Studies on the Sarkisov program and minimal model theory

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Contents

CHAPTER 1

Introduction

In this thesis, the author provides three results related to minimal model theory. Throughout this thesis, we will work over an arbitrary algebraically closed field k of any characteristic unless otherwise mentioned. In this thesis, a *scheme* means a separated scheme of finite type over k. We call such a scheme a *variety* if it is reduced and irreducible.

The aim of minimal model theory is to establish the following statement.

CONJECTURE 1.0.1. *Let* (Z, B) *be a* $\mathbb{Q}$*-factorial projective log canonical pair. Then there is a birational map* $f : Z \dashrightarrow X$ *such that either*

(i) (Minimal model) $K_X + B_X = f_*(K_Z + B)$ *is nef, or*
(ii) (Mori fiber space) *for a suitable morphism* $\phi : X \to S$, $-(K_X + B_X)$ *is* ϕ*-ample,* $\rho(X/S) = 1$ *and* $\dim X > \dim S$.

Conjecture 1.0.1 was proved in [**3**] when (Z, B) is a kawamata log terminal pair such that $K_Z + B$ is pseudo-effective with either B or $K_Z + B$ big, and that $K_Z + B$ is not pseudo-effective.

We are interested in the structure of birational map between above models. A birational map between two minimal models is isomorphic in codimension one (for example, minimal models for surfaces are uniquely determined). Moreover, two minimal models of any kawamata log terminal pair with any big boundary divisor are connected by finitely many flops ([**3**, Corollary 1.1.3]). On the other hand, Kawamata [**24**] proved that a birational map between two terminal pairs, whose log canonical divisors, that is, the sum of the canonical divisor and a boundary divisor, are nef, is the composition of a sequence of flops.

In contrast, Sarkisov [**41**] introduced a birational map between 3-fold Mori fiber spaces called an *elementary link* and announced a proof that every birational map between 3-fold Mori fiber spaces is connected by finitely many elementary links, which is called the Sarkirov program. Reid reviewed the Sarkisov program and outlined some key ideas involved in the proof in [**40**]. Unfortunately, the author could not get their original papers and he found these information in [**7**]. Eventually, the Sarkisov program for 3-folds with $\mathbb{Q}$-factorial terminal singularities was completed by Corti [**7**]. Matsuki established the toric Sarkisov program for 3-folds with $\mathbb{Q}$-factorial terminal singularities ([**33**, Chapter 14] and [**44**]). This proof is based on the original idea by Sarkisov (cf. [**7**]). On the other hand, Hacon–M^{c}Kernan [**21**] proved the Sarkisov program for n-folds with $\mathbb{Q}$-factorial terminal singularities by using a brand new method. Roughly speaking, in the original proof, we keep track of three invariants, called the Sarkisov degree, associated with the singularities and we need to prove that the Sarkisov degree satisfies the ascending chain condition. This proof heavily depends on a detailed study of the singularities. On the other hand, Hacon–M^{c}Kernan's approach is quite different from this. Hacon–M^{c}Kernan used "the geography of (log) models" established in [**3**] and [**6**], instead of the Sarkisov degree, and proved the following slightly weak Sarkisov program, called the log Sarkisov program, for kawamata log terminal pairs of any dimensional.

Theorem 1.0.2 (Log Sarkisov program for kawamata log terminal pairs, cf. [**21**, Theorem 1.4]). *Let (Z, B) be a projective $\mathbb{Q}$-factorial kawamata log terminal pair. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the $(K_X + B)$-MMP. Then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many elementary links.*

As a result, by combining Theorem 1.0.2 and the following easy result, Hacon–M^{c}Kernan completed the Sarkisov program for n-folds with $\mathbb{Q}$-factorial terminal singularities.

Proposition 1.0.3 ([**4**]). *Let $\phi : X \to S$ and $\psi : X \to Y$ be two Mori fiber spaces with $\mathbb{Q}$-factorial terminal singularities. If X and Y are birational, then there is a smooth projective variety Z such that ϕ and ψ are outputs of the K_Z-MMP.*

The log Sarkisov program was first introduced by Bruno–Matsuki [**4**], and they called the above relation between two Mori fiber spaces, that is, there is a smooth projective variety Z such that these are outputs of the K_Z-MMP, the Sarkisov relation. Unfortunately, the log Sarkisov program for log canonical pairs is still open.

