Measure theory on general linear groups over a higher dimensional local field

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Introduction

One-dimensional local fields are complete locally compact fields. In algebraic geometry, one-dimensional local fields appear as completions of fractional fields of arithmetic curves. These one-dimensional local fields play important roles in arithmetic geometry. In this thesis, we will study higher dimensional local fields. Zero-dimensional local fields are finite fields and n-dimensional local fields are complete discrete valuation fields whose residue fields are $(n-1)$-dimensional local fields. It is well-known that n-dimensional local fields are finite extensions of

$$\mathbb{F}_p((t_1))\dots((t_n)) \text{ or } \mathbb{Q}_p\{\{t_1\}\}\dots\{\{t_{r-1}\}\}((t_{r+1}))\dots((t_n)).$$

Therefore, higher dimensional local fields appear as the completions of fractional fields of arithmetic varieties at full flag subvarieties. Let X be a scheme over $\mathbb{Z}$ of dimension n. A full flag subvarieties of X is a sequence of closed integral subschemes of X

$$X_0 \subset X_1 \subset \dots \subset X_n = X$$

such that $\dim X_i = i$.

K. Kato([K]) and I. Fesenko([F1]) constructed a class field theory on higher dimensional local fields. So arithmetic properties of these fields are of interest. In this thesis, we construct a measure theory on higher dimensional local fields. Research of this area was initially constructed by Fesenko([F2]) , whose aim was to Iwasawa-Tate theory to two-dimensional local fields.

In [F3], Fesenko constructed a theory of measures and integration over higher dimensional local fields. For an n-dimensional local field F, Fesenko regarded the valuation ring O_F of rank n as a “compact subset”. Fesenko called a subset of F which is either of form $a + t^\alpha O_F$ or the empty set a distinguished subset, and call the set difference $\bigcup_i A_i \backslash \bigcup_j B_j$ such that each A_i, B_j are distinguished subsets a dd-set. The set of distinguished subsets becomes a totally ordered set. Let $f\colon F \longrightarrow \mathbb{C}$ be a function which is expressed as a countable linear combination $f = \sum c_i \mathbb{I}_{A_i}$ with $c_i \in \mathbb{C}$ of characteristic functions $\mathbb{I}_{A_i}$ of mutually disjoint dd-dets A_i. Suppose the series $\sum c_i \mu_F(A_i)$ absolutely converges. Then Fesenko proved that

$$\int_F f(x) d\mu_F(x) = \sum c_i \mu_F(A_i)$$

is well-defined in the sense that the right hand side does not depend on the choice of c_i, A_i. By using this property, Fesenko established a higher dimensional generalization of harmonic analysis including the theory of Fourier transform. Further, in the last part of the paper [F3], Fesenko expected that there exists a theory of measures and integration on generalized loop groups, for example, on $GL_d(\mathbb{Q}_p((t_2))\dots((t_n)))$. In the papers [KL] and [L], Kim and Lee considered measures and convolutions on $SL_2(F_2)$ where F_2 is a two-dimensional local field. We will extend Fesenko's measures to these groups. Morrow and Waller already established the theories of measures and integrations in [M2] and in [W]. In [M1] and [M2], Morrow considered integration on GL_d over higher dimensional local fields. To define a measure on a higher dimensional local field, Morrow used a measure on a one-dimensional residue field. On the other hand, we are also interested in higher dimensional information that is not reflected in the structure of the residue fields. For example as Waller indicated, refined countable additivities of measures needed for the study of higher zeta integral has not been discussed in Morrow's approach. His paper provides details for the so-called lifting approach to the integral. Waller constructed a measure theory on $GL_2(F)$ where F is a two-dimensional local field. He defined the volumes of congruence subgroups of $GL_2(F)$. For example, in his approach, one has

$$\begin{aligned}&\mu_{GL_2(\mathbb{F}_q((t_1))((t_2)))}\left(1+t_1^{i_1}t_2^{i_2}M_2(\mathbb{F}_q[[t_1]]+t_2\mathbb{F}_q((t_1))[[t_2]])\right)\\&=\frac{q^3}{(q^2-1)(q-1)}q^{-4i_1}X^{4i_2}\end{aligned}$$

and

$$\mu_{GL_2(\mathbb{F}_q((t_1))((t_2)))}(GL_2(\mathbb{F}_q[[t_1]]+t_2\mathbb{F}_q((t_1))[[t_2]]))=1.$$

Similar to Fesenko's definition, Waller called a subset of the form gK for some $g\in GL_2(F)$ and some congruence subgroup K of the form $K=1+t^\alpha M_2(O_F)$ for some $\alpha\in\mathbb{Z}^2_{>0}$ a distinguished subset. In his definition, measurable subsets are not stable under some basic operations such as taking intersections and conjugations. For any $g\in GL_2(F)$ and any congruence subgroup K, a measure of the subset gKg^{-1} is not necessarily defined. Further since the family of fundamental systems of distinguished subsets becomes a totally ordered set, the operations of the intersections are rigid. In other words, the intersection of two measurable subsets is one or the other. In our approach, an intersection of measurable sets is not necessarily equal to any of the original measurable sets. We overcome these problems in this thesis.

Now we outline the contents of this thesis.

In the first Section, we will define the family of measurable subsets of F^m by

$$\mathcal{J}(F^m)=\bigcup_{s\geq m}\bigcup_{\substack{T:F^s\to F^m,\\ \mathrm{rank}(T)=m}}T\left(\mathcal{J}(F)^{\oplus s}\right).$$

Here $\mathcal{J}(F)$ is the set of measurable subsets of F defined by Fesenko [F3]. Further denote by $C_c^\infty(F^m)$ the $\mathbb{C}_{(n)}$-module generated by the characteristic functions of measurable subsets, we will construct the integration

$$I_{F^m}:C_c^\infty(F^m)\longrightarrow\mathbb{C}_{(n)};f\longmapsto\int_{F^m}f(x)f\mu_{F^m}(x).$$

Moreover Fubini's theorem holds for this integral:

Theorem 0.1. *For any $f\in C_c^\infty(F^m)$, we have*

$$\int_{F^m}f(x)d\mu_{F^m}(x)=\int_F\cdots\int_F f(x_1,\ldots,x_m)d\mu_F(x_1)\cdots d\mu_F(x_m).$$

On the other hand, we will not discuss non-linear changes of variables and Fubini's theorem is considered in by Morrow [M3]. So innovating Morrow's argument on our measure theory is left as a future work in progress.

In the next section, we will establish a measure theory on $GL_d(F)$. These results are generalizations of Waller's works. We will show the existence of translation-invariant integration

$$I_{GL_d(F)} \colon C_c^\infty(GL_d(F)) \longrightarrow \mathbb{C}_{(n)}.$$

Moreover we will consider the theory of measures and integration on standard parabolic subgroups of $GL_d(F)$. Measures on these groups are introduced in [Mori1]. As Morrow remarked in [M2], harmonic analysis on other groups than $GL_d(F)$ is one of the concerns in this area. For any standard parabolic subgroup $P(F)$ of $GL_d(F)$, we will construct the left-invariant integration

$$I^l_{P(F)} \colon C_c^\infty(P(F)) \longrightarrow \mathbb{C}_{(n)}$$

and the right-invariant integration

$$I^r_{P(F)} \colon C_c^\infty(P(F)) \longrightarrow \mathbb{C}_{(n)}.$$

These two functions are linked by a modular character $\delta_P \colon P(F) \longrightarrow \mathbb{C}^\times_{(n)}$.

Almost all of the contents of this thesis are based on [Mori2] and a part of [Mori1], which has been accepted for publication earlier.

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1 A measure on F^m

Notations 1.1. *(1) Let F be an n-dimensional local field and $t_1, \dots, t_n$ its system of local parameters. We denote by*

$$v_F \colon F \longrightarrow \mathbb{Z}^n \cup \{\infty\}$$

the valuation of rank n associated to $t_1, \dots, t_n$, where $\mathbb{Z}^n$ has the lexicographic order. We denote by O_F the valuation ring of rank n, $\{x \in F \mid v_F(x) \geq 0\}$.

(2) Let us write $\mathbb{C}_{(n)} = \mathbb{C}((X_2)) \dots ((X_n))$. This is a field with $n-1$ variables added to $\mathbb{C}$ in order.

(3) For any multiple index $\alpha = (\alpha_n, \dots, \alpha_1) \in \mathbb{Z}^n$, we write $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ and $X^\alpha = q_1^{-\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ respectively. Here $q = \mathrm{char}(F_0)$ is an order of the last residue field of F. For any $x \in F$, $||x|| = X^{v_F(x)}$.

In this section, we will consider translation invariant measure and integration on a vector space F^m.

Definition 1.2. *(1) We say that a subset of F is measurable if it is of the form $a+t^\alpha O_F$ for some $a \in F$ and $\alpha \in \mathbb{Z}^n$ is measurable and we let $\mathcal{J}(F)$ denote the family of measurable subsets of F.*

(2) We denote by $C_c^\infty(F)$ the $\mathbb{C}_{(n)}$-module generated by the characteristic functions of measurable subsets of F.

The below proposition immediately follows from the properties of the measurable subsets.

Proposition 1.3. *For any measurable subset $A = a + t^\alpha O_F$ of F, we put $\mu_F(A) = X^\alpha$. Then*

$$I_F\colon C_c^\infty(F) \longrightarrow \mathbb{C}_{(n)}; \sum_{i=1}^{k} c_i \mathbb{I}_{A_i} \longmapsto \sum_{i=1}^{k} c_i \mu_F(A_i)$$

is well-defined. Here each $\mathbb{I}_{A_i}$ is the characteristic function of A_i.

The measure does depend on the choice of $t_2, \ldots, t_n$. For example, In two-dimensional local field $F = \mathbb{F}_p((t_1))((t_2))$, the elements $t_1, t_2' = t_1 t_2$ also form a system of local parameters. Then

$$a + t_1^{\alpha_1} t_2^{\alpha_2} O_F \longmapsto (1/p)^{\alpha_1 - \alpha_2} X_2^{\alpha_2}$$

becomes another measure on F. This is the measure associated to the system of local parameters t_1, t_2'.

Our measure theory on F is a higher dimensional analogue of the measure theory on a locally compact field.

Definition 1.4. *We say that a subset A of F^m is measurable if we can take a linear epimorphism $T\colon F^s \longrightarrow F^m$, $a_1, \ldots, a_s \in F$ and $\alpha_1 \ldots, \alpha_s \in \mathbb{Z}^n$ such that*

$$A = T \begin{pmatrix} a_1 + t^{\alpha_1} O_F \\ \vdots \\ a_s + t^{\alpha_s} O_F \end{pmatrix}.$$

Let $\mathcal{J}(F^m)$ denote the set of measurable subsets of F^m. Then by definition, we have

$$\mathcal{J}(F^m) = \bigcup_{s \geq m} \bigcup_{\substack{T\colon F^s \to F^m, \\ \mathrm{rank}(T)=m}} T\left(\mathcal{J}(F)^{\oplus s}\right).$$

as sets.

Any linear epimorphism $T\colon F^s \longrightarrow F^m$ induces the map

$$T_*\colon \mathcal{J}(F^s) \longrightarrow \mathcal{J}(F^m); A \longmapsto T(A).$$

Example 1.5. *An example of measurable subset is*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 + t^{\alpha_1} O_F \\ a_2 + t^{\alpha_2} O_F \end{pmatrix}$$

in F^2. This does not belong to $\mathcal{J}(F)^2$. So $\mathcal{J}(F^2)$ is strictly larger than $\mathcal{J}(F)^2$.

Remark 1.6. *For any A, B of $\mathcal{J}(F^m)$, the set $A+B = \{a+b \mid a \in A, b \in B\}$ lies in $\mathcal{J}(F^m)$. Indeed, let $T\colon F^s \longrightarrow F^m$ and $S\colon F^r \longrightarrow F^m$ be linear epimorphisms such that*

$$A = T \begin{pmatrix} a_1 + t^{\alpha_1} O_F \\ \vdots \\ a_s + t^{\alpha_s} O_F \end{pmatrix}, B = S \begin{pmatrix} b_1 + t^{\beta_1} O_F \\ \vdots \\ b_s + t^{\beta_s} O_F \end{pmatrix}.$$

Then we have

$$A + B = (T|S) \begin{pmatrix} a_1 + t^{\alpha_1} O_F \\ \vdots \\ a_s + t^{\alpha_s} O_F \\ b_1 + t^{\beta_1} O_F \\ \vdots \\ b_s + t^{\beta_s} O_F \end{pmatrix}.$$

So we can consider the additive map $\mathcal{J}(F^m) \times \mathcal{J}(F^m) \longrightarrow \mathcal{J}(F^m); (A, B) \longmapsto A + B$.

We will define the volumes of measurable subsets of F^m.

Lemma 1.7. *For any matrix $T \in M_{m,s}(F)$ of rank m, there exist $P \in GL_m(O_F)$ and $Q \in GL_s(O_F)$ such that*

$$PTQ = \begin{pmatrix} t^{\alpha_1} & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & t^{\alpha_m} & 0 & \dots & 0 \end{pmatrix}.$$

Then $\alpha_1, \dots, \alpha_m$ do not depend, up to permutations, on the choice of P, Q.

Proof. Let us write $T = (x_{ij})$. Then there exist i_1 and j_1 such that $v_F(x_{i_1 j_1}) = \min\{v_F(x_{ij})\}$. By multiplying some element $P_1 \in GL_m(O_F)$ from the left and by multiplying some element $Q_1 \in GL_s(O_F)$ from the right, we may assume that the $(1,1)$ entry of $P_1 T Q_1$ is t^{α_1} where $\alpha_1 = v_F(x_{i_1 j_1})$ and other entries in the first column and row vanish. By induction m, we obtain the equality

$$PTQ = \begin{pmatrix} t^{\alpha_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & t^{\alpha_m} & 0 & \cdots & 0 \end{pmatrix}.$$

If there exist $P_1 \in GL_m(O_F), Q_1 \in GL_s(O_F)$ and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$ such that $\alpha_1 \leq \ldots \leq \alpha_m, \beta_1 \leq \ldots \leq \beta_m$ and

$$P_1 \begin{pmatrix} t^{\alpha_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & t^{\alpha_m} & 0 & \cdots & 0 \end{pmatrix} Q_1 = \begin{pmatrix} t^{\beta_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & t^{\beta_m} & 0 & \cdots & 0 \end{pmatrix}.$$

Let us write $P_1 = (p_{ij})$. Then $(p_{ij}t^{\alpha_j - \beta_i}) \in GL_m(O_F)$. Suppose that $\alpha_i < \beta_i$ for some i. Then for $i \leq k \leq m$ and for $1 \leq \ell \leq i$, the (k, l)-entry of P_1 belongs to the maximal ideal of O_F. This implies that P_1 modulo the maximal ideal is not an invertible matrix.

In particular, we have a standard Cartan decomposition

$$\mathrm{GL}_m(F) = \bigcup_{\alpha_1 \leq \cdots \leq \alpha_m} \mathrm{GL}_m(O_F) \begin{pmatrix} t^{\alpha_1} & & \\ & \ddots & \\ & & t^{\alpha_m} \end{pmatrix} \mathrm{GL}_m(O_F).$$

In the cases of $n = 1$, such decompositions are supported by the theory of affine buildings. Geometric foundations for higher dimensional local fields are given in our preprint [Mori1], where the notation of "Babel building" is introduced and studied. For the general linear group $GL_d(F)$ or other reductive groups over F, a generalized version of Iwahori factorization holds. For example, we have

$$GL_d(F) = BW_n(A_{d-1})B$$

and $W_n(A_{d-1})$ bijectively corresponds to $B\backslash GL_d(F)/B$. Here

$$B = \begin{pmatrix} O_F^\times & \cdots & O_F \\ \vdots & \ddots & \vdots \\ t_1 O_F & \cdots & O_F^\times \end{pmatrix},$$

N is the monomial subgroup and $W_n(A_{d-1}) = N/B \cap N$. There exists a natural isomorphism $W_n(A_{d-1}) \simeq W(A_{d-1}) \ltimes \mathbb{Z}^n(A_{d-1})$. Unlike the cases of $n = 0, 1$, the Weyl group $W_n(A_{d-1})$ is not a Coxeter group. The above Cartan decomposition can be interpreted using the theory of Babel buildings as follows:

$$GL_d(F) = GL_d(O_F)\mathbb{Z}^n_{\geq 0}(A^+_{d-1})GL_d(O_F)$$

where A^+_{d-1} is a positive system of A_{d-1}. Note that

$$\mathbb{Z}^n_{\geq 0}(A^+_{d-1}) \simeq W(A_{d-1})\backslash W_n(A_{d-1})/W(A_{d-1}) \simeq GL_d(O_F)\backslash GL_d(F)/GL_d(O_F).$$

Let us go back to measure theory on F^m.

Let $T \in M_{m,s}(F)$ be a matrix of rank m. Then we have

$$PTQ = \begin{pmatrix} t^{\alpha_1} & & & 0 & \ldots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & t^{\alpha_m} & 0 & \ldots & 0 \end{pmatrix} \tag{1.1}$$

for some $P \in GL_m(O_F)$ and $Q \in GL_s(O_F)$.

Definition 1.8. *For any $T \in M_{m,s}(F)$ of rank m, let us choose $P \in GL_m(O_F)$ and $GL_s(O_F)$ as in* (1.1) *and write $v_F(\mathrm{Det}(T)) = \alpha_1 + \cdots + \alpha_m$. We call $\alpha_1, \ldots, \alpha_m$ the singular exponents of T.*

Lemma 1.9. *For any $T \in M_{m,s}(F)$, the followings hold.*

(i) For any $P \in GL_m(F)$, we have $v_F(\mathrm{Det}(PT)) = v_F(\mathrm{Det}(P)) + v_F(\mathrm{Det}(T))$.

(ii) For any $Q \in GL_s(O_F)$, we have $v_F(\mathrm{Det}(TQ)) = v_F(\mathrm{Det}(T))$.

Proof. (i) If P belongs to $GL_m(O_F)$, the claim is trivial. By Lemma 1.7, we assume that $P = \mathrm{diag}(t^{\gamma_1}, \ldots, t^{\gamma_m})$ for some $\gamma_1, \ldots, \gamma_m \in \mathbb{Z}^n$. By Lemma 1.7, there exists a matrix $Q^{'} \in GL_s(O_F)$ such that

$$TQ^{'} = \begin{pmatrix} x_{11} & \cdots & x_{1m} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm} & 0 & \cdots & 0 \end{pmatrix}.$$

Then we have $v_F(\mathrm{Det}(T)) = v_F(\mathrm{Det}(x_{ij}))$ and $v_F(\mathrm{Det}(PT)) = v_F(\mathrm{Det}(t^{\gamma_i} x_{ij}))$. Here the right hand sides of these two equalities are the valuations of the usual determinants of square matrices. Therefore we obtain the result.

(ii) This is trivial.

Lemma 1.10. *Let $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_r$ be elements of $\mathbb{Z}^n$. For any linear epimorphisms $T\colon F^s \longrightarrow F^m$ and $S\colon F^r \longrightarrow F^m$ such that we have*

$$T \begin{pmatrix} t^{\alpha_1} O_F \\ \vdots \\ t^{\alpha_s} O_F \end{pmatrix} = S \begin{pmatrix} t^{\beta_1} O_F \\ \vdots \\ t^{\beta_r} O_F \end{pmatrix},$$

as subsets of F^m, we have

$$||\mathrm{Det}(T\mathrm{diag}(t^{\alpha_1}, \ldots, t^{\alpha_s}))|| = ||\mathrm{Det}(S\mathrm{diag}(t^{\beta_1}, \ldots, t^{\beta_r}))||.$$

Proof. By replacing T and S with $T\mathrm{diag}(t^{\alpha_1}, \ldots, t^{\alpha_s})$ and $S\mathrm{diag}(t^{\beta_1}, \ldots, t^{\beta_r})$ respectively, we assume that $\alpha_1 = \cdots = \alpha_s = \beta_1 = \cdots \beta_r = 0$. As matrices we write $S = (x_{ij})$ and $T = (y_{ij})$. For any $Q \in GL_s(O_F)$, we have

$$Q \begin{pmatrix} O_F \\ \vdots \\ O_F \end{pmatrix} = \begin{pmatrix} O_F \\ \vdots \\ O_F \end{pmatrix}.$$

Hence, by multiplying T and S by elements of $GL_s(O_F)$ and $GL_r(O_F)$ from the right respectively, we may assume that all but one entries are 0 in the first rows of T and S. By assumption the two sets

$$S\begin{pmatrix} O_F \\ \vdots \\ O_F \end{pmatrix} = \left\{ \left. \begin{pmatrix} x_{1j_1}c_{j_1} \\ \sum_{j=1}^r x_{2j}c_j \\ \vdots \\ \sum_{j=1}^r x_{mj}c_j \end{pmatrix} \right| c_1,\ldots,c_r \in O_F \right\}$$

and

$$T\begin{pmatrix} O_F \\ \vdots \\ O_F \end{pmatrix} = \left\{ \left. \begin{pmatrix} y_{1k_1}d_{k_1} \\ \sum_{k=1}^s y_{2k}d_k \\ \vdots \\ \sum_{k=1}^s y_{mk}d_k \end{pmatrix} \right| d_1,\ldots,d_s \in O_F \right\}.$$

are equal. Thus we have $v_F(x_{1j_1}) = v_F(y_{1k_1})$. Let $S^{'}$ and $T^{'}$ be the minor matrices at $(1,j_1)$ and $(1,k_1)$ in S and T respectively. By induction on size, one has $v_F(\mathrm{Det}(S^{'})) = v_F(\mathrm{Det}(T^{'}))$. Therefore we obtain the equality

$$\begin{aligned} v_F(\mathrm{Det}(S)) &= v_F(x_{1j_1}) + v_F(\mathrm{Det}(S^{'})) \\ &= v_F(y_{1k_1}) + v_F(\mathrm{Det}(T^{'})) \\ &= v_F(\mathrm{Det}(T)). \end{aligned}$$

The following proposition implies the existence and uniqueness of the measure on F^m.

Proposition 1.11. *Let us choose $c \in \mathbb{C}_{(n)}$. For any measurable subset*

$$A = T\begin{pmatrix} a_1 + t^{\alpha_1}O_F \\ \vdots \\ a_s + t^{\alpha_s}O_F \end{pmatrix}$$

of F^m, we define $\mu_{F^m}(A)$ to be $c||\mathrm{Det}(T\mathrm{diag}(t^{\alpha_1},\ldots,t^{\alpha_s}))||$. This is well-defined.

Proof. By definition, we have

$$\mathcal{J}(F^m) = \bigcup_{s\geq m} \bigcup_{\substack{T:F^s\to F^m, \\ \mathrm{rank}(T)=m}} T \circ \mathcal{J}(F)^{\oplus s}$$

as sets. For any element $(s,T) \in \{(s,T) \mid s \geq m, T \in M_{m,s}(F)\}$, we define

$$\mu_{F^m} : T \circ \mathcal{J}(F)^{\oplus s} \longrightarrow \mathbb{C}_{(n)}$$

to be

$$T\begin{pmatrix} a_1 + t^{\alpha_1}O_F \\ \vdots \\ a_s + t^{\alpha_s}O_F \end{pmatrix} \longmapsto c||\mathrm{Det}(T\mathrm{diag}(t^{\alpha_1},\ldots,t^{\alpha_s}))||.$$

Then by Lemma 1.10, this is well-defined and these maps can be glued to a map from the whole $\mathcal{J}(F^m)$. Therefore our claim is implied from the case of $m = 1$.

Remark 1.12. *Of course, we cannot claim the uniqueness of a measure unless we fix the parameter system as in the case of $m = 1$.*

We consider a measure

$$\mathcal{J}(F^m) \longrightarrow \mathbb{C}_{(n)}$$

in the case of $c = 1$ in the above proposition. This assumption is necessary for us to use Fubini's theorem.

As in the case of $m = 1$, we denote by $C_c^\infty(F^m)$ the $\mathbb{C}_{(n)}$-module generated by the characteristic functions of the measurable subsets of F^m. We will define integration

$$I_{F^m} : C_c^\infty(F^m) \longrightarrow \mathbb{C}_{(n)}.$$

Theorem 1.13. *Let $T \colon F^s \longrightarrow F^m$ be a linear epimorphism and* $\mathrm{pr} \colon F^m \longrightarrow F^d$ *any projection. Then for any $A \in \mathcal{J}(F)^{\oplus s}$, the image $\mathrm{pr}(T(A))$ is a measurable subset of F^d and for any $u \in \mathrm{pr}(T(A))$, $\mathrm{pr}^{-1}(u) \cap T(A)$ is measurable as a subset of the fiber space F^{m-d}. Moreover, we have an equality*

$$\mu_{F^m}(T(A)) = \mu_{F^d}(\mathrm{pr}(T(A)))\mu_{F^{m-d}}(\mathrm{pr}^{-1}(u) \cap T(A)).$$

Proof. It is clear that $\mathrm{pr}(T(A))$ is a measurable subset of F^d. Let us write

$$A = \begin{pmatrix} a_1 + t^{\alpha_1} O_F \\ \vdots \\ a_s + t^{\alpha_s} O_F \end{pmatrix}.$$

By replacing T with $T\mathrm{diag}(t^{-\alpha_1}, \dots, t^{-\alpha_s})$, we assume that $\alpha_1 = \cdots = \alpha_s = 0$. For any $u \in \mathrm{pr}(T(A))$, there exists $u^{'} \in \mathrm{pr}T(O_F^{\oplus s})$ such that $u = \mathrm{pr}T(a_1, \dots, a_s) + u^{'}$. Then $\mathrm{pr}^{-1}(u) \cap T(A) = T(a_1, \dots, a_s) + \mathrm{pr}^{-1}(u^{'}) \cap T(O_F^{\oplus s})$. Therefore without loss of generality, we assume that $A = O_F^{\oplus s}$. As a matrix we write $T = (x_{ij})$. Then $\mathrm{pr} \circ T = (x_{i_k j})$. Let $\beta_1, \dots, \beta_m$ be singular exponents of T and $\gamma_1, \dots, \gamma_d$ singular exponents of $\mathrm{pr} \circ T$. Let us write $u = (u_1, \dots, u_d)$. Then one has

$$\mathrm{pr}^{-1}(u) \cap T(A) = \left\{ \begin{pmatrix} \sum x_{1j} c_j \\ \vdots \\ \sum x_{mj} c_j \end{pmatrix} \middle| \begin{array}{c} c_1, \dots, c_s \in O_F, \\ u_k = \sum x_{i_k j} c_j, k = 1, \dots, d \end{array} \right\}.$$

Sort the rows of matrix T, we have a new matrix

$$\tilde{T} = \left(\begin{array}{c|c} S & U \\ \hline S^{'} & U^{'} \end{array} \right) = \left(\begin{array}{ccc|ccc} x_{i_1 1} & \cdots & x_{i_1 d} & x_{i_1 d-1} & \cdots & x_{i_1 s} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{i_d 1} & \cdots & x_{i_d d} & x_{i_d d-1} & \cdots & x_{i_d s} \\ \hline x_{i_{d-1} 1} & \cdots & x_{i_{d-1} d} & x_{i_{d-1} d-1} & \cdots & x_{i_{d-1} s} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{i_m 1} & \cdots & x_{i_m d} & x_{i_m d-1} & \cdots & x_{i_m s} \end{array} \right).$$

Here $\{i_{d-1},\ldots,i_m\}$ is the complement of $\{i_1,\ldots,i_d\}$ in $\{1,\ldots,m\}$. By assumption, we can take $P\in GL_d(O_F)$ and $Q\in GL_s(O_F)$ such that

$$P(S|U)Q=\begin{pmatrix} t^{\gamma_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & t^{\gamma_d} & 0 & \cdots & 0 \end{pmatrix}.$$

Swapping the coordinates appropriately, we can identify $\mathrm{pr}^{-1}(u)\cap T(A)$ with the set

$$\left\{ \left(\begin{array}{c|c} S & U \\ \hline S' & U' \end{array}\right)\begin{pmatrix} c_1 \\ \vdots \\ c_d \\ c_{d-1} \\ \vdots \\ c_s \end{pmatrix} \,\middle|\, \begin{array}{c} c_1,\ldots,c_s\in O_F, \\ P\begin{pmatrix} u_1 \\ \vdots \\ u_d\end{pmatrix} = PS\begin{pmatrix} c_1 \\ \vdots \\ c_d\end{pmatrix} + PU\begin{pmatrix} c_{d-1} \\ \vdots \\ c_s\end{pmatrix} \end{array} \right\}.$$

Note that $Q\in GL_s(O_F)$. Put

$$\begin{pmatrix} c_1 \\ \vdots \\ c_s\end{pmatrix} = Q\begin{pmatrix} c_1' \\ \vdots \\ c_s'\end{pmatrix},$$

the above set equals to

$$\left\{ \left(\begin{array}{c|c} S & U \\ \hline S' & U' \end{array}\right) Q\begin{pmatrix} c_1' \\ \vdots \\ c_s'\end{pmatrix} \,\middle|\, \begin{array}{c} c_1',\ldots,c_s'\in O_F, \\ P\begin{pmatrix} u_1 \\ \vdots \\ u_d\end{pmatrix} = \mathrm{diag}(t^{\gamma_1},\ldots,t^{\gamma_d})\begin{pmatrix} c_1' \\ \vdots \\ c_d'\end{pmatrix} \end{array} \right\}.$$

As a subset of $\mathrm{pr}^{-1}(u)\simeq F^{m-d}$, $\mathrm{pr}^{-1}(u)\cap T(A)$ is regarded as

$$\left\{ \left(\begin{array}{c} Y \\ \hline Y' \end{array}\right)\mathrm{diag}(t^{-\gamma_1},\ldots,t^{-\gamma_d})P\begin{pmatrix} u_1 \\ \vdots \\ u_d\end{pmatrix} + Z'\begin{pmatrix} c_{d-1}' \\ \vdots \\ c_s'\end{pmatrix} \,\middle|\, c_{d-1}',\ldots,c_s'\in O_F \right\},$$

where one has

$$\left(\begin{array}{c|c} Y & 0 \\ \hline Y' & Z' \end{array}\right) = \tilde{T}Q.$$

This is measurable in $\mathrm{pr}^{-1}(u)\simeq F^{m-d}$ and its volume is $||\mathrm{Det}(Z')||$. Moreover since

$$\left(\begin{array}{c|c} \mathrm{diag}(t^{\gamma_1},\ldots,t^{\gamma_d}) & 0 \\ \hline 0 & Z' \end{array}\right) = \left(\begin{array}{c|c} P & 0 \\ \hline -Y'Y^{-1} & 1 \end{array}\right)\left(\begin{array}{c|c} Y & 0 \\ \hline Y' & Z' \end{array}\right),$$

by Lemma 1.9, we have $v_F(\mathrm{Det}(Z'))+\gamma_1+\cdots\gamma_d=\beta_1+\cdots+\beta_m$. Thus we obtain the consequence.

Proposition 1.14. *We define $I_F\colon C_c^\infty(F^m) \longrightarrow \mathbb{C}_{(n)}$ to be*

$$I_{F^m}(\sum_{i=1}^k a_i \mathbb{I}_{A_i}) = \sum_{i=1}^k a_i \mu_{F^m}(A_i).$$

Then I_{F^m} is well-defined. In other words, if we have $\sum_i a_{i=1}^k \mathbb{I}_{A_i} = \sum_{j=1}^\ell b_j \mathbb{I}_{B_j}$, then we have

$$\sum_{i=1}^k a_i \mu_{F^m}(A_i) = \sum_{j=1}^\ell b_j \mu_{F^m}(B_j).$$

.

Proof. We apply Theorem 1.13 for $d = 1$. For any $x \in F$, by assumption we have the equality

$$\sum_{i=1}^k a_i \mathbb{I}_{\mathrm{pr}_1^{-1}(x)\cap A_i} = \sum_{j=1}^\ell b_j \mathbb{I}_{\mathrm{pr}_1^{-1}(x)\cap B_j}$$

$C_c^\infty(F^{m-1})$. By induction on m, we have

$$\sum_{i=1}^k a_i \mu_{F^{m-1}}(\mathrm{pr}_1^{-1}(x) \cap A_i) = \sum_{j=1}^\ell b_j \mu_{F^{m-1}}(\mathrm{pr}_1^{-1}(x) \cap B_j).$$

By varying $x \in F$, we regard the both sides as functions of $x \in F$. Integrating with respect to $x \in F$, we have

$$\sum_{i=1}^k a_i \mu_{F^m}(A_i) = \sum_{j=1}^\ell b_j \mu_{F^m}(B_j).$$

For any $f \in C_c^\infty(F^m)$, we denote $I_{F^m}(f)$ by

$$\int_{F^m} f(x) d\mu_{F^m}(x).$$

Corollary 1.15. *For any $f \in C_c^\infty(F^m)$, we have*

$$\int_{F^m} f(z) d\mu_{F^m}(z) = \int_{F^d} \int_{F^{m-d}} f(x,y) d\mu_{F^d}(x) d\mu_{F^{m-d}}(y).$$

Proof. This follows from Theorem 1.13 and Proposition 1.14. By the linearity, we may assume that $f = \mathbb{I}_A$ for some $A \in \mathcal{J}(F^m)$. For any $x \in F^d$, the function

$$f(x,-)\colon F^{m-d} \longrightarrow \mathbb{C}_{(n)}; y \longmapsto f(x,y)$$

is measurable by Theorem 1.13. Here when $x \in F^d$ varies, the function

$$x \longmapsto \int_{F^{m-d}} f(x,y) d\mu_{F^{m-d}}(y)$$

belongs to $C_c^\infty(F^d)$. Integrating with respect to $x \in F^d$, we have

$$\int_{F^m} f(z) d\mu_{F^m}(z) = \int_{F^d} \int_{F^{m-d}} f(x,y) d\mu_{F^d}(x) d\mu_{F^{m-d}}(y)$$

by Proposition 1.14.

By the sequential discussion on m, we obtain the equality

$$\begin{aligned}&\int_{F^m} f(x_1, \dots, x_m) d\mu_{F^m}((x_1, \dots, x_m)) \\ &= \int_F \int_F \cdots \int_F f(x_1, \dots, x_m) d\mu_F(x_1) \cdots d\mu_F(x_m)\end{aligned}$$

for any $f \in C_c^\infty(F^m)$.

This corollary is Fubini's theorem on F^m. In [M3], Morrow showed that for any polynomial over O_F or any linear polynomial over F, its repeated integral is independent of the order of integralation in his theory. On the other hand, in §7 and §8 in [M3], Morrows gave some examples whose repeated integrals depends on the order of integralation. We explained that all repeated integrals of measurable functions are independent on the order of integralation. But we have not verified that non-linear functions does not depend on the order of integralation.

Recall the definition of absolute convergence of a series $\mathbb{C}_{(n)}$ in [F3]. We say that $\sum c_i, c_i \in \mathbb{C}_{(n)}$ absolutely converges if the series $\sum \mathrm{res}_{X_n^j} c_j$ absolutely converges in $\mathbb{C}_{(n-1)}$ for all j.

Proposition 1.16. *Let $\{A_i\}_{i \in \mathbb{N}}$ be a family of countably many disjoint measurable sets of F^m such that $A = \bigcup_{i=1}^\infty A_i \in \mathcal{J}(F^m)$. If $\sum_{i=1}^\infty \mu_{F^m}(A_i)$ is an absolutely convergent series in $\mathbb{C}_{(n)}$ then we have*

$$\mu_{F^m}(A) = \sum_{i=1}^\infty \mu_{F^m}(A_i).$$

Remark 1.17. *In [F3], Fesenko showed that μ_F is countably additive in a refined sense. In [W], Waller showed that $\mu_{GL_2(F)}$ has the same property by using the Fesenko's result. We will prove the above proposition by the same way.*

Proof. Since $\{A_i\}_{i \in \mathbb{N}}$ are piecewise disjoint,

$$\mathbb{I}_A = \sum_{i=1}^\infty \mathbb{I}_{A_i}.$$

So we have

$$\begin{aligned}\mu_{F^m}(A) &= \int_{F^m} \mathbb{I}_A(x) d\mu_{F^m}(x) \\ &= \int_{F^m} \sum_{i=1}^{\infty} \mathbb{I}_{A_i}(x) d\mu_{F^m}(x) \\ &= \int_F \cdots \int_F \sum_{i=1}^{\infty} \mathbb{I}_{A_i}(x_1, \dots, x_m) d\mu_F(x_1) \cdots d\mu_F(x_m).\end{aligned}$$

Because the measure μ_F has countable additivity, this integral equals to

$$\sum_{i=1}^{\infty} \int_F \cdots \int_F \mathbb{I}_{A_i}(x_1, \dots, x_m) d\mu_F(x_1) \cdots d\mu_F(x_m).$$

By definition this is $\sum_{i=1}^{\infty} \mu_{F^m}(A_i)$.

In his approach to higher measure theory given in [M1] and [M2], Morrow did not show that their measure has countable additivity. So this result is an advantage to [M1] and [M2].

Proposition 1.18. *For any $A, B \in \mathcal{J}(F^m)$, the intersection $A \cap B$ is measurable in F^m or empty.*

Proof. Let us consider a linear isomorphism $M\colon F^{2m} \longrightarrow F^{2m}; (x, y) \longmapsto (x, x-y)$ and a projection $\mathrm{pr}_2\colon F^{2m} \longrightarrow F^m; (x, y) \longmapsto y$. Then the intersection $A \cap B$ equals to $\mathrm{pr}_2^{-1}(0) \cap M(A, B)$ as a subset of the fiber space $\mathrm{pr}_2^{-1}(0) \simeq F^m$. Hence the claim follows from Theorem 1.13. Note that if $0 \notin \mathrm{pr}_2 M(A, B)$, $A \cap B$ is empty.

Corollary 1.19. *For any $f_1, f_2 \in C_c^{\infty}(F^m)$, the product function $f_1 f_2; x \longmapsto f_1(x) f_2(x)$ lies in $C_c^{\infty}(F^m)$.*

Proof. Indeed, for any measurable subsets A, B of F^m, the product of $\mathbb{I}_A$ and $\mathbb{I}_B$ is $\mathbb{I}_{A \cap B}$.

Definition 1.20. *We can define a convolution product $C_c^{\infty}(F^m) \times C_c^{\infty}(F^m) \longrightarrow C_c^{\infty}(F^m)$ to be*

$$f_1 * f_2(x) := \int_{F^m} f_1(y) f_2(x-y) d\mu_{F^m}(y).$$

Proposition 1.21. *The convolution algebra $C_c^{\infty}(F^m)$ is commutative, associative and does not have a unit element.*

Proof. Since the measure on F^m is additive-invariant, one has

$$\begin{aligned} f_1 * f_2(x) &= \int_{F^m} f_1(y) f_2(x-y) d\mu_{F^m}(y) \\ &= \int_{F^m} f_1(x-z) f_2(z) d\mu_{F^m}(z) \\ &= f_2 * f_1(x) \end{aligned}$$

for any $f_1, f_2 \in C_c^\infty(F^m)$. Let f_1, f_2, f_3 be elements of $C_c^\infty(F^m)$. Then we have

$$\begin{aligned} (f_1 * f_2) * f_3(x) &= \int_{F^m} f_1 * f_2(y) f_3(x-y) d\mu_{F^m}(y) \\ &= \int_{F^m} \int_{F^m} f_1(z) f_2(y-z) f_3(x-y) d\mu_{F^m}(z) d\mu_{F^m}(y) \\ &= \int_{F^m} \int_{F^m} f_1(z) f_2(y-z) f_3(x-y) d\mu_{F^m}(y) d\mu_{F^m}(z) \\ &= \int_{F^m} f_1(z) f_2 * f_3(x-z) d\mu_{F^m}(z) \\ &= f_1 * (f_2 * f_3)(x). \end{aligned}$$

Suppose that there exists an element $e \in C_c^\infty(F^m)$ such that $e * f = f$ for any $f \in C_c^\infty(F^m)$. Then we have

$$f(x) = \int_{F^m} e(y) f(x-y) d\mu_{F^m}(y).$$

Especially for the characteristic function $f = \mathbb{I}_{t^\alpha O_F^m}$,

$$1 = \int_{t^\alpha O_F^m} e(y) d\mu_{F^m}(y).$$

The right side depends on α, so this is not possible.

2 A measure on $GL_d(F)$

In this section, we will construct a measure and a theory of integration on $GL_d(F)$ by using those on $M_d(F) \simeq F^{d^2}$.

Proposition 2.1. *Let A be a measurable subset of $M_d(F)$ and $g \in GL_d(F)$. Then we have*

$$\mu_{M_d(F)}(gA) = \mu_{M_d(F)}(Ag) = ||\det(g)||^d \mu_{M_d(F)}(A).$$

Proof. Let us regard $M_d(F)$ as a d^2-dimensional vector space. Then the determinant of the multiplication by g from the left or right $M_d(F) \longrightarrow M_d(F)$ is $\det(g)^d$. Hence the proposition is immediately follows from Lemma 1.9 and Proposition 1.11.

Lemma 2.2. *For any element $i \in \mathbb{Z}^n_{>0}$, $K_i = 1+t^i M_d(O_F)$ is a normal subgroup of $GL_d(O_F)$.*

Proof. Indeed we have $K_i = \mathrm{Ker}(GL_d(O_F) \longrightarrow GL_d(O_F/t^iO_F))$.

Lemma 2.3. *For any element g of $GL_d(F)$ and $i \in \mathbb{Z}^n_{>0}$, there exists an element j of $\mathbb{Z}^n_{>0}$ such that we have $g^{-1}K_jg \subset K_i$.*

Proof. By Lemma 1.7, we can take $g_1, g_2 \in GL_d(O_F)$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Z}^n$ such that $g = g_1\mathrm{diag}(t^{\alpha_1}, \dots, t^{\alpha_d})g_2$. By Lemma 2.2, since $g_2K_ig_2^{-1} = K_i$ and $g_1^{-1}K_jg_1 = K_j$, it is sufficient to show the case of $g = \mathrm{diag}(t^{\alpha_1}, \dots, t^{\alpha_d})$. Then an index $j \in \mathbb{Z}^n$ over $\max_{p,q}(\alpha_p - \alpha_q + i)$ satisfies $g^{-1}K_jg \subset K_i$.

Proposition 2.4. *The group $GL_d(F)$ has a structure of a Hausdorff topological group whose fundamental system of neighborhoods of 1 is given by $\{K_i \mid i \in \mathbb{Z}^n_{>0}\}$.*

Proof. First, we will prove that the product $GL_d(F) \times GL_d(F) \longrightarrow GL_d(F)$ is continuous. For any $x, y \in GL_d(F)$ and $k \in \mathbb{Z}^n_{>0}$, we can take $i, j \in \mathbb{Z}^n_{>0}$ such that $i > k$ and $xK_jx^{-1} \subset K_k$ by the above lemma. Since

$$K_ixK_jy = K_i(xK_jx^{-1})xy \subset K_kxy,$$

the product $GL_d(F) \times GL_d(F) \longrightarrow GL_d(F)$ is continuous. Next, let $j^{'}$ be an element of $\mathbb{Z}^n_{>0}$ such that $x^{-1}K_{j'}x \subset K_k$, we have $(K_{j'}x)^{-1} = (x^{-1}K_{j'}x)x^{-1} \subset K_kx^{-1}$. Therefore the inverse involution on the group $GL_d(F)$ is continuous. As the intersection of all K_i is $\{1\}$, the group $GL_d(F)$ becomes a Hausdorff topological group.

Definition 2.5. *Let us consider a measure on $GL_d(F)$. We say that a $\mathbb{C}_{(n)}$-valued function f on $GL_d(F)$ is measurable if the function*

$$M_d(F) \longrightarrow \mathbb{C}_{(n)}; x \longmapsto f(x)||\det(x)||^{-d}$$

(vanishing on $M_d(F)\backslash GL_d(F)$) lies in $C_c^\infty(M_d(F))$. We say that a subset A of $GL_d(F)$ is measurable if its characteristic function is measurable. We define $\mathcal{J}(GL_d(F))$ to be the set of measurable subsets of $GL_d(F)$ and $C_c^\infty(GL_d(F))$ to be the $\mathbb{C}_{(n)}$-module generated by $\{\mathbb{I}_A \mid A \in \mathcal{J}(GL_d(F))\}$.

For any $f \in C_c^\infty(GL_d(F))$, we define $I_{GL_d(F)}(f)$ to be

$$\int_{M_d(F)} f(x)||\det(x)||^{-d} d\mu_{M_d(F)}(x).$$

We also denote this by

$$\int_{GL_d(F)} f(x) d\mu_{GL_d(F)}(x).$$

Proposition 2.6. *The integration*

$$I_{GL_d(F)}\colon C_c^\infty(GL_d(F)) \longrightarrow \mathbb{C}_{(n)}; f \longmapsto \int_{GL_d(F)} f(x) d\mu_{GL_d(F)}(x)$$

is left and right $GL_d(F)$-invariant.

Proof. Let $f \in C_c^\infty(GL_d(F))$. By the linearity, we may assume that the zero extension to $M_d(F)$ of

$$GL_d(F) \longrightarrow \mathbb{C}_{(n)}; f \longmapsto f(x)||\det(x)||^{-d}$$

is the characteristic function of some measurable subset A of $M_d(F)$. By Proposition 2.1, for any element g of $GL_d(F)$, we have

$$d\mu_{M_d(F)}(xg) = d\mu_{M_d(F)}(gx) = ||\det(g)||^d d\mu_{M_d(F)}(x).$$

Hence we have

$$\begin{aligned}\int_{M_d(F)} f(xg)||\det(x)||^{-d} d\mu_{M_d(F)}(x) &= \int_{M_d(F)} f(xg)||\det(xg)||^{-d}||\det(g)||^d d\mu_{M_d(F)}(x)\\ &= \int_{M_d(F)} f(xg)||\det(xg)||^{-d} d\mu_{M_d(F)}(xg)\\ &= \mu_{M_d(F)}(A)\end{aligned}$$

and

$$\begin{aligned}\int_{M_d(F)} f(gx)||\det(x)||^{-d} d\mu_{M_d(F)}(x) &= \int_{M_d(F)} f(gx)||\det(gx)||^{-d}||\det(g)||^d d\mu_{M_d(F)}(x)\\ &= \int_{M_d(F)} f(gx)||\det(gx)||^{-d} d\mu_{M_d(F)}(gx)\\ &= \mu_{M_d(F)}(A).\end{aligned}$$

Thus

$$C_c^\infty(GL_d(F)) \longrightarrow \mathbb{C}_{(n)}; f \longmapsto \int_{M_d(F)} f(x)||\det(x)||^{-d} d\mu_{M_d(F)}(x)$$

is left and right-invariant.

Remark 2.7. *For any element A of $\mathcal{J}(GL_d(F))$, a set $\{v_F(\det(x)) \mid x \in A\}$ is finite set. Indeed, by definition, we can take $a_1, \ldots, a_r \in \mathbb{C}_{(n)}$ and $A_1, \ldots, A_r \in \mathcal{J}(M_d(F))$ such that*

$$||\det(x)||^{-d} = \sum_{i=1}^{r} a_i \mathbb{I}_{A_i}(x)\,(x \in A).$$

Then the right hand side takes at most a finite number of values. In particular, since there is no finite subgroup of $\mathbb{Z}^n$ other than 0, we have $v_F(\det(K)) = 0$ for any measurable subgroup K of $GL_d(F)$.

Remark 2.8. *The uniqueness up to a constant multiple of the left-right-invariant integration*

$$I_{GL_d(F)} \colon C_c^\infty(GL_d(F)) \longrightarrow \mathbb{C}_{(n)}$$

follows from that of the integration on $M_d(F)$.

The family of measurable subgroups is much larger than the family of measurable subgroups which are considered in [W]. For example, subgroup

$$\begin{pmatrix} 1+t^\gamma O_F & t^{\alpha-\beta+\gamma} O_F \\ t^{-\alpha+\beta+\gamma} O_F & 1+t^\gamma O_F \end{pmatrix} = \begin{pmatrix} t^\alpha & \\ & t^\beta \end{pmatrix} K_\gamma \begin{pmatrix} t^{-\alpha} & \\ & t^{-\beta} \end{pmatrix}$$

is not measurable in the sense of [W], however this subgroup has a measure in this theory.

Since we define the measure $\mu_{GL_d(F)}$ by using the measure on F^{d^2}, we can prove that $\mu_{GL_d(F)}$ is countably additive by the same argument as Proposition 1.16. This is a generalization of [W, Corollary 5.8].

Proposition 2.9. *Let $\{A_i\}_{i\in\mathbb{N}}$ be a family of countably many disjoint measurable sets of $GL_d(F)$ such that $A = \bigcup_{i=1}^\infty A_i \in \mathcal{J}(GL_d(F))$. If $\sum_{i=1}^\infty \mu_{GL_d(F)}(A_i)$ is an absolutely convergent series in $\mathbb{C}_{(n)}$ then we have*

$$\mu_{GL_d(F)}(A) = \sum_{i=1}^\infty \mu_{GL_d(F)}(A_i)$$

Proof. See the proof of Proposition 1.16.

Let $d = d_1 + \cdots + d_r$ be a partition of d. Then the restriction to diagonal blocks matrices

$$M_d(F) \longrightarrow \prod_{i=1}^r M_{d_i}(F)$$

is a linear epimorphism. The restriction of a measurable subset of $GL_d(F)$ to $\prod_{i=1}^r GL_{d_i}(F)$ is measurable in $\prod_{i=1}^r GL_{d_i}(F)$. So we have natural maps

$$\mathcal{J}(GL_d(F)) \longrightarrow \mathcal{J}\left(\prod_{i=1}^r GL_{d_i}(F)\right)$$

and

$$C_c^\infty(GL_d(F)) \longrightarrow C_c^\infty\left(\prod_{i=1}^r GL_{d_i}(F)\right).$$

Here $\mathcal{J}\left(\prod_{i=1}^r GL_{d_i}(F)\right)$ is the set of measurable subsets of $\prod_{i=1}^r GL_{d_i}(F)$ and $C_c^\infty\left(\prod_{i=1}^r GL_{d_i}(F)\right)$ is the $\mathbb{C}_{(n)}$-module generated by the characteristic functions of the measurable subsets of $\prod_{i=1}^r GL_{d_i}(F)$. When F is a one-dimensional local field, it is well-known that we have an isomorphism

$$C_c^\infty\left(\prod_{i=1}^r GL_{d_i}(F)\right) \simeq \bigotimes_{i=1}^r C_c^\infty(GL_{d_i}(F)).$$

However when $n > 1$, we probably do not have such an isomorphism. We will consider the repeated integral on the product space $\prod_{i=1}^r GL_{d_i}(F)$.

Theorem 2.10. *For any measurable function $f \in C_c^\infty\left(\prod_{i=1}^r GL_{d_i}(F)\right)$, $s = 1, \ldots, r$ and $x_j \in GL_{d_j}(F)$ $(j \neq s)$, the function*

$$GL_{d_s}(F) \longrightarrow \mathbb{C}_{(n)}; x_s \longmapsto f(x_1, \ldots, x_s, \ldots, x_r)$$

is an element of $C_c^\infty(GL_{d_s}(F))$. Moreover, the integration

$$\int_{GL_{d_1}(F)} \cdots \int_{GL_{d_r}(F)} f(x_1, \ldots, x_r) d\mu_{GL_{d_1}(F)}(x_1) \cdots d\mu_{GL_{d_r}(F)}(x_r)$$

on $\prod_{i=1}^r GL_{d_i}(F)$ does not depend on the order of integration.

Proof. By linearity, we may assume $f = \mathbb{I}_A$ for some $A \in \mathcal{J}\left(\prod_{i=1}^r GL_{d_i}(F)\right)$. The projection $\mathrm{pr}_s \colon \prod_{i=1}^r GL_{d_i}(F) \longrightarrow GL_{d_s}(F)$ induces

$$\mathrm{pr}_{s,*} \colon \mathcal{J}\left(\prod_{i=1}^r GL_{d_i}(F)\right) \longrightarrow \mathcal{J}(GL_{d_s}(F)).$$

By Corollary 1.15,

$$GL_{d_s}(F) \longrightarrow \mathbb{C}_{(n)}; x_s \longmapsto f(x_1, \ldots, x_s, \ldots, x_r)$$

lies in $C_c^\infty(GL_{d_s}(F))$. Moreover, for any $a \in \mathrm{pr}_s(A)$, $\mathrm{pr}_s^{-1}(a) \cap A$ is measurable in $\prod_{i \neq s} GL_{d_i}(F)$ and we have

$$\mu_{\prod_{i=1}^r GL_{d_i}(F)}(A) = \mu_{GL_{d_s}(F)}(\mathrm{pr}_s(A)) \cdot \mu_{\prod_{i \neq s} GL_{d_i}(A)}(\mathrm{pr}_s^{-1}(a) \cap A).$$

Therefore

$$\int_{GL_{d_1}(F)} \cdots \int_{GL_{d_r}(F)} f(x_1, \ldots, x_r) d\mu_{GL_{d_1}(F)}(x_1) \cdots d\mu_{GL_{d_r}(F)}(x_r)$$

on $\prod_{i=1}^r GL_{d_i}(F)$ does not depend on the order of integration.

This measure is, of course, also countably additive. By the above proposition, for any $s = 1, \ldots, r$ and $f \in C_c^\infty(\prod_{i=1}^r GL_{d_i}(F))$, we have

$$I_{GL_{d_s}(F)} \colon C_c^\infty\left(\prod_{i=1}^r GL_{d_i}(F)\right) \longrightarrow C_c^\infty\left(\prod_{i \neq s} GL_{d_i}(F)\right).$$

For any substitution $\sigma \in S_r$,

$$I_{GL_{d_{\sigma(1)}}(F)} \circ \cdots \circ I_{GL_{d_{\sigma(r)}}(F)} \colon C_c^\infty(\prod_{i=1}^r GL_{d_i}(F)) \longrightarrow \mathbb{C}_{(n)}$$

does not depend on σ.

Example 2.11. *For the punctured unit disc $D = \{g \in M_d(O_F) \mid \det(g) \neq 0\}$ and a positive integer $s \in \mathbb{Z}_{>0}$, we consider the integral*

$$I(n) = \int_D ||\det(g)||^s d\mu_{GL_d(F)}(g).$$

For any $\alpha \in \mathbb{Z}^n_{\geq 0}$, we write $D_\alpha = \{g \in D \mid v_F(\det(g)) = \alpha\}$. Since we have

$$D = \coprod_{\alpha \in \mathbb{Z}^n_{\geq 0}} D_\alpha, \int_{D_\alpha} ||\det(g)||^s dg = X^{\alpha s},$$

$$I(n) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} X^{\alpha s} = \sum_{(\alpha_n, \ldots, \alpha_1) \geq 0} (\frac{1}{q})^{\alpha_1 s} X_2^{\alpha_2 s} \cdots X_n^{\alpha_n s}$$

When $n = d = 2$, this is an example considered in [W].

3 A measure on a parabolic subgroup

Let $d = d_1 + \cdots + d_r$ be a partition of d. let $M_d^{(d_1,\ldots,d_r)}(F)$ denote the block upper matrix algebra associated to the partition $d = d_1 + \cdots + d_r$. Then $P(F) = GL_d(F) \cap M_d^{(d_1,\ldots,d_r)}(F)$ is a standard parabolic subgroup of $GL_d(F)$. Let $P = MN$ be a standard Levi decomposition. Then the Levi subgroup $M(F)$ is isomorphic to $GL_{d_1}(F) \times \cdots \times GL_{d_r}(F)$ and N is the unipotent radical of P. In particular, so note that for any $n \in N$, we have $\det(n) = 1$.

Definition 3.1. *We say a subset of $P(F)$ is measurable if it is an intersection of a measurable subset of $GL_d(F)$ and $P(F)$. We denote by $\mathcal{J}(P(F))$ the set of measurable subsets of $P(F)$.*

We define $C_c^\infty(P(F))$ to be the $\mathbb{C}_{(n)}$-module generated by $\{\mathbb{I}_A \mid A \in \mathcal{J}(P(F))\}$. We will construct a left $P(F)$-invariant $\mathbb{C}_{(n)}$-linear map

$$I_{P(F)} = I^l_{P(F)} \colon C_c^\infty(P(F)) \longrightarrow \mathbb{C}_{(n)}.$$

Since the multiplication $g = (g_1, \ldots, g_r) \in M(F) \simeq \prod GL_{d_i}(F)$ by the left on $M_d^{(d_1,\ldots,d_r)}(F)$ is

$$\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \times \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ & \ddots & \vdots \\ & & a_{rr} \end{pmatrix} = \begin{pmatrix} g_1 a_{11} & \cdots & g_1 a_{1r} \\ & \ddots & \vdots \\ & & g_r a_{rr} \end{pmatrix},$$

for A is a measurable subset of $P(F)$, we have

$$\mu_{M_d^{(d_1,\ldots,d_r)}(F)}(gA) = \prod_{i=1}^r ||\det(g_i)||^{d - \sum_{j<i} d_j} \cdot \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(A).$$

Let us put $D_l(g) = \prod_{i=1}^{r} ||\det(g_i)||^{d-\sum_{j<i} d_j}$ for any $g = (g_1, \ldots, g_r) \in M(F)$. Then we define $I_{P(F)} \colon C_c^\infty(P(F)) \longrightarrow \mathbb{C}_{(n)}$ to be

$$I_{P(F)}(f) := \int_{M_d^{(d_1,\ldots,d_r)}(F)} f(xy) D_l(x)^{-1} \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(xy),$$

where $x = (x_1, \ldots, x_r) \in M(F)$ and $y \in N(F)$. We will denote $I_{P(F)}(f)$ by

$$\int_{P(F)} f(z) d\mu_{P(F)}(z).$$

Proposition 3.2. *The morphism $I_{P(F)} \colon C_c^\infty(P(F)) \longrightarrow \mathbb{C}_{(n)}$ is left $P(F)$-invariant.*

Proof. Let $m \in M(F)$ and $n \in N(F)$. Let $\lambda_g f$ denote the function $x \longmapsto f(g^{-1}x)$. Since $(mn)^{-1} = m^{-1}(mn^{-1}m^{-1})$, we have

$$\begin{aligned} I_{P(F)}(\lambda_{mn} f) &= \int_{M_d^{(d_1,\ldots,d_r)}(F)} f(m^{-1}(mn^{-1}m^{-1})xy) D(x)^{-1} \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(xy) \\ &= \int_{M_d^{(d_1,\ldots,d_r)}(F)} f((m^{-1}x)(x^{-1}mn^{-1}m^{-1}x)y) D(x)^{-1} \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(xy). \end{aligned}$$

We put $x_1 = m^{-1}x \in M(F), y_1 = (x^{-1}mn^{-1}m^{-1}x)y = x_1^{-1}n^{-1}x_1 y \in N(F)$, then $(mn)^{-1}xy = x_1 y_1$. Since

$$\begin{aligned} \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(xy) &= \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(mnx_1y_1) \\ &= D(m) \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(x_1 y_1), \end{aligned}$$

We have an equality

$$\begin{aligned} I_{P(F)}(\lambda_{mn} f) &= \int_{M_d^{(d_1,\ldots,d_r)}(F)} f(x_1 y_1) D(x)^{-1} D(m) \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(x_1 y_1) \\ &= \int_{M_d^{(d_1,\ldots,d_r)}(F)} f(x_1 y_1) D(x_1)^{-1} \mu_{M_d^{(d_1,\ldots,d_r)}(F)}(x_1 y_1) \\ &= I_{P(F)}(f). \end{aligned}$$

As a consequence, $I_{P(F)} \colon C_c^\infty(P(F)) \longrightarrow \mathbb{C}_{(n)}$ is left $P(F)$-invariant.

Remark 3.3. *Let us put $D_r(g) = \prod_{i=1}^{r} ||\det(g_i)||^{d-\sum_{j>i} d_j}$ for any $g = (g_1, \ldots, g_r) \in M(F)$. Then we can prove that for any $f \in C_c^\infty(P(F))$,*

$$I^r_{P(F)}(f) := \int_{M_d^{(d_1,\ldots,d_r)}(F)} f(xy) D_r(x)^{-1} d\mu_{M_d^{(d_1,\ldots,d_r)}(F)}(xy)$$

is right $P(F)$-invariant as well as the above proposition.

Remark 3.4. *We defined the left invariant measure and the right invariant measure of $P(F)$ by using the measure on the vector space $M_d^{(d_1,\dots,d_r)}(F)$. Hence we can prove countable additivity of the left invariant measure and right invariant measure of $P(F)$.*

Definition 3.5. *We define a homomorphism $\delta_P \colon P(F) \longrightarrow \mathbb{C}_{(n)}^\times$ to be*

$$\delta_P(xy) := D_r(x)D_l(x)^{-1}$$

for any $x \in M(F)$ and $y \in N(F)$. We call the homomorphism δ_P a modular character of $P(F)$.

Proposition 3.6. *Under the above notations, we have*

$$d\mu_{P(F)}(zp) = \delta_P(p)d\mu_{P(F)}(z).$$

Proof. By definition, we have

$$\int_{P(F)} f(z)d\mu_{P(F)}(z) = \delta_P(p)\int_{P(F)} f(zp)d\mu_{P(F)}(z)$$

for any $f \in C_c^\infty(P(F))$. Hence we obtain the claim.

Example 3.7. *Let $f = \mathbb{I}_{K_\gamma} \in C_c^\infty(GL_2(F))$ be the characteristic function of measurable subgroup*

$$K_\gamma = 1 + t^\gamma M_2(O_F)$$

and P is the standard Borel subgroup of GL_2, that is, P is the upper triangular subgroup. Then since

$$P(F) \cap K_\gamma = \begin{pmatrix} 1+t^\gamma O_F & t^\gamma O_F \\ & 1+t^\gamma O_F \end{pmatrix},$$

we have

$$\int_{P(F)} \mathbb{I}_{P(F)\cap K_\gamma}(z)d\mu_{P(F)}(z) = X^{3\gamma}.$$

Further let us put

$$x = \begin{pmatrix} t^\alpha & \\ & t^\beta \end{pmatrix}.$$

Then one has

$$xK_\gamma x^{-1} = \begin{pmatrix} 1+t^\gamma O_F & t^{\alpha-\beta+\gamma}O_F \\ t^{-\alpha+\beta+\gamma}O_F & 1+t^\gamma O_F \end{pmatrix}, P(F)\cap xK_\gamma x^{-1} = \begin{pmatrix} 1+t^\gamma O_F & t^{\alpha-\beta+\gamma}O_F \\ & 1+t^\gamma O_F \end{pmatrix}.$$

So we have

$$\int_{P(F)} \mathbb{I}_{P(F)\cap K_\gamma}(x^{-1}zx)d\mu_{P(F)}(z) = X^{\alpha-\beta+3\gamma}.$$

So we obtain $\delta_P(x) = X^{\alpha-\beta}$.

4 Future works

Convolution products

For any measurable sets A, B, we could not show whether $AB = \{ab \mid a \in A, b \in B\}, A^{-1} = \{a^{-1} \mid a \in A\}$ are measurable. So we still cannot define the convolution product

$$C_c^\infty(GL_d(F)) \times C_c^\infty(GL_d(F)) \longrightarrow C_c^\infty(GL_d(F)).$$

If we can construct this product, we consider a higher dimensional generalization of Hecke algebra $\mathcal{H}(GL_d(F))$. In the cases where F is a one-dimensional local field, $\mathcal{H}(GL_d(F))$ has strong ties to representation theory of $GL_d(F)$. On the other hand, for an n-dimensional local field F, representations of $GL_d(F)$ are not yet well understood. In the future we will address this theme. For a two-dimensional local field F, $\mathcal{H}(SL_2(F))$ was treated in [KL]. In their study, Kim and Lee defined a convolution product of the spherical Hecke algebra of $\mathcal{H}(SL_2(F), SL_2(O_F))$ by a combinatorial way. The main claim of [KL] is the Satake isomorphism. They constructed the isomorphism

$$\mathcal{S} \colon \mathcal{H}(SL_2(F), SL_2(O_F)) \longrightarrow \mathcal{H}(T(F), T(O_F))^{W(A_1)},$$

where $T(F) \simeq F^\times$ is the diagonal torus of $SL_2(F)$. We hope to find a more natural definition of a convolution product than that given in [KL] in a combinatorial way and construct the Satake isomorphism over $GL_d(F)$.

Quotient measures on the flag varieties

Let $G = GL_d(F)$ and P be a standard parabolic subgroup of G. The Iwasawa decomposition

$$G = KP$$

where $K = GL_d(O_F)$ is proved in my preprint [Mori1]. Now we would like to construct a quotient measure on the flag varieties G/P. The natural method is to use a measure on K. So we show that for any $f \in C_c^\infty(G)$,

$$\int_G f(g) d\mu_G(g) = \int_K \int_P f(pk) d\mu_P(p) d\mu_K(k).$$

The issue of this approach is that we do not know if $P \longrightarrow \mathbb{C}_{(n)}; p \longmapsto f(pk)$ is measurable on P when $k \in K$ is fixed.

These problems are caused by our measure theory on higher dimensional local fields being exotic. Unlike measures on locally compact groups, our measure does not have much flexibility in treating. In the future, a new measure theory should be constructed in which a broader kind of sets are allowed as measurable sets.

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