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Parametrizations of generalized Teichmüller spaces

Katsuhiro Miguchi

Abstract

In this paper, we introduce a new variant of the Teichmüller space, namely the deformation space of hyperbolic structures on a surface with both enhancement and decoration. We construct the parameterization of this deformation space, which is a common generalization of the shear coordinates and the λ-length coordinates. Furthermore, we introduce the lamination space corresponding to this new deformation space, and show a certain compatibility of the shear coordinates and the λ-length coordinates.

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Chapter 1

Introduction

The enhanced Teichmüller spaces are considered when we transform the triangulated hyperbolic surfaces by shearing along the edges. The decorated Teichmüller spaces are introduced by R. C. Penner [10]. V. V. Fock and A. B. Goncharov improved the theory of these moduli spaces [5]. There are the canonical correspondence of these Teichmüller spaces and the corresponding lamination spaces. The aim of this paper is to construct a common generalization of the enhanced and decorated Teichmüller spaces, and establish a correspondence between the generalized Teichmüller space and a corresponding lamination space. There are other known generalizations of Teichmüller spaces in the theory of the cluster varieties. The symplectic doubles of V. V. Fock and A. B. Goncharov [6] and the cluster variety with principal coefficients of A. B. Goncharov and L. Shen [4] differ from the generalized Teichmüller spaces introduced in this paper because they have no part corresponding to the decoration parameters.

1.1 Main results

Let S be an oriented connected compact surface of genus $g \geq 0$ with $c \geq 0$ boundary components. Remove $p \geq 0$ points from the interior of S and call them *puncture vertices*. Remove $s \geq 0$ points from the boundaries of S and call them *spike vertices*. We assume that all boundary components have at least one spike vertex and that $4 - 4g - 2c - 2p - s < 0$. Then S admits a hyperbolic structure with crown-shaped boundaries.

An *enhancement* is an assignment of signs to the closed geodesic boundaries (Definition 2.2.1). We define the *decoration* as a collection of horocycles around the cusps and the spikes (Definition 2.3.1) and moreover equidistant curves around the closed geodesic boundaries (Definition 3.1.2). Then, we introduce the *decorated enhanced Teichmüller space* $\mathcal{T}^{ax}(S)$ of S as the deformation space of all marked hyperbolic surfaces homeomorphic to S with both enhancement and decoration (Definition 3.1.3).

Let Γ be a triangulation obtained by connecting the $p + s$ vertices on S. Let V denote the set of the vertices, V^i denote the set of the puncture vertices, E denote the set of the edges, and E^i denote the set of the edges which are in the interior of S. A *shear parameter* of an interior edge is the signed length parameter for gluing

the ideal triangles along the edge. It is known that the enhanced Teichmüller space $\mathcal{T}^x(S)$ of S is parameterized by the shear coordinates [5] (Proposition 2.2.4):

$$\varphi_x : \mathcal{T}^x(S) \xrightarrow{\text{diffeo.}} \mathbb{R}^{E^i}.$$

Let

$$\rho : \mathbb{R}^{E^i} \to \mathbb{R}^V$$

be a piecewise-smooth function. Such ρ is called a *normalizing function*. In this paper, we introduce decoration parameters normalized by ρ for the vertices, and give a parameterization of the decorated enhanced Teichmüller space of S:

Theorem A. (Theorem 3.1.5) Let

$$\psi_{x,\rho} : \mathcal{T}^{ax}(S) \to \mathbb{R}^{E^i} \times \mathbb{R}^V$$

be the map giving the shear parameters and the decoration parameters normalized by ρ. Then, $\psi_{x,\rho}$ is a homeomorphism.

A λ-*length* of an edge is the signed length between the horocycles around the endpoints of the edges. It is also known that the decorated Teichmüller space $\mathcal{T}^a(S)$ of S is parameterized by the λ-length coordinates [5] (Proposition 2.3.6):

$$\varphi_a : \mathcal{T}^a(S) \xrightarrow{\text{diffeo.}} \mathbb{R}^E.$$

Let

$$\psi_a : \mathcal{T}^{ax}(S) \to \mathbb{R}^E \times \mathbb{R}^{V^i}$$

be the map giving the λ-lengths and the sighed boundary lengths. In this paper, we calculate the map

$$\tau_\rho := \psi_a \circ \psi_{x,\rho}^{-1} : \mathbb{R}^{E^i} \times \mathbb{R}^V \to \mathbb{R}^E \times \mathbb{R}^{V^i}$$

from the shear parameters and the decoration parameters to the λ-lengths and the signed boundary lengths for researching the map ψ_a, and analyze the generalized λ-length coordinates of the decorated enhanced Teichmüller space. If S has a spike, there are triangulations whose puncture vertices are 1-valent. For such a triangulation, we give another parameterization of the decorated enhanced Teichmüller space of S:

Theorem B. (Theorem 3.5.1) The map

$$\psi_a : \mathcal{T}^{ax}(S) \to \mathbb{R}^E \times \mathbb{R}^{V^i}$$

giving the λ-lengths and the sighed boundary lengths is a homeomorphism if the puncture vertices of the triangulation are 1-valent.

There are some fomulae in this paper. We give new relations between the decoration parameters and other parameters (Proposition 3.4.3 and 3.5.5). We define the decoration parameters so that the derivatives of the λ-lengths with respect to

the decoration parameters are constants (Proposition 3.2.1). We generalize the relation between shear parameters and λ-length parameters (Proposition 3.5.4). These fomulae are useful when we prove that τ_ρ is homeomorphism concretely.

Next, we consider certain lamination spaces corresponding to the shear parameters and the λ-lengths. There are a homeomorphism between the space $T^x(S,\mathbb{R})$ of real $\mathcal{X}$-laminations and $\mathbb{R}^{E^i}$ [5] (Proposition 4.1.6)

$$\Phi_x : T^x(S,\mathbb{R}) \xrightarrow{\text{homeo.}} \mathbb{R}^{E^i},$$

and a homeomorphism between the space $T^a(S,\mathbb{R})$ of real $\mathcal{A}$-laminations and $\mathbb{R}^E$ [5] (Proposition 4.1.3)

$$\Phi_a : T^a(S,\mathbb{R}) \xrightarrow{\text{homeo.}} \mathbb{R}^E.$$

In this paper, we introduce the space $T^{ax}(S,\mathbb{R})$ of real $\mathcal{AX}$-laminations and give its canonical generalized parametrization

$$\Psi_x : T^{ax}(S,\mathbb{R}) \xrightarrow{\text{homeo.}} \mathbb{R}^{E^i} \times \mathbb{R}^V.$$

There are ingredients of real $\mathcal{AX}$-laminations that is corresponding to the enhancements (Proposition 4.2.4). Moreover, there is a normalizing function ρ_1 (in Example 3.3.2) of decoration parameters so that we can show the compatibility of Φ_x and Φ_a:

Theorem C. (Theorem 4.3.1) Let

$$\Psi_{a,\rho_1} : T^{ax}(S,\mathbb{R}) \to \mathbb{R}^E \times \mathbb{R}^{V^i}$$

be the composition of Ψ_x and τ_{ρ_1}. Then the equations

$$\Psi_x \circ I_x = \iota_x \circ \Phi_x \text{ and } \Psi_{a,\rho_1} \circ I_a = \iota_a \circ \Phi_a$$

hold, where ι_x is the inclusion map from $\mathbb{R}^{E^i}$ to $\mathbb{R}^{E^i} \times \mathbb{R}^V$, ι_a is the inclusion map from $\mathbb{R}^E$ to $\mathbb{R}^E \times \mathbb{R}^{V^i}$, I_x is the inclusion map from $T^x(S,\mathbb{R})$ to $T^{ax}(S,\mathbb{R})$, and I_a is the inclusion map from $T^a(S,\mathbb{R})$ to $T^{ax}(S,\mathbb{R})$ as in Figure 1.1.

Theorem A and Theorem B imply the upper side of the diagram in Figure 1.1, and Theorem C implies the lower side of the diagram in Figure 1.1. There is not obvious choice of the normalizing function of the decoration parameter. We carefully pick an normalizing function ρ_1 so that Theorem C holds. The relation between the maps Φ_x and Φ_a of the lamination spaces is similar to the relation between the shear and λ-length parametrization of the Teichmüller spaces, when the decoration parameters are normalized by ρ_1. With this choice of the normalizing function, the map τ_{ρ_1} is only a piecewise diffeomorphism, since ρ_1 is given by the minimum function that is used in tropical analogues. For another choice of the normalizing function, we can improve the conclusion of Theorem A and B. For example, there is a normalizing function ρ_2 (in Example 3.3.4) of decoration parameters so that the map ψ_{x,ρ_2} in Theorem A and the map ψ_a in Theorem B are diffeomorphisms.

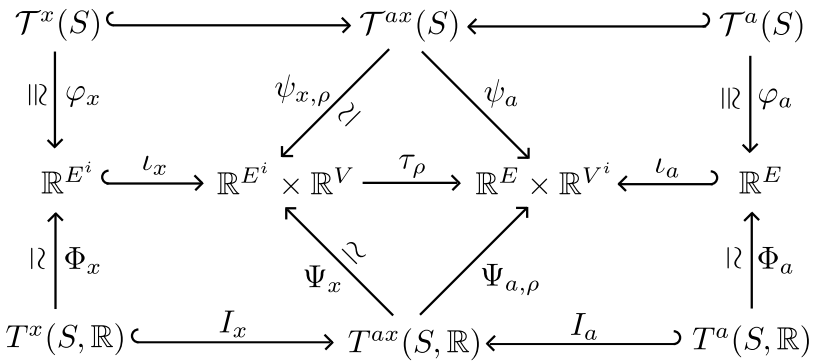

Figure 1.1: The commutative diagram describing that $\mathcal{T}^{ax}(S)$ is a common generalization of $\mathcal{T}^x(S)$ and $\mathcal{T}^a(S)$, and the compatibility of Φ_x and Φ_a. The symbol $\simeq$ means homeomorphisms and the symbol $\cong$ means diffeomorphisms.

1.2 Organization of this thesis

The organization of this paper is as follows. In §2.1, we we introduce the terminologies and the symbols used in this paper. In §2.1 and §2.2, we review the definitions of enhanced and decorated Teichmüller spaces, respectively. In §3.1, we introduce the decorated enhanced Teichmüller space, parametrize the decorations and prove Theorem A. In §3.2, we analyze the properties about derivatives of decoration parameters. In §3.3, we introduce certain normalizing functions of decoration parameters. In §3.4, we give some calculations associated with generalized shear coordinates. In §3.5, we prove Theorem B. In §4.1, we review the definitions of the spaces of laminations corresponding to generalized Teichmüller spaces. In §4.2, we introduce the space of laminations corresponding to the decorated enhanced Teichmüller space. In §4.3, we prove Theorem C. In §5.1, we note that the assumption in Theorem B is a sufficient condition.

1.3 Acknowledgements

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Chapter 2

Preliminaries

In this chapter, we introduce the Teichmüller space and its variations. These generalized Teichmüller spaces have the coordinate spaces, which will be generalized to the coordinate space of the decorated enhanced Teichmüller space.

2.1 Hyperbolic geometry

In this section, we introduce the terminologies and the symbols used in this paper. We will define the standard deformation space of the marked hyperbolic surfaces, which is called the *Teichmüller space*.

The space

$$\mathbb{H}^2 := \{x + iy \in \mathbb{C} \mid y > 0\}$$

with the metric

$$(ds_{\mathbb{H}^2})^2 := \frac{(dx)^2 + (dy)^2}{y^2}$$

is a Riemannian manifold of constant curvature -1. This space is called the *upper half plane*, and used for calculation in the hyperbolic geometry. Geodesics of $\mathbb{H}^2$ are half circles and lines orthogonal to the x-axis. Circles tangent to the x-axis and lines parallel to the x-axis are called *horocycles* of $\mathbb{H}^2$. Points of the x-axis and the point ∞ are called *ideal point* of $\mathbb{H}^2$, and triangles whose vertices are ideal are called *ideal triangles* (Figure 2.1).

The projective special linear group $\mathrm{PSL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H}^2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d} \quad \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R}), z \in \mathbb{H}^2 \right).$$

The projective special linear group is the group of the orientation preserving isometries of $\mathbb{H}^2$. For an isometry $A \in \mathrm{PSL}(2,\mathbb{R})$ of $\mathbb{H}^2$, the absolute value of the trace of a matrix representing A is defined. Then it is classified into three cases:

- If $|\mathrm{tr}(A)| < 2$, then A is called *elliptic* and fixes one point in $\mathbb{H}^2$.
- If $|\mathrm{tr}(A)| = 2$, then A is called *parabolic* and fixes one ideal point x of $\mathbb{H}^2$ and horocycles tangent at x.

- If $|\mathrm{tr}(A)| > 2$, A is called *hyperbolic* and fixes two ideal points x_1, x_2 of $\mathbb{H}^2$ and the geodesic which has x_1, x_2 as endpoints. This geodesic is called the *axis* of A.

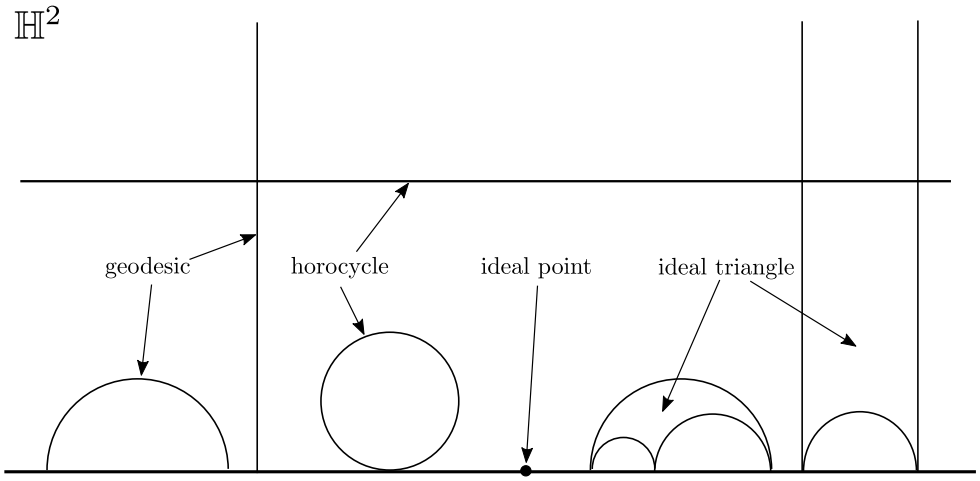

Figure 2.1: Geodesics, horocycles, ideal points and ideal triangles of $\mathbb{H}^2$.

Let G be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{H}^2$ freely. The quotient space $\mathbb{H}^2/G$ is a 2-dimensional hyperbolic Riemannian manifold. The surface $\mathbb{H}^2/G$ may have funnel-shaped subsurfaces and have an infinite area. Replace funnel-shaped subsurfaces with closed geodesic boundaries or crown-shaped subsurfaces, then the obtained surface is a complete surface with totally geodesic boundaries which has a finite area.

Let $\hat{S}$ be an oriented connected compact surface of genus $g \geq 0$ with $c \geq 0$ boundary components, and boundary components of $\hat{S}$ are called *crowns*. Take $p \geq 0$ marked points on the interior of $\hat{S}$ and call them *punctures*. Take $s \geq 0$ marked points on the boundary of $\hat{S}$ and call them *spikes*. We denote V^i the set of the punctures, V^b the set of the spikes, and V the union of V^i and V^b.