Conjecture 1.0.4 (Log Sarkisov program). *Let (Z, B) be a projective log canonical pair. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the $(K_X + B)$-MMP. Then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many elementary links.*

Conjecture 1.0.4 for Z (not necessarily $\mathbb{Q}$-factorial) surface was proved in [**34**].

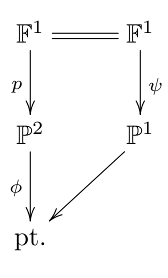
The first purpose of the present thesis is to establish the generalization of Conjecture 1.0.4 for toric varieties as follows. We remark that this result was established in [**35**].

Theorem 1.0.5 ([**35**, Theorem 1.2]). *Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be an $\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP. Then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many Sarkisov links.*

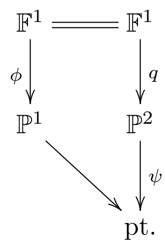
We note that D_Z is not the log canonical divisor but arbitrary (not necessarily effective) $\mathbb{R}$-divisor and there are no assumptions about singularities except $\mathbb{Q}$-factoriality.

Elementary links in [**41**] are known as Sarkisov links. The Sarkisov links for smooth surfaces are very simple as follows:

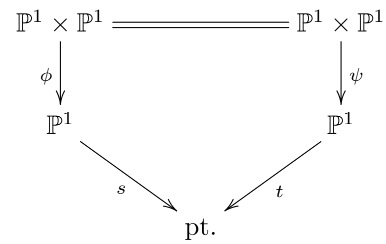
Type (I)



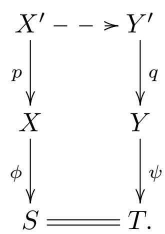
Type (III)



Type (IV)



Type (II)



In a link of Type (I) (resp. (III)), $p : \mathbb{F}^1 \to \mathbb{P}^2$ (resp. $q : \mathbb{F}^1 \to \mathbb{P}^2$) is a blow-up of a smooth point of $\mathbb{P}^2$. A link of Type (II) is an elementary transformation of $\mathbb{P}^1$-bundles over smooth curves. In a link of Type (IV), ϕ and ψ are projections to $\mathbb{P}^1$. We remark that Sarkisov links for 4 or higher-dimensional Mori fiber spaces are slightly different from the above links. More precisely, links of Type (VI) are separated into two types (see Definition 2.0.1).

The second one is to give a new characterization of projective spaces. Now there are various characterizations of of projective spaces. Using Kodaira vanishing theorem, Kobayashi-Ochiai established the characterization of projective spaces from the viewpoint of complete Ricci-flat Kähler metrics as follows.

Theorem 1.0.6 ([**27**, Theorem 1.1]). *Let X be an n-dimensional complex projective manifold and let H be an ample Cartier divisor. Assume that $-K_X \equiv (n+1)H$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.*

Fujita generalized Kobayashi-Ochiai's characterization by using his own theory, that is, the theory of Δ-genus (see Section 3.2 for definition of Δ-genus).

Theorem 1.0.7 ([**19**, Theorem 1]). *Let X be an n-dimensional complex normal projective variety with rational Gorenstein singularities and let H be an ample Cartier divisor. Assume that $K_X + nH$ is not nef. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.*

We establish a new characterization of projective spaces from the viewpoint of minimal model theory by using the theory of quasi-log schemes (see Section 3.1 for definition of quasi-log schemes) and the theory of Δ-genus. We remark that this result was established in [**15**].

Theorem 1.0.8 ([**15**, Theorem 1.4]). *Let $[X, \omega]$ be a projective quasi-log canonical pair where X is defined over $\mathbb{C}$ and connected. Assume that ω is not nef and that $\omega \equiv rD$ for some Cartier divisor D on X with $r > n = \dim X$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$. Moreover, there are no qlc centers of $[X, \omega]$.*

For the reader's convenience, we give Theorem 1.0.8 for log canonical pairs.

