transient performance of the proposed H_∞ adaptive control. It says that L_2 type gain from the disturbance and the initial values to the tracking error of the adaptive control is bounded by γ as that of the corresponding nominal closed-loop system (3.19) is. In addition, if we use large learning rate η, the transient performance becomes better, which will be confirmed in the numerical example below.

Regarding the conservativeness of the performance evaluation if perfect tracking is achieved ($x_{e0} = x_{p0} - x_{m0} = 0$) and there is no estimation error ($W_{e0}(t) = \hat{W}_0(t) - W = 0$), then the tracking error is bounded the same as the nominal system.

3.6 Numerical Example

In this section, we demonstrate the proposed adaptive control through a numerical example. Referring [44, 45], we selected the known part of the system (3.1) as

$$A = \begin{bmatrix} -1.0189 & 0.9051 \\ 0.8223 & -1.0774 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0022 \\ -0.1756 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T, \quad \alpha(x_p(t)) = x_p(t),$$

while we chose the unknown part of the system (3.1) as

$$\Lambda = 0.5, \qquad F = \begin{bmatrix} 1 & 2 \end{bmatrix}, \qquad d(t) = 9e^{-t/10}\sin t,$$

where Assumption 3.1 is satisfied. When we solve the LMIs (3.11) with $\gamma = 1$ and $R = 3I$, we obtained

$$X = \begin{bmatrix} 1.1469 & -0.1202 \\ -0.1202 & 2.9155 \end{bmatrix},$$
$$G = \begin{bmatrix} 21.9789 & -10.1665 \end{bmatrix},$$

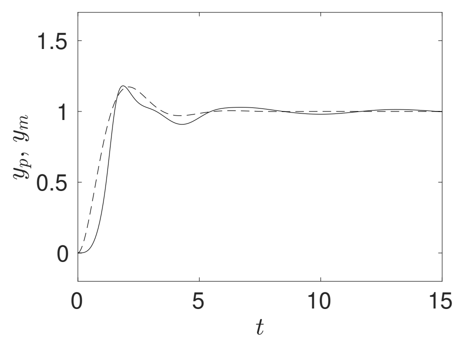

FIGURE 3.1: Output with $\eta = 700$

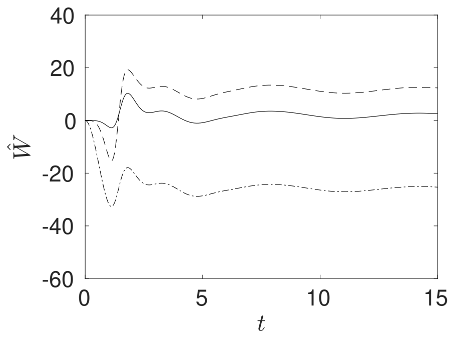

FIGURE 3.2: Adaptive gains with $\eta = 700$

which gives the tracking gains are

$$K = \begin{bmatrix} 18.8795 & -2.7087 \end{bmatrix}, \qquad H = -18.0361.$$

Also, with (3.13), we have

$$P = \begin{bmatrix} 0.8757 & 0.0361 \\ 0.0361 & 0.3445 \end{bmatrix},$$

which is used in the proposed adaptive control gain (3.23), where we set

$$\sigma(x_p(t), r(t)) = \begin{bmatrix} x_p(t) \\ r(t) \end{bmatrix}.$$

Based on the setting above, we performed numeral simulations, where all of the initial conditions were set as zero. Figs. 3.1, 3.2, 3.3, and 3.4 show the step responses and the gain behaviors of the proposed adaptive control, where Figs. 3.1 and 3.2 are with $\eta = 700$, while Figs. 3.3 and 3.4 are with $\eta = 3500$. Note that $y_p(t)$ and $y_m(t)$ are indicated as solid and dashed lines, respectively, in Figs. 3.1 and 3.3. The elements $\hat{W}_1(t), \hat{W}_2(t)$ and $\hat{W}_3(t)$ of the adaptive gain

$$\hat{W}(t) = \begin{bmatrix} \hat{W}_1(t) & \hat{W}_2(t) & \hat{W}_3(t) \end{bmatrix}$$

are indicated as solid, dashed, and dash-dotted lines, respectively, in Figs. 3.2 and 3.4. In these figures, we see that all signals are bounded and $y_p(t)$ tends to $y_m(t)$ as t tends to infinity, which is consistent with Theorem3.2. Furthermore, comparing Figs. 3.2 and 3.4, we see that a larger η gives a better performance, which is consistent with Theorem 3.4. Overall, the proposed adaptive control shows good transient responses.

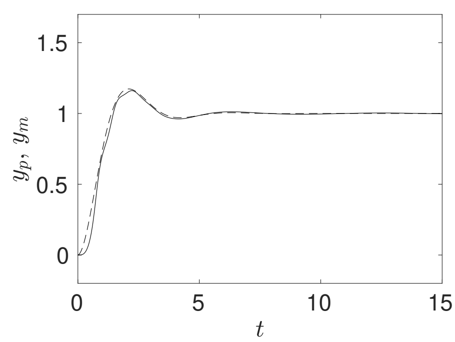

FIGURE 3.3: Output with $\eta = 3500$

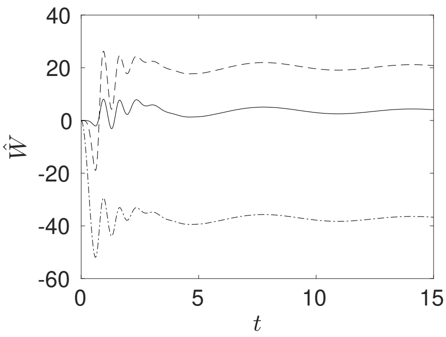

FIGURE 3.4: Adaptive gains with $\eta = 3500$

3.7 Summary

In this chapter, an adaptive control for H_∞ tracking of uncertain dynamical systems is proposed. Particularly, an H_∞ type control[39–41] for a suitable reference model selection is employed. Then an adaptive control law for uncertain dynamical systems with L_2 disturbance is developed. Finally it is proved that the proposed adaptive control guarantees the boundedness of the error dynamics behaviors in the presence of disturbances and uncertainties, where it achieves zero tracking error in the steady state as well. Furthermore, we have established an explicit error bound of tracking, which enables us to evaluate a transient performance of the control system.

Chapter 4

Distributed adaptive control for optimal LQ tracking of uncertain interconnected dynamical systems

4.1 Overview

The work in chapter two [24], introduced an adaptive control for optimal tracking of a single plant is considered for a step-type reference signal. On the other hand, this chapter deals with a distributed adaptive control; for optimal tracking of an interconnected system for a general reference signal, which clarifies a possible performance guarantee for a type of adaptive control law.

In this chapter a distributed model reference adaptive control scheme for optimal tracking to a class of large-scale dynamical systems with uncertain interconnection between the subsystems is considered in which each controller applied to each subsystem uses information about its neighbors. A reference model selection which achieves an optimal tracking for the nominal system is introduced by using linear quadratic regulator theory. Then an adaptive control law is developed for the uncertain interconnected dynamical system, where it employs the specified reference model.

It is shown that the proposed control law achieves the desired behavior such that the output of the system asymptotically tracks the output of the reference model in the presence of the uncertainties. In addition, the boundedness of all signals and establishment of an explicit error bound with respect to the nominal optimal tracking, where a role of the learning rate of the update rule is clarified. Numerical examples illustrate that the theoretical results developed in this chapter are useful.

4.2 Problem formulation

Let us consider an interconnected system consisting of N uncertain dynamical subsystems with uncertain interconnection.

The topology of the interconnection is expressed by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of the nodes each of which corresponds to a subsystem, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of the edges which represents the interaction among the subsystems. The set of neighborhood of the i-th subsystem is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$, where the i-th subsystem is affected by the j-th subsystem through uncertain interconnection if $j \in \mathcal{N}_i$. Here we assume that $\mathcal{G}$ is *known* and time-invariant.

With the graph $\mathcal{G}$, we describe the dynamics of the i-th subsystem as

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i[\Lambda_i u_i(t) + \sum_{j \in \mathcal{N}_i} \Delta_{i,j}(x_j(t)) + \Delta_{i,i}(x_i(t))], \qquad x_i(0) = x_{i0}, \\ y_i(t) &= C_i x_i(t), \end{aligned} \tag{4.1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input restricted to the class of admissible controls consisting of measurable functions, $y_i(t) \in \mathbb{R}^{m_i}$ is the controlled output. Throughout this chapter, the subscripts i and j correspond to the i-th and the j-th subsystems, respectively, i.e., $i \in \mathcal{V}$ and $j \in \mathcal{N}_i \subset \mathcal{V}$. The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, and $C_i \in \mathbb{R}^{m_i \times n_i}$ represent the *nominal* part of the subsystem, where the pair (A_i, B_i) is controllable and the pair (C_i, A_i) is observable. On the other hand, the matrix $\Lambda_i \in \mathbb{R}^{m_i \times m_i}$ and the vector-valued function $\Delta_{i,j} : \mathbb{R}^{n_j} \to \mathbb{R}^{m_i}$ represents the *uncertain* part of the subsystem as well as the *uncertain* interaction among the subsystems. That is, $\Delta_{i,i}(x_i(t))$ expresses the uncertain part of the i-th subsystem itself, while $\Delta_{i,j}(x_j(t))$ $(j \neq i)$ expresses the uncertain influence from the j-th subsystem to the i-th subsystem.

In this regard, we introduce the following assumption for Λ_i and $\Delta_{i,j}(x_j(t))$.

Assumption 4.1. *The control effectiveness Λ_i of the i-th subsystem is an unknown symmetric and positive definite matrix. The state dependent uncertainty $\Delta_{i,j}(x_i)$ of the i-th subsystem is linearly parameterized as*

$$\Delta_{i,j}(x_j) = F_{i,j} \alpha_{i,j}(x_j),$$

for all $j \in \mathcal{N}_i \cup \{i\}$, where $F_{i,j} \in \mathbb{R}^{m_i \times s_{i,j}}$ is an unknown weight matrix and $\alpha_{i,j} : \mathbb{R}^{n_j} \to \mathbb{R}^{s_{i,j}}$ is a given basis function.

For each subsystem (4.1), we define the corresponding nominal system as

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \qquad\qquad x_i(0) = x_{i0}, \\ y_i(t) &= C_i x_i(t). \end{aligned} \tag{4.2}$$

That is, when $\Lambda_i = I$ and $\Delta_{i,j}(x_j(t)) \equiv 0$ for all $j \in \mathcal{N}_i \cup \{i\}$, the i-th subsystem (4.1) takes its nominal behavior.

In this chapter, we consider a reference signal $r_i(t) \in \mathbb{R}^{m_i}$ generated by

$$\begin{aligned} \dot{x}_{ri}(t) &= A_{ri} x_{ri}(t), \qquad\qquad x_{ri}(0) = x_{ri0}, \\ r_i(t) &= C_{ri} x_{ri}(t), \end{aligned} \tag{4.3}$$

where $x_{ri}(t) \in \mathbb{R}^{r_i}$, the eigenvalues of A_{ri} are on the imaginary axis and all distinct from one another, and the pair (C_{ri}, A_{ri}) is observable. That is, $x_{ri}(t)$ and thus $r_i(t)$ of (4.3) are bounded signals which are represented as a linear combination of a constant signal and sinusoidal signals having several frequencies and phases. The boundedness of $x_{ri}(t)$ will be used for establishing the zero steady state tracking error. We define the initial time $t = 0$ at the time when the reference signal is applied. The initial state x_{ri0} of (4.3) is arbitrary.

It is known that the controlled output $y_i(t)$ of the nominal subsystem (4.2) can follow any reference signal $r_i(t)$ in the steady state if

$$\operatorname{rank} \begin{bmatrix} A_i - \lambda_{ri} I & B_i \\ C_i & 0 \end{bmatrix} = n_i + m_i \tag{4.4}$$

for all eigenvalues λ_{ri} of A_{ri}. We assume this condition for all subsystems (4.1).

The objective of this chapter is to construct a distributed adaptive control law for $u_i(t)$ of the form

$$u_i(t) = f_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \tag{4.5}$$

such that the output $y_i(t)$ of the given system (4.1) asymptotically tracks the reference signal $r_i(t)$ of (4.3) in the presence of the system uncertainty described by Λ_i and $\Delta(x_{i,j})$ satisfying Assumption 4.1. Here, $x_{\mathcal{N}_i}(t)$ denotes the set of $x_j(t)$ with $j \in \mathcal{N}_i$, and thus the control law (4.5) utilizes the knowledge of the i-th subsystem itself and its surrounding neighbors only. In this sense, we call the form (4.5) a *distributed* adaptive control law. Then, we employ an optimal tracking control law for each nominal subsystem (4.2) and derive a distributed adaptive tracking control law for the uncertain interconnected system (4.1) with a performance guarantee related to the nominal optimality.

4.3 Reference model selection

In this section, we select suitable reference models for our distributed adaptive control. To this end, we consider the *nominal* system (4.2) and revisit a standard optimal tracking for the reference signal (4.3) [46].

Under the assumption (4.4), there exist a unique state $x_{si}(t)$ and a unique control input $u_{si}(t)$ described by

$$\begin{aligned} x_{si}(t) &= L_{xi}x_{ri}(t), \qquad\qquad x_{si0} = L_{xi}x_{ri0}, \\ u_{si}(t) &= L_{ui}x_{ri}(t) \end{aligned} \tag{4.6}$$

for which the controlled output $y_i(t)$ is identical to the reference signal $r_i(t)$ [47], where L_{xi} and L_{ui} are defined by

$$L_{xi}A_{ri} = A_iL_{xi} + B_iL_{ui}, \qquad\qquad C_{ri} = C_iL_{xi}. \tag{4.7}$$

We denote the variations of $x_i(t)$ and $u_i(t)$ from $x_{si}(t)$ and $u_{si}(t)$ by

$$\begin{aligned} \tilde{x}_i(t) &= x_i(t) - x_{si}(t), \qquad\qquad \tilde{x}_{i0} = x_{i0} - x_{si0}, \\ \tilde{u}_i(t) &= u_i(t) - u_{si}(t), \end{aligned} \tag{4.8}$$

and the tracking error of the controlled output $y_{ni}(t)$ by

$$e_i(t) = r_i(t) - y_i(t).$$

Using these notations with (4.7), we have the variation system

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= A_i\tilde{x}_i(t) + B_i\tilde{u}_i(t), \qquad\qquad \tilde{x}_i(0) = \tilde{x}_{i0} \\ e_i(t) &= -C_i\tilde{x}_i(t). \end{aligned} \tag{4.9}$$

To obtain a good transient behavior of tracking to the reference signal $r_i(t)$, we apply linear quadratic regulator theory to the variation system (4.9) with the performance index

$$J_i = \int_0^\infty \{e_i^T(t)Q_ie_i(t) + \tilde{u}_i^T(t)R_i\tilde{u}_i(t)\}dt, \tag{4.10}$$

where $Q_i \in \mathbb{R}^{m_i \times m_i}$ and $R_i \in \mathbb{R}^{m_i \times m_i}$ are symmetric and positive definite matrices. Then the optimal control law which minimizes J_i with respect to (4.9) is given by

$$\tilde{u}_i(t) = K_i \tilde{x}_i(t), \tag{4.11}$$

where

$$K_i = -R_i^{-1} B_i^T P_i \tag{4.12}$$

and $P_i \in \mathbb{R}^{n_i \times n_i}$ is a symmetric and positive definite solution of the Riccati equation

$$A_i^T P_i + P_i A_i - P_i B_i R_i^{-1} B_i^T P_i + C_i^T Q_i C_i = 0. \tag{4.13}$$

When the control law (4.11) is applied to the variation system (4.9), the resultant closed loop system is stable, and thus $\tilde{x}_i(t) \to 0$, $e_i(t) \to 0$ as $t \to \infty$, where the minimum value of J_i is given by

$$\min_{\tilde{u}_i} J_i = \tilde{x}_{i0}^T P_i \tilde{x}_{i0}. \tag{4.14}$$

Using (4.6) and (4.8), we rewrite the control law (4.11) as

$$u_i(t) = K_i x_i(t) + H_i x_{ri}(t) \tag{4.15}$$

for the nominal system (4.2), where

$$H_i = -K_i L_{xi} + L_{ui}. \tag{4.16}$$

That is, the optimal tracking control law for the nominal system (4.2) is composed of a feedback from the state $x_i(t)$ and a feedforward from the state $x_{ri}(t)$. The resultant control system with (4.2) and (4.15) is described by

$$\begin{aligned} \dot{x}_i(t) &= (A_i + B_i K_i) x_i(t) + B_i H_i x_{ri}(t), \qquad x_i(0) = x_{i0}, \\ y_i(t) &= C_i x_i(t). \end{aligned} \tag{4.17}$$

When there is no uncertainty in the system (4.1), the optimal tracking (4.17) represents the best achievable behavior of each subsystem. We therefore consider a distributed adaptive control framework for (4.1) which asymptotically realizes the optimal tracking

(4.17). To this end, we employ

$$\begin{aligned} \dot{x}_{mi}(t) &= (A_i + B_i K_i) x_{mi}(t) + B_i H_i x_{ri}(t), \qquad\qquad x_{mi}(0) = x_{mi0}, \\ y_{mi}(t) &= C_i x_{mi}(t). \end{aligned} \tag{4.18}$$

as the reference model for each subsystem, where $x_{mi}(t) \in \mathbb{R}^{n_i}$ is the state, K_i is given by (4.12) based on the performance index J_i of (4.10), and H_i is given by (4.16) with this K_i.

4.4 Distributed adaptive control scheme

Let us go back to the uncertain interconnected system (4.1). According to the selected reference model (4.18), we rewrite each subsystem (4.1) as

$$\begin{aligned} \dot{x}_i(t) &= (A_i + B_i K_i) x_i(t) + B_i H_i x_{ri}(t) + B_i \Lambda_i [u_i(t) + \delta_i(t)], \qquad x_i(0) = x_{i0}, \\ y_i(t) &= C_i x_i(t), \end{aligned} \tag{4.19}$$

where we define

$$\begin{aligned} \delta_i(t) &= -\Lambda_i^{-1} \left[K_i x_i(t) + H_i x_{ri}(t) - \sum_{j \in \mathcal{N}_i} \Delta_{i,j}(x_j(t)) - \Delta_{i,i}(x_i(t)) \right] \\ &= -W_i \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \end{aligned}$$

with $x_{\mathcal{N}_i}(t)$ used in (4.5). In fact, from Assumption 4.1, the signal $\delta_i(t)$ must be linearly parameterized by using an unknown weight $W_i \in \mathbb{R}^{m_i \times q_i}$ and the corresponding basis function $\sigma_i : \mathbb{R}^{n_i + n_{\mathcal{N}_i} + r_i} \to \mathbb{R}^{q_i}$ which contains $x_i(t)$, $x_{ri}(t)$, and $\alpha_{i,j}(x_j(t))$ $(j \in \mathcal{N}_i \cup \{i\})$, where $q_i \leq n_i + r_i + \sum_{j \in \mathcal{N}_i \cup \{i\}} s_{i,j}$.

Then we introduce a distributed adaptive control law

$$u_i(t) = \hat{W}_i(t) \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \tag{4.20}$$

where we define the update rule of the adaptive gain $\hat{W}_i(t) \in \mathbb{R}^{m_i \times q_i}$ as

$$\dot{\hat{W}}_i(t) = -\eta_i B_i^T P_i (x_i(t) - x_{mi}(t)) \sigma_i^T (x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \qquad \hat{W}_i(0) = \hat{W}_{i0}. \tag{4.21}$$

The signal $x_{mi}(t)$ of (4.21) is given by (4.18). The learning rate η_i is a positive real number and P_i is the symmetric and positive definite solution of the Riccati equation (4.13).

We see that the control law (4.20) with (4.21) is *distributed* indeed. In fact, the basis $\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t))$ is distributed. Thus, if the interconnection of the overall system (4.1) is *sparse*, we enjoy a *sparse* structure of the control, where the control law (4.20) with (4.21) utilizes the knowledge of the i-th subsystem itself and its surrounding neighbors only. See also the numerical example in Section 4.6 for further details.

Now, let us define the errors from the ideal case as

$$\begin{aligned} x_{ei}(t) &= x_i(t) - x_{mi}(t), & x_{ei0} &= x_{i0} - x_{mi0}, \\ y_{ei}(t) &= y_i(t) - y_{mi}(t), \\ W_{ei}(t) &= \hat{W}_i(t) - W_i, & W_{ei0} &= \hat{W}_{i0} - W_i, \end{aligned}$$

where $x_{mi}(t)$ and $y_{mi}(t)$ are defined in (4.18). With (4.18), (4.19), (4.20), and (4.21), we have

$$\dot{x}_{ei}(t) = (A_i + B_iK_i)x_{ei}(t) + B_i\Lambda_iW_{ei}(t)\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \qquad x_{ei}(0) = x_{ei0},$$
$$y_{ei}(t) = C_ix_{ei}(t), \tag{4.22}$$
$$\dot{W}_{ei}(t) = -\eta_iB_i^TP_ix_{ei}(t)\sigma_i^T(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \qquad W_{ei}(0) = W_{ei0}, \tag{4.23}$$

which describes the error dynamics from the reference model (4.18).

The next theorem presents the result of this section.

Theorem 4.2. *Consider the uncertain interconnected dynamical system described by* (4.1) *subject to Assumption 4.1. Consider, in addition, the reference model given by* (4.18) *and the distributed adaptive control given by* (4.20) *and* (4.21). *Then, all of the solutions* $(x_{ei}(t), W_{ei}(t))$ $(i = 1, 2, \dots, N)$ *given by* (4.22) *and* (4.23) *are bounded. Furthermore, all of the tracking errors* $y_{ei}(t)$ $(i = 1, 2, \dots, N)$ *satisfy*

$$\lim_{t\to\infty} y_{ei}(t) = 0$$

for any $(x_{ei}(0), W_{ei}(0))$ $(i = 1, 2, \dots, N)$.

Proof. Consider a candidate of Lyapunov function

$$V = \sum_{i=1}^{N} V_i(x_{ei}, W_{ei}), \qquad V_i(x_{ei}, W_{ei}) = x_{ei}^TP_ix_{ei} + \frac{1}{\eta_i}\mathrm{tr}W_{ei}^T\Lambda_iW_{ei},$$

where η_i and P_i are taken from (4.21) and Λ_i of (4.1) satisfies Assumption 4.1, which means that $P_i = P_i^T > 0$ and $\Lambda_i = \Lambda_i^T > 0$. Thus the function $V_i(x_{ei}, W_{ei})$ is a continuously differentiable function such that $V_i(0,0) = 0$ and $V_i(x_{ei}, W_{ei}) > 0$ for all $(x_{ei}, W_{ei}) \neq (0,0)$, which implies that V is also positive definite.

Differentiating each component $V_i(x_{ei}, W_{ei})$ of this candidate V along the trajectories of (4.22) and (4.23), we have

$$\begin{aligned}
\dot{V}_i(x_{ei}(t), W_{ei}(t)) &= \dot{x}_{ei}^T(t)P_i x_{ei}(t) + x_{ei}^T(t)P\dot{x}_{ei}(t) + \frac{2}{\eta_i}\mathrm{tr}\dot{W}_{ei}^T(t)\Lambda_i W_{ei}(t) \\
&= x_{ei}^T(t)\left((A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i)\right)x_{ei}(t) \\
&\quad + 2x_{ei}^T(t)P_iB_i\Lambda_i W_{ei}(t)\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \\
&\quad - 2\mathrm{tr}\left(\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t))x_{ei}^T(t)P_iB_i\Lambda_i W_{ei}(t)\right) \\
&= x_{ei}^T(t)\left((A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i)\right)x_{ei}(t) \\
&= -x_{ei}^T(t)\left(C_i^T Q_i C_i + K_i^T R_i K_i\right)x_{ei}(t),
\end{aligned} \tag{4.24}$$

where we use the fact that the Riccati equation (4.13) can be rewritten as

$$(A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i) = -(C_i^T Q_i C_i + K_i^T R_i K_i)$$

with K_i of (4.12). Since $Q_i = Q_i^T > 0$ and $R_i = R_i^T > 0$, we see that

$$\dot{V}_i(x_{ei}(t), W_{ei}(t)) \leq 0,$$

which implies that

$$V = \sum_{i=1}^N V_i(x_{ei}(t), W_{ei}(t)) \leq \sum_{i=1}^N V_i(x_{ei0}, W_{ei0}) < \infty$$

holds true for all $t \geq 0$. Hence all of the solutions $(x_{ei}(t), W_{ei}(t))$ $(i = 1, 2, \ldots, N)$ given by (4.22) and (4.23) are bounded.

Now, let us recall a standard fact

$$\int_0^t \dot{V}_i(x_{ei}(\tau), W_{ei}(\tau))d\tau = V_i(x_{ei}(t), W_{ei}(t)) - V_i(x_{ei0}, W_{ei0}),$$

which holds for all $t \geq 0$. With (4.24), we have

$$\begin{aligned}
-\dot{V}_i(x_{ei}(t), W_{ei}(t)) &= x_{ei}^T(t)\left(C_i^T Q_i C_i + K_i^T R_i K_i\right)x_{ei}(t) \\
&\geq y_{ei}^T(t)Q_i y_{ei}(t).
\end{aligned}$$

Note also that $V_i(x_{ei}(t), W_{ei}(t)) \geq 0$. Thus we obtain

$$\begin{aligned}\int_0^t y_{ei}^T(\tau)Q_i y_{ei}(\tau)d\tau &\leq -\int_0^t \dot{V}_i(x_{ei}(\tau), W_{ei}(\tau))d\tau \\ &\leq -\int_0^t \dot{V}_i(x_{ei}(\tau), W_{ei}(\tau))d\tau + V_i(x_{ei}(t), W_{ei}(t)) \\ &= V_i(x_{ei0}, W_{ei0}) \end{aligned} \tag{4.25}$$

for all $t \geq 0$. We therefore see that

$$\int_0^\infty \left(\sum_{i=1}^N y_{ei}^T(t)Q_i y_{ei}(t) \right) dt = \sum_{i=1}^N \int_0^\infty y_{ei}^T(t)Q_i y_{ei}(t)dt \leq \sum_{i=1}^N V_i(x_{ei0}, W_{ei0}). \tag{4.26}$$

Furthermore, since $\dot{y}_{ei}(t) = C_i\dot{x}_{ei}(t)$ and $\dot{x}_{ei}(t)$ of (4.22) is represented by the signals $x_{ei}(t)$, $W_{ei}(t)$, $x_i(t)$, $x_{\mathcal{N}_i}(t)$ and $x_{ri}(t)$, we can see that

$$\sup_{t\geq 0} \left(\sum_{i=1}^N \dot{y}_{ei}^T(t)Q_i\dot{y}_{ei}(t) \right) \leq \sum_{i=1}^N \sup_{t\geq 0} \dot{y}_{ei}^T(t)Q_i\dot{y}_{ei}(t) < \infty. \tag{4.27}$$

In fact, $x_{ei}(t)$ and $W_{ei}(t)$ are bounded as we have proved above, while $x_{ri}(t)$ of (4.3) is bounded as we have assumed. The boundedness of $x_i(t)$ and $x_{\mathcal{N}_i}(t)$ follows the boundedness of $x_{ei}(t)$ $(i = 1, 2, \ldots, N)$ and $x_{mi}(t)$ $(i = 1, 2, \ldots, N)$ of (4.18). Using (4.26), (4.27), and [43] with $Q_i = Q_i^T > 0$, we conclude that all of the tracking errors $y_{ei}(t)$ $(i = 1, 2, \ldots, N)$ satisfy

$$\lim_{t\to\infty} y_{ei}(t) = 0$$

for any $(x_{ei}(0), W_{ei}(0))$ $(i = 1, 2, \ldots, N)$. □

Remark 4.3. Theorem 4.2 establishes the boundedness of the error signals generated by the proposed distributed adaptive control in the presence of the system/interconnection uncertainties for each subsystem. The theorem also shows that zero steady state tracking error is achieved by this control for all the subsystems.

4.5 Performance evaluation

Since the distributed adaptive control given by (4.20) and (4.21) employs the *optimal* tracking system (4.18) as the reference model, one of our interests should be to evaluate the performance degradation from the optimal response.

To this end, let us rewrite the minimum value (4.14) of the performance index (4.10) for the nominal system as

$$\min_{\tilde{u}_i} J_i = \tilde{x}_{i0}^T P_i \tilde{x}_{i0} = \int_0^\infty \tilde{x}_i^T(t) \left(C_i^T Q_i C_i + K_i^T R_i K_i \right) \tilde{x}_i(t) dt,$$

where K_i is the optimal gain (4.12). For the reference model (4.18), this means that

$$\begin{aligned} J_{mi} &= \int_0^\infty (x_{mi}(t) - x_{si}(t))^T \left(C_i^T Q_i C_i + K_i^T R_i K_i \right) (x_{mi}(t) - x_{si}(t))\, dt \\ &= (x_{mi0} - x_{si0})^T P_i (x_{mi0} - x_{si0}). \end{aligned} \tag{4.28}$$

Referring the above, we define a performance index for the adaptive control as

$$J_{ei} = \int_0^\infty (x_i(t) - x_{mi}(t))^T \left(C_i^T Q_i C_i + K_i^T R_i K_i \right) (x_i(t) - x_{mi}(t))\, dt. \tag{4.29}$$

This index (4.29) is reasonable for evaluating the degradation caused by adaptation since its weight $(C_i^T Q_i C_i + K_i^T R_i K_i)$ coincides with that of (4.28). In fact, when we evaluate the tracking error from the preferable state $x_{si}(t)$ which achieves $y_i(t) = r_i(t)$ rather than the tracking error from the reference model state $x_{mi}(t)$, if we introduce the performance index

$$J_{esi} = \int_0^\infty (x_i(t) - x_{si}(t))^T \left(C_i^T Q_i C_i + K_i^T R_i K_i \right) (x_i(t) - x_{si}(t))\, dt,$$

we immediately obtain its evaluation as

$$J_{esi} \le \left(J_{mi}^{1/2} + J_{ei}^{1/2} \right)^2$$

by employing the triangle inequality. Notice also that the value J_{ei} becomes 0 (i.e., the value J_{esi} becomes J_{mi}) if the perfect tracking $x_i(t) = x_{mi}(t)$ is achieved.

For this index (4.29), we have the following result.

Theorem 4.4. *Consider the uncertain interconnected dynamical system described by* (4.1) *subject to Assumption 4.1. Consider, in addition, the reference model given by* (4.18) *and the distributed adaptive control given by* (4.20) *and* (4.21)*. Then, all of the indices* (4.29) $(i = 1, 2, \ldots, N)$ *are bounded as*

$$J_{ei} \le x_{ei0}^T P_i x_{ei0} + \frac{1}{\eta_i} \mathrm{tr} W_{ei0}^T \Lambda_i W_{ei0}. \tag{4.30}$$

Proof. In the proof of Theorem 4.2, we have established (4.24) and (4.25), which implies that

$$\int_0^t x_{ei}^T(\tau)\left(C_i^T Q_i C_i + K_i^T R_i K_i\right) x_{ei}(\tau) d\tau \le V(x_{ei0}, W_{ei0})$$

holds true for all $t \ge 0$. Thus it turns out that

$$\begin{aligned} J_{ei} &= \int_0^\infty x_{ei}^T(t)\left(C_i^T Q_i C_i + K_i^T R_i K_i\right) x_{ei}(t) dt \\ &\le V(x_{ei0}, W_{ei0}) = x_{ei0}^T P_i x_{ei0} + \frac{1}{\eta_i} \mathrm{tr} W_{ei0}^T \Lambda_i W_{ei0}, \end{aligned}$$

which establishes the bound (4.30). □

Remark 4.5. The upper bound given by this theorem guarantees the transient performance of the proposed adaptive control. It shows that the performance of the distributed adaptive control applied to the uncertain interconnected dynamical system captured by $x_{ei}(t)$ cannot be more than the right hand side of (4.30) at each subsystem. In addition, if we make the learning rate η_i large, the transient performance will be better, which will be confirmed in the numerical example in the following section.

Regarding the conservativeness of the performance evaluation if perfect tracking is achieved ($x_{ei0}(t) = x_{i0}(t) - x_{mi0}(t) = 0$) and there is no estimation error ($W_{ei0}(t) = \hat{W}_{i0}(t) - W_{i0} = 0$), then the performance index $J_{ei} \le 0$.

4.6 Numerical example

Let us consider a mass-spring-damper system having N masses in one line shown in Fig. 4.1. Each mass m_i ($1 < i < N$) is possibly connected with its neighbors m_{i-1} and m_{i+1} by springs $k_{i-1,i}$, $k_{i,i+1}$ and dampers $c_{i-1,1}$, $c_{i,i+1}$, that is,

$$\begin{aligned} m_i \ddot{q}_i(t) = &-k_{i-1,i}\left(q_i(t) - q_{i-1}(t)\right) - c_{i-1,i}\left(\dot{q}_i(t) - \dot{q}_{i-1}(t)\right) \\ &- k_{i,i+1}\left(q_i(t) - q_{i+1}(t)\right) - c_{i,i+1}\left(\dot{q}_i(t) - \dot{q}_{i+1}(t)\right) + u_i(t), \end{aligned} \tag{4.31}$$

where $q_i(t) \in \mathbb{R}$ is the position of the mass m_i to be controlled, and $\dot{q}_i(t)$ and $\ddot{q}_i(t)$ are its velocity and acceleration, respectively. The force $u_i(t) \in \mathbb{R}$ is applied to the mass m_i as the control input, while we assume that all physical parameters $m_i > 0$, $k_{i-1,i} \ge 0$, $k_{i,i+1} \ge 0$, $c_{i-1,i} \ge 0$, and $c_{i,i+1} \ge 0$ are *unknown.* The cases $i = 1$ and $i = N$ having one neighbor are not explicitly stated in this section, though it is clear that they can be described in a similar way.

The mass-spring-damper system (4.31) stated above is consistent with the system description (4.1). In fact, we can rewrite (4.31) as

$$\dot{x}_i(t) = A_i x_i(t) + B_i[\Lambda_i u_i(t) + \sum_{j=i-1}^{i+1} \Delta_{i,j}(x_j(t))],$$
$$y_i(t) = C_i x_i(t), \tag{4.32}$$

where we define

$$x_i(t) = \begin{bmatrix} q_i(t) \\ \dot{q}_i(t) \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \Lambda_i = \frac{1}{m_i},$$
$$\Delta_{i,i-1}(x_{i-1}(t)) = \begin{bmatrix} \frac{k_{i-1,i}}{m_i} & \frac{c_{i-1,i}}{m_i} \end{bmatrix} x_{i-1}(t),$$
$$\Delta_{i,i}(x_i(t)) = \begin{bmatrix} -\frac{k_{i-1,i}+k_{i,i+1}}{m_i} & -\frac{c_{i-1,i}+c_{i,i+1}}{m_i} \end{bmatrix} x_i(t),$$
$$\Delta_{i,i+1}(x_{i+1}(t)) = \begin{bmatrix} \frac{k_{i,i+1}}{m_i} & \frac{c_{i,i+1}}{m_i} \end{bmatrix} x_{i+1}(t).$$

That is, all unknown parameters are included in Λ_i and $\Delta_{i,j}(x_j(t))$. Apparently, (A_i, B_i) is controllable, (C_i, A_i) is observable, and the uncertainties Λ_i and $\Delta_{i,j}(x_j(t))$ satisfy Assumption 4.1. That is, the mass-spring-damper system (4.31) can be represented as (4.1), where we define $\mathcal{N}_i = \{i-1, i+1\}$. For this system, we consider a sinusoidal reference signal such as $\sin \omega_i t$, i.e., we define the coefficient matrices of (4.3) as

$$A_{ri} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix}, \qquad C_{ri} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where $\omega_i > 0$. We see that (C_{ri}, A_{ri}) is observable. Also, the rank condition (4.4) is satisfied for the eigenvalues of A_{ri}. Actually, we have the solutions of (4.7) as

$$L_{xi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad L_{ui} = \begin{bmatrix} -\omega_i^2 & 0 \end{bmatrix}.$$

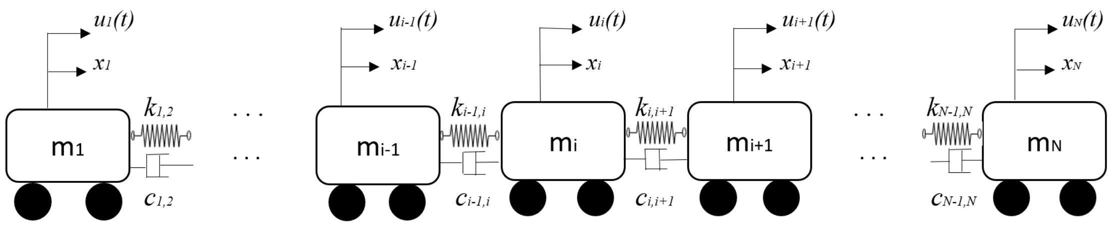

FIGURE 4.1: Interconnected mass-spring-damper system

Regarding the performance index J_i of (4.10) with for $Q_i = 1$ and $R_i = 1$, we obtain the positive definite solution of the Riccati equation (4.13) as

$$P_i = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}.$$

Then the optimal tracking gains (4.12) and (4.16) are

$$K_i = \begin{bmatrix} -1 & -\sqrt{2} \end{bmatrix}, \qquad H_i = \begin{bmatrix} 1-\omega_i^2 & \sqrt{2} \end{bmatrix}.$$

In this way, we can construct the reference model (4.18) which achieves the optimal tracking for the sinusoidal reference signal.

According to this reference model, we can rewrite the system (4.32) as (4.19), i.e.,

$$\begin{aligned} \dot{x}_i(t) &= (A_i + B_iK_i)x_i(t) + B_iH_ix_{ri}(t) + B_i\Lambda_i[u_i(t) + \delta_i(t)], \\ y_i(t) &= C_ix_i(t). \end{aligned}$$

It should be noted that its uncertainty is described as

$$\begin{aligned} \delta_i(t) &= -\Lambda_i^{-1}\left[K_ix_i(t) + H_ix_{ri}(t) - \sum_{j=i-1}^{i+1}\Delta_{i,j}(x_j(t))\right] \\ &= -W_i\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \end{aligned}$$

where

$$\begin{aligned} W_i = \Big[&-k_{i-1,i} \quad -c_{i-1,i} \quad k_{i-1,i} + k_{i,i+1} - m_i \quad c_{i-1,i} + c_{i,i+1} - \sqrt{2}m_i \\ &-k_{i,i+1} \quad -c_{i,i+1} \quad (1-\omega_i^2)m_i \quad \sqrt{2}m_i\Big] \\ \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) &= \begin{bmatrix} x_{i-1}^T(t) & x_i^T(t) & x_{i+1}^T(t) & x_{ri}^T(t) \end{bmatrix}^T. \end{aligned}$$

That is, due to the sparse structure of the system (4.31) of Fig. 4.1, the basis $\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t))$ contains only *the states of its neighbors*, i.e., $x_{i-1}(t)$ and $x_{i+1}(t)$. Since the adaptive control law (4.20) and the update rule of the adaptive gain (4.21) have the form

$$\begin{aligned} u_i(t) &= \hat{W}_i(t)\sigma_i(x_i(t), x_{i+1}(t), x_{ri}(t)), \\ \dot{\hat{W}}_i(t) &= -\eta_iB_i^TP_i(x_i(t) - x_{mi}(t))\sigma_i^T(x_i(t), x_{i+1}(t), x_{ri}(t)), \end{aligned}$$

these become in fact a *distributed* control law thanks to the sparse structure of the basis $\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t))$.

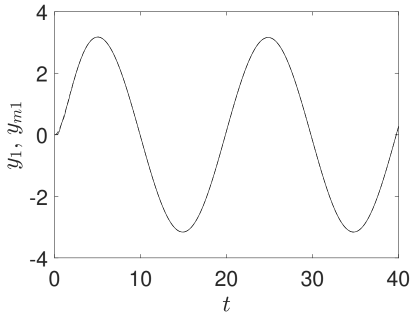

FIGURE 4.2: Output tracking for $y_1, y_{m1}(\eta_i = 50)$

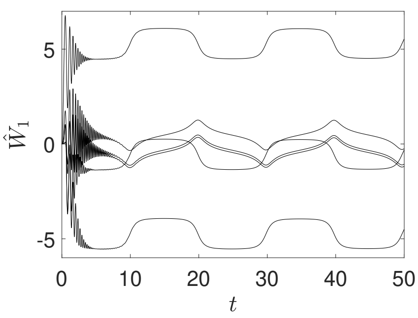

FIGURE 4.3: Adaptive gains for $\hat{W}_1(\eta_i = 50)$

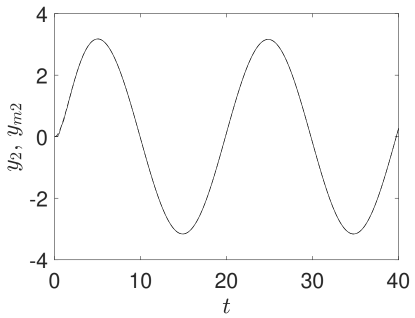

FIGURE 4.4: Output tracking for y_2, y_{m2} $(\eta_i = 50)$

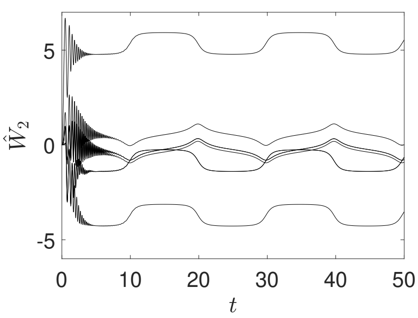

FIGURE 4.5: Adaptive gains for $\hat{W}_2$ $(\eta_i = 50)$

Now, let us consider a numerical simulation. We investigate the case $N = 3$, where we set the unknown and uncertain parameters as $m_1 = m_2 = m_3 = 3$, $k_{1,2} = k_{2,3} = 2$, and $c_{1,2} = c_{2,3} = 1$. We chose all initial states are zero except for the reference signal generators, where we used $x_{r1} = x_{r2} = x_{r3} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. We set $\omega_1^2 = \omega_2^2 = \omega_3^2 = 0.1$.

Figs. 4.2-4.7 show the tracking responses and the gain behaviors of the proposed distributed adaptive control for subsystems 1, 2 and 3, respectively, where we set the learning rates as $\eta_1 = \eta_2 = \eta_3 = 50$. On the other hand, Figs. 4.8-4.13 show the tracking responses and the gain behaviors of the proposed adaptive control, where we use $\eta_1 = \eta_2 = \eta_3 = 15$. Note that $y_1(t)$, $y_2(t)$, $y_3(t)$ are indicated as solid lines and $y_{m1}(t), y_{m2}(t), y_{m3}(t)$ are indicated as dashed lines in Figs. 4.2, 4.4, 4.6, 4.8, 4.10, and 4.12. The elements of the adaptive gains $\hat{W}_1(t)$, $\hat{W}_2(t)$ and $\hat{W}_3(t)$ for subsystems 1, 2 and 3 are indicated as solid lines in Figs. 4.3, 4.5, 4.7, 4.9, 4.11, and 4.13.

In Figs. 4.2-4.7, all signals are bounded and $y_i(t)$ tends to $y_{mi}(t)$ $(i = 1, 2, 3)$ as t tends to infinity, which is consistent with Theorem 4.2. Furthermore, comparing Figs. 4.2, 4.4, and 4.6 with Figs 4.8, 4.10, and 4.12, we see that a larger learning rate η_i gives a better performance, which is consistent with Theorem 4.4.

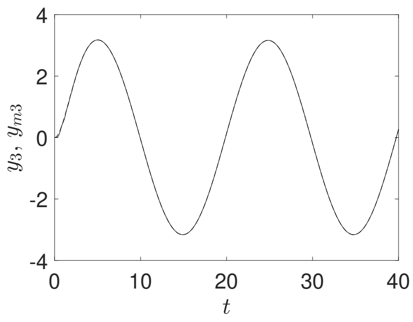

FIGURE 4.6: Output tracking for y_3, y_{m3} ($\eta_i = 50$)

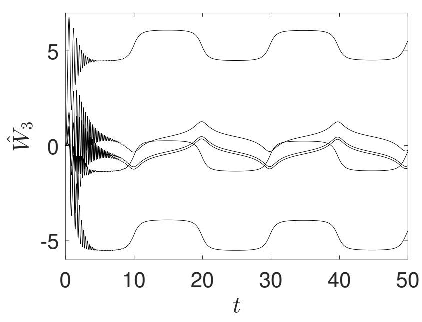

FIGURE 4.7: Adaptive gains $\hat{W}_3$ ($\eta_i = 50$)

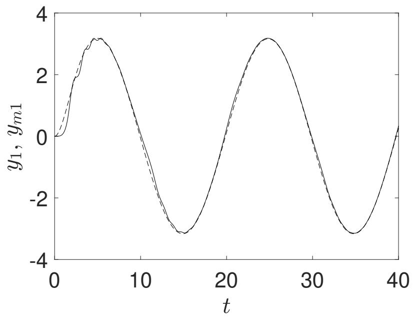

FIGURE 4.8: Output tracking for $y_1, y_{m1}(\eta_i = 15)$

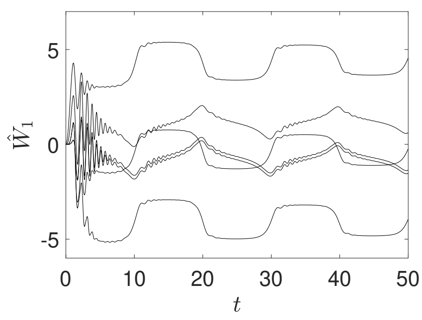

FIGURE 4.9: Adaptive gains $\hat{W}_1$ ($\eta_i = 15$)

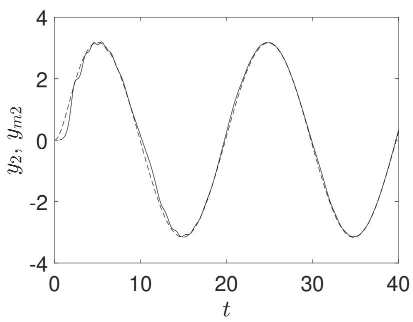

FIGURE 4.10: Output tracking for y_2, y_{m2} ($\eta_i = 15$)

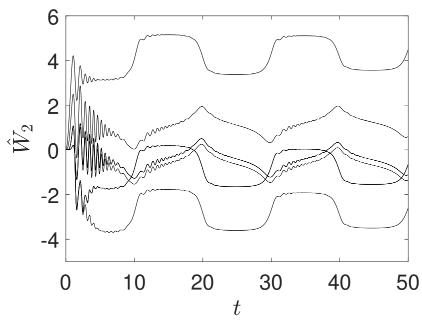

FIGURE 4.11: Adaptive gains $\hat{W}_2$ ($\eta_i = 15$)

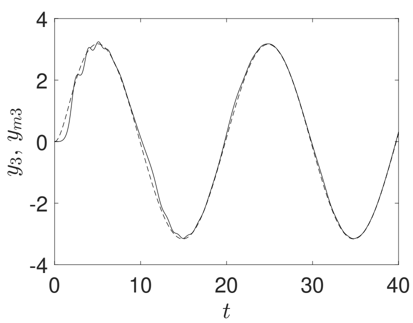

FIGURE 4.12: Output tracking for y_3, y_{m3} ($\eta_i = 15$)

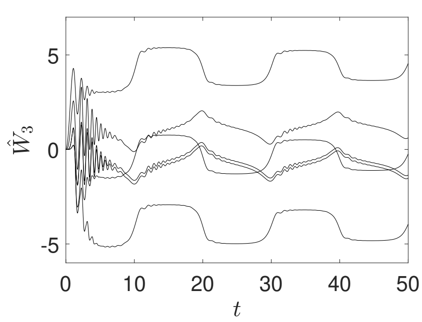

FIGURE 4.13: Adaptive gains $\hat{W}_3$ ($\eta_i = 15$)

4.7 Summary

In this chapter, a distributed adaptive control such that the output of an uncertain interconnected dynamical system asymptotically tracks the output of a reference model in the presence of system/interconnection uncertainties is investigated. The reference model is constructed as optimal tracking for the nominal system, where a Riccati solution of an LQ regulator determines the model. Then this Riccati solutionis employed in the update rule of the adaptive gain, and have shown that the proposed adaptive control law actually achieves the desirable tracking as well as the boundedness of all signals. In addition, an explicit error bound regarding optimal tracking is also established. The numerical examples have shown that the theoretical results developed in this chapter are useful. Although the state feedback case under no disturbance has been investigated in this chapter, the extension to the output feedback case in the presence of disturbance can be an important future work. In this regard, a preliminary result [48] has been obtained, where L_2 disturbance is considered for a single plant in an H_∞ tracking setting.

Chapter 5

Distributed adaptive control for H_∞ tracking of uncertain interconnected dynamical systems

5.1 Overview

The work in Chapter 2 [48], introduced an adaptive H_∞ tracking of a single plant is considered for a step-type reference signal. On the other hand, the present chapter deals with a distributed adaptive H_∞ tracking of an interconnected system for a general reference signal, which clarifies a possible performance guarantee for a type of adaptive control law.

In this chapter a novel method is proposed for distributed adaptive control for realizing a robust tracking of a class of interconnected dynamical systems that is characterized by sets of uncertain dynamics with an unknown physical interconnection between these dynamics. A reference model which achieves a robust tracking in the presence of L_2 disturbances is introduced by using H_∞ control with transients. Then a distributed adaptive control law is developed for uncertain dynamical systems, where it employs the specified reference model.

It is shown that the boundedness of the error dynamics behaviors as well as zero tracking error in the steady state is guaranteed by the proposed distributed adaptive control law in the presence of disturbances and uncertainties. In addition, an explicit error bound of tracking, which enables us to evaluate a transient performance of the control system is established. Numerical examples were discussed to show applicability of the theoretical findings.

5.2 Problem formulation

Let us consider an interconnected system consisting of N uncertain dynamical subsystems with uncertain interconnection. The topology of the interconnection is expressed by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of the nodes each of which corresponds to a subsystem, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of the edges which represents the interaction among the subsystems. The set of neighborhood of the i-th subsystem is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$, where the i-th subsystem is affected by the j-th subsystem through uncertain interconnection if $j \in \mathcal{N}_i$. Here we assume that $\mathcal{G}$ is *known* and time-invariant.

With the graph $\mathcal{G}$, we describe the dynamics of the i-th subsystem as

$$\dot{x}_i(t) = A_i x_i(t) + B_i[\Lambda_i u_i(t) + \sum_{j \in \mathcal{N}_i} \Delta_{i,j}(x_j(t)) + \Delta_{i,i}(x_i(t))] + E_i d_i(t), \quad x_i(0) = x_{i0},$$
$$y_i(t) = C_i x_i(t), \tag{5.1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input restricted to the class of admissible controls consisting of measurable functions, $y_i(t) \in \mathbb{R}^{m_i}$ is the controlled output. Throughout this chapter, the subscripts i and j correspond to the i-th and the j-th subsystems, respectively, i.e., $i \in \mathcal{V}$ and $j \in \mathcal{N}_i \subset \mathcal{V}$. In addition, $d_i(t) \in \mathbb{R}^{p_i}$ is an *unknown* disturbance. Here we assume that $d_i \in L_\infty \cap L_2$ and $\dot{d}_i \in L_\infty$. In other words, we assume that $d_i(t)$ satisfies

$$\sup_{t \geq 0} d_i^T(t) d_i(t) < \infty, \qquad \int_0^\infty d_i^T(t) d_i(t) dt < \infty, \qquad \sup_{t \geq 0} \dot{d}_i^T(t) \dot{d}_i(t) < \infty.$$

Notice here that $\lim_{t \to \infty} d_i(t) = 0$ if $d_i \in L_2$ and $\dot{d}_i \in L_\infty$ [43]. The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $C_i \in \mathbb{R}^{m_i \times n_i}$, and $E_i \in \mathbb{R}^{n_i \times p_i}$ represent the nominal *known* part of the subsystem, where the pair (A_i, B_i) is controllable and the pair (C_i, A_i) is observable. On the other hand, the matrix $\Lambda_i \in \mathbb{R}^{m_i \times m_i}$ and the vector-valued function $\Delta_{i,j} : \mathbb{R}^{n_j} \to \mathbb{R}^{m_i}$ represents the *uncertain* part of the subsystem as well as the *uncertain* interaction among the subsystems. That is, $\Delta_{i,i}(x_i(t))$ expresses the uncertain dynamics of the i-th subsystem itself, while $\Delta_{i,j}(x_j(t))$ $(j \neq i)$ expresses the uncertain influence from the j-th subsystem to the i-th subsystem.

Here we introduce the following assumption for Λ_i and $\Delta_{i,j}(x_j(t))$.

Assumption 5.1. *The control effectiveness Λ_i of the i-th subsystem is an unknown symmetric and positive definite matrix. The state dependent uncertainty $\Delta_{i,j}(x_j)$ of the* *i-th subsystem is linearly parameterized as*

$$\Delta_{i,j}(x_j) = F_{i,j}\alpha_{i,j}(x_j) \tag{5.2}$$

for all $j \in \mathcal{N}_i \cup \{i\}$, where $F_{i,j} \in \mathbb{R}^{m_i \times s_{i,j}}$ is an unknown weight matrix and $\alpha_{i,j} : \mathbb{R}^{n_j} \to \mathbb{R}^{s_{i,j}}$ is a given basis function.

For this system (5.1), we define its nominal system as

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + E_i d_i(t), \qquad && x_i(0) = x_{i0} \\ y_i(t) &= C_i x_i(t). \end{aligned} \tag{5.3}$$

That is, when $\Lambda_i = I$ and $\Delta_{i,j}(x_j(t)) \equiv 0$ for all $j \in \mathcal{N}_i \cup \{i\}$, the i-th subsystem (5.1) takes its nominal behavior.

In this chapter, we consider a reference signal $r_i(t) \in \mathbb{R}^{m_i}$ generated by

$$\begin{aligned} \dot{x}_{ri}(t) &= A_{ri} x_{ri}(t), \qquad && x_{ri}(0) = x_{ri0}, \\ r_i(t) &= C_{ri} x_{ri}(t), \end{aligned} \tag{5.4}$$

where $x_{ri}(t) \in \mathbb{R}^{r_i}$ is the state of the reference signal generator, the eigenvalues of A_{ri} are on the imaginary axis and all distinct from one another, and the pair (C_{ri}, A_{ri}) is observable. That is, we can deal with a sinusoidal reference signal as well as a step-type reference signal, while both of them are bounded. We define the initial time $t = 0$ at the time when the reference signal is applied. The initial state x_{ri0} of (5.4) is arbitrary.

It is known that the controlled output $y_i(t)$ of the nominal subsystem (5.3) can follow any reference signal $r_i(t)$ of (5.4) in the steady state if

$$\operatorname{rank}\begin{bmatrix} A_i - \lambda_{ri} I & B_i \\ C_i & 0 \end{bmatrix} = n_i + m_i \tag{5.5}$$

for all eigenvalues λ_{ri} of A_{ri}. We assume this condition for all subsystems (5.1).

The objective of this chapter is to construct a robust *distributed* adaptive control law of the form

$$u_i(t) = f_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \tag{5.6}$$

such that the output $y_i(t)$ of the given system (5.1) asymptotically tracks the reference signal $r_i(t)$ of (5.4) in the presence of the L_2 disturbance $d_i(t)$ and the system uncertainty described by Λ_i and $\Delta_{i,j}(x_i)$ satisfying Assumption 5.1. Here, $x_{\mathcal{N}_i}(t)$ denotes the set of $x_j(t)$ with $j \in \mathcal{N}_i$, and thus the control law (5.6) utilizes the knowledge of the i-th subsystem itself and its surrounding neighbors only. In this sense, we call the form (5.6) a *distributed* adaptive control law. Then we provide a reference model via an H_∞ type robust tracking control for the nominal system (5.3) and derive a robust distributed adaptive tracking control law for the interconnected uncertain system (5.1) with a performance guarantee related to an H_∞ type measure. Here, the performance measure is the induced norm of the tracking error over all possible disturbances and initial states for each subsystem.

5.3 Reference model selection

In this section, we select a suitable reference model for our adaptive control. To this end, we employ an H_∞ type control [39–41] in order to design a robust tracking control law for the nominal system (5.3) to the reference signal (5.4).

We first introduce a variation system for tracking control. Under the assumption (5.5), there exist a unique state $x_{si}(t)$ and a unique control input $u_{si}(t)$ described by

$$\begin{aligned} x_{si}(t) &= L_{xi}x_{ri}(t), \qquad & x_{si0} &= L_{xi}x_{ri0}, \\ u_{si}(t) &= L_{ui}x_{ri}(t) \end{aligned} \tag{5.7}$$

for which the controlled output $y_i(t)$ is identical to the reference signal $r_i(t)$ [47], where L_{xi} and L_{ui} are defined by

$$L_{xi}A_{ri} = A_iL_{xi} + B_iL_{ui}, \qquad C_{ri} = C_iL_{xi}. \tag{5.8}$$

We denote the variations of $x_i(t)$ and $u_i(t)$ from $x_{si}(t)$ and $u_{si}(t)$ by

$$\begin{aligned} \tilde{x}_i(t) &= x_i(t) - x_{si}(t), \qquad & \tilde{x}_{i0} &= x_{i0} - x_{si0}, \\ \tilde{u}_i(t) &= u_i(t) - u_{si}(t), \end{aligned} \tag{5.9}$$

and the tracking error of the controlled output $y_i(t)$ by

$$e_i(t) = r_i(t) - y_i(t).$$

Using these notations, the variation system is defined by

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= A_i\tilde{x}_n(t) + B_i\tilde{u}_i(t) + E_id_i(t), \qquad & \tilde{x}_i(0) &= \tilde{x}_{i0}, \\ e_i(t) &= -C_i\tilde{x}_i(t). \end{aligned} \tag{5.10}$$

In this way, we can recast the original *tracking* problem as a *stabilization* problem of the variation system. That is, if a feedback control law

$$\tilde{u}_i(t) = K_i \tilde{x}_i(t) \tag{5.11}$$

stabilizes the variation system given by (5.10), it turns out that $\lim_{t\to\infty} \tilde{x}_i(t) = 0$, i.e., $\lim_{t\to\infty} y_i(t) = r_i(t)$.

In order to select a suitable control law (5.11) for tracking in the presence of L_2 disturbance $d_i(t)$, we employ a robust control design which is called H_∞ *control with transients* [39]. That is, in this paper, we utilize a performance specification

$$\sup \left\{ \frac{\|e_i\|^2}{\|d_i\|^2 + \tilde{x}_{i0}^T R_i \tilde{x}_{i0}} \right\}^{1/2} < \gamma_i, \tag{5.12}$$

where $\gamma_i \in \mathbb{R}$ is a specified positive number and $R_i > 0$ is a positive definite matrix which is weight for the initial uncertainties. The supremum is taken over all $\tilde{x}_{i0} \in \mathbb{R}^{n_i}$ and $d_i \in L_2$ which satisfy $\|d_i\|^2 + \tilde{x}_{i0}^T R_i \tilde{x}_{i0} \neq 0$. Then, we see that there exists a state feedback (5.11) which stabilizes (5.10) and achieves (5.12) if and only if there exist $X_i = X_i^T \in \mathbb{R}^{n_i \times n_i}$ and $G_i \in \mathbb{R}^{m_i \times n_i}$ which satisfy the linear matrix inequalities

$$\begin{bmatrix} A_i X_i + X_i A_i^T + B_i G_i + G_i^T B_i^T & E_i & X_i C_i^T \\ E_i^T & -\gamma_i^2 I & 0 \\ C_i X_i & 0 & -I \end{bmatrix} < 0, \quad \begin{bmatrix} X_i & I \\ I & \gamma_i^2 R_i \end{bmatrix} > 0, \tag{5.13}$$

where such a state feedback gain K_i of (5.11) is obtained by

$$K_i = G_i X_i^{-1}. \tag{5.14}$$

This is a direct consequence of the existing work [41]. Throughout this chapter, we assume that such an X_i exists for a given $\gamma_i > 0$.

As a matter of fact, the performance (5.12) is guaranteed by the feedback gain (5.14) as follows. When we define

$$P_i = X_i^{-1} \tag{5.15}$$

and use (5.14), we can rewrite (5.13) as

$$P_i(A_i + B_i K_i) + (A_i + B_i K_i)^T P_i + \frac{1}{\gamma_i^2} P_i E_i E_i^T P_i + C_i^T C_i < 0, \tag{5.16}$$

$$\gamma_i^2 R_i > P_i = P_i^T > 0, \tag{5.17}$$

where we see that $A_i + B_iK_i$ is Hurwitz, i.e., the resultant closed-loop system is stable. The inequality (5.16) together with (5.10) and (5.11) implies that

$$\begin{aligned}
\frac{d}{dt}\left(\tilde{x}_i^T(t)P_i\tilde{x}_i(t)\right) &= \dot{\tilde{x}}_i^T(t)P_i\tilde{x}_i(t) + \tilde{x}_i^T(t)P_i\dot{\tilde{x}}_i(t) \\
&= \tilde{x}_i^T(t)\left(P_i(A_i + B_iK_i) + (A_i + B_iK_i)^TP_i\right)\tilde{x}_i(t) \\
&\qquad + \tilde{x}_i^T(t)P_iE_id_i(t) + d_i^T(t)E_i^TP_i\tilde{x}_i(t) \\
&< -\frac{1}{\gamma_i^2}\tilde{x}_i^T(t)P_iE_iE_i^TP_i\tilde{x}_i(t) - \tilde{x}_i^T(t)C_i^TC_i\tilde{x}_i(t) \\
&\qquad + \tilde{x}_i^T(t)P_iE_id_i(t) + d_i^T(t)E_i^TP_i\tilde{x}_i(t) \\
&= -\left(\gamma_id_i(t) - \frac{1}{\gamma_i}E_i^TP_i\tilde{x}_i(t)\right)^T\left(\gamma_id_i(t) - \frac{1}{\gamma_i}E_i^TP_i\tilde{x}_i(t)\right) \\
&\qquad - \tilde{x}_i^T(t)C_i^TC_i\tilde{x}_i(t) + \gamma_i^2d_i^T(t)d_i(t) \\
&\leq -e_i^T(t)e_i(t) + \gamma_i^2d_i^T(t)d_i(t)
\end{aligned} \tag{5.18}$$

for any $\tilde{x}_i(t) \neq 0$. Integrating this inequality from 0 to ∞, with (5.17), we have

$$\|e_i\|^2 - \gamma_i^2\|d_i\|^2 < \tilde{x}_{i0}^TP_i\tilde{x}_{i0} < \gamma^2\tilde{x}_{i0}^TR_i\tilde{x}_{i0},$$

where we use $\lim_{t\to\infty}\tilde{x}_i^T(t)P_i\tilde{x}_i(t) = 0$ which is guaranteed by the closed-loop stability. The above inequality shows that the performance specification (5.12) holds.

Using (5.7) and (5.9), we rewrite the control law (5.11) as

$$\begin{aligned}
u_i(t) &= K_ix_i(t) - K_ix_{si}(t) + u_{si}(t) \\
&= K_ix_i(t) + H_ix_{ri}(t)
\end{aligned} \tag{5.19}$$

for the nominal system (5.3), where K_i is given by (5.14) and H_i is represented as

$$H_i = -K_iL_{xi} + L_{ui}. \tag{5.20}$$

That is, the H_∞ tracking control law for the nominal system (5.3) is composed of a feedback from $x_i(t)$ and a feedforward from $x_{ri}(t)$.

The resultant control system with (5.3) and (5.19) is described by

$$\begin{aligned}
\dot{x}_i(t) &= (A_i + B_iK_i)x_i(t) + B_iH_ix_{ri}(t) + E_id_i(t), \qquad x_i(0) = x_{i0} \\
y_i(t) &= C_ix_i(t).
\end{aligned} \tag{5.21}$$

Since it achieves the performance specification (5.12), it is suitable as a reference model for adaptive control. However, the *unknown* $d_i(t)$ should be excluded in the reference model. Also, the initial states of the given system and the reference model may be different. Thus, in this paper, we employ the system

$$\begin{aligned} \dot{x}_{mi}(t) &= (A_i + B_iK_i)x_{mi}(t) + B_iH_ix_{ri}(t), \qquad x_{mi}(0) = x_{mi0} \\ y_{mi}(t) &= C_ix_{mi}(t) \end{aligned} \tag{5.22}$$

as the reference model for adaptive control, where $x_{mi}(t) \in \mathbb{R}^{n_i}$ is the state, K_i is given by (5.14) based on the LMIs (5.13), and H_i is given by (5.20) with this K_i. Then, we develop a model reference adaptive control and derive its performance guarantee.

5.4 Distributed adaptive control scheme

Let us go back to the uncertain dynamical system (5.1). Referring to the selected reference model (5.22), we rewrite each subsystem (5.1) as

$$\begin{aligned} \dot{x}_i(t) &= (A_i + B_iK_i)x_i(t) + B_iH_ix_{ri}(t) + E_id_i(t) + B_i\Lambda_i[u_i(t) + \delta_i(t)], \quad x_i(0) = x_{i0} \\ y_i(t) &= C_ix_i(t), \end{aligned} \tag{5.23}$$

where we define

$$\begin{aligned} \delta_i(t) &= -\Lambda_i^{-1}\left[K_ix_i(t) + H_ix_{ri}(t) - \sum_{j\in\mathcal{N}_i}\Delta_{i,j}(x_j(t)) - \Delta_{i,i}(x_i(t))\right] \\ &= -W_i\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \end{aligned}$$

with $x_{\mathcal{N}_i}(t)$ used in (5.6). In fact, from Assumption 5.1, the signal $\delta_i(t)$ must be linearly parameterized by using an unknown weight $W_i \in \mathbb{R}^{m_i\times q_i}$ and the corresponding basis function $\sigma_i : \mathbb{R}^{n_i+n_{\mathcal{N}_i}+r_i} \to \mathbb{R}^{q_i}$ which contains $x_i(t)$, $x_{ri}(t)$, and $\alpha_{i,j}(x_j(t))$ ($j \in \mathcal{N}_i \cup \{i\}$), where $q_i \le n_i + p_i + \sum_{j\in\mathcal{N}_i\cup\{i\}} s_{i,j}$. Then we introduce a distributed adaptive feedback control law

$$u_i(t) = \hat{W}_i(t)\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \tag{5.24}$$

where we define the update rule of the adaptive control gain $\hat{W}_i(t) \in \mathbb{R}^{m_i\times q_i}$ as

$$\dot{\hat{W}}_i(t) = -\frac{\eta_i}{\gamma_i^2}B_i^TP_i(x_i(t) - x_{mi}(t))\sigma_i^T(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \qquad \hat{W}_i(0) = \hat{W}_{i0}. \tag{5.25}$$

Notice that $x_{mi}(t)$ of (5.25) is generated by (5.22). The learning rate η_i is any positive real number, the performance specification γ_i for the nominal system is the one in (5.12), and P_i is a symmetric and positive definite matrix defined by (5.15), where X_i is a solution of the linear matrix inequalities (5.13).

We see that the control law (5.24) is *distributed* indeed. In fact, the basis $\sigma_i(x(t), x_{\mathcal{N}_i}(t), x_{ri}(t))$ is distributed and $x_{mi}(t)$ of (5.22) is given by $x_{ri}(t)$. Thus, if the interconnection of the overall system (5.1) is *sparse*, we enjoy a *sparse* structure of the control, where the control law (5.24) utilizes the knowledge of the i-th subsystem itself and its surrounding neighbors only. See also the numerical example in Section 5.6 for further details.

Now, let us define the errors from the ideal case as

$$\begin{aligned} x_{ei}(t) &= x_i(t) - x_{mi}(t), & x_{ei0} &= x_{i0} - x_{mi0} \\ y_{ei}(t) &= y_i(t) - y_{mi}(t), & & \\ W_{ei}(t) &= \hat{W}_i(t) - W_i, & W_{ei0} &= \hat{W}_{i0} - W_i \end{aligned}$$

where $x_{mi}(t)$ and $y_{mi}(t)$ are defined in (5.22). With (5.22), (5.23), (5.24), and (5.25), we have

$$\begin{aligned} \dot{x}_{ei}(t) &= (A_i + B_iK_i)x_{ei}(t) + E_id_i(t) + B_i\Lambda_iW_{ei}(t)\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \quad x_{ei}(0) = x_{ei0}, \\ y_{ei}(t) &= C_ix_{ei}(t), \end{aligned} \tag{5.26}$$

$$\dot{W}_{ei}(t) = -\frac{\eta_i}{\gamma_i^2}B_i^TP_ix_{ei}(t)\sigma_i^T(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \qquad W_{ei}(0) = W_{ei0}, \tag{5.27}$$

which describes the error dynamics from the reference model (5.22).

The next theorem presents the result of this section.

Theorem 5.2. *Consider the uncertain interconnected dynamical system described by* (5.1) *subject to Assumption 5.1. Consider, in addition, the reference model given by* (5.22) *and the distributed adaptive controller given by* (5.24) *and* (5.25)*. Then, all of the solutions* $(x_{ei}(t), W_{ei}(t))$ $(i = 1, 2, \dots, N)$ *given by* (5.26) *and* (5.27) *are bounded. Furthermore, all of the tracking errors* $y_{ei}(t)$ $(i = 1, 2, \dots, N)$ *satisfy*

$$\lim_{t\to\infty} y_{ei}(t) = 0$$

for any (x_{ei0}, W_{ei0}) $(i = 1, 2, \dots, N)$*.*

Proof. Consider a candidate of Lyapunov function

$$V = \sum_{i=1}^{N} V_i(x_{ei}, W_{ei}), \qquad V_i(x_{ei}, W_{ei}) = x_{ei}^TP_ix_{ei} + \frac{\gamma_i^2}{\eta_i}\mathrm{tr}W_{ei}^T\Lambda_iW_{ei},$$

where η_i and P_i are taken from (5.25) and Λ_i of (5.1) satisfies Assumption 5.1, which means that $P_i = P_i^T > 0$ and $\Lambda_i = \Lambda_i^T > 0$. Thus the function $V_i(x_{ei}, W_{ei})$ is in fact a continuously differentiable function such that $V_i(0,0) = 0$ and $V_i(x_{ei}, W_{ei}) > 0$ for all $(x_{ei}, W_{ei}) \neq (0,0)$, which implies V is also positive definite.

Differentiating this candidate along the trajectories of (5.26) and (5.27), we have

$$\begin{aligned}
\dot{V}_i(x_{ei}(t), W_{ei}(t)) &= \dot{x}_{ei}^T(t) P x_{ei}(t) + x_{ei}^T(t) P \dot{x}_{ei}(t) + \frac{2\gamma_i^2}{\eta_i} \mathrm{tr} \dot{W}_{ei}^T(t) \Lambda_i W_{ei}(t) \\
&= x_{ei}^T(t) \left(P_i(A_i + B_i K_i) + (A_i + B_i K_i)^T P_i \right) x_{ei}(t) \\
&+ x_{ei}^T(t) P_i E_i d_i(t) + d_i^T(t) E_i^T P_i x_{ei}(t) \\
&\quad + 2 x_{ei}^T(t) P_i B_i \Lambda_i W_{ei}(t) \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \\
&- 2\mathrm{tr} \left(\sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) x_{ei}^T(t) P_i B_i \Lambda_i W_{ei}(t) \right) \\
&= x_{ei}^T(t) \left((A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) \right) x_{ei}(t) \\
&+ x_{ei}^T(t) P_i E_i d_i(t) + d_i^T(t) E_i^T P_i x_{ei}(t) \\
&< -y_{ei}^T(t) y_{ei}(t) + \gamma_i^2 d_i^T(t) d_i(t)
\end{aligned}$$

for any $\tilde{x}_{ei}(t) \neq 0$, which partially follows completing the square used in (5.18). Integrating this inequality from 0 to any $T > 0$, we have

$$\int_0^T y_{ei}^T(t) y_{ei}(t) dt + V_i(x_{ei}(T), W_{ei}(T)) \leq \int_0^T \gamma_i^2 d_i^T(t) d_i(t) dt + V_i(x_{ei0}, W_{ei0}). \tag{5.28}$$

Since $d_i \in L_2$, the right-hand side of the inequality is bounded. Thus, $V_i(x_{ei}(T), W_{ei}(T))$ is bounded for any $T > 0$, which implies that the solution $(x_{ei}(t), W_{ei}(t))$ is bounded. Also, the boundedness of the right-hand side of (5.28) implies that

$$\lim_{T \to \infty} \int_0^T y_{ei}^T(t) y_{ei}(t) dt < \infty. \tag{5.29}$$

Furthermore, since $\dot{y}_{ei}(t) = C_i \dot{x}_{ei}(t)$ and $\dot{x}_{ei}(t)$ of (5.26) is represented by the signals $x_{ei}(t)$, $d_i(t)$, $W_{ei}(t)$, $x_i(t)$, $x_{\mathcal{N}_i}(t)$, and $x_{ri}(t)$, we can see that

$$\sup_{t \geq 0} \dot{y}_{ei}^T(t) \dot{y}_{ei}(t) < \infty. \tag{5.30}$$

In fact, $x_{ei}(t)$ and $W_{ei}(t)$ are bounded as we have proved above, while $d_i \in L_\infty$ is bounded and $x_{ri}(t)$ of (5.4) is bounded. The boundedness of $x_i(t)$ and $x_{\mathcal{N}_i}(t)$ follows the boundedness of $x_{ei}(t)$ $(i = 1, 2, \cdots, N)$ and $x_{mi}(t)$ $(i = 1, 2, \cdots, N)$ of (5.22). With (5.29), (5.30), and a version of the Barbălat lemma[43], we conclude that all of the tracking errors $y_{ei}(t)$ $(i = 1, 2, \ldots, N)$ satisfy

$$\lim_{t \to \infty} y_{ei}(t) = 0.$$

This completes the proof of the theorem. □

Remark 5.3. Theorem 5.2 establishes boundedness of the error dynamics behaviors via the proposed distributed adaptive control in the presence of disturbances and uncertainties in each subsystem. The theorem also shows that zero steady state tracking error is achieved by this control.

5.5 Performance evaluation

Since the proposed adaptive control given by (5.24) and (5.25) employs an H_∞ control (5.22) for the nominal system (5.3) as the reference model, one of our interests is to evaluate the performance degradation due to introduction of adaptive mechanism.

In this regard, let us recall the performance specification (5.12) for the nominal system (5.3). It says that

$$\|r_i - y_i\| \leq \gamma_i \left(\|d_i\|^2 + (x_{i0} - x_{si0})^T R_i (x_{i0} - x_{si0})\right)^{1/2} \tag{5.31}$$

holds for any $d_i \in L_2$, $x_{i0} \in \mathbb{R}^{n_i}$, and $x_{si0} \in \mathbb{R}^{n_i}$. On the other hand, when we use the adaptive control of (5.22), (5.24), and (5.25), we can evaluate the tracking error $\|r_i - y_i\|$ of the uncertain systems (5.1) similarly as follows.

Theorem 5.4. *Consider the uncertain interconnected dynamical system described by* (5.1) *subject to Assumption 5.1. Consider, in addition, the reference model given by* (5.22) *and the distributed adaptive controller given by* (5.24) *and* (5.25)*. Then, the tracking error is bounded as*

$$\begin{aligned}\|r_i - y_i\| \leq \gamma_i \Big\{ &\left(\|d_i\|^2 + (x_{i0} - x_{mi0})^T R_i (x_{i0} - x_{mi0}) + \frac{1}{\eta_i}\mathrm{tr}(\hat{W}_{i0} - W_i)^T \Lambda_i (\hat{W}_{i0} - W_i)\right)^{1/2} \\ &+ \left((x_{mi0} - x_{si0})^T R_i (x_{mi0} - x_{si0})\right)^{1/2} \Big\}\end{aligned} \tag{5.32}$$

for any $d_i \in L_2$, $x_{i0} \in \mathbb{R}^{n_i}$, $x_{mi0} \in \mathbb{R}^{n_i}$, $x_{si0} \in \mathbb{R}^{n}$, and $\hat{W}_{i0} \in \mathbb{R}^{m_i \times q_i}$.

Proof. Since (5.31) holds for (5.21), we have

$$\|r_i - y_{mi}\| \leq \gamma_i \left((x_{mi0} - x_{si0})^T R_i (x_{mi0} - x_{si0})\right)^{1/2} \tag{5.33}$$

for the reference model (5.22) which does not contain $d_i \in L_2$. Regarding the adaptive control, we have established (5.28) for any $T > 0$, which means that

$$\|y_{ei}\|^2 \leq \gamma_i^2 \|d_i\|^2 + V_i(x_{ei0}, W_{ei0}).$$

Thus we have

$$\|y_i - y_{mi}\| \leq \gamma_i\big(\|d_i\|^2 + (x_{i0} - x_{mi0})^T R_i(x_{i0} - x_{mi0}) + \frac{1}{\eta_i}\mathrm{tr}(\hat{W}_{i0} - W_i)^T \Lambda_i(\hat{W}_{i0} - W_i)\big)^{1/2}, \tag{5.34}$$

where we use (5.17). These evaluations (5.33) and (5.34) together with the triangle inequality

$$\|r_i - y_i\| \leq \|r_i - y_{mi}\| + \|y_{mi} - y_i\| \tag{5.35}$$

gives the tracking error bound of the theorem. □

Remark 5.5. This bound given by Theorem 5.4 shows a transient performance of the proposed distributed H_∞ adaptive control. It says that L_2 type gain from the disturbance and the initial values to the tracking error of the distributed adaptive control is bounded by γ_i as that of the corresponding nominal closed-loop system (5.21) is. In addition, if we use large learning rate η_i, the transient performance becomes better, which will be confirmed in the numerical example below.

Regarding the conservativeness of this performance evaluation if perfect tracking is achieved ($x_{ei0} = x_{i0} - x_{mi0} = 0$) and there is no estimation error ($W_{ei0}(t) = \hat{W_i}0(t) - W_i = 0$), then the tracking error is bounded the same as the nominal system.

5.6 Numerical Example

In this section, we demonstrate the proposed distributed adaptive control through a numerical example. Let us consider a mass-spring-damper system having N masses in one line shown in Figure 5.1. Each mass m_i $(1 < i < N)$ is possibly connected with its neighbors m_{i-1} and m_{i+1} by springs $k_{i-1,i}$, $k_{i,i+1}$ and dampers $c_{i-1,1}$, $c_{i,i+1}$, that is,

$$\begin{aligned} m_i\ddot{q}_i(t) = &-k_{i-1,i}\left(q_i(t) - q_{i-1}(t)\right) - c_{i-1,i}\left(\dot{q}_i(t) - \dot{q}_{i-1}(t)\right) \\ &- k_{i,i+1}\left(q_i(t) - q_{i+1}(t)\right) - c_{i,i+1}\left(\dot{q}_i(t) - \dot{q}_{i+1}(t)\right) + u_i(t) + d_i(t), \end{aligned} \tag{5.36}$$

where $q_i(t) \in \mathbb{R}$ is the position of the mass m_i to be controlled, and $\dot{q}_i(t)$ and $\ddot{q}_i(t)$ are its velocity and acceleration, respectively. The force $u_i(t) \in \mathbb{R}$ is applied to the mass m_i as the control input, while we assume the *unknown* disturbance force $d_i(t) \in \mathbb{R}$ to m_i satisfies $d_i \in L_\infty \cap L_2$ and $\dot{d}_i \in L_\infty$. We also assume that all physical parameters $m_i > 0$, $k_{i-1,i} \geq 0$, $k_{i,i+1} \geq 0$, $c_{i-1,i} \geq 0$, and $c_{i,i+1} \geq 0$ are *unknown.* The cases $i = 1$ and $i = N$ having one neighbor are not explicitly stated in this section, though it is clear that they can be described in a similar way.

The mass-spring-damper system (5.36) stated above is consistent with the system description (5.1). In fact, we can rewrite (5.36) as

$$\dot{x}_i(t) = A_i x_i(t) + B_i[\Lambda_i u_i(t) + \sum_{j=i-1}^{i+1} \Delta_{i,j}(x_j(t))] + E_i d_i(t),$$
$$y_i(t) = C_i x_i(t), \tag{5.37}$$

where we define

$$x_i(t) = \begin{bmatrix} q_i(t) \\ \dot{q}_i(t) \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad E_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\Lambda_i = \frac{1}{m_i}, \quad \Delta_{i,i}(x_i(t)) = \begin{bmatrix} -\frac{k_{i-1,i} + k_{i,i+1}}{m_i} & -\frac{c_{i-1,i} + c_{i,i+1}}{m_i} \end{bmatrix} x_i(t),$$
$$\Delta_{i,i-1}(x_{i-1}(t)) = \begin{bmatrix} \frac{k_{i-1,i}}{m_i} & \frac{c_{i-1,i}}{m_i} \end{bmatrix} x_{i-1}(t), \quad \Delta_{i,i+1}(x_{i+1}(t)) = \begin{bmatrix} \frac{k_{i,i+1}}{m_i} & \frac{c_{i,i+1}}{m_i} \end{bmatrix} x_{i+1}(t).$$

That is, all unknown parameters are included in Λ_i and $\Delta_{i,j}(x_j(t))$. Apparently, (A_i, B_i) is controllable, (C_i, A_i) is observable, and the uncertainties Λ_i and $\Delta_{i,j}(x_j(t))$ satisfy Assumption 5.1. That is, the mass-spring-damper system (5.36) can be represented as (5.1), where we define $\mathcal{N}_i = \{i-1, i+1\}$.

For this system, we consider a sinusoidal reference signal such as $\sin \omega_i t$, i.e., we define the coefficient matrices of (5.4) as

$$A_{ri} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix}, \qquad C_{ri} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where $\omega_i > 0$. We see that (C_{ri}, A_{ri}) is observable. Also, the rank condition (5.5) is satisfied for the eigenvalues of A_{ri}. Actually, we have the solutions of (5.8) as

$$L_{xi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad L_{ui} = \begin{bmatrix} -\omega_i^2 & 0 \end{bmatrix}.$$

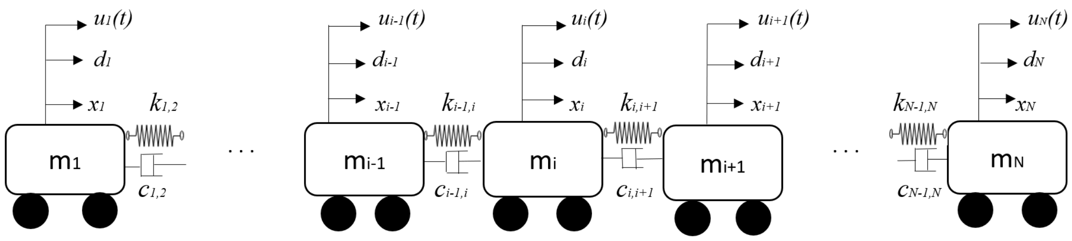

FIGURE 5.1: Interconnected mass-spring-damper system

In the following, we set $\omega_i^2 = 0.1$ for all $i \in \mathcal{V}$.

With $\gamma = 1$ and $R_i = 3I$, we solved the LMIs (5.13). Then we obtained the H_∞ tracking gains (5.14) and (5.20) as

$$K_i = \begin{bmatrix} -4.1538 & -1.5385 \end{bmatrix}, \qquad H_i = \begin{bmatrix} 4.0538 & 1.5385 \end{bmatrix},$$

which determines the reference model (5.22). We also have P_i of (5.15) as

$$P_i = \begin{bmatrix} 1.3012 & 0.3253 \\ 0.3253 & 0.4337 \end{bmatrix},$$

which is used in the update rule (5.25) of the adaptive control gain.

Referring to the above K_i and H_i, we rewrite the system (5.37) as the form of (5.23), i.e.,

$$\begin{aligned} \dot{x}_i(t) &= (A_i + B_i K_i) x_i(t) + B_i H_i x_{ri}(t) + E_i d_i(t) + B_i \Lambda_i [u_i(t) + \delta_i(t)], \\ y_i(t) &= C_i x_i(t). \end{aligned}$$

It should be noted that its uncertainty is described as

$$\begin{aligned} \delta_i(t) &= -\Lambda_i^{-1} \left[K_i x_i(t) + H_i x_{ri}(t) - \sum_{j=i-1}^{i+1} \Delta_{i,j}(x_j(t)) \right] \\ &= -W_i \sigma_i(x_{i-1}(t), x_i(t), x_{i+1}(t), x_{ri}(t)), \end{aligned}$$

where

$$\begin{aligned} W_i &= \begin{bmatrix} -\begin{bmatrix} k_{i-1,i} & c_{i-1,i} \end{bmatrix} & \begin{bmatrix} k_{i-1,i} + k_{i,i+1} & c_{i-1,i} + c_{i,i+1} \end{bmatrix} + m_i K_i & -\begin{bmatrix} k_{i,i+1} & c_{i,i+1} \end{bmatrix} & m_i H_i \end{bmatrix} \\ \sigma_i(x_{i-1}(t), x_i(t), x_{i+1}(t), x_{ri}(t)) &= \begin{bmatrix} x_{i-1}^T(t) & x_i^T(t) & x_{i+1}^T(t) & x_{ri}^T(t) \end{bmatrix}^T. \end{aligned}$$

That is, due to the sparse structure of the system (5.36) of Figure 5.1, the basis $\sigma_i(x_{i-1}(t), x_i(t), x_{i+1}(t), x_{ri}(t))$ contains only *the states of its neighbors*, i.e., $x_{i-1}(t)$ and $x_{i+1}(t)$. Since the adaptive control law (5.24) and the update rule of the adaptive gain (5.25) have the form

$$\begin{aligned} u_i(t) &= \hat{W}_i(t) \sigma_i(x_{i-1}(t), x_i(t), x_{i+1}(t), x_{ri}(t)), \\ \dot{\hat{W}}_i(t) &= -\frac{\eta_i}{\gamma_i^2} B_i^T P_i (x_i(t) - x_{mi}(t)) \sigma_i^T(x_{i-1}(t), x_i(t), x_{i+1}(t), x_{ri}(t)), \end{aligned}$$

these become in fact a *distributed* control law thanks to the sparse structure of the basis $\sigma_i(x_{i-1}(t), x_i(t), x_{i+1}(t), x_{ri}(t))$.

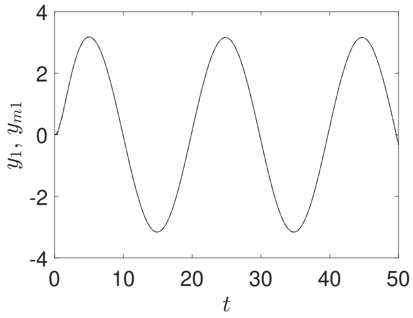

FIGURE 5.2: Output tracking for $y_1, y_{m1}(\eta_i = 100)$

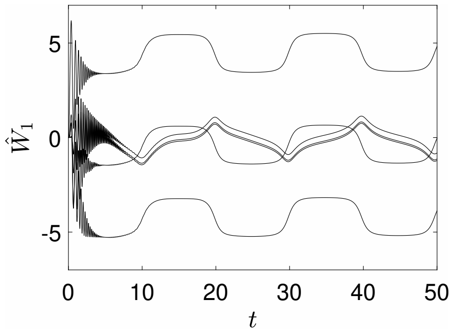

FIGURE 5.3: Adaptive gains for $\hat{W}_1(\eta_i = 100)$

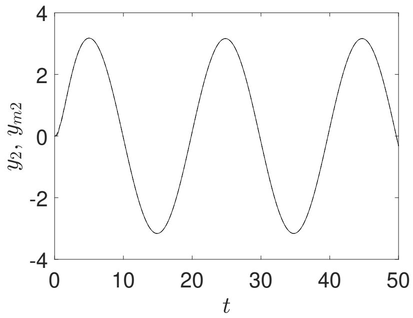

FIGURE 5.4: Output tracking for y_2, y_{m2} $(\eta_i = 100)$

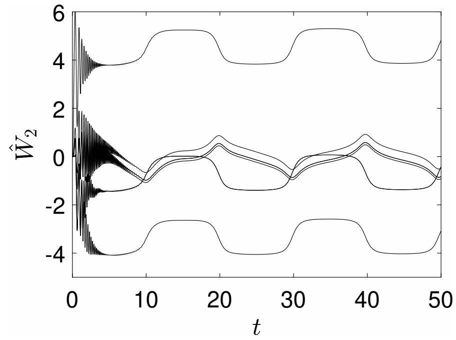

FIGURE 5.5: Adaptive gains for $\hat{W}_2$ $(\eta_i = 100)$

Now, let us consider a numerical simulation. We investigate the case $N = 3$, i.e., $\mathcal{V} = \{1, 2, 3\}$, where we set the unknown and uncertain parameters as $m_1 = m_2 = m_3 = 3$, $k_{1,2} = k_{2,3} = 2$, and $c_{1,2} = c_{2,3} = 1$. We chose all initial states are zero except for the reference signal generators, where we used $x_{ri0} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ for all $i \in \mathcal{V}$. We chose the disturbance as $d_i(t) = e^{-t/10} \sin t$, i.e., $d_i \in L_\infty \cap L_2$ and $\dot{d}_i \in L_\infty$ for all $i \in \mathcal{V}$.

Figures 5.2–5.7 show the tracking responses and the gain behaviors of the proposed distributed adaptive control for subsystems 1, 2, and 3, respectively, where we set the learning rates as $\eta_1 = \eta_2 = \eta_3 = 100$. On the other hand, Figures 5.8–5.13 show the tracking responses and the gain behaviors of the proposed adaptive control, where we use $\eta_1 = \eta_2 = \eta_3 = 10$. Note that $y_1(t)$, $y_2(t)$, $y_3(t)$ are indicated as solid lines and $y_{m1}(t), y_{m2}(t), y_{m3}(t)$ are indicated as dashed lines in Figures 5.2, 5.4, 5.6, 5.8, 5.10, and 5.12. The elements of the adaptive gains $\hat{W}_1(t)$, $\hat{W}_2(t)$, and $\hat{W}_3(t)$ for subsystems 1, 2 and 3 are indicated as solid lines in Figures 5.3, 5.5, 5.7, 5.9, 5.11, and 5.13.

In Figures 5.2–5.7, all signals are bounded and $y_i(t)$ tends to $y_{mi}(t)$ $(i \in \mathcal{V})$ as t tends to infinity, which is consistent with Theorem 5.2. Furthermore, comparing Figures 5.2, 5.4, and 5.6 with Figures 5.8, 5.10, and 5.12, we see that a larger learning rate η_i gives a better performance, which is consistent with Theorem 5.4.

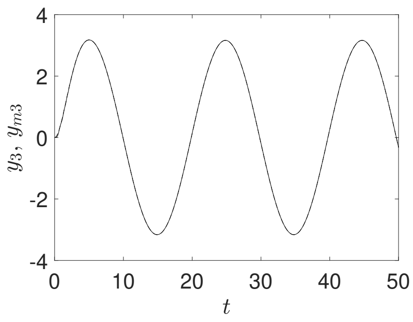

FIGURE 5.6: Output tracking for y_3, y_{m3} ($\eta_i = 100$)

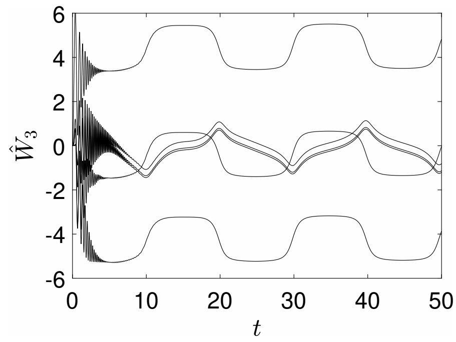

FIGURE 5.7: Adaptive gains $\hat{W}_3$ ($\eta_i = 100$)

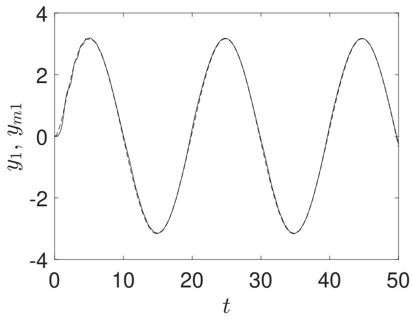

FIGURE 5.8: Output tracking for $y_1, y_{m1}(\eta_i = 10)$

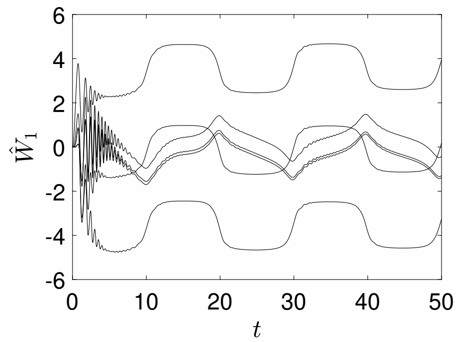

FIGURE 5.9: Adaptive gains $\hat{W}_1$ ($\eta_i = 10$)

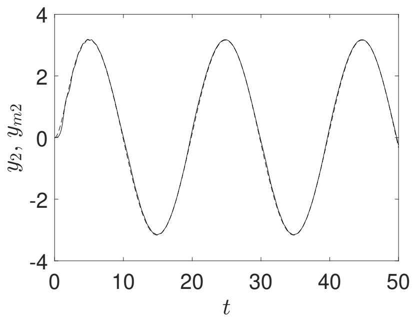

FIGURE 5.10: Output tracking for y_2, y_{m2} ($\eta_i = 10$)

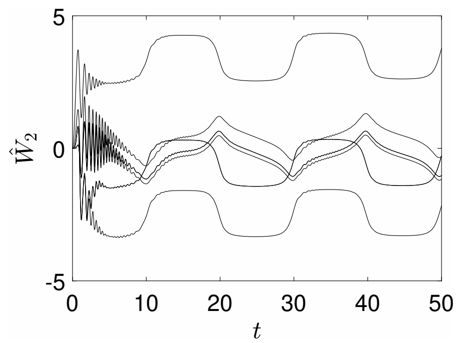

FIGURE 5.11: Adaptive gains $\hat{W}_2$ ($\eta_i = 10$)

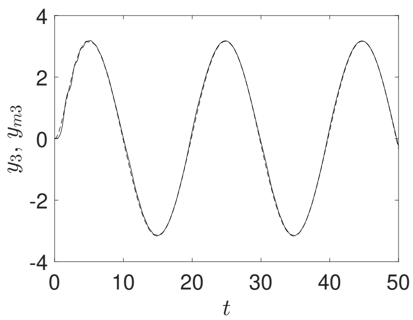

FIGURE 5.12: Output tracking for y_3, y_{m3} ($\eta_i = 10$)

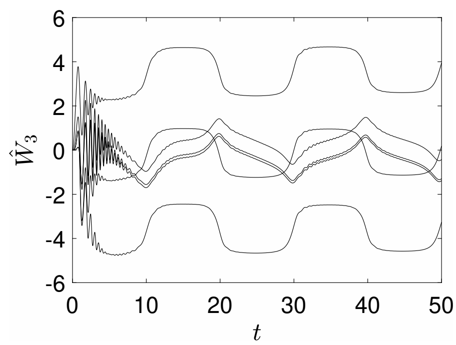

FIGURE 5.13: Adaptive gains $\hat{W}_3$ ($\eta_i = 10$)

5.7 Summary

In this chapter, a distributed adaptive control for H_∞ tracking of uncertain interconnected dynamical system is proposed that asymptotically tracks the output of a reference model in the presence of system/interconnection uncertainties. An H_∞ type control[39–41] for a suitable reference model selection is employed. Then a distributed adaptive control law for uncertain dynamical systems with L_2 disturbance is developed. It is proved that the proposed distributed adaptive control guarantees the boundedness of the error dynamics behaviors in the presence of disturbances and uncertainties, where it achieves zero tracking error in the steady state as well. Furthermore, an explicit error bound of tracking, which enables us to evaluate a transient performance of the control system is established. The numerical examples have shown that the theoretical results developed in this paper are useful. The extension of this work to the output feedback can be considered as future work.

Chapter 6

Conclusion

6.1 Summary

The number of uncertain dynamical systems in the presence of parametric uncertainties and uncertain nonlinearities (e.g., disturbances) operating under changing conditions has been increasing due to the advances in technology and the wide range of possible uses, such as energy, economy, market competition or healthcare. This is why control systems with high accuracy, robustness and fast response are needed for tracking problems. Having an adaptive system is crucial when our process is unknown or when there are changes in the dynamics of a system, in particular, due to non-linear effects, which can imply a modification in the type of response. If the modifications are unpredictable, there has to be a mechanism that can compensate them. As systems are becoming more complex and connected in large networks, the growth in complexity requires the use of decision-making processes that can be centralized or distributed.

Previous works have been developed for various complex systems such as interconnected and/or large-scale systems under different perspectives. However, they do not guarantee an optimal solution while estimating the unknown uncertainty parameters of the system dynamics. As a result, poor transient performance, system instability or high frequency oscillation can be introduced to the system. Performance guarantee related to nominal tracking is also an issue in research works for the development of distributed adaptive control architectures for realizing a desirable tracking.

In this dissertation, some of the limitations in earlier adaptive control approaches for tracking problems are addressed. In particular, the objective is to select an appropriate control input which incorporates stability analysis and the evaluation of performance degradation from the nominal system with an explicit error bound and zero steady state error.

Adaptive LQ tracking control provides an optimal reference model whose parameters are identical with the original tracking problem but without uncertainties. The performance evaluation of the proposed control and closed loop stability analysis of the uncertain dynamical system are demonstrated.

Adaptive control for H_∞ tracking control with transients is introduced for a robust performance in the presence of system uncertainties. Zero tracking error in steady state for a step type reference signal and an explicit error bound is established in the presence of uncertainties and disturbances.

Distributed adaptive LQ control law which utilizes a specified reference model is investigated for a class of interconnected uncertain dynamical system. The proposed adaptive control law achieves the desirable optimal tracking asymptotically and achieves boundedness of all signals. In addition, an explicit error bound with respect to the nominal optimal tracking and evaluation of performance guarantee is established. The role of the update rule for the learning rate is also explained.

A distributed adaptive control for a class of interconnected dynamical system employing H_∞ control with transients in the precedence of L_2 disturbances for a general reference input is also proposed. Thus, boundedness of the error dynamics and steady state error is guaranteed. Performance degradation is also evaluated. However, the extension to the output feedback case in the presence/absence of disturbance can be considered as an important work.

While there are several adaptive control approaches not covered by the above proposed techniques in this dissertation, they addressed some of the most important themes in adaptive control particularly for tracking problems, namely: robustness, consensus, convergence, and optimization. Additionally, this dissertation has shown that the stability analysis, boundedness of the error dynamics and performance evaluation of previous works can still be attained by applying the novel approaches described in this study.

6.2 Future works

The proposed adaptive control approaches can still be modified to application of practical uncertain dynamical systems. Although the distributed adaptive control using state feedback case with/without disturbance has been investigated in this dissertation, the extension to the output feedback case in the presence/absence of disturbance can be an important future work. For the interconnected uncertain dynamical systems, the effect of using more complex interconnection schemes can be explored. Additional analyses can also be performed to further characterize the stability and evaluation of performance degradation properties of the proposed approaches.

The effectiveness of these theoretical findings can be further applied to electrical power systems like adaptive generator exciter control and adaptive load-frequency control, aircraft control systems, and active control of vibration suppression. In addition the results can also be used as an effective approach for mechanical robot manipulator controller design due to the presence of nonlinearties and uncertainties in robot dynamic models. More generally, to distributed coordinated tracking Control for a class of uncertain multiagent systems and so on.

In general, the advancements of microprocessors and highspeed computing technology shall enable further development and practical implementation of adaptive techniques to uncertain dynamical systems. Considering this issue, adaptive control for multi-input, multioutput (MIMO) systems also deserves further attention and may worth considering for further study. Some possible future work in this area includes multi-model adaptive Control and safe-switching, robust-adaptive and adaptive-robust controls, and model-free control systems like fuzzy or neural networks to minimize the effect of uncertainty in the absence of complex mathematical burden.

Appendix A

Barbalat Lemma

Lemma A.1. *This Lemma gives a simple proof of the property that if a signal is square integrable and has a bounded derivative, then the signal converges to zero asymptotically. That is if* $\epsilon(t) \in L_2$ *and* $\dot{\epsilon}(t) \in L_\infty$ *then* $\lim_{x\to\infty} \epsilon(t) = 0$.

Appendix B

Standard adaptive control

Consider an uncertain dynamical system given by

$$\begin{aligned} S: \dot{x}(t) &= Ax(t) + B[\Lambda u(t) + \Delta(x(t))], \\ y_p(t) &= Cx_p(t), \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector which is assumed to be available for control, $u(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, and $y(t) \in \mathbb{R}^m$ is the controlled output. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ represent the nominal *known* part of this system, where the pair (A, B) is controllable and the pair (C, A) is observable. On the other hand, the matrix $\Lambda \in \mathbb{R}^{m\times m}$ and the vector-valued function $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ represent the uncertain *unknown* part of the system.

In this regard, the following assumption for Λ and $\Delta(x_p)$ is introduced.

Assumption B.1. *The control effectiveness Λ is an unknown symmetric and positive definite matrix. The state dependent matched uncertainty $\Delta(x)$ is linearly parameterized as*

$$\Delta(x) = F\alpha(x),$$

where $F \in \mathbb{R}^{m\times s}$ is an unknown weight matrix and $\alpha : \mathbb{R}^n \to \mathbb{R}^s$ is the corresponding basis function.

Next consider the reference system capturing a desired closed loop performance given by:

$$S_r : x_r(t) = A_r x_r(t) + B_r c(t)$$
$$y_r(t) = C_r x_r(t), \tag{B.1}$$

where, $x_r(t) \in \mathbb{R}^n$ is the reference state vector, $c(t) \in \mathbb{R}^m$ is a given bounded command, $A_r(t) \in \mathbb{R}^{n\times n}$ is the reference system matrix, and $B_r(t) \in \mathbb{R}^{n\times m}$ is the command input matrix.

Assumption B.2. *There exists $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$ such that $A_r = A + BK_1$ and $B_r = BK_2$ is satisfied with A_r being Hurwitz.*

In Chapter 2, the LQR control theory and the MRAC based on the Lyapunov stability theory are combined. LQR is applied to the nominal system (2.2) to optimize its response using the tracking control law (2.12) and achieve optimal reference model (2.15) tracking, which is then used as the target for the controlled system to track the reference model. Whereas, the MRAC is designed to automatically tune the control parameters to track the response of the uncertain dynamical system (2.1) to the reference model (2.15) that exhibits the desired behavior. Furthermore, the proposed adaptive law (2.17) is able to tune the parameters of the controller to compensate the uncertainties and track the response to the desired reference model.

Now, the statement of the control problem is given as follows. The aim is to design a control input $u(t)$ for the uncertain dynamical system to asymptotically drive the mismatch between S and S_r to zero in the presence of uncertainty in control effectiveness Λ and model uncertainties $\Delta(x(t))$. Then S can be rewritten as

$$x(t) = A_r x(t) + B_r c(t) + B\Lambda[u(t) + \delta(t)],$$
$$y(t) = Cx(t), \tag{B.2}$$

where $\delta(t)$ is defined as

$$\delta(t) = -\Lambda^{-1}[K_1 x(t) + K_2 c(t) - \Delta(x(t))]$$
$$= -W\sigma(x(t), c(t)).$$

In fact, from Assumption B.1, the signal $\delta(t)$ must be parameterized by using an unknown weight $W \in \mathbb{R}^{m\times q}$ and the corresponding basis function $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^q$ which contains $x(t)$, $c(t)$, and $\alpha(x(t))$, where $q \leq n + m + s$.

Then an adaptive feedback control law is introduced as

$$u(t) = \hat{W}(t)\sigma(x(t), c(t)), \tag{B.3}$$

where $\hat{W}(t)$ is an estimate of W which is defined with the update rule of the adaptive control gain $\hat{W}(t) \in \mathbb{R}^{m \times q}$ as

$$\dot{\hat{W}}(t) = -\gamma B^T P(x(t) - x_r(t))\sigma^T(x(t), c(t)). \tag{B.4}$$

It should be noted that $x_r(t)$ of (B.4) is generated by (B.1). The learning rate γ is a positive real number and P is the symmetric and positive definite solution of the Lyapunov equation (B.5)

$$A_r^T P + P A_r + R = 0, \tag{B.5}$$

where R is a given positive definite matrix. Now, the errors from the ideal case are defined as

$$\begin{aligned} x_e(t) &= x(t) - x_r(t), & y_e(t) &= y(t) - y_r(t), \\ W_e(t) &= \hat{W}(t) - W, \end{aligned}$$

where $x_r(t)$ and $y_r(t)$ are from (B.1). With (B.1), (B.2), (B.3), and (B.4), the error dynamics from the reference model can be rewritten as

$$\begin{aligned} \dot{x}_e(t) &= (A + BK_1)x_e(t) + B\Lambda W_e(t)\sigma(x(t), c(t)), \\ y_e(t) &= Cx_e(t), \\ \dot{W}_e(t) &= -\gamma B^T P x_e(t)\sigma^T(x(t), c(t)). \end{aligned}$$

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