We consider hyperbolic surfaces homeomorphic to the topological surface $S = \hat{S} \setminus V$. The hyperbolicity of S is confirmed by the Euler characteristic. If $c = 0$, the Euler characteristic of S is

$$\chi(S) = 2 - 2g - p.$$

If $c \geq 1$, we consider the doubled surface S_D of S. The number of gluing 1-dimensional simplices equals s and the genus of S_D is $2g + c - 1$. Then, the Euler characteristic of S_D is

$$\begin{aligned}\chi(S_D) &= 2 - 2(2g + c - 1) - 2p - s \\ &= 4 - 4g - 2c - 2p - s.\end{aligned}$$

Throughout the paper, we assume that all crowns have at least one spike and that $4 - 4g - 2c - 2p - s < 0$. Then S admits a hyperbolic structure with crown-shaped boundaries.

Let X be a complete 2-dimensional hyperbolic Riemannian manifold with totally geodesic boundaries, and $\mathring{X}$ denote the subsurface of X obtained by removing all closed geodesic boundaries. Such a manifold X is called a *hyperbolic surface* of the reference surface S if X has a finite area and the subsurface $\mathring{X}$ is homeomorphic to S. A hyperbolic surface X may have cusps, closed geodesic boundaries and crown-shaped boundaries that are composed of asymptotic infinite geodesic boundaries, and X has no funnel-shaped subsurfaces (Figure 2.2).

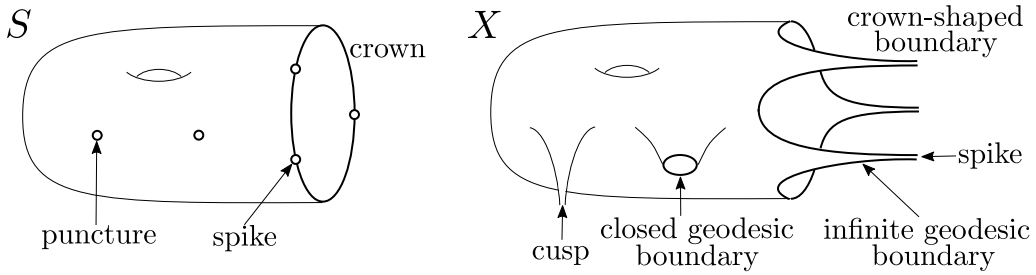

Figure 2.2: A reference surface S (left) and a hyperbolic surface X (right).

Definition 2.1.1. (Triangulations) A graph Γ on $\hat{S}$ is called a *triangulation* of S if the vertex set of Γ is V and each component of the complement of Γ is a triangle bounded by three edges and three vertices which are not necessarily distinct.

Edges included in the interior of $\hat{S}$ are called *interior edges* and edges included in the boundary of $\hat{S}$ are called *boundary edges*. We denote $E(\Gamma)$ the set of edges, $E^i(\Gamma)$ the set of interior edges and $E^b(\Gamma)$ the set of boundary edges of Γ.

A triangulation Γ of S makes an ideal triangulation of X. Map the edges of Γ by a homeomorphism from S to $\mathring{X}$, and deform them to be geodesics by an isotopic deformation, then an ideal triangulation is obtained. This ideal triangulation depends on the homeomorphisms that are the *markings* and the deformations that are the *enhancements* around the closed geodesic boundaries of X (Figure 2.3).

Definition 2.1.2. (Markings) Let X be a hyperbolic surface and $\mathring{X}$ denote the subsurface of X obtained by removing all closed geodesic boundaries. An orientation preserving homeomorphism $f : S \to \mathring{X}$ is called a *marking* of X and a pair (X, f) is called a *marked hyperbolic surface* of S.

Marked hyperbolic surfaces $(X_1, f_1), (X_2, f_2)$ are *Teichmüller equivalent* and denoted by $(X_1, f_1) \sim_T (X_2, f_2)$, if there is an orientation preserving isometry φ from X_1 to X_2 such that the maps $\varphi \circ f_1, f_2$ are homotopic. Let us define the deformation space of the marked hyperbolic structures on S.

Definition 2.1.3. (Teichmüller spaces) The set of Teichmüller equivalence classes of marked hyperbolic surfaces of S is called the *Teichmüller space* of S, which is denoted by

$$\mathcal{T}(S) := \{\text{Marked hyperbolic surfaces of } S\}/\sim_T .$$

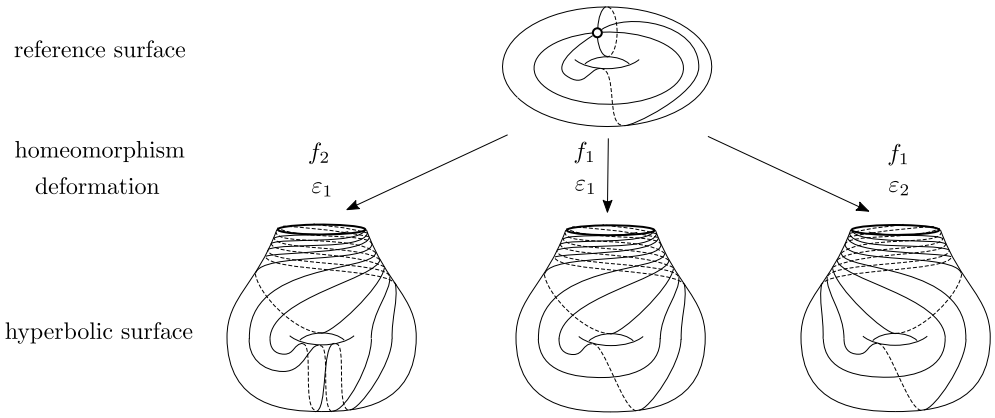

Figure 2.3: Ideal triangulations depend on the marking $f_i : S \to \mathring{X}$ and the enhancement ε_i.

Sometimes the Teichmüller space means the quotient space of the marked hyperbolic surfaces without closed geodesic boundaries. To avoid confusion, this is denoted by

$$\mathcal{T}_0(S) := \{\text{Marked hyperbolic surfaces of } S \text{ without closed geodesic boundaries}\}/\sim_T$$

in this paper.

2.2 Enhanced Teichmüller spaces and shear parameters

In this section, we consider the additional datum on the closed geodesic boundaries of the hyperbolic surfaces, which is called the *enhancement*. We will define the deformation space respecting the marking and the enhancement, and construct the *shear coordinates* of this space.

Definition 2.2.1. (Enhancements) Let X be a hyperbolic surface and f be a marking of X. A map $\varepsilon : V^i \to \{0, \pm 1\}$ is called an *enhancement* of (X, f), if it maps a vertex v to 0 if and only if the vertex v corresponds to a cusp of X. Then a triplet (X, f, ε) is called an *enhanced marked hyperbolic surface* of S.

When we consider a ideal triangulation of an enhanced marked hyperbolic surface, the enhancement of a puncture vertex v determines a direction of the spiral of the edges which have v as endpoints. The enhancement ε_i which maps the vertex to the sign $(-1)^{i-1}$ determines the direction as in Figure 2.3.

Enhanced marked hyperbolic surfaces $(X_1, f_1, \varepsilon_1), (X_2, f_2, \varepsilon_2)$ are *enhanced equivalent* and denoted by $(X_1, f_1, \varepsilon_1) \sim_E (X_2, f_2, \varepsilon_2)$, if $(X_1, f_1) \sim_T (X_2, f_2)$ and $\varepsilon_2 \circ \varphi_* = \varepsilon_1$ where the map φ_* is induced from the isometry φ of the Teichmüller equivalence. Since the enhancement is thought of as the signs of the lengths of the closed geodesic

boundaries and cusps are regarded as the closed geodesic boundaries whose length are 0, the enhancement at the cusps can be ignored. Let us define the deformation space of the marked hyperbolic structures with the enhancement on S.

Definition 2.2.2. (Enhanced Teichmüller spaces) The set of enhanced equivalence classes of enhanced marked hyperbolic surfaces of S is called the *enhanced Teichmüller space* of S, which is denoted by

$$\mathcal{T}^x(S) := \{\text{Enhanced marked hyperbolic surfaces of } S\}/\sim_E .$$

For a map $\varepsilon_0 : V^i \to \{0, \pm 1\}$, let

$$\mathcal{T}^x_{\varepsilon_0}(S) := \{[X, f, \varepsilon] \in \mathcal{T}^x(S) \mid \varepsilon = \varepsilon_0\}$$

be the subspace of $\mathcal{T}^x(S)$. The subspace $\mathcal{T}^x_{\varepsilon_0}(S)$ is identified with $\mathcal{T}_0(S)$ if $\varepsilon_0(V^i) = \{0\}$, and $\mathcal{T}^x_{\varepsilon_0}(S)$ is identified with $\mathcal{T}(S)$ if $\varepsilon_0(V^i) = \{\pm 1\}$ by forgetting the enhancement. For maps $\varepsilon_1, \varepsilon_2 : V^i \to \{\pm 1\}$, $\mathcal{T}^x_{\varepsilon_1}(S)$ and $\mathcal{T}^x_{\varepsilon_2}(S)$ are glued along the subspace $\mathcal{T}^x_{\frac{1}{2}(\varepsilon_1+\varepsilon_2)}(S)$ in $\mathcal{T}^x(S)$.

Since the marking and the enhancement determine the triangulation of the hyperbolic surface, the parameters about gluing triangles at interior edges are defined as follows.

Definition 2.2.3. (Shear parameters) Consider an interior edge e of a triangulation Γ. Let T, T' denote the triangles sharing e. Let v_1, v_2, v_3 denote the vertices of T, and v_3, v_4, v_1 denote the vertices of T'. Lift them to the upper half plane $\mathbb{H}^2$ with respect to a given marked hyperbolic structure. The lift of e is denoted by $\widetilde{e}$, and so on. The foot of the perpendicular from $\widetilde{v_2}$ to $\widetilde{e}$ is called the *base point* of $\widetilde{e}$ with respect to $\widetilde{T}$. The *shear parameter* of e is the signed hyperbolic length along $\widetilde{e}$ from the base point with respect to $\widetilde{T}$ to the base point with respect to $\widetilde{T'}$, which is measured along the orientation induced from $\widetilde{T}$ (Figure 2.4).

Sometimes we consider the shear parameters of the doubled surface S_D of S. The shear parameter of a boundary edge of S is defined as 0, and the shear parameter of the copy e' of an interior edge e of S is defined as the additive inverse of the shear parameter x_e of e, since the doubled triangulation is invariant by the involution.

The enhancement of v appears as the sign of the sum of the shear parameters of the edges which have v as endpoints (Proposition 3.4.2), where the shear parameters of the edges which have v as both of the endpoints are added two times.

It is known that measuring shear parameters is a global parametrization of $\mathcal{T}^x(S)$.

Proposition 2.2.4. (Shear coordinates [5, §4.1]) Let Γ be a triangulation of S. Then the map

$$\mathcal{T}^x(S) \to \mathbb{R}^{E^i(\Gamma)}$$

giving the shear parameters of the interior edges of Γ is a bijection.

These coordinates are called the *shear coordinates*, and we give $\mathcal{T}^x(S)$ the differential structure of the coordinate space $\mathbb{R}^{E^i(\Gamma)}$.

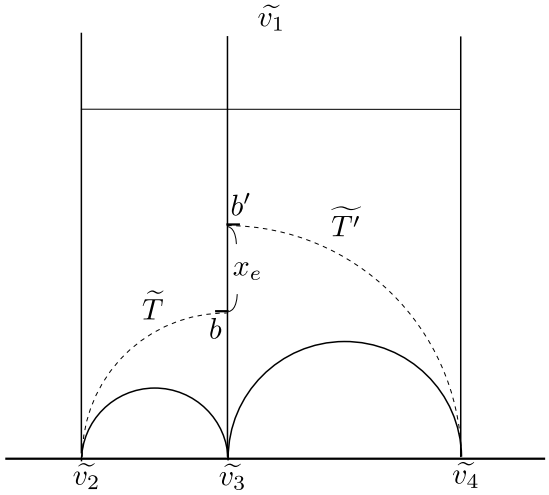

Figure 2.4: The shear parameter x_e of an edge e, and the base point b of $\widetilde{e}$ with respect to $\widetilde{T}$.

2.3 Decorated Teichmüller spaces and λ-lengths

In this section, we consider another datum on the ends of the hyperbolic surfaces, which is called the *decoration*. We will define the deformation space respecting the marking and the decoration, and construct the λ-*length coordinates* of this space.

We assume that hyperbolic surfaces in this section have no closed geodesic boundaries. Let X be a hyperbolic surface. For a puncture or a spike vertex v, lift the cusp or the spike C_v corresponding to v to the ideal point of the upper half plane $\mathbb{H}^2$, and take a horocycle tangent to the lift of C_v. The projection of this horocycle to X is called a *horocycle* or a *horocyclic arc* around v. A *decoration curve* around v is a horocycle around the cusp or a horocyclic arc around the spike corresponding to v. If a decoration curve around v is a horocyclic arc, its endpoints are in the infinite geodesic boundaries which are asymptotic to the spike corresponding to v.

Definition 2.3.1. (Decorations) Let X be a hyperbolic surface, f be a marking of X and $D = \bigcup_{v \in V} \gamma_v$ be the union of decoration curves γ_v around the cusps or the spikes corresponding to the vertices v. Then D is called a *decoration* of X and a triplet (X, f, D) is called a *decorated marked hyperbolic surface* of S.

Decorated marked hyperbolic surfaces $(X_1, f_1, D_1), (X_2, f_2, D_2)$ are *decorated equivalent* and denoted by $(X_1, f_1, D_1) \sim_D (X_2, f_2, D_2)$, if $(X_1, f_1) \sim_T (X_2, f_2)$ and $\varphi(D_1) = D_2$ where φ is the isometry of the Teichmüller equivalence. Let us define the deformation space of the marked hyperbolic structures with the decoration on S.

Definition 2.3.2. (Decorated Teichmüller spaces) The set of decorated equivalence classes of decorated marked hyperbolic surfaces of S is called the *decorated*

Teichmüller space of S, which is denoted by

$$\mathcal{T}^a(S) := \{\text{Decorated marked hyperbolic surfaces of } S \text{ without closed geodesic boundaries}\}/\sim_D .$$

The length of the horocycle around a cusp C increases when the horocycle goes away from C monotonously. Since the decoration is determined by the lengths of the decoration curves, there is an injection:

$$\mathcal{T}^a(S) \ni [X, f, D] \mapsto ([X, f], (r_v)_{v\in V}) \in \mathcal{T}_0(S) \times \mathbb{R}^V,$$

where r_v is the length of the decoration curve around the vertex v.

Remark 2.3.3. (Length of horocycles) Only a part of the horocycle may be contained in X. For example, on the once punctured monogon in the left of Figure 2.5, the decoration curve γ_1 around v_1 is a part of horocycle. We define the length of the decoration curve as the length r_1 of the reconstructed horocycle obtained as follows: glue the hyperbolic half planes along the infinite geodesic boundaries which intersects the decoration curve, and reconstruct the decoration curve by adding the horocyclic arcs on the glued half planes.

Remark 2.3.4. (Length of horocyclic arcs) Horocyclic arcs around spikes are regarded as the half of the horocycles on the doubled surface. For example, on the ideal triangle in the right of Figure 2.5, the decoration curve γ_1 around v_1 is a half of horocycle on the three punctured sphere. We define the length of the decoration curve as the length r_1 of the horocycle obtained by doubling γ_1 (Figure 2.5).

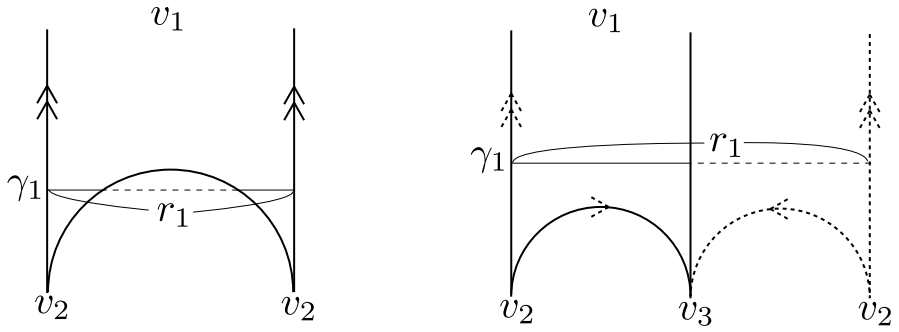

Figure 2.5: A part of horocycle (left), a horocyclic arc (right) and their reconstructing.

The marking determines the triangulation of the hyperbolic surface without closed geodesic boundaries, and the parameters about the distance between decoration curves along edges are defined as follows.

Definition 2.3.5. (λ-lengths) Consider an edge e of a triangulation Γ. Let T denote the triangle which has e as a boundary. Let v_1, v_2, v_3 denote the vertices of T, and γ_1, γ_3 denote the decoration curves around v_1, v_3, respectively. Lift them to the upper

half plane $\mathbb{H}^2$ with respect to a given marked hyperbolic structure. The lift of e is denoted by $\widetilde{e}$, and so on. Let h denote the half-edge of e incident to v_3. The λ-*length* of the half-edge h is the signed hyperbolic length along $\widetilde{e}$ from the intersection with $\widetilde{\gamma_3}$ to the base point of $\widetilde{e}$ with respect to $\widetilde{T}$, which is measured along the orientation induced from $\widetilde{T}$. The λ-*length* of the edge e is the signed hyperbolic lengths along $\widetilde{e}$ from the intersection with $\widetilde{\gamma_3}$ to the intersection with $\widetilde{\gamma_1}$, which is measured along the orientation induced from $\widetilde{T}$ (Figure 2.6).

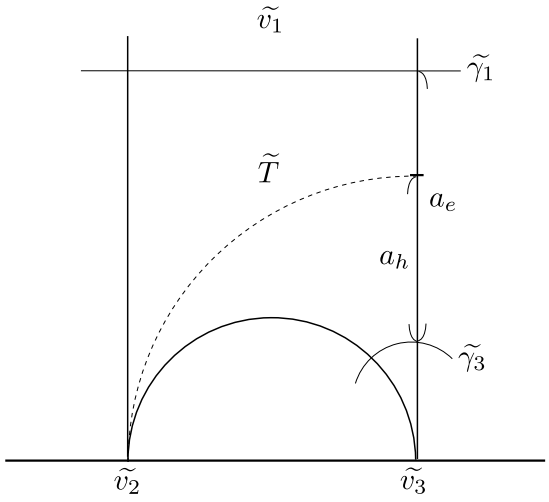

Figure 2.6: The λ-length a_e of an edge e, and the λ-length a_h of a half-edge h.

It is known that measuring λ-lengths is a global parametrization of $\mathcal{T}^a(S)$.

Proposition 2.3.6. (λ-length coordinates [5, §4.2]) Let Γ be a triangulation of S. Then the map

$$\mathcal{T}^a(S) \to \mathbb{R}^{E(\Gamma)}$$

giving the λ-lengths of the edges of Γ is a bijection.

These coordinates are called the λ-*length coordinates* and, we give $\mathcal{T}^a(S)$ the differential structure of the coordinate space $\mathbb{R}^{E(\Gamma)}$.

Chapter 3

Decorated enhanced Teichmüller spaces

In this chapter, we consider both of the additional data which are introduced before. We will define the deformation space respecting the marking, the enhancement and the decoration, and generalize the shear coordinates by adding the *decoration parameters*.

3.1 Shear-decoration coordinates

The *decorated enhanced Teichmüller space* is defined as the quotient space of the enhanced marked hyperbolic surfaces which have the decorations. We need to define the decoration curves around the closed geodesic boundaries.

Let X be a hyperbolic surface. For a puncture vertex v which corresponds to the closed geodesic boundary ∂_v of X, lift ∂_v to the upper half plane $\mathbb{H}^2$, and take a curve which is equidistant from a lift $\widetilde{\partial}_v$ of ∂_v. The projection of this curve to X is called an *equidistant curve* around ∂_v. We assume that equidistant curves in this paper have positive distance from the corresponding closed geodesic boundaries. Similarly to Remark 2.3.3, decoration curves may be parts of equidistant curves. The length of the part of equidistant curve is also defined as the length of the reconstructed curve.

Proposition 3.1.1. (The limit of the equidistant curves) Let v be a puncture vertex of S, l be a positive number and X_l be a hyperbolic surface on reference surface S whose closed geodesic boundary ∂_l corresponding to v has length l. Let r be a positive number and γ_l denote the length r equidistant curve on X_l around ∂_l. Then γ_l converges to the horocycle of length r around the cusp C corresponding to v when l tends to 0.

Proof. Lift ∂_l and γ_l to the upper half plane $\mathbb{H}^2$, and we can suppose that a lift $\widetilde{\partial}_l$ of ∂_l and a lift $\widetilde{\gamma}_l$ of γ_l have $-\frac{1}{l}$ and ∞ as the common endpoints (Figure 3.1). Let h_l denote the hyperbolic isometry of $\mathbb{H}^2$ corresponding to ∂_l. Then,

$$h_l = \begin{bmatrix} e^{\frac{l}{2}} & \frac{1}{l}(e^{\frac{l}{2}} - e^{-\frac{l}{2}}) \\ 0 & e^{-\frac{l}{2}} \end{bmatrix} \quad \xrightarrow{l \to 0} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =: h_0.$$

The limit h_0 is the parabolic isometry of $\mathbb{H}^2$ which fixes ∞ and the limit of ∂_l is the cusp corresponding to v.

Let θ denote the angle of $\widetilde{\partial}_l$ and $\widetilde{\gamma}_l$. Take a parametrized curve

$$z(t) = (e^l - 1)e^{i(\frac{\pi}{2}-\theta)}t + e^{i(\frac{\pi}{2}-\theta)} - \frac{1}{l} \quad (0 \leq t \leq 1)$$

and calculate r by the line integral

$$r = \int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \frac{l}{\cos\theta}.$$

Then, the curve $\widetilde{\gamma}_l$ is expressed as follows:

$$\widetilde{\gamma}_l : y = \frac{l}{\sqrt{r^2 - l^2}}x + \frac{1}{\sqrt{r^2 - l^2}} \quad \xrightarrow{l\to 0} \quad \widetilde{\gamma}_0 : y = \frac{1}{r}.$$

Therefore, the limit curve of γ_l is the projection γ_0 of $\widetilde{\gamma}_0$. Take a parametrized curve

$$z(t) = t + \frac{i}{r} \quad (0 \leq t \leq 1)$$

and the length of γ_0 is

$$\int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = r.$$

□

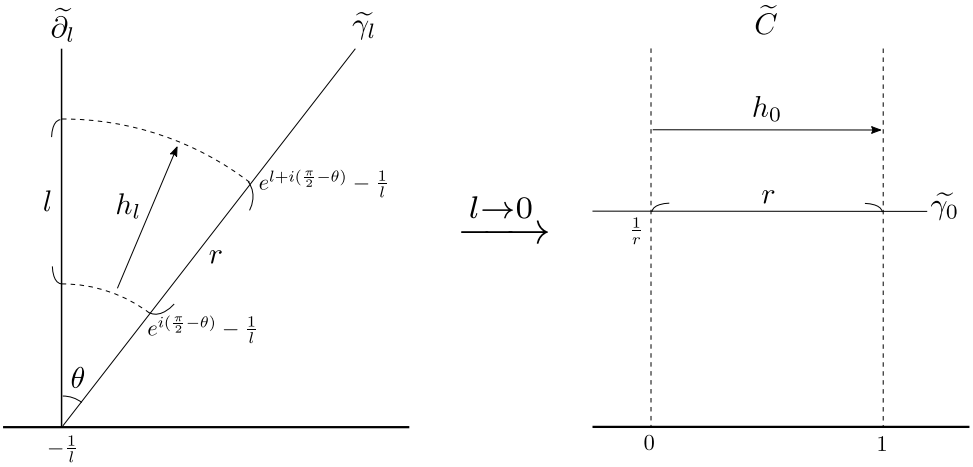

Figure 3.1: The lift of an equidistant curve (left) and its limit curve (right).

As seen from this proposition, equidistant curves play the role of the horocycles for the closed geodesic boundaries. For a vertex v, a *decoration curve* around v is a horocycle around the cusp, a horocyclic arc around the spike or an equidistant curve around the closed geodesic boundary corresponding to v.

Definition 3.1.2. (Generalized decorations) Let X be a hyperbolic surface, f be a marking of X, ε be an enhancement of X, and $D = \bigcup_{v \in V} \gamma_v$ be the union of the decoration curves γ_v around the cusps, the spikes or the closed geodesic boundaries corresponding to the vertices v. Then D is called a *decoration* of X and a quadruple (X, f, ε, D) is called a *decorated enhanced marked hyperbolic surface*.

Decorated enhanced marked hyperbolic surfaces $(X_1, f_1, \varepsilon_1, D_1), (X_2, f_2, \varepsilon_2, D_2)$ are *decorated enhanced equivalent* and denoted by $(X_1, f_1, \varepsilon_1, D_1) \sim_{DE} (X_2, f_2, \varepsilon_2, D_2)$, if $(X_1, f_1, \varepsilon_1) \sim_E (X_2, f_2, \varepsilon_2)$ and $(X_1, f_1, D_1) \sim_D (X_2, f_2, D_2)$. Let us define the deformation space of the marked hyperbolic structures with the enhancement and the decoration on S.

Definition 3.1.3. (Decorated enhanced Teichmüller spaces) The set of decorated enhanced equivalence classes of decorated enhanced marked hyperbolic surfaces of S is called the *decorated enhanced Teichmüller space* of S, which is denoted by

$$\mathcal{T}^{ax}(S) = \{\text{Decorated enhanced marked hyperbolic surfaces of } S\}/\sim_{DE} .$$

The length of the equidistant curve around a closed geodesic boundary ∂ increases when the equidistant curve goes away from ∂ monotonously. Since the decoration is determined by the lengths of the decoration curves, there is an injection:

$$\mathcal{T}^{ax}(S) \ni [X, f, \varepsilon, D] \mapsto ([X, f, \varepsilon], (r_v)_{v \in V}) \in \mathcal{T}^{x}(S) \times \mathbb{R}^V,$$

where r_v is the length of the decoration curve around the vertex v. We give $\mathcal{T}^{ax}(S)$ the differential structure of the product structure of $\mathcal{T}^{x}(S) \times \mathbb{R}^V$.

The set of the decoration curves around a vertex v is parametrized by their length, but this parameter has a lower bound which depends on the length of the boundary component corresponding to v. We rescale this parameters of the decoration so that its range becomes $\mathbb{R}$, and define the *decoration parameters* in generalized setting.

Let v be a vertex and r_v denote the length of the decoration curve around v. If v is a spike vertex, consider the shear parameters of edges of the doubled triangulation Γ_D of Γ. Let n denote the number of edges which have v as an endpoint. Choose an edge e_1 among them, and label the edges incident to v as $e_1, \cdots, e_n, \cdots$ in the counter-clockwise order. Let x_k denote the shear parameter of e_k.

Definition 3.1.4. (Decoration parameters) Let

$$\rho : \mathbb{R}^{E^i} \ni (x_e)_{e \in E^i} \mapsto (\rho_v((x_e)_{e \in E^i}))_{v \in V} \in \mathbb{R}^V$$

be a piecewise-smooth function. Then the *decoration parameter* d_v of v is defined as follows.

- If v corresponds to a closed geodesic boundary,

$$d_v = \frac{1}{2} \log \frac{\rho_v((x_e)_{e \in E^i})^2 - l_v^2}{r_v^2 - l_v^2},$$

- If v corresponds to a cusp or a spike,

$$d_v = \log \frac{\rho_v((x_e)_{e \in E^i})}{r_v},$$

where l_v is the length of the closed geodesic boundary corresponding to v and $(x_e)_{e\in E^i}$ is the shear parameter of the enhanced marked hyperbolic surface.

The cusp is thought of as the closed geodesic boundary whose length is 0. We observe that

$$\frac{1}{2}\log\frac{\rho_v((x_e)_{e\in E^i})^2 - l_v^2}{r_v^2 - l_v^2} \xrightarrow{l_v\to 0} \log\frac{\rho_v((x_e)_{e\in E^i})}{r_v}.$$

For the enhanced marked hyperbolic surface whose shear coordinates are $(x_e)_{e\in E^i}$, the function ρ_v in Definition 3.1.4 designates the decoration curve around v whose length is $\rho_v((x_e)_{e\in E^i})$ as the decoration curve around v whose decoration parameter is 0. The function ρ is called the *normalizing function* of the decoration parameters.

By the decoration parameters, we obtain the generalized shear coordinates of the decorated enhanced Teichmüller space.

Theorem 3.1.5. (Shear-decoration coordinates) Let Γ be a triangulation of S and ρ be a normalizing function. Then the map

$$\psi_{x,\rho} : \mathcal{T}^{ax}(S) \to \mathbb{R}^{E^i(\Gamma)} \times \mathbb{R}^V.$$

giving the shear parameters of the interior edges of Γ and the decoration parameters around the vertices is a homeomorphism.

Proof. Let us regard $\mathcal{T}^{ax}(S)$ as the subset of $\mathbb{R}^{E^i} \times \mathbb{R}^V$ by the corresponding to the shear parameters of the interior edges of Γ and the lengths of the decoration curves around the vertices. We show that the map

$$\psi_{x,\rho} : \mathcal{T}^{ax}(S) \ni ((x_e)_{e\in E^i}, (r_v)_{v\in V}) \mapsto ((x_e)_{e\in E^i}, (d_v)_{v\in V}) \in \mathbb{R}^{E^i} \times \mathbb{R}^V$$

is a piecewise-diffeomorphism.

Let v be a vertex. If v is corresponding to a closed geodesic boundary, the range of the parameter r_v is the interval (l_v, ∞) and d_v is the diffeomorphism from (l_v, ∞) to $\mathbb{R}$, where l_v is the length of the closed geodesic boundary corresponding to v. If v is corresponding to a cusp or a spike, the range of the parameter r_v is the interval $(0, \infty)$ and d_v is the diffeomorphism from $(0, \infty)$ to $\mathbb{R}$. Therefore, $\psi_{x,\rho}$ is a bijection.

Consider the Jacobian matrix $D\psi_{x,\rho}$ of $\psi_{x,\rho}$. We can calculate the entries:

$$\frac{\partial x_e}{\partial x_{e'}} = \begin{cases} 1 & (e = e') \\ 0 & (e \neq e'), \end{cases}$$

$$\frac{\partial x_e}{\partial r_v} = 0,$$

$$\frac{\partial d_v}{\partial r_{v'}} = \begin{cases} -\frac{r_v}{r_v^2 - l_v^2} & (v = v' \text{ corresponds to a closed geodesic boundary}) \\ -\frac{1}{r_v} & (v = v' \text{ corresponds to a cusp or a spike}) \\ 0 & (v \neq v'), \end{cases} \tag{3.1}$$

and $D\psi_{x,\rho}$ is a lower triangular matrix. Since $\psi_{x,\rho}$ is bijective, it is a piecewise-diffeomorphism. □

These coordinates of $\mathcal{T}^{ax}(S)$ are called the *shear-decoration coordinates*.

Remark 3.1.6. (Differentiability of $\psi_{x,\rho}$) The decoration parameter introduced in Definition 3.1.4 is in generalized setting for the theorem in §4.3. If ρ is a diffeomorphism, the map $\psi_{x,\rho}$ in Theorem 3.1.5 is also a diffeomorphism.

3.2 Derivatives of decoration parameters

The definition of the decoration parameters in this paper is one of the rescaling the parameters of the decoration. In this section, we research the derivatives of the decoration parameters and confirm its utilities.

Proposition 3.2.1. (The derivative of λ-lengths by decoration parameters) Let v be a vertex and h be a half-edge. Then,

$$\frac{da_h}{dd_v} = \begin{cases} 1 & (h \text{ is incident to } v) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. First, suppose that v is corresponding to a closed geodesic boundary ∂_v. Let l_v denote the length of ∂_v and γ_v be an equidistant curve around ∂_v. Lift them to the upper half plane $\mathbb{H}^2$ such that a lift $\widetilde{\partial_v}$ of ∂_v and a lift $\widetilde{\gamma_v}$ of γ_v have 0 and ∞ as their common endpoints (Figure 3.2). Let w denote the hyperbolic distance from ∂_v to γ_v, r_v denote the length of γ_v, and θ denote the angle of $\widetilde{\partial_v}$ and $\widetilde{\gamma_v}$ at 0. Take a parametrized curve

$$z(t) = e^{it} \quad \left(\frac{\pi}{2} - \theta \leq t \leq \frac{\pi}{2}\right)$$

and calculate w by the line integral

$$w = \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \log \frac{1+\sin\theta}{\cos\theta}. \tag{3.2}$$

Take a parametrized curve

$$z(t) = (e^{l_v} - 1)e^{i(\frac{\pi}{2}-\theta)}t + e^{i(\frac{\pi}{2}-\theta)} \quad (0 \leq t \leq 1)$$

and calculate r_v by the line integral

$$r_v = \int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \frac{l_v}{\cos\theta}. \tag{3.3}$$

By Equations (3.2) and (3.3), it follows that

$$r_v = l_v \cosh w. \tag{3.4}$$

Let g_e be the geodesic corresponding to the edge e which has h as a half-edge. There is a lift $\widetilde{g_e}$ of g_e which has ∞ as an endpoint (Figure 3.2). Similarly, the other endpoint $x \in \mathbb{R}$ of $\widetilde{g_e}$ is corresponding to the endpoint $v' \in V$ of e. The lift $\widetilde{g_e}$ intersects $\widetilde{\gamma_v}$ at $x + \frac{x}{\tan\theta}i$. The λ-length a_h of the half-edge h is the length between the intersection and the base point. We consider the infinitesimal deformation of r_v. If r_v increases, θ increases, the imaginary part of the intersection of $\widetilde{g_e}$ and $\widetilde{\gamma_v}$ decreases, and a_h decreases. The change of the amount of a_h is

$$\triangle a_h = \log \frac{\tan\theta}{\tan(\theta + \triangle\theta)} = \frac{1}{2} \log \frac{r_v^2 - l_v^2}{(r_v + \triangle r_v)^2 - l_v^2}.$$

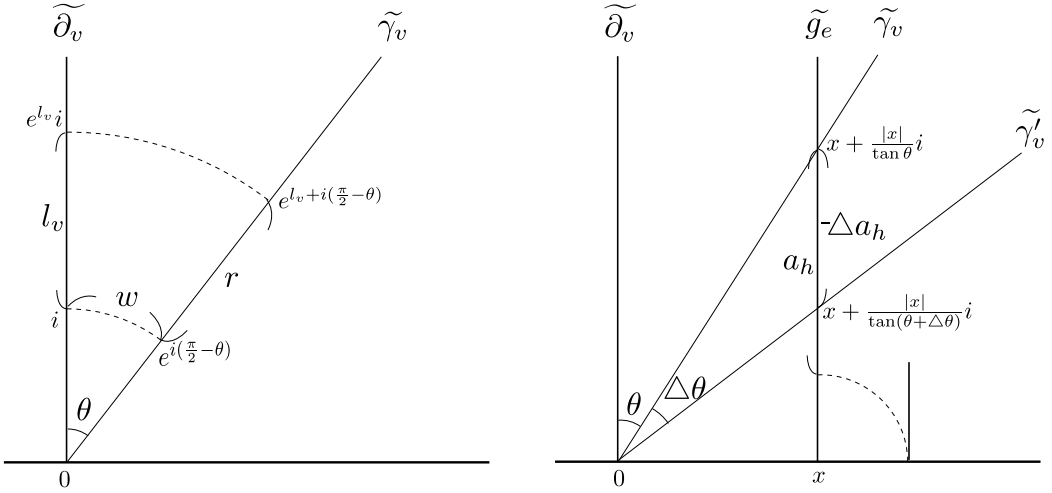

Figure 3.2: The relation between $\triangle\theta$ and $\triangle a_h$ for the case of closed geodesic boundaries.

Then the derivative of a_h by r is

$$\frac{da_h}{dr_v} = -\frac{r_v}{r_v^2 - l_v^2}.$$

In comparison with Equation (3.1), the asserted equation follows.

Second, suppose that v is corresponding to a cusp. Let C_v denote the cusp corresponding to v and γ_v be a horocycle. Lift them to the upper half plane $\mathbb{H}^2$ such that a lift $\widetilde{C_v}$ of C_v is at ∞ (Figure 3.3). Then there is a lift $\widetilde{\gamma_v}$ of γ_v which is parallel to the real axis. Let g_e be the geodesic corresponding to the edge e which has h as a half-edge. There is a lift $\widetilde{g_e}$ of g_e which has ∞ as an endpoint and we can suppose that $\widetilde{g_e}$ and $f_v(\widetilde{g_e})$ have 0 and 1 as the other endpoint, respectively, where f_v is the parabolic isometry of $\mathbb{H}^2$ corresponding to C_v. Let y denote the imaginary part of $\widetilde{\gamma_v}$. Take a parameterized curve

$$z(t) = t + iy \quad (0 \leq t \leq 1)$$

and calculate the length r_v of γ_v by the line integral

$$r_v = \int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \frac{1}{y}. \tag{3.5}$$

We consider the λ-length a_h of the half-edge h for the infinitesimal deformation of r_v. If r_v increases, the imaginary part y of the intersection of $\widetilde{g_e}$ and $\widetilde{\gamma_v}$ decreases, and a_h decreases. The change of the amount of a_h is

$$\triangle a_h = \log \frac{y + \triangle y}{y} = \log \frac{r_v}{r_v + \triangle r_v}.$$

Then the derivative of a_h by r_v is

$$\frac{da_h}{dr_v} = -\frac{1}{r_v}.$$

Finally, suppose that v is corresponding to a spike. This case is similar to the case of the cusp. Let C_v denote the spike corresponding to v and γ_v be a horocyclic arc. Lift them to the upper half plane $\mathbb{H}^2$ such that a lift $\widetilde{C_v}$ of C_v is at ∞ (Figure 3.3). Then there is a lift $\widetilde{\gamma_v}$ of γ_v which is parallel to the real axis. Let g_0 and g_1 denote the infinite geodesics of X ending at C_v, there are lifts $\widetilde{g_0}$ of g_0 and $\widetilde{g_1}$ of g_1 which have ∞ as an endpoint, and we can suppose that $\widetilde{g_0}$ and $\widetilde{g_1}$ have 0 and 1 as the other endpoint, respectively. Let g_e be the geodesic corresponding to the edge e which has h as a half-edge, then there is a lift $\widetilde{g_e}$ of g_e which has ∞ as an endpoint. Let y be the imaginary part of $\widetilde{\gamma_v}$. Calculate the length r_v of γ_v by the line integral

$$r_v = 2\int_0^1 \frac{1}{y(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \frac{2}{y}.$$

By the same calculation as the case of the cusp, the derivative of the λ-length a_h of the half-edge h by r_v is

$$\frac{da_h}{dr_v} = -\frac{1}{r_v}.$$

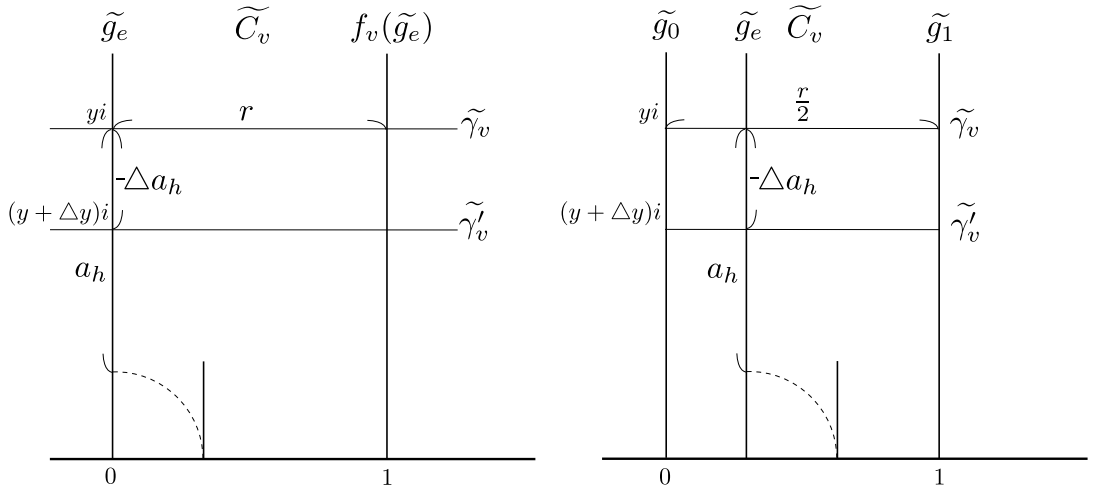

Figure 3.3: Relation between $\triangle y$ and $\triangle a_h$ for the case of the cusps (left) and the spikes (right).

In comparison with Equation (3.1), the asserted equation follows.

The λ-length a_h of a half-edge h which does not have a vertex v as the endpoint is independent of the decoration parameter d_v of v. Then, we have that

$$\frac{da_h}{dd_v} = 0$$

for such a half-edge h. □

3.3 Normalizations of decoration parameters

In this section, we introduce two normalizing functions. The former gives the decoration parameters which satisfy the compatibility of parametrizations in §4.3. The latter gives the decoration parameters whose origin corresponds to the origin of λ-length parameters.

Let v be a vertex and r_v denote the length of the decoration curve around v. If v is a spike vertex, consider the shear parameters of edges of the doubled triangulation Γ_D of Γ. Let n denote the number of edges which have v as an endpoint. Choose an edge e_1 among them, and label the edges incident to v as $e_1, \cdots, e_n, \cdots$ in the counter-clockwise order. Let x_k denote the shear parameter of e_k and g_k denote the geodesic corresponding to e_k. Lift these geodesics to the upper half plane $\mathbb{H}^2$ such that a lift $\widetilde{g_k}$ of g_k has ∞ as an endpoint.

Let $R_{v,k}$ denote the value defined as follows:

$$R_{v,k} = \sum_{i=1}^{n} \exp\left(\sum_{j=1}^{i-1} x_{k+j}\right).$$

The value $R_{v,k}$ equals the length of a certain decoration curve.

Lemma 3.3.1. (Lengths represented by $R_{v,k}$) Let u_k denote the other endpoint of $\widetilde{g_k}$. Suppose that $u_k = 0$ and $u_{k+1} = 1$. Then,

$$u_{n+k} = R_{v,k}.$$

Proof. By definition, x_j is the signed length between the base point of $\widetilde{g_j}$ with respect to the triangle which has u_{j-1} as a vertex and the base point of $\widetilde{g_j}$ with respect to the triangle which has u_{j+1} as a vertex. Then, for an integer j,

$$x_j = \log \frac{u_{j+1} - u_j}{u_j - u_{j-1}}. \tag{3.6}$$

This is the recurrence formula of $\{u_j\}_{j\in\mathbb{Z}}$, and

$$\begin{aligned} u_{n+k} &= u_k + \sum_{i=1}^{n} (u_{k+i} - u_{k+i-1}) \\ &= u_k + \sum_{i=1}^{n} (u_{k+1} - u_k) \exp\left(\sum_{j=1}^{i-1} x_{k+j}\right) \\ &= R_{v,k}. \end{aligned}$$

□

Example 3.3.2. (Normalizing function ρ_1) The function ρ_1 defined as follows is a normalizing function of decoration parameters.

$$\rho_{1,v}((x_e)_{e\in E^i}) = \min_{1\le k\le n}\{r_{v,k}\} \quad (v \in V),$$

where the value $r_{v,k}$ is as follows.

- If v corresponds to a closed geodesic boundary,

$$r_{v,k} = \frac{l_v}{e^{l_v} - 1}\sqrt{R_{v,k}^2 + (e^{l_v} - 1)^2},$$

  where l_v is the length of the closed geodesic boundary corresponding to v.

- If v corresponds to a cusp or a spike,

$$r_{v,k} = R_{v,k}.$$

The cusp is thought of as the closed geodesic boundary whose length is 0. We observe that

$$\frac{l_v}{e^{l_v} - 1}\sqrt{R_{v,k}^2 + (e^{l_v} - 1)^2} \xrightarrow{l_v \to 0} R_{v,k}.$$

Decoration curves are the boundaries of the equidistant neighborhoods around closed geodesic boundaries, and the boundaries of the cusp neighborhoods around cusps and spikes. The value $r_{v,k}$ in Example 3.3.2 is the length of the boundary curve of the equidistant or cusp neighborhoods whose boundary curve passes through the base point of e_k with respect to the triangle which has e_k and e_{k+1} as boundaries. By Definition 3.1.4, the decoration parameter of v is 0 if and only if the length of the curve r_v equals the shortest boundary curve of such neighborhoods. Consequently, following proposition follows.

Proposition 3.3.3. (Decoration parameters normalized by ρ_1) Let v be a vertex. For the decoration parameter d_v normalized by ρ_1, the following conditions are equivalent.

- $d_v(r_v) = 0$.
- The decoration curve around v is the boundary curve of the equidistant or cusp neighborhood which is intersection of the equidistant or cusp neighborhoods whose boundary curve passes through the base point of the edge e_k which has v as a vertex with respect to the triangle which has e_k and e_{k+1} as boundaries.

Proof. We prove that $r_{v,k}$ is the length of the decoration curve passing through the base point of e_k with respect to the triangle which has e_k and e_{k+1} as boundaries.

First, suppose that v is corresponding to a closed geodesic boundary. By Equation (3.3), the length r of the decoration curve passing through the base point is

$$r = \frac{l_v}{\cos\theta},$$

where θ is the angle of the lift of the boundary curve and the lift of the decoration curve. By Lemma 3.3.1,

$$\cos\theta = \frac{e^{l_v} - 1}{\sqrt{R_{v,k}^2 + (e^{l_v} - 1)^2}}.$$

Then,

$$r = \frac{l_v}{e^{l_v} - 1}\sqrt{R_{v,k}^2 + (e^{l_v} - 1)^2} = r_{v,k}.$$

Second, suppose that v is corresponding to a cusp. By Lemma 3.3.1, the length r of the decoration curve passing through the base point is the line integral along the parameterized curve

$$z(t) = t + i \quad (0 \leq t \leq R_{v,k}).$$

Then,

$$r = \int_0^{R_{v,k}} \frac{1}{y(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = R_{v,k} = r_{v,k}.$$

□

Example 3.3.4. (Normalizing function ρ_2) The function ρ_2 defined as follows is a normalizing function of decoration parameters.

- If v is a vertex which corresponds to a closed geodesic boundary,

$$\rho_{2,v}((x_e)_{e\in E^i}) = \frac{l_v \cosh\frac{l_v}{2}}{\sinh\frac{l_v}{2}},$$

  where l_v is the length of the closed geodesic boundary corresponding to v.

- If v is a vertex which corresponds to a cusp or a spike,

$$\rho_{2,v}((x_e)_{e\in E^i}) = 2.$$

The cusp is thought of as the closed geodesic boundary whose length is 0. We observe that

$$\frac{l_v \cosh\frac{l_v}{2}}{\sinh\frac{l_v}{2}} \xrightarrow{l_v \to 0} 2.$$

Decoration curves are the boundaries of the equidistant neighborhoods around closed geodesic boundaries, and the boundaries of the cusp neighborhoods around cusps and spikes. It is known that there exist special neighborhoods of closed geodesic boundaries.

Lemma 3.3.5. (Collar lemma [3, Theorem 4.1.1]) Let γ be a simple closed geodesic on the hyperbolic surface X. Then, there exists an annular neighborhood of γ which is called the *collar neighborhood* of γ, whose width is w_0 given as follows:

$$w_0 = \operatorname{arcsinh}\frac{1}{\sinh\frac{l_\gamma}{2}},$$

where l_γ is the length of γ.

By Lemma 3.3.5 and Equation (3.4), the following corollary follows.

Corollary 3.3.6. (Lengths of the collar boundary) The length r_0 of the collar boundary of ∂_v is as follows:

$$r_0 = \frac{l_v \cosh \frac{l_v}{2}}{\sinh \frac{l_v}{2}}.$$

The following corollary is thought of as the limit case of Corollary 3.3.6 when l_v tends to 0.

Corollary 3.3.7. (Cusp neighborhoods) Let C be a cusp on the hyperbolic surface X. Then, there exists a cusp neighborhood of C whose boundary length r_0 is 2. For a spike C', there exists a half of the cusp neighborhood of C' whose boundary length r_0 is 2 as the length of the decoration curve.

The lengths of these neighborhoods are sharp. Namely, there is a critical case as follows. In the left of Figure 3.4, the boundaries γ_1, γ_2 of the neighborhood of γ are tangent. See the center of Figure 3.4, and regard this critical case as the case that the boundaries γ_1, γ_2 of the neighborhoods of two boundary curves ∂_1, ∂_2 are tangent.

Suppose that there exists an edge e corresponding to a geodesic g_e such that g_e is homotopic to a curve which connects ∂_1 to ∂_2 in the connecting neighborhoods of ∂_1 and ∂_2, and the enhancements $\varepsilon(v_i)$ of the vertices v_i corresponding to ∂_i are same. Lift them to the upper half plane $\mathbb{H}^2$. We can suppose that a lift $\widetilde{\partial_1}$ of ∂_1 and a lift $\widetilde{\gamma_1}$ of γ_1 have 0 and ∞ as their common endpoints, a lift $\widetilde{\partial_2}$ of ∂_2 and a lift $\widetilde{\gamma_2}$ of γ_2 have $x_0 - c$ and $x_0 + c$ as their common endpoints, and a lift $\widetilde{g_e}$ of g_e has ∞ and $x_0 - c$ as the endpoints (Figure 3.4). The hyperbolic geodesic $\widetilde{\gamma_2}$ is the Euclidian circle whose center is $x_0 + y_0 i$ expressed as follows:

$$\widetilde{\gamma_2} : (x - x_0)^2 + (y - y_0)^2 = y_0^2 + c^2.$$

Let θ denote the angle of $\widetilde{\partial_2}$ and $\widetilde{\gamma_2}$. Calculate the slope of the tangential line of $\widetilde{\gamma_2}$ at $x_0 + c$, then the following equation is obtained:

$$\tan\theta = \frac{y_0}{c}.$$

By the assumption that the size of neighborhoods of ∂_1 and ∂_2 are same, θ equals the angle of $\widetilde{\partial_1}$ and $\widetilde{\gamma_1}$, and $\widetilde{\gamma_1}$ is expressed as follows:

$$\widetilde{\gamma_1} : y = \frac{c}{y_0} x.$$

When $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ are tangent, the real part of their tangent point is $x_0 - c$. Consequently, $\widetilde{g_e}$ passes the tangent point and the λ-length of e is 0. Similarly, in the critical case that two cusp neighborhoods are tangent, the λ-length of such an edge is 0.

In the critical case, the boundaries of these neighborhoods give the origin of the λ-length of such an edge. The following proposition says that ρ_2 is the normalizing function of the decoration parameters such that the origins correspond to the boundary lengths r_0 of these neighborhoods.

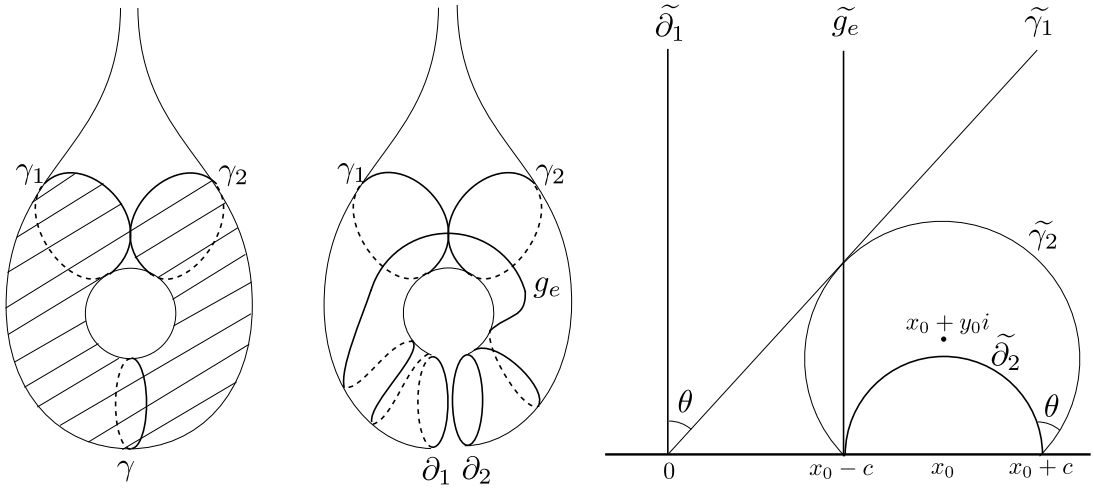

Figure 3.4: The case that collar neighborhoods are tangent (left) and the corresponding case that decoration curves are tangent (center and right).

Proposition 3.3.8. (Decoration parameters normalized by ρ_2) Let v be a vertex. For the decoration parameter d_v normalized by ρ_2, the following conditions are equivalent.

- $d_v(r_v) = 0$.
- The decoration curve around v is the boundary curve of the neighborhood in Corollaries 3.3.6 and 3.3.7.

Proof. By Example 3.3.4 and Corollaries 3.3.6 and 3.3.7, the assertion is immediately follows. □

3.4 λ-lengths and boundary lengths

For a decorated enhanced marked hyperbolic surface, the λ-length is defined as in the decorated Teichmüller space and the signed boundary length is defined as in the enhanced Teichmüller space. In this section, we study the relation between these parameters and the shear-decoration coordinates, and prepare to construct the generalized λ-length coordinates of the decorated enhanced Teichmüller space.

We calculate the λ-lengths and the signed boundary lengths from the shear parameters and the decoration parameters. These calculations make the map from $\mathbb{R}^{E^i(\Gamma)} \times \mathbb{R}^V$ to $\mathbb{R}^{E(\Gamma)} \times \mathbb{R}^{V^i}$.

Let v be a vertex and r_v denote the length of the decoration curve around v. If v is a spike vertex, consider the shear parameters of edges of the doubled triangulation Γ_D of Γ. Let n denote the number of edges which have v as an endpoint. Choose an edge e_1 among them, and label the edges incident to v as $e_1, \cdots, e_n, \cdots$ in the counter-clockwise order. Let x_k denote the shear parameter of e_k and g_k denote the

geodesic corresponding to e_k. Lift these geodesics to the upper half plane $\mathbb{H}^2$ such that a lift $\widetilde{g_k}$ of g_k has ∞ as an endpoint.

Lemma 3.4.1. (Real parts of endpoints) For $1 \leq k \leq n+1$, the other endpoint u_k of $\widetilde{g_k}$ satisfies

$$u_k = u_1 + (u_{n+1} - u_1) R_{v,1}^{-1} \sum_{i=1}^{k-1} \exp\left(\sum_{j=1}^{i-1} x_{j+1}\right).$$

Proof. By the recurrence formula (3.6),

$$\begin{aligned}
u_{n+1} - u_1 &= \sum_{i=1}^{n} (u_{i+1} - u_i) \\
&= \sum_{i=1}^{n} (u_2 - u_1) \exp\left(\sum_{j=1}^{i-1} x_{j+1}\right) \\
&= (u_2 - u_1) R_{v,1}, \\
u_k &= u_1 + \sum_{i=1}^{k-1} (u_{i+1} - u_i) \\
&= u_1 + \sum_{i=1}^{k-1} (u_2 - u_1) \exp\left(\sum_{j=1}^{i-1} x_{j+1}\right) \\
&= u_1 + (u_{n+1} - u_1) \frac{\sum_{i=1}^{k-1} \exp\left(\sum_{j=1}^{i-1} x_{j+1}\right)}{\sum_{i=1}^{n} \exp\left(\sum_{j=1}^{i-1} x_{j+1}\right)} \\
&= u_1 + (u_{n+1} - u_1) R_{v,1}^{-1} \sum_{i=1}^{k-1} \exp\left(\sum_{j=1}^{i-1} x_{j+1}\right).
\end{aligned}$$

□

To claim the relation between the shear parameters and the signed boundary lengths, suppose that v is corresponding to a closed geodesic boundary and n-valent. Let ∂_v denote the closed geodesic boundary corresponding to v, and l_v denote the signed length of ∂_v, where the sign of l_v is the enhancement $\varepsilon(v)$ of v. Then, $f_v(z) = e^{l_v} z$ is the hyperbolic isometry of $\mathbb{H}^2$ corresponding to ∂_v. For $i \in \{0, 1\}$, $f_v(\widetilde{g_i}) = \widetilde{g_{n+i}}$ and the geodesics g_i and g_{n+i} are identified.

Proposition 3.4.2. (Signed boundary lengths from shear parameters) The shear parameters and the signed boundary lengths satisfy

$$l_v = \sum_{i=1}^{n} x_i.$$

Proof. The endpoint u_k of the lift $\widetilde{g_k}$ of g_k is positive if and only if $\varepsilon(v)$ is positive.

By Equation (3.6),

$$e^{l_v} = \frac{e^{l_v}u_1 - e^{l_v}u_0}{u_1 - u_0} = \frac{f_v(u_1) - f_v(u_0)}{u_1 - u_0} = \frac{u_{n+1} - u_n}{u_1 - u_0}$$
$$= \prod_{i=1}^{n} \frac{u_{i+1} - u_i}{u_i - u_{i-1}} = \prod_{i=1}^{n} e^{x_i} = \exp\left(\sum_{i=1}^{n} x_i\right)$$

and the asserted equation follows. □

For a vertex v corresponding to a cusp, the parabolic isometry f_v of $\mathbb{H}^2$ corresponding to the cusp is the parallel translation. Then

$$\exp\left(\sum_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} e^{x_i} = \prod_{i=1}^{n} \frac{u_{i+1} - u_i}{u_i - u_{i-1}}$$
$$= \frac{u_{n+1} - u_n}{u_1 - u_0} = \frac{f_v(u_1) - f_v(u_0)}{u_1 - u_0} = 1,$$

and Proposition 3.4.2 holds.

Proposition 3.4.3. (λ-lengths from shear parameters and decoration parameters) For a half-edge h incident to v, the λ-length of h is as follows.

- If v corresponds to a closed geodesic boundary,

$$a_h = d_v + \log \frac{R_{v,1} \sum_{i=1}^{n} x_i}{(\exp\left(\sum_{i=1}^{n} x_i\right) - 1)\sqrt{\rho_v((x_e)_{e\in E^i})^2 - \left(\sum_{i=1}^{n} x_i\right)^2}},$$

- If v corresponds to a cusp or a spike,

$$a_h = d_v + \log \frac{R_{v,1}}{\rho_v((x_e)_{e\in E^i})},$$

where d_v is the decoration parameter of v. In particular, if the normalizing function ρ of d_v is ρ_1 in Example 3.3.2, the λ-length of h is as follows:

$$a_h = d_v + \log \frac{R_{v,1}}{\min_{1\le k\le n}\{R_{v,k}\}},$$

Then, the λ-length of an edge e is as follows:

$$a_e = x_e + \sum_{h\in E^h(e)} a_h,$$

where $E^h(e)$ is the set of the half-edges of e.

Proof. First, consider the case that v is corresponding to a closed geodesic boundary ∂_v. Suppose that h is a half-edge of e_1 in the beginning of this section. Lift the geodesic g_k corresponding to e_k to the upper half plane $\mathbb{H}^2$ as given in Lemma 3.4.1. We can suppose that a lift $\widetilde{\partial_v}$ of ∂_v and a lift $\widetilde{\gamma_v}$ of the decoration curve γ_v have

the common endpoints 0 and ∞, $u_1 = \varepsilon(v)$ and $u_{n+1} = \varepsilon(v)e^{l_v}$, where ε is the enhancement and l_v is the signed length of ∂_v. Let θ denote the angle of $\widetilde{\partial}_v$ and $\widetilde{\gamma}_v$. Then,

$$\begin{aligned} a_h &= \log \frac{u_1 \frac{1}{\tan\theta}}{u_2 - u_1} && \text{(by Figure 3.2)}, \\ &= \log \frac{l_v R_{v,1}}{(e^{l_v}-1)\sqrt{r_v^2 - l_v^2}} && \text{(by Equation (3.3) and Lemma 3.4.1)}, \\ &= d_v + \log \frac{l_v R_{v,1}}{(e^{l_v}-1)\sqrt{\rho_v((x_e)_{e\in E^i})^2 - l_v^2}} && \text{(by Definition 3.1.4)} \end{aligned}$$

and asserted equation is follows by Proposition 3.4.2.

The argument for the cases of the cusps and the spikes are similar. Lift the geodesic g_k corresponding to e_k to the upper half plane $\mathbb{H}^2$ as given in Lemma 3.4.1, and we can suppose that $u_1 = 0$ and $u_{n+1} = 1$. Let y denote the imaginary part of the lift $\widetilde{\gamma}$ of the decoration curve γ. Then,

$$\begin{aligned} a_h &= \log \frac{y}{u_2 - u_1} && \text{(by Figure 3.3)}, \\ &= \log \frac{R_{v,1}}{r_v} && \text{(by Equation (3.5) and Lemma 3.4.1)}, \\ &= d_v + \log \frac{R_{v,1}}{\rho_v((x_e)_{e\in E^i})} && \text{(by Definition 3.1.4)}. \end{aligned}$$

Finally, we research the calculation of a_e. By definition, the shear parameter x_e of e is the signed length between two base points of e, and two λ-lengths of half-edges are the signed lengths between the respective base point and the intersection with the respective decoration curve. Then, the sum of these three lengths equals the λ-length a_e. □

By Proposition 3.4.2 and 3.4.3, we can construct the map

$$\tau_\rho : \mathbb{R}^{E^i(\Gamma)} \times \mathbb{R}^V \ni ((x_e)_{e\in E^i(\Gamma)}, (d_v)_{v\in V}) \mapsto ((a_e)_{e\in E(\Gamma)}, (l_v)_{v\in V^i}) \in \mathbb{R}^{E(\Gamma)} \times \mathbb{R}^{V^i},$$

where $(a_e)_{e\in E(\Gamma)}$ and $(l_v)_{v\in V^i}$ are the λ-lengths and signed boundary lengths of the decorated enhanced marked hyperbolic surface whose shear-decoration coordinates are $((x_e)_{e\in E^i(\Gamma)}, (d_v)_{v\in V})$.

3.5 λ-boundary-length coordinates

In this section, we calculate the shear parameters and the decoration parameters from the λ-lengths and the signed boundary lengths. These calculations make the inverse map from $\mathbb{R}^{E(\Gamma)} \times \mathbb{R}^{V^i}$ to $\mathbb{R}^{E^i(\Gamma)} \times \mathbb{R}^V$ and give $\mathcal{T}^{ax}(S)$ the generalized λ-length coordinates.

Theorem 3.5.1. (λ-boundary-length coordinates) For a triangulation Γ_0 whose puncture vertices are 1-valent, the map

$$\psi_a : \mathcal{T}^{ax}(S) \to \mathbb{R}^{E(\Gamma_0)} \times \mathbb{R}^{V^i}.$$

giving the λ-lengths of the edges of Γ_0 and the signed boundary lengths of the puncture vertices is a homeomorphism.

These coordinates of $\mathcal{T}^{ax}(S)$ are called the λ-*boundary-length coordinates*. Theorem 3.5.1 is the main theorem of this paper. In the remainder of this section, we prove this theorem.

Let e be an interior edge. We research the relation between the shear parameter of e and the λ-lengths. Let T, T' denote the triangles sharing e. Let v_1, v_2, v_3 denote the vertices of T, v_3, v_4, v_1 denote the vertices of T', and $e_{i,j}$ denote the edges of T or T' which have v_i and v_j as the endpoints. Lift them to the upper half plane $\mathbb{H}^2$ with respect to a given marked hyperbolic structure. Conveniently, regard these subscripts as the numbers modulo 4. The lift of e is denoted by $\widetilde{e}$, and so on. We define the *gaps* t_i to be the signed length along $\widetilde{e_{i-1,i}}$ between the intersections with the horocyclic arc and with the decoration curve around v_i, which is measured along the orientation induced from the triangle $\widetilde{T}$ or $\widetilde{T'}$ having $\widetilde{e_{i-1,i}}$ as an edge (Figure 3.5).

Lemma 3.5.2. (Gaps from shear parameters) For $1 \leq k \leq 4$, let n_k denote the valency of v_k, $x_{k,1}$ denote the shear parameter of $e_{k,k+1}$, and $x_{k,j+1}$ denote the shear parameters next to $x_{k,j}$ for counter-clockwise inductively. If v_k is corresponding to a closed geodesic boundary and k is even,

$$t_k = x_{k,2} + \log \frac{R_{v_k,2}}{R_{v_k,1}}.$$

If v_k is corresponding to a closed geodesic boundary and k is odd,

$$t_k = x_{k,2} + x_{k,3} + \log \frac{R_{v_k,3}}{R_{v_k,1}}.$$

If v_k is corresponding to a cusp or a spike,

$$t_k = 0.$$

Proof. By definition, if v_k is corresponding to a cusp or a spike, the decoration curve around v_k is a horocyclic arc and the gap t_k is 0.

We consider the case of the closed geodesic boundaries. Lift the geodesics g_j corresponding to the edge which gives the shear parameters $x_{k,j}$ to the upper half plane $\mathbb{H}^2$ as given in Lemma 3.4.1 so that v_k corresponds to ∞. We can suppose that a lift $\widetilde{\partial_k}$ of the boundary ∂_k and a lift $\widetilde{\gamma_k}$ of the decoration curve γ_k around v_k have the common endpoints 0 and ∞, $u_1 = \varepsilon(v_k)$ and $u_{n+1} = \varepsilon(v_k)e^{l_k}$, where ε is the enhancement and l_k is the signed length of ∂_k. Let θ denote the angle of $\widetilde{\partial_k}$ and $\widetilde{\gamma_k}$. If k is even, then $e_{k-1,k}$ corresponds to g_2 and

$$\begin{aligned} t_k &= \log \frac{u_2 \frac{1}{\tan\theta}}{|\frac{1}{\tan\theta}|} \\ &= x_{k,2} + \log \frac{R_{v_k,2}}{R_{v_k,1}}. \end{aligned}$$

If k is odd, then $e_{k-1,k}$ corresponds to g_3 and

$$\begin{aligned} t_k &= \log \frac{u_3 \frac{1}{\tan\theta}}{|\frac{1}{\tan\theta}|} \\ &= x_{k,2} + x_{k,3} + \log \frac{R_{v_k,3}}{R_{v_k,1}}. \end{aligned}$$

□

Lemma 3.5.3. (Shear parameters from λ-lengths and gaps) The shear parameter of e is

$$x_e = \frac{1}{2}(a_{1,2} - a_{2,3} + a_{3,4} - a_{4,1} + t_1 - t_2 + t_3 - t_4),$$

where $a_{i,j}$ are the λ-lengths of $e_{i,j}$.

Proof. For $1 \leq k \leq 4$, the λ-length $a_{k,k+1}$ of $e_{k,k+1}$ is the sum of two signed lengths from the base points of $\widetilde{T}$ or $\widetilde{T}'$ on $\widetilde{e_{k,k+1}}$ to the intersections with the lifts of the decoration curves around v_k and v_{k+1}. Let a_k denote a certain term of the sum $a_{k,k+1}$ as Figure 3.5. Then,

$$\begin{aligned} a_{1,2} &= a_1 + (a_2 + t_2), \\ a_{2,3} &= a_2 + (a_3 + t_3 - x_e), \\ a_{3,4} &= a_3 + (a_4 + t_4), \\ a_{4,1} &= a_4 + (a_1 + t_1 - x_e), \end{aligned}$$

and the asserted equation follows. □

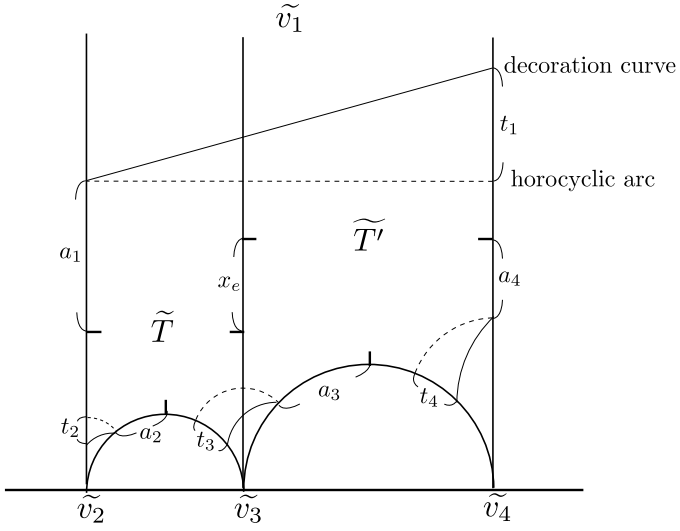

Figure 3.5: Gaps between the horocyclic arcs and the decoration curves.

Lemma 3.5.3 gives the relation between the shear parameters and the λ-lengths, while we need to solve the equations because the gaps t_k are expressed by the shear

parameters. By Proposition 3.4.2 and Lemma 3.5.2, t_k are 0 if v_k are corresponding to cusps or spikes, and t_k are the signed boundary lengths of v_k if the valencies of v_k are 1. The equations are solvable for the shear parameters when the triangulation is special.

If S has a spike, there exists a triangulation of S such that any vertex of V^i is 1-valent. Actually, take a spike vertex v_0, join the puncture vertices v to v_0 by edges e_v, surround v and e_v with the edges e'_v which have v_0 as both of the endpoints, triangulate the remaining area, and such a triangulation is obtained.

Proposition 3.5.4. (Shear parameters from λ-lengths and signed boundary lengths) Suppose that any vertex of V^i is 1-valent, then it is classified into three cases:

- v_1 is a puncture vertex, v_2, v_3, v_4 are spike vertices, and

$$x_e = \frac{1}{2}(a_{1,2} - a_{2,3} + a_{3,4} - a_{4,1} + 2l_1),$$

  where l_1 is the signed boundary length of v_1.

- v_2 is a puncture vertex, v_1, v_3, v_4 are spike vertices, and

$$x_e = \frac{1}{2}(a_{1,2} - a_{2,3} + a_{3,4} - a_{4,1} - l_2),$$

  where l_2 is the signed boundary length of v_2.

- v_1, v_2, v_3, v_4 are spike vertices, and

$$x_e = \frac{1}{2}(a_{1,2} - a_{2,3} + a_{3,4} - a_{4,1}).$$

Proof. First, suppose that v_1 is a puncture vertex. By the assumption, the valency of v_1 is 1, then $e = e_{1,2} = e_{4,1}$, $v_2 = v_3 = v_4$ and $e \neq e_{2,3} = e_{3,4}$. Since the valency of v_2 is more than 1, v_2 is a spike vertex. By Proposition 3.4.2 and Lemmas 3.5.2 and 3.5.3, $t_1 = 2x_e = 2l_1$, $t_2 = t_3 = t_4 = 0$ and the asserted equation follows.

Second, suppose that v_2 is a puncture vertex. By the assumption, the valency of v_2 is 1, then $e_{1,2} = e_{2,3}$ and $v_1 = v_3$. Since the valency of v_1 is more than 1, v_1 is a spike vertex. By Proposition 3.4.2 and Lemmas 3.5.2 and 3.5.3, $t_2 = x_{1,2} = l_2$ where $x_{1,2}$ is the shear parameter of $e_{1,2}$, $t_1 = t_3 = t_4 = 0$ and the asserted equation follows.

Finally, suppose that v_1, v_2, v_3, v_4 are spike vertices. By Lemmas 3.5.2 and 3.5.3, $t_1 = t_2 = t_3 = t_4 = 0$ and the asserted equation follows. □

Let v be a vertex. We consider the calculation of the decoration parameters from the λ-lengths. Take a triangle T which has v as a vertex, let v_1, v_2, v_3 denote the vertices of T in this counter-clockwise order such that $v = v_1$, and $e_{i,j}$ denote the edges of T which have v_i and v_j as the endpoints. Let t_1, t_2, t_3 denote the gaps between the horocyclic arcs and the lifts of the decoration curves along $\widetilde{e_{3,1}}, \widetilde{e_{1,2}}, \widetilde{e_{2,3}}$, respectively.

Proposition 3.5.5. (Decoration parameters from shear parameters and λ-lengths)

- If v corresponds to a closed geodesic boundary, the decoration parameter of v is

$$d_v = \frac{1}{2}\Bigg(a_{1,2} - a_{2,3} + a_{3,1} - t_2 + t_3 - x_{e_{3,1}} \\ - \log \frac{R_{v,1}R_{v,2}\left(\sum_{i=1}^{n} x_i\right)^2}{\left(\exp\left(\sum_{i=1}^{n} x_i\right) - 1\right)^2 \left(\rho_v((x_e)_{e\in E^i})^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right)} \Bigg),$$

- If v corresponds to a cusp or a spike, the decoration parameter of v is

$$d_v = \frac{1}{2}\left(a_{1,2} - a_{2,3} + a_{3,1} - t_2 + t_3 - x_{e_{3,1}} - \log \frac{R_{v,1}R_{v,2}}{\rho_v((x_e)_{e\in E^i})^2} \right),$$

where $e_{1,2} = e_1$ and $x_1, \cdots, x_n$ are labeled as §3.4. In particular, if the normalizing function ρ of d_v is ρ_1 in Example 3.3.2, the decoration parameter of v is as follows:

$$d_v = \frac{1}{2}\left(a_{1,2} - a_{2,3} + a_{3,1} - t_2 + t_3 - x_{e_{3,1}} - \log \frac{R_{v,1}R_{v,2}}{\min_{1\le k\le n}\{R_{v,k}\}^2} \right),$$

Proof. Use the equations of $a_{1,2}, a_{2,3}, a_{3,1}$ in Proposition 3.4.3, eliminate the terms of the decoration parameters d_{v_2}, d_{v_3} of v_2, v_3, and the equation of d_v is obtained. Then, the terms

$$-x_{e_{1,2}} - \log \frac{R_{v_2,2}}{R_{v_2,1}}$$

are deformed to $-t_2$, the terms

$$-x_{e_{2,3}} - \log \frac{R_{v_3,2}}{R_{v_3,1}}$$

are deformed to $-t_3$, and the asserted equation follows. □

We can confirm the independence of d_v from the choice of the triangle T. Let T' be the triangle adjacent to T along $e_{3,1}$, v_3, v_4, v_1 denote the vertices of T', t'_i denote the gaps for T', and d'_v denote the asserted formula in Proposition 3.5.5 for T'. By Lemmas 3.5.2 and 3.5.3,

$$d_v - d'_v = -x_{e_{3,1}} + \frac{1}{2}(a_{1,2} - a_{2,3} + a_{3,4} - a_{4,1} + t_1 + t'_1 - t_2 + t_3 + t'_3 - t'_4) = 0,$$

where $t_k + t'_k$ are the gaps of v_k for odd numbers k in Lemma 3.5.3. Proposition 3.5.5 implies that we can calculate the decoration parameters from the λ-lengths and the signed boundary lengths if the shear parameters are calculated.

Proof of Theorem 3.5.1. By Propositions 3.4.2 and 3.4.3, the map τ_ρ is continuous. Moreover, τ_ρ has the continuous inverse by Proposition 3.5.4 and 3.5.5. Since the map $\psi_{x,\rho}$ is a homeomorphism, the map $\psi_a = \tau_\rho \circ \psi_{x,\rho}$ is a homeomorphism. □

Chapter 4

Lamination spaces

There are spaces of laminations corresponding to the generalized Teichmüller spaces. In this chapter, we recall these spaces of laminations, and introduce the generalized spaces of laminations corresponding to the decorated enhanced Teichmüller spaces. Then, we find the common properties to the specific curves of laminations.

4.1 Teichmüller spaces and lamination spaces

$\mathcal{X}$-laminations (or unbounded measured laminations) and $\mathcal{A}$-laminations (or bounded measured laminations) are defined by Fock and Goncharov in [5]. The former are corresponding to the enhanced Teichmüller spaces, and the latter are corresponding to the decorated Teichmüller spaces. For the purpose of the definition of $\mathcal{AX}$-laminations, we define these known laminations in other words.

In this chapter, simple closed curves on S and simple arcs on S whose endpoints are in V or the boundaries of S are called *curves* on S.

- A curve γ is *contractible* if it satisfies one of the following:

  (C_1-1) there exists a boundary edge $e \in E^b(\Gamma)$ such that the endpoints of γ are in the interior of e, and there is a subarc δ of e such that γ and δ bound a disk on S,

  (C_1-2) γ bounds a disk on S,

  (C_2-1) there exist a spike vertex $v \in V^b$ and a boundary edge $e \in E^b(\Gamma)$ which has v as an endpoint such that an endpoint of γ is v and the other is in the interior of e, and there is a subarc δ of e such that γ, δ and v bound a disk on S,

  (C_2-2) there exists a vertex $v \in V$ such that the endpoints of γ are v and that γ and v bound a disk on S.

- A curve γ is called an $\mathcal{A}$-*curve* if γ is not contractible and satisfies either:

  (A-1) there exist boundary edges $e^+, e^- \in E^b(\Gamma)$ such that the endpoints of γ are in the interior of $e^\pm$, and there are subarcs $\delta^\pm$ of $e^\pm$ such that γ and $\delta^\pm$ bound a disk on S.

(A-2) there exists a puncture vertex $v \in V^i$ such that γ and v bound an annulus on S.

- A curve γ is called an $\mathcal{X}_D$-*curve* if γ is not contractible and

  (X-1) there exists a spike vertex $v \in V^b$ such that an endpoint of γ is v.

- A curve γ is called an $\mathcal{X}$-*curve* if γ is neither contractible nor $\mathcal{X}_D$-curve and

  (X-2) there exists a puncture vertex $v \in V^i$ such that an endpoint of γ is v.

- Otherwise, a curve γ is *general*.

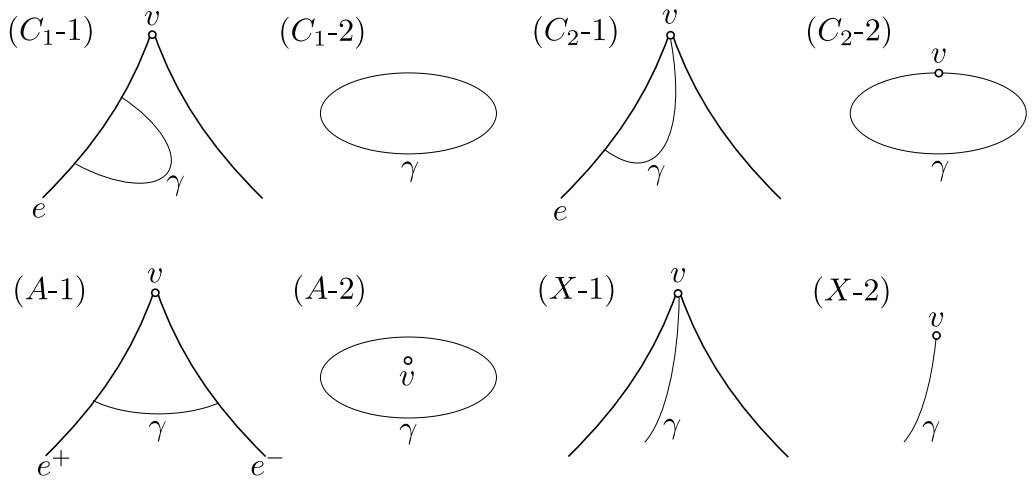

Figure 4.1: features of curves

Consider the doubled surface S_D of S, Conditions (C_1-1), (C_2-1), (A-1) and (X-1) are regard as Conditions (C_1-2), (C_2-2), (A-2) and (X-2), respectively, for the arcwise connected components of the doubled curve γ_D of γ.

Definition 4.1.1. (Laminations) A *rational measured lamination* on S is a finite collection of curves on S with rational weights, and satisfies the following conditions:

- A curve of negative weight is $\mathcal{A}$-curve.
- If two curves are intersecting, one of these curves is an $\mathcal{A}$-curve and the other is an $\mathcal{X}$-curve or an $\mathcal{X}_D$-curve.

Moreover, it is subject to the following equivalence relations:

- A lamination containing a contractible curve or a curve of weight 0 is equivalent to the lamination with these curves removed.

- A lamination containing two homotopy (rel endpoints) equivalent curves γ_1 and γ_2 of weight w_1 and w_2, respectively, is equivalent to the lamination with the weight $w_1 + w_2$ on γ_1 and with γ_2 removed.

Definition 4.1.2. ($\mathcal{A}$-laminations) A *rational $\mathcal{A}$-lamination* (or a *rational bounded measured lamination*) is a rational measured lamination which has no $\mathcal{X}$-curves and no $\mathcal{X}_D$-curves. The set of rational $\mathcal{A}$-laminations on S is denoted by $T^a(S, \mathbb{Q})$.

For an $\mathcal{A}$-lamination L and an edge e, let $a_e(L)$ denote the sum of the weights of curves intersected with e, where the representative of L is minimal intersected with edges.

Proposition 4.1.3. (Coordinates of $\mathcal{A}$-lamination space [5, §3.2]) The map

$$\Phi_{\mathbb{Q},a} : T^a(S, \mathbb{Q}) \ni L \mapsto (a_e(L))_{e \in E} \in \mathbb{Q}^E$$

from rational $\mathcal{A}$-laminations to rational weights of edges is a bijection.

The topology of $T^a(S, \mathbb{Q})$ is obtained by $\mathbb{Q}^E$, and we can define the space of *real $\mathcal{A}$-laminations* $T^a(S, \mathbb{R})$ as the completion of $T^a(S, \mathbb{Q})$. Let

$$\Phi_a : T^a(S, \mathbb{R}) \to \mathbb{R}^E$$

denote the resulting homeomorphism. Proposition 4.1.3 implies that the space of real $\mathcal{A}$-laminations is parametrized by the product of $\mathbb{R}$ as the decorated Teichmüller space.

Definition 4.1.4. (Orientation maps) For a rational measured lamination L, a map $\epsilon : V \to \{0, \pm 1\}$ is called an *orientation map* if $\epsilon^{-1}(\{\pm 1\}) = V_L$, where V_L is the set of vertices that are endpoints of curves of non-zero weight.

Definition 4.1.5. ($\mathcal{X}$-laminations) A *rational $\mathcal{X}$-lamination* (or a *rational unbounded measured lamination*) (L, ϵ) is a pair of a rational measured lamination L which has no $\mathcal{A}$-curves and no $\mathcal{X}_D$-curves and an orientation map ϵ. The set of rational $\mathcal{X}$-laminations on S is denoted by $T^x(S, \mathbb{Q})$.

Let (L, ϵ) be an $\mathcal{X}$-lamination, e be an interior edge, and edges $e_{1,2}, e_{2,3}, e_{3,4}, e_{4,1}$ denote the edges as the right of Figure 4.2. Take the representative of L spiraling at endpoints infinitely as shown in the left of Figure 4.2. A curve intersects with e *positively* if the intersection is interposed between the intersections with $e_{1,2}$ and $e_{3,4}$ as the curve γ in Figure 4.2. A curve intersects with e *negatively* if the intersection is interposed between the intersections with $e_{2,3}$ and $e_{4,1}$ as the curve γ' in Figure 4.2. Let $x_e(L, \epsilon)$ denote the sum of the weights of curves intersected with e positively minus the sum of the weights of curves intersected with e negatively. Other curves as the curve δ and δ' in Figure 4.2 do not effect on $x_e(L, \epsilon)$.

Proposition 4.1.6. (Coordinates of $\mathcal{X}$-lamination space [5, §3.1]) The map

$$\Phi_{\mathbb{Q},x} : T^x(S, \mathbb{Q}) \ni (L, \epsilon) \mapsto (x_e(L, \epsilon))_{e \in E^i} \in \mathbb{Q}^{E^i}$$

from rational $\mathcal{X}$-laminations to rational weights of internal edges is a bijection.

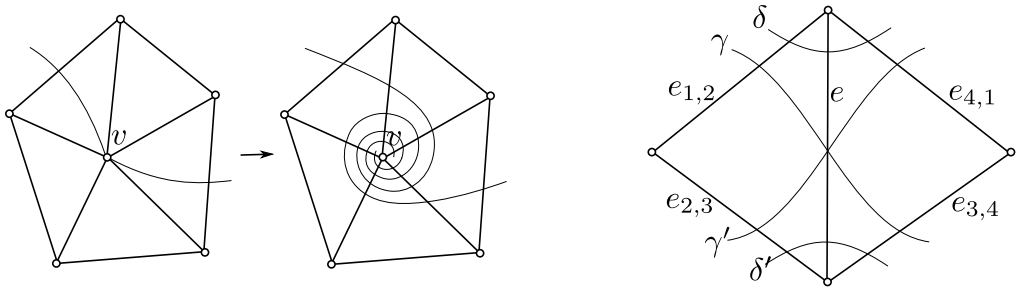

Figure 4.2: Spiraling representative in the case that $\epsilon(v) = 1$ (left), and curves intersect with edges positively and negatively (right).

The topology of $T^x(S, \mathbb{Q})$ is obtained by $\mathbb{Q}^{E^i}$, and we can define the space of *real $\mathcal{X}$-laminations* $T^x(S, \mathbb{R})$ as the completion of $T^x(S, \mathbb{Q})$. Let

$$\Phi_x : T^x(S, \mathbb{R}) \to \mathbb{R}^{E^i}$$

denote the resulting homeomorphism. Proposition 4.1.6 implies that the space of real $\mathcal{X}$-laminations is parametrized by the product of $\mathbb{R}$ as the enhanced Teichmüller space.

4.2 $\mathcal{AX}$-lamination spaces

In this section, we introduce the generalized spaces of laminations corresponding to the decorated enhanced Teichmüller spaces. Then, we research the meaning of curves of each type.

Definition 4.2.1. ($\mathcal{AX}$-laminations) A *rational $\mathcal{AX}$-lamination* (L, ϵ) is a pair of a rational measured lamination L which has no $\mathcal{X}_D$-curves and an orientation map ϵ. The set of rational $\mathcal{AX}$-laminations on S is denoted by $T^{ax}(S, \mathbb{Q})$.

Let (L, ϵ) be a rational $\mathcal{AX}$-lamination and L_a denote the lamination composed of $\mathcal{A}$-curves of L. Let $d_v(L_a)$ denote the weight of the $\mathcal{A}$-curve corresponding to v, where the correspondence is shown in Figure 4.1. Since $L \setminus L_a$ has no $\mathcal{A}$-curves, $x_e(L \setminus L_a, \epsilon)$ can be calculated as rational $\mathcal{X}$-laminations. By Proposition 4.1.6, the following proposition follows.

Proposition 4.2.2. (Coordinates of $\mathcal{AX}$-lamination space) This map

$$\Psi_{\mathbb{Q},x} : T^{ax}(S, \mathbb{Q}) \ni (L, \epsilon) \mapsto ((x_e(L \setminus L_a, \epsilon))_{e \in E^i}, (d_v(L_a))_{v \in V}) \in \mathbb{Q}^{E^i} \times \mathbb{Q}^V$$

from rational $\mathcal{AX}$-laminations to rational weights of internal edges and rational weights of $\mathcal{A}$-curves is a bijection.

The topology of $T^{ax}(S, \mathbb{Q})$ is obtained by $\mathbb{Q}^{E^i} \times \mathbb{Q}^V$, and we can define the space of *real $\mathcal{AX}$-laminations* $T^{ax}(S, \mathbb{R})$ as the completion of $T^{ax}(S, \mathbb{Q})$. Let

$$\Psi_x : T^{ax}(S, \mathbb{R}) \to \mathbb{R}^{E^i} \times \mathbb{R}^V$$

denote the resulting homeomorphism. Proposition 4.2.2 implies that the space of real $\mathcal{AX}$-laminations is parametrized by the product of $\mathbb{R}$ as the decorated enhanced Teichmüller space, and weights of $\mathcal{A}$-curves correspond to decoration parameters.

We can also define x_e for a boundary edge e, as follows. Consider the doubled surface S_D of S, the doubled lamination L_D of L and the doubled orientation ϵ_D of ϵ such that these are invariant under the involution. The value $x_e(L, \epsilon)$ of the boundary edge e of S is defined by $x_e(L_D, \epsilon_D)$ of the interior edge e of S_D. This definition corresponds to considering the shear parameters of boundary edges.

Definition 4.2.3. ($\mathcal{X}_D$-laminations and $\mathcal{AX}_D$-laminations) A *rational $\mathcal{X}_D$-lamination* (L, ϵ) is a pair of a rational measured lamination L which has no $\mathcal{A}$-curves and an orientation map ϵ. The set of rational $\mathcal{X}_D$-laminations on S is denoted by $T^x_D(S, \mathbb{Q})$. A *rational $\mathcal{AX}_D$-lamination* (L, ϵ) is a pair of a rational measured lamination L and an orientation map ϵ. The set of rational $\mathcal{AX}_D$-laminations on S is denoted by $T^{ax}_D(S, \mathbb{Q})$.

By the argument similar to Proposition 4.1.6 and 4.2.2, the maps

$$\Phi_{\mathbb{Q},x,D} : T^x_D(S, \mathbb{Q}) \ni (L, \epsilon) \mapsto (x_e(L, \epsilon))_{e \in E} \in \mathbb{Q}^E \text{ and}$$

$$\Psi_{\mathbb{Q},x,D} : T^{ax}_D(S, \mathbb{Q}) \ni (L, \epsilon) \mapsto ((x_e(L \setminus L_a, \epsilon))_{e \in E}, (d_v(L_a))_{v \in V}) \in \mathbb{Q}^E \times \mathbb{Q}^V$$

are bijective, and their completions

$$\Phi_{x,D} : T^x_D(S, \mathbb{R}) \to \mathbb{R}^E \text{ and } \Psi_{x,D} : T^{ax}_D(S, \mathbb{R}) \to \mathbb{R}^E \times \mathbb{R}^V$$

are defined.

For a real $\mathcal{AX}$-lamination $\mathcal{L}$, there is a Cauchy sequence $(L_n, \epsilon_n)_{n \in \mathbb{N}}$ in $T^{ax}(S, \mathbb{Q})$ such that its equivalence class in $T^{ax}(S, \mathbb{R})$ equals $\mathcal{L}$. Following proposition implies that orientations of $\mathcal{AX}$-laminations and enhancements of decorated enhanced marked hyperbolic surfaces are corresponding.

Proposition 4.2.4. (Orientations and enhancements) Let $\mathcal{L} = [(L_n, \epsilon_n)_{n \in \mathbb{N}}]$ be a real $\mathcal{AX}$-lamination, $[X, f, \varepsilon, D]$ be a decorated enhanced marked hyperbolic surface, and v be an interior vertex. If $[(L_n, \epsilon_n)_{n \in \mathbb{N}}]$ and $[X, f, \varepsilon, D]$ are parametrized to the same point of $\mathbb{R}^{E^i} \times \mathbb{R}^V$ and $\epsilon_n(v)$ converges to 0, $\varepsilon(v)$ equals 0.

Proof. Since $\epsilon_n(v)$ converges to 0, for sufficiently large n, $\epsilon_n(v)$ equals 0. Let N_v be the union of triangles which have v as a vertex. Take a component γ of the intersection of N_v and curves of L_n whose weight w is not 0. By Proposition 3.4.2, $\varepsilon(v)$ is the sign of the signed boundary length l_v that is the sum of shear parameters of edges which have v as an endpoint. Consider the effect of γ on l_v. The arc γ intersects at most once with edges which have v as an endpoint, and γ does not have v since $\epsilon_n(v)$ is 0. When one goes along γ clockwise direction, γ adds w at the first intersection and subtracts w at the last intersection. In the total result, γ has no effect on l_v. Therefore, $\varepsilon(v)$ equals 0. □

However the converse proposition of Proposition 4.2.4 is not applied.

4.3 Compatibility of parametrizations

Weighted general curves are corresponding to the shear coordinates of a marked hyperbolic surface by the map Φ_x. On the other hand, they are corresponding to the λ-length coordinates of a marked hyperbolic surface by the map Φ_a. The following theorem states that these marked hyperbolic surfaces are the same. Although $T^x(S,\mathbb{R})$ and $T^a(S,\mathbb{R})$ are defined independently, their common curves have the same meaning.

Let

$$\Psi_{a,\rho} := \tau_\rho \circ \Psi_x : T^{ax}(S,\mathbb{R}) \to \mathbb{R}^E \times \mathbb{R}^{V^i}$$

be the composition of Ψ_x and τ_ρ.

Theorem 4.3.1. (Compatibility of parametrization) The equations

$$\Psi_x \circ I_x = \iota_x \circ \Phi_x \text{ and } \Psi_{a,\rho_1} \circ I_a = \iota_a \circ \Phi_a$$

hold, where ι_x is the inclusion map from $\mathbb{R}^{E^i}$ to $\mathbb{R}^{E^i} \times \mathbb{R}^V$, ι_a is the inclusion map from $\mathbb{R}^E$ to $\mathbb{R}^E \times \mathbb{R}^{V^i}$, I_x is the inclusion map from $T^x(S,\mathbb{R})$ to $T^{ax}(S,\mathbb{R})$, and I_a is the inclusion map from $T^a(S,\mathbb{R})$ to $T^{ax}(S,\mathbb{R})$ as in Figure 4.3.

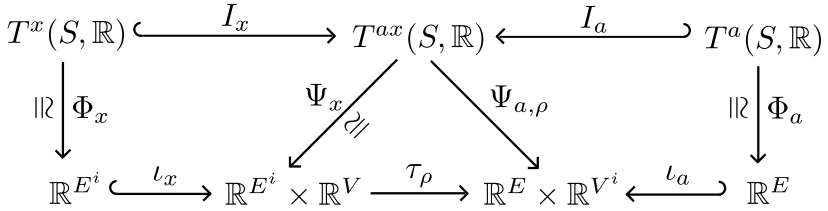

Figure 4.3: The commutative diagram describing that general curves have the same meaning in $T^x(S,\mathbb{R})$ and $T^a(S,\mathbb{R})$.

Moreover, Theorem 4.3.1 implies that the weight of the $\mathcal{A}$-curve of a lamination L around a vertex v is the decoration parameter d_v of the decorated marked hyperbolic surface whose λ-length coordinates are $\Phi_a(L)$. In the remainder of this section, we prove this theorem.

By the construction of Φ_x and Ψ_x, the former commutative equation immediately follows. Since the maps in the statement are continuous, it is sufficient to show that

$$\tau_{\rho_1} \circ \Psi_x \circ I_a(L) = \iota_a \circ \Phi_a(L)$$

for any rational $\mathcal{A}$-lamination L.

Let e be an edge, and v, v' denote the endpoints of e. Lift the edges which have v as an endpoint to the upper half plane $\mathbb{H}^2$ such that a lift $\tilde{v}$ of v is ∞. Suppose that there is no curve of L intersecting with e negatively. (Otherwise, consider the mirror image.) Lift edges which have v as an endpoint to the upper half plane $\mathbb{H}^2$ such that a lift $\tilde{v}$ of v is ∞. Let $\tilde{\gamma}_1, \cdots, \tilde{\gamma}_n$ denote the lifts of curves of L intersecting with e positively, $\tilde{\delta}, \tilde{\delta}'$ denote the lifts of $\mathcal{A}$-curves of L around v, v' respectively,

and $\tilde{\delta}_1, \cdots, \tilde{\delta}_m, \tilde{\delta}'_1, \cdots, \tilde{\delta}'_{m'}$ denote the lifts of remain general curves of L intersecting with e as shown in Figure 4.4. Let $\tilde{e}$ denote the lift of e, and $\tilde{e}_k$ denote the lift of the edge which has v as an endpoint and intersects with δ_k positively as shown in Figure 4.4. It may occur that $\tilde{e}_i = \tilde{e}_j$ for distinct i, j. Take the base point b of $\tilde{e}$ with respect to the triangle on the right side of $\tilde{e}$, the base point b_1 of $\tilde{e}_1$ with respect to the triangle on the right side of $\tilde{e}_1$, and the lift $\tilde{\alpha}_1$ of the horocycle α_1 around v passing through b_1.

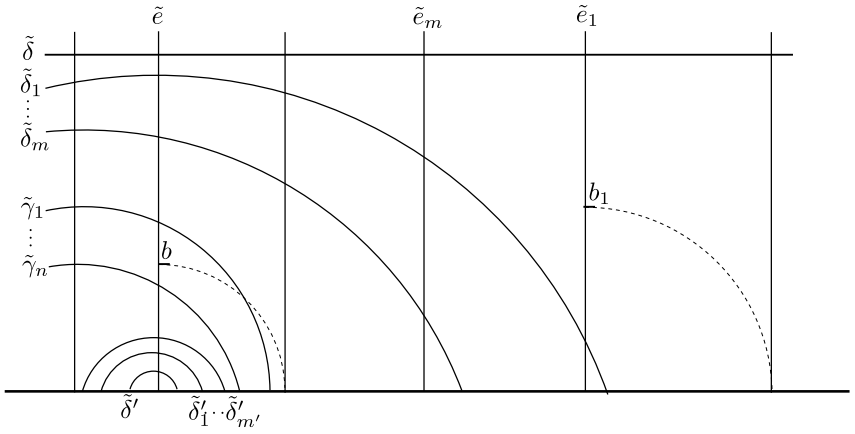

Figure 4.4: The labels of curves of L. (In the case that there is a curve of L intersecting with e negatively, consider the mirror image.)

Lemma 4.3.2. (Decoration curves passing through b_1) The horocycle α_1 is the decoration curve around v when the shear coordinates of the decorated marked hyperbolic surface are $\Psi_x \circ I_a(L)$ and decoration parameter of v is 0.

Proof. If there exist a curve γ of L and an edge e_- such that they are intersecting negatively and the real part of $\tilde{e}_-$ is lower than the real part of $\tilde{e}_1$, there must be an edge e_+ which intersects with γ positively and the real part of $\tilde{e}_+$ is not larger than the real part of $\tilde{e}_1$. That is to say, the negative factor of the shear parameter of an edge $\tilde{e}_-$ on the left side of $\tilde{e}_1$ must be canceled by the positive factor of the shear parameter of $\tilde{e}_1$ or the edge $\tilde{e}_+$ on the left side of $\tilde{e}_1$. It may occur that the positive factor of the shear parameter of an edge on the left side of $\tilde{e}_1$ is not canceled, when v is spike vertex. Similarly, the positive factor of the shear parameter of an edge on the right side of $\tilde{e}_1$ must be canceled by the negative factor of the shear parameter of $\tilde{e}_1$ or an edge on the right side of $\tilde{e}_1$.

These considerations imply that the imaginary part of b_1 is larger than the imaginary parts of other base points. By the definition of the normalizing function of decoration parameters, the decoration curve around v passes through b_1 when the decoration parameter of v is 0. □

Lemma 4.3.3. (The signed length from b to $\tilde{\alpha}_1$) The signed length from b to $\tilde{\alpha}_1$ is

$$\sum_{k=1}^{m} w_{\delta_k},$$

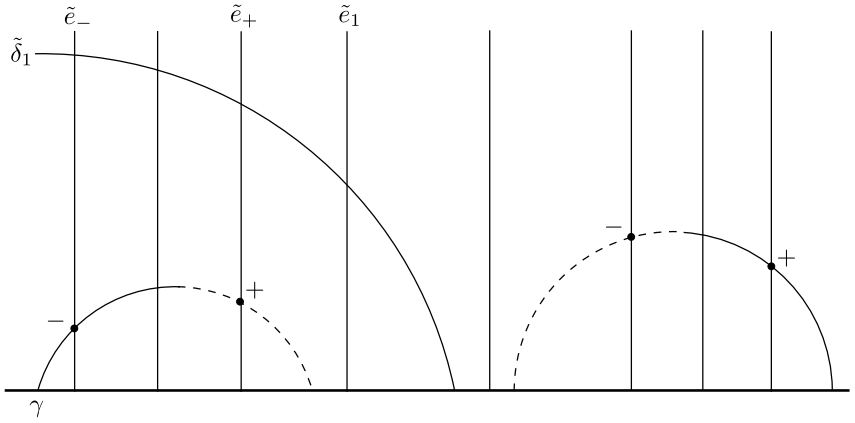

Figure 4.5: Factors of shear parameters which decrease the imaginary part of b_1 are canceled.

when w_γ denotes the weight of γ for a curve γ of L.

Proof. By Lemma 4.3.2, the negative factor of the shear parameter of an edge on the left side of $\tilde{e}_1$ must be canceled by the positive factor of the shear parameter of $\tilde{e}_1$ or an edge on the left side of $\tilde{e}_1$. If there exist a curve γ of L other than $\delta_1, \cdots, \delta_m$ and an edge e_+ such that they are intersecting positively and the real part of $\tilde{e}_+$ is not larger than the real part of $\tilde{e}_k$, there must be an edge e_- which intersects with γ negatively and the real part of $\tilde{e}_-$ is larger than the real part of $\tilde{e}_{k+1}$, where $\tilde{e}_{m+1}$ is defined as e. That is to say, the positive factor of the shear parameter of $\tilde{e}_k$ or an edge $\tilde{e}_+$ on the left side of $\tilde{e}_k$ obtained from γ must be canceled by the negative factor of the shear parameter of the edge $\tilde{e}_-$ on the right side of $\tilde{e}_{k+1}$.

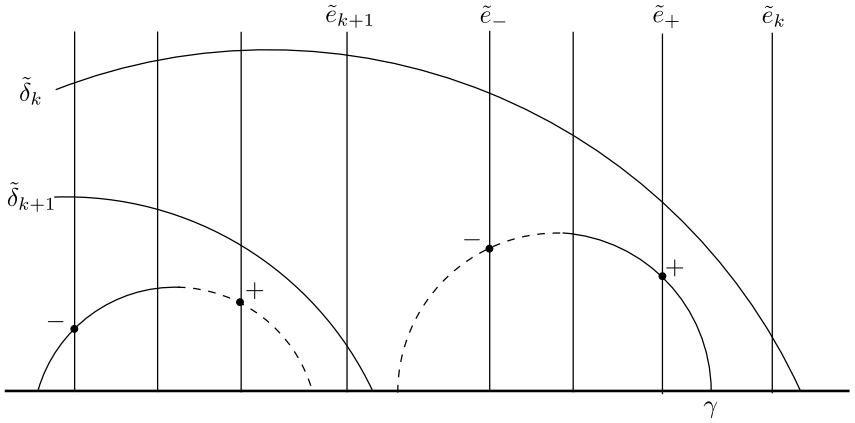

Figure 4.6: Factors of shear parameters which are obtained from curves other than $\delta_1, \cdots, \delta_m$ are canceled.

These considerations imply that factors of shear parameters which are not canceled are positive factors obtained from $\delta_1, \cdots, \delta_m$. Then, the asserted statement follows. □

Consider the lifting to the upper half plane $\mathbb{H}^2$ such that a lift $\tilde{v}'$ of v' is ∞, and we have results about v' similar to Lemma 4.3.2 and Lemma 4.3.3.

Proof. (Theorem 4.3.1) The tuple $\tau_{\rho_1} \circ \Psi_x \circ I_a(L)$ consists of the λ-lengths and the signed boundary lengths of the decorated enhanced hyperbolic surface whose shear-decoration coordinates are $\Psi_x \circ I_a(L)$. By Lemma 4.3.2 and Lemma 4.3.3,

$$\begin{aligned} p_e \circ \tau_{\rho_1} \circ \Psi_x \circ I_a(L) =& \pi_v \circ \Psi_x \circ I_a(L) + \sum_{k=1}^{m} w_{\delta_k} + \pi_{v'} \circ \Psi_x \circ I_a(L) \\ &+ \sum_{k=1}^{m'} w_{\delta'_k} + \pi_e \circ \Psi_x \circ I_a(L) \\ =& w_\delta + \sum_{k=1}^{m} w_{\delta_k} + w_{\delta'} + \sum_{k=1}^{m'} w_{\delta'_k} + \sum_{k=1}^{n} w_{\gamma_k} \\ =& p_e \circ \iota_a \circ \Phi_a(L), \end{aligned}$$

where π_e, π_v and $\pi_{v'}$ are the projection maps from $\mathbb{R}^{E^i} \times \mathbb{R}^V$ to $\mathbb{R}$ corresponding to the subscripts, and p_e is the projection map from $\mathbb{R}^E \times \mathbb{R}^{V^i}$ to $\mathbb{R}$ corresponding to the subscript. By Proposition 4.2.4,

$$\begin{aligned} p_v \circ \tau_{\rho_1} \circ \Psi_x \circ I_a(L) &= 0 \\ &= p_v \circ \iota_a \circ \Phi_a(L), \end{aligned}$$

where p_v is the projection map from $\mathbb{R}^E \times \mathbb{R}^{V^i}$ to $\mathbb{R}$ corresponding to the subscript. Therefore, Theorem 4.3.1 follows. □

Chapter 5

Appendix

5.1 2-valent triangulations

For a certain normalizing function of decoration parameters, the assertion of Theorem 3.5.1 also holds true when the assumption is that any puncture vertex is 2-valent and any spike vertex is 3-valent, because the equation of the λ-length for several shear variables can be deformed to the equation for one shear variable. Actually, there are only two topological types of the reference surface, which has a triangulation satisfying this assumption and no triangulation satisfying the assumption of Theorem 3.5.1. In this section, we consider the coordinate transformation between the shear-decoration coordinates and the λ-boundary-length coordinates of these two surfaces for the normalizing function ρ_1 in Example 3.3.2.

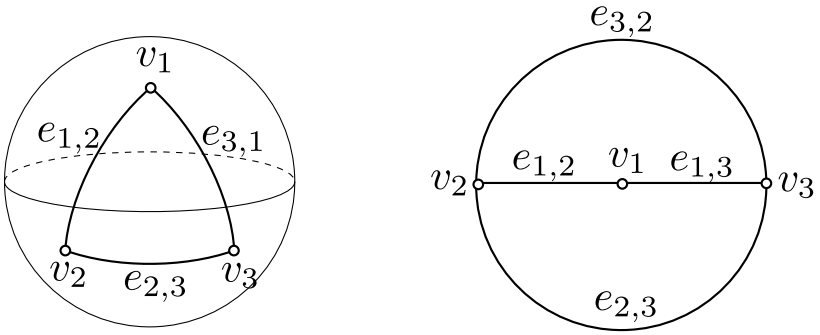

Figure 5.1: Three-punctured sphere (left) and once punctured bigon (right).

Example 5.1.1. (Three-punctured sphere) Let S be the three-punctured sphere. For $i, j \in \{1, 2, 3\}$, let v_i and $e_{i,j}$ denote the vertices and the edges as the left of Figure 5.1. Let $x_{i,j}$ denote the shear parameters of $e_{i,j}$, d_i denote the decoration parameters of v_i, $a_{i,j}$ denote the λ-lengths of $e_{i,j}$, and l_i denote the signed boundary

lengths of v_i. By Propositions 3.4.2 and 3.4.3,

$$\begin{aligned} a_{i,j} &= x_{i,j} + \sum_{l\in\{i,j\}} \left(d_l + \log \frac{1+e^{x_{k,l}}}{1+\min\{e^{x_{i,j}}, e^{x_{k,l}}\}} \right), \\ l_i &= x_{i,j} + x_{i,k}, \end{aligned}$$

for i, j, k with $\{i, j, k\} = \{1, 2, 3\}$. Because of the valency of the vertices, we can solve the equation for the shear parameters and the decoration parameters:

$$\begin{aligned} x_{i,j} =& \frac{1}{2}(l_i + l_j - l_k), \\ d_i =& \frac{1}{2}(a_{i,j} + a_{i,k} - a_{j,k} - l_i) - \log 2\cosh \frac{1}{4}(l_i - l_j - l_k) \\ &+ \log(1 + \min\{e^{\frac{1}{2}(l_i+l_j-l_k)}, e^{\frac{1}{2}(l_i+l_k-l_j)}\}), \end{aligned}$$

for i, j, k with $\{i, j, k\} = \{1, 2, 3\}$.

Example 5.1.2. (Once punctured bigon) Let S be the once punctured bigon. For $i, j \in \{1, 2, 3\}$, let v_i and $e_{i,j}$ denote the vertices and the edges as the right of Figure 5.1. Set the parameters as Example 5.1.1. By Propositions 3.4.2 and 3.4.3,

$$\begin{aligned} a_{1,i} &= d_1 + d_i + \log \frac{1+e^{x_{1,j}}}{1+\min\{e^{x_{1,i}}, e^{x_{1,j}}\}} + \log \frac{1+e^{x_{1,i}}}{1+\min\{e^{x_{1,i}}, e^{-x_{1,i}}\}}, \\ a_{i,j} &= d_i + d_j + \log \frac{1+e^{x_{1,i}}}{1+\min\{e^{x_{1,i}}, e^{-x_{1,i}}\}} + \log \frac{1+e^{-x_{1,j}}}{1+\min\{e^{x_{1,j}}, e^{-x_{1,j}}\}}, \\ l_1 &= x_{1,2} + x_{1,3}, \end{aligned}$$

for i, j with $\{i, j\} = \{2, 3\}$. Because of the valency of the vertices, we can solve the equation for the shear parameters and the decoration parameters:

$$\begin{aligned} x_{1,i} =& \frac{1}{2}(l_1 + a_{i,j} - a_{j,i}), \\ d_1 =& \frac{1}{2}(a_{1,2} + a_{1,3} - \log(e^{\frac{1}{2}(l_1+a_{i,j}+a_{j,i})} + \max\{e^{l_1+a_{2,3}}, e^{l_1+a_{3,2}}\}), \\ d_i =& \frac{1}{2}(a_{1,i} - a_{1,j} + a_{i,j} + \log(1 + \min\{e^{\frac{1}{2}(l_1+a_{i,j}-a_{j,i})}, e^{-\frac{1}{2}(l_1+a_{i,j}-a_{j,i})}\}) \\ &- \log(1 + e^{\frac{1}{2}(l_1+a_{i,j}-a_{j,i})})(1 + e^{-\frac{1}{2}(l_1+a_{j,i}-a_{i,j})})), \end{aligned}$$

for i, j with $\{i, j\} = \{2, 3\}$.

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