THEOREM 1.0.9. *Let (X, B) be a projective log canonical pair where X is defined over $\mathbb{C}$. Assume that $K_X + B$ is not nef and that $-(K_X + B) \equiv rD$ for some Cartier divisor D on X with $r > n = \dim X$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. Moreover, there are no lc centers of (X, B), that is, (X, B) is kawamata log terminal.*

This theorem directly follows from Theorem 1.0.8 because arbitary log canonical pair (X, B) has a natural quasi-log canonical structure $[X, K_X+B]$ (cf. Example 3.1.3). This is joint work with his supervisor Osamu Fujino.

The third one is to give the generalization of the following ampleness criterion, known as the Nakai–Moishezon criterion.

THEOREM 1.0.10 ([**22**, Theorem 5.1]). *Let X be a complete scheme and let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subscheme $Z \subset X$.*

Theorem 1.0.10 was proved for smooth surfaces by Nakai [**37**] and for smooth varieties of any dimensional by Moishezon [**36**]. After that, Kleiman clarified these proofs and gave the simple proof of Theorem 1.0.10 in [**26**]. Theorem 1.0.10 was generalized to $\mathbb{R}$-line bundles on projective schemes by Campana–Peternell [**5**] by using Kleiman's method.

Our purpose is to relax the projectivity in Campana–Peternell's result as follows. We remark that this result was established in [**16**].

THEOREM 1.0.11 (Nakai–Moishezon criterion for real line bundles on complete schemes, [**16**, Theorem 1.3]). *Let X be a complete scheme and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subscheme $Z \subset X$.*

Unfortunately, Campana–Peternell's arguments do not work for complete nonprojective schemes because they need an ample line bundle, equivalently, the projectivity of schemes. Moreover, Kleiman's ampleness criterion does not always hold for complete nonprojective schemes (see [**26**, Section 3] and [**13**, Example 12.1]). We can explicitly construct a complete nonprojective toric threefold X and a line bundle $\mathcal{L}$ on X such that $\mathcal{L}$ is positive on $\overline{\mathrm{NE}}(X)\setminus\{0\}$, where $\overline{\mathrm{NE}}(X)$ is the Kleiman–Mori cone of X (see [**26**, Section 3]). Thus, we need some new idea to generalize to $\mathbb{R}$-line bundles on complete schemes.

For the proof of Theorem 1.0.11, we use the characterization of the augmented base loci for $\mathbb{R}$-line bundles on projective schemes established by Birkar [**2**]. Hence, our approach is quite different from those of [**26**] and [**5**]. Although we can not directly apply geometric arguments to $\mathbb{R}$-line bundles, we can generalize Theorem 1.0.11 for proper morphisms.

THEOREM 1.0.12 (Relative Nakai–Moishezon criterion for real line bundles on complete algebraic spaces, [**16**, Theorem 1.5]). *Let $\pi : X \to S$ be a proper morphism between schemes and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Then $\mathcal{L}$ is π-ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subscheme $Z \subset X$ such that $\pi(Z)$ is a point.*

In Section 4.5, we generalize Theorem 1.0.11 to algebraic spaces. It plays a crucial role in Kollár's projectivity criterion for moduli spaces. For details, see [**13**] and [**28**]. This is also joint work with his supervisor Osamu Fujino.

The contents of this thesis mainly combine results obtained in [**35**], [**15**] and [**16**].

1.1. Preliminaries

In this section, we quickly review the basic definition and results of minimal model theory.

1.1.1. Fundamentals of algebraic geometry. Let us start with the definition of $\mathbb{R}$-Weil divisors, $\mathbb{R}$-Cartier divisors and $\mathbb{R}$-line bundles.

Definition 1.1.1 ($\mathbb{R}$-Weil divisors). Let X be a complete scheme. An $\mathbb{R}$-*Weil divisor* (resp. $\mathbb{Q}$-*Weil divisor*) D is an element of $\mathrm{WDiv}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ (resp. $\mathrm{WDiv}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$), where $\mathrm{WDiv}(X)$ denotes the group of Weil divisors on X. For simplicity, $\mathbb{R}$-Weil divisors are usually called by $\mathbb{R}$-divisors.

Definition 1.1.2 ($\mathbb{R}$-Cartier divisors). Let X be a complete scheme. An $\mathbb{R}$-*Cartier divisor* (resp. *a* $\mathbb{Q}$-*Cartier divisor*) D is an element of $\mathrm{CDiv}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ (resp. $\mathrm{CDiv}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$), where $\mathrm{CDiv}(X)$ denotes the group of Cartier divisors on X.

Definition 1.1.3 ($\mathbb{R}$-line bundles). Let X be a complete scheme. An $\mathbb{R}$-*line bundle* (resp. a $\mathbb{Q}$-*line bundle*) $\mathcal{L}$ is an element of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ (resp. $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$), where $\mathrm{Pic}(X)$ is the Picard group of X.

Remark 1.1.4. In minimal model theory, we usually confuse $\mathbb{R}$-Cartier divisors with $\mathbb{R}$-line bundles. This is because the following natural homomorphism

$$\psi : \mathrm{CDiv}(X)\otimes_{\mathbb{Z}}\mathbb{R} \to \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$$

is surjective if a complete scheme X is reduced.

On the other hand, in Chapter 4, we discuss the ampleness for $\mathbb{R}$-line bundles on a (not necessarily reduced) complete scheme. Thus, we carefully define the nefness, ampleness, and semi-ampleness for $\mathbb{R}$-line bundles as follows.

For simplicity of notation, we write the group law of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ additively.

Definition 1.1.5. Let $\pi : X \to U$ be a proper morphism between complete schemes.

An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is *nef over* U (or π-*nef*) if $\mathcal{L}\cdot C > 0$ for any curve C contained in a fiber of π.

An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is *ample over* U (or π-*ample*) if we can write $\mathcal{L} = \sum_i l_i\mathcal{L}_i$ such that l_i is a positive real number and $\mathcal{L}_i$ is an ample line bundle on X over U for every i.

An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is *semi-ample over* U (or π-*semi-ample*) if we can write $\mathcal{L} = \sum_i l_i\mathcal{L}_i$ such that l_i is a positive real number and $\mathcal{L}_i$ is a semi-ample line bundle on X over U for every i, that is, the natural homomorphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective.

Definition 1.1.6. Let $\pi : X \to U$ be a proper morphism between complete schemes. Let ψ be a natural homomorphism as in Remark 1.1.4. We note that ψ is not necessarily surjective. Let D be an $\mathbb{R}$-Cartier divisor on X. If the image of D by ψ is π-nef, π-ample, and π-semi-ample, then D is said to be π-*nef*, π-*ample*, and π-*semi-ample*, respectively.

Definition 1.1.7. Let $\pi : X \to U$ be a proper morphism between normal quasi-projective varieties.

Two $\mathbb{R}$-divisors D and D' on X are $\mathbb{R}$-*linearly equivalent over* U (denoted by $D \sim_{\mathbb{R},U} D'$) if there is an $\mathbb{R}$-Cartier divisor B on U such that $D - D' \sim_{\mathbb{R}} \pi_* B$.

Two $\mathbb{R}$-divisors D and D' on X are *numerically equivalent over* U (denoted by $D \equiv_U D'$) if $D - D'$ is an $\mathbb{R}$-Cartier divisor and $(D - D') \cdot C = 0$ for any curve C contained in a fiber of π.

An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is *nef over* U (or π-*nef*) if $\mathcal{L} \cdot C > 0$ for any curve C contained in a fiber of π.

An $\mathbb{R}$-Cartier divisor D on X is *ample over* U (or π-*ample*) if we can write $D \sim_{\mathbb{R},U} \sum_i d_i D_i$ such that d_i is a positive real number and D_i is an ample Cartier divisor on X over U for every i.

An $\mathbb{R}$-Cartier divisor D on X is *semi-ample over* U (or π-*semi-ample*) if we can write $D \sim_{\mathbb{R},U} \sum_i d_i D_i$ such that d_i is a positive real number and D_i is a semi-ample Cartier divisor on X over U for every i.

An $\mathbb{R}$-divisor D on X is *big over* U (or π-*big*) if there are an ample $\mathbb{R}$-Cartier divisor A and an $\mathbb{R}$-divisor $B \geq 0$ such that $D \sim_{\mathbb{R}} A + B$.

An $\mathbb{R}$-divisor D on X is *pseudo-effective over* U (or π-*pseudo-effective*) if its numerical class belongs to the closure of the cone of big divisors over U.

We make similar definition for $\mathbb{Q}$-divisors.

Next, let us recall the basic operations and notations for $\mathbb{R}$-divisors.

Definition 1.1.8 (Operation of $\mathbb{R}$-divisors, [**15**, Definition 3.2]). Let V be an equidimensional reduced scheme. Let D be an $\mathbb{R}$-divisor on V and let $D = \sum d_i D_i$ be its prime decomposition, that is, each D_i is an irreducible reduced closed subscheme of V of pure codimension one with $D_i \neq D_i$ for $i \neq j$ and d_i is a real number for every i. Then we put

$$D^{<1} = \sum_{d_i<1} d_i D_i, D^{=1} = \sum_{d_i=1} d_i D_i \text{ and } D^{>1} = \sum_{d_i>1} d_i D_i,$$

and

$$\lceil D \rceil = \sum \lceil d_i \rceil D_i \text{ and } \lfloor D \rfloor = -\lceil -D \rceil,$$

where $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$.

For any subset $S \subset \mathbb{R}$, we denote $D \in S$ if $d_i \in S$ for every i.

Definition 1.1.9 (Relative Picard numbers). Let $f : X \to Y$ be a proper morphism between schemes. We define

$$N^1(X/Y) = \{\mathrm{Pic}(X)/\equiv_Y\} \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$N_1(X/Y) = \{Z_1(X/Y)/\equiv_Y\} \otimes_{\mathbb{Z}} \mathbb{R},$$

where $Z_1(X/Y)$ is the free abelian group of 1-cycles of X over Y. These are inducing the following non-degenerate bilinear pairing:

$$N^1(X/Y) \times N_1(X/Y) \to \mathbb{R}.$$

It is well-known that $N^1(X/Y)$ and $N_1(X/Y)$ are finite-dimensional $\mathbb{R}$-vector spaces. We write

$$\rho(X/Y) = \dim_{\mathbb{R}} N^1(X/Y) = \dim_{\mathbb{R}} N_1(X/Y)$$

and call it the *relative Picard number* of X over Y. We write $\rho(X) = \rho(X/Y)$, $N^1(X) = N^1(X/Y)$ and $N_1(X) = N_1(X/Y)$ when Y is a point. We simply call $\rho(X)$ the *Picard number* of X.

DEFINITION 1.1.10 (Kleiman-Mori cones). Let $f : X \to Y$ be a proper morphism between schemes. The *Kleiman-Mori cone* $\overline{\mathrm{NE}}(X/Y)$ is defined as the convex cone in $N_1(X/Y)$ generated by 1-cycles of X over Y.

The nef cone $\mathrm{Nef}(X/Y)$ is defined as the convex cone of all nef $\mathbb{R}$-Cartier divisors over Y. We write $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X/Y)$ and $\mathrm{Nef}(X) = \mathrm{Nef}(X/Y)$ when Y is a point. It is well-known that $\overline{\mathrm{NE}}(X/Y)$ is dual to $\mathrm{Nef}(X/Y)$.

DEFINITION 1.1.11 (Linear systems). Let $\pi : X \to U$ be a projective morphism between normal varieties. Let D be an $\mathbb{R}$-divisor on X. The *real linear system* associated to D over U is defined as

$$|D/U|_{\mathbb{R}} = \{B \geq 0 \mid B \sim_{\mathbb{R},U} D\}.$$

The *stable base locus* of an $\mathbb{R}$-divisor D over U is defined as

$$\mathbf{B}(D/U) = \bigcap_{D' \in |D/U|_{\mathbb{R}}} D'.$$

We consider that $\mathbf{B}(D/U)$ with the reduced scheme structure.

1.2. Minimal model theory

In this section, we collect some basic definition and results of minimal model theory.

1.2.1. Singularities of pairs. First of all, we review singularities of pairs in minimal model theory.

DEFINITION 1.2.1. Let X be a normal variety and let D be an effective $\mathbb{R}$-divisor on X. We say that (X, D) is a *pair* if $K_X + \Delta$ is $\mathbb{R}$-Cartier, where K_X is the canonical divisor of X.

Let (X, D) be a pair. Let $f : Y \to X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + D) + \sum a_i E_i.$$

Then we say that (X, D) is

(i) *kawamata log terminal* (*klt*, for short) if $a_i > -1$ for any f and i, or
(ii) *log canonical* (*lc*, for short) if $a_i \geq -1$ for any f and i.

We say that X is *$\mathbb{Q}$-factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. In addition, we say that a pair (X, D) is *$\mathbb{Q}$-factorial* if so is X.

DEFINITION 1.2.2 (cf. [**3**, Definition 3.6.1]). Let $f : X \dashrightarrow Y$ be a proper birational contraction of normal varieties and let D be an $\mathbb{R}$-Cartier divisor on X such that f_*D is also $\mathbb{R}$-Cartier. Then we say that f is *D-non-positive* (resp. *D-negative*) if there is a common resolution $p : W \to X$ and $q : W \to Y$ such that

$$p^*D = q^*f_*D + E,$$

where $E \geq 0$ is q-exceptional (resp. $E \geq 0$ is q-exceptional and whose support contains the strict transform of the f-exceptional divisors).

Next, let us define some models of minimal model theory.

Definition 1.2.3. Let $f : X \dashrightarrow Y$ be a birational contraction of normal projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X such that f_*D is also $\mathbb{R}$-Cartier.

We say that f is a *weak log canonical model* of D if

- f is D-non-positive and
- f_*D is nef.

We say that f is a *minimal model* of D if

- Y is $\mathbb{Q}$-factorial,
- f is D-negative and
- f_*D is nef.

Let $\phi : X \to S$ be a contraction morphism of normal projective varieties.

We say that ϕ is a *Mori fiber space* of D if

- X is $\mathbb{Q}$-factorial,
- $-D$ is ϕ-ample,
- $\rho(X/S) = \rho(X) - \rho(S) = 1$ and
- $\dim S < \dim X$.

We say that ϕ has a *Mori fiber structure* if ϕ is a Mori fiber space of some $\mathbb{R}$-Cartier divisor.

We say that f is the *output* of the D-MMP if it is a minimal model of D or a Mori fiber space of D. On the other hand, we say that f is the *result* of running the D-MMP if it is any sequence of steps (i.e. divisorial contractions and flips) for the D-MMP. We emphasize that the result of running the D-MMP is not necessarily a minimal model of D or a Mori fiber space of D.

Definition 1.2.4 (Semi-ample models and Ample models, [**3**, Definition 3.6.5]). Let X be a normal projective variety and let D be an $\mathbb{R}$-Cartier divisor on X.

We say that a birational contraction $f : X \dashrightarrow Y$ is a *semi-ample model* of D if

- f is D-non-positive,
- Y is normal and projective, and
- f_*D is semi-ample.

We say that a rational contraction $g : X \dashrightarrow Y$ is the *ample model* of D if

- Y is normal and projective and
- there is an ample $\mathbb{R}$-Cartier divisor H such that if $p : W \to X$ and $q : W \to Y$ are a common resolution, then we can write $p^*D \sim_{\mathbb{R}} q^*H + E$, where $E \geq 0$, then $B \geq E$ for any $B \in |p^*D|_{\mathbb{R}}$.

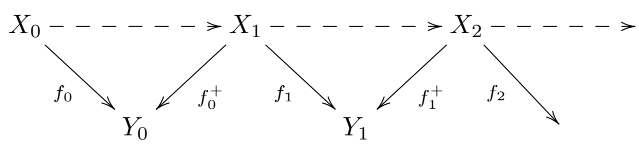
We close this section with explaining a special kind of the MMP with scaling called a 2-*ray game*. For details, see [**30**, Chapter 6].

Definition 1.2.5 (2-ray games). Let $(X = X_0, B = B_0)$ be a $\mathbb{Q}$-factorial log canonical pair and $\pi : X \to U$ be a projective morphism of normal varieties. Assume that $\rho(X/U) = 2$. This means that the Klaiman-Mori cone $\overline{\mathrm{NE}}(X_0/U)$ is a 2-dimensional closed cone in $N_1(X/S)$ and so it is generated by two extremal rays R_0 and R_0'. If $K_X + B$ is not nef over U, then we may assume that $(K_{X_0} + B_0) \cdot R_0 < 0$. If the $(K_{X_0} + B_0)$-negative extremal contraction $f_0 : X_0 \to Y_0$ corresponding to R_0 is divisorial, then we finish.

Assume that f_0 is small and let $f_0^+ : X_1 \to Y_0$ be its flip. Then $\rho(X_1/Y_0) = \rho(X_0/Y_0) = 2$ and so $\overline{\mathrm{NE}}(X/S)$ is also generated by two extremal rays R_1 and

R_1'. Let R_1' be the extremal ray consisted of the curves contracted by f_0^+. Then $(K_{X_1}+B_1)\cdot R_1' > 0$. If $K_{X_1}+B_1$ is not nef, then the $(K_{X_1}+B_1)$-negative extremal ray is uniquely determined and it is R_1, where B_1 is the strict transform of B_0. If the $(K_{X_1}+B_1)$-negative extremal contraction $f_1 : X_1 \to Y_1$ corresponding to R_1 is divisorial, then we finish.

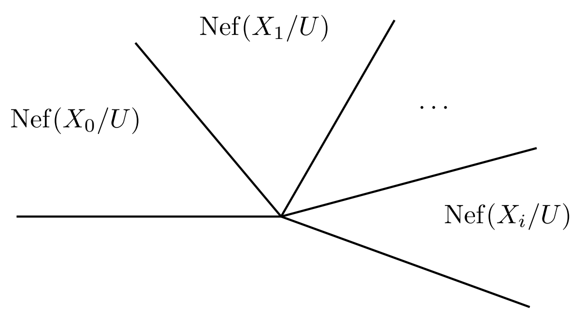
Assume that f_1 is small and let $f_1^+ : X_2 \to Y_1$ be its flip. By repeating this discussion, we uniquely obtain the following sequence of steps of the $(K_{X_0}+B_0)$-MMP:



The 2-ray game is also explained from the viewpoint of cones of divisors. The induced birational map $X_i \dashrightarrow X_{i+1}$ is isomorphic in codimension one. Thus there are the following natural isomorphisms:

$$N^1(X_0/U) \cong N^1(X_1/U) \cong \cdots N^1(X_i/U) \cong \cdots .$$

The nef cone $\mathrm{Nef}(X_i/U)$ is a dual cone of $\overline{\mathrm{NE}}(X_i/U)$ and so it is generated by two extremal rays L_i and L_i'. We may assume that L_i corresponds to f_i^+, that is, $L_i\cdot R_i{=}0$, where R_i is the extremal ray corresponding to f_i^+. Then L_i is the pullback of $\mathrm{Nef}(Y_i/U)$. Similarly, L_i' is also the pullback of $\mathrm{Nef}(Y_{i-1}/U)$. Thus $\mathrm{Nef}(X_i/U)$ and $\mathrm{Nef}(X_{i+1}/U)$ have the same edge coming from the pullback of $\mathrm{Nef}(Y_i/U)$.



1.2.2. Basic results of the minimal model program. The following lemma is well-known.

LEMMA 1.2.6. *Let (X,B) be a log canonical pair. Let $D_1,\cdots,D_k \geq 0$ be Cartier divisors passing through a closed point $x \in X$. If $(X, B+\sum D_i)$ is log canonical, then $k \geq \dim X$.*

PROOF. We proceed by induction on $\dim X$. If $\dim X = 1$, then it is clear. Let $\nu : Y \to D_k$ be the normalization of D_k and we put

$$K_Y + B_Y = \nu^*(K_X + B + D_k).$$

Since $(X, B+\sum_{i=1}^k D_i)$ is log canonical, (Y, B_Y) is also log canonical by adjunction and D_k has no common components with D_i for $1 \leq i \leq k-1$. Thus by adjunction again, $(Y, B_Y + \sum_{i=1}^{k-1} \nu^* D_i)$ is log canonical. We take a closed point $y \in \nu^{-1}(x)$.

Since $x \in \operatorname{Supp} \nu^* D_i$ for $1 \leq i \leq k-1$, it follows from the induction hypothesis that

$$k - 1 \leq \dim Y = \dim X - 1.$$

This means that $k \leq \dim X$. □

The next one is the basic properties of ample models.

LEMMA 1.2.7 ([**3**, Lemma 3.6.6]). *Let X be a projective variety and let D be an $\mathbb{R}$-divisor on X.*

(i) *If $g_i : X \dashrightarrow X_i$ are two ample models of D ($i = 1, 2$), then there is an isomorphism $\chi : X_1 \to X_2$ such that $g_2 = \chi \circ g_1$.*
(ii) *If $f : X \dashrightarrow Y$ is a semi-ample model of D, then there is the ample model $g : X \to Z$ of D and a contraction morphism $h : Y \to Z$ such that $g = h \circ f$ with $f_* D \sim_{\mathbb{R}} h^* H$, where H is an ample $\mathbb{R}$-Cartier divisor corresponding to g.*
(iii) *If $f : X \dashrightarrow Y$ is a birational map, then f is the ample model of D if and only if f is a semi-ample model of D such that $f_* D$ is ample.*

1.3. Toric varieties

In this section, we quickly review minimal model theory for toric varieties.

1.3.1. Toric varieties. First of all, let us recall the definition of toric varieties. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n and let $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice.

DEFINITION 1.3.1 (cf. [**14**, 2.1]). A *toric variety* $X(\Delta)$ is associated to a fan $\Delta = \{\sigma\}$, a finite collection of convex cones $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying:

(i) Each convex cone σ is rational polyhedral in the sense that there are finitely many $v_1, \ldots, v_s \in N \subset N_{\mathbb{R}}$ such that
$$\sigma = \{r_1 v_1, \ldots, r_s v_s | r_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}$$
and it is strongly convex in the sense
$$\sigma \cap -\sigma = \{0\}.$$
(ii) Each face τ of a convex cone $\sigma \in \Delta$ is again contained in Δ.
(iii) The intersection of two cones in Δ is a face of each.

DEFINITION 1.3.2. The *dimension* $\dim \sigma$ of a polyhedral cone σ is the dimension of the span of σ, that is, the smallest subspace of $N_{\mathbb{R}}$ containing σ.

THEOREM 1.3.3 ([**8**, Theorem 6.3.12]). *Let D be $\mathbb{R}$-Cartier divisor on a normal toric variety X. The following are equivalent:*

(i) *D is semi-ample.*
(ii) *D is nef.*
(iii) *$D \cdot C \geq 0$ for all torus-invariant irreducible complete curve $C \subset X$.*

The following lemma is the combinatorial characterization of $\mathbb{Q}$-factoriality.

LEMMA 1.3.4. *[*[**8**, Proposition 4.2.7]*] Let $X = X(\Delta)$ be a toric variety. Then X is $\mathbb{Q}$-factorial if and only if each of $\sigma \in \Delta$ is simplicial, that is, there are $v_1, \ldots, v_s \in \sigma \cap N$ such that $\sigma = \{\sum r_i v_i \mid r_i \geq 0\}$ where $s = \dim \sigma$.*

The following lemma is a well-known criteria for singularities of toric pairs.

LEMMA 1.3.5 ([**8**, Proposition 11.4.24]). *Let (X, D) be a pair where X is toric variety. If $D \in [0, 1]$, then (X, D) is log canonical. Moreover if $D \in [0, 1)$, then it is kawamata log terminal.*

1.3.2. The minimal model theory for toric varieties. In this subsection, we collect the basic results of the toric minimal model theory without proof.

The toric minimal model theory was first established by Reid [**39**] for projective toric varieties with $\mathbb{Q}$-factorial terminal singularities. His proof depends on the combinatorics for toric varieties. On the other hand, Fujino–Sato [**14**] gave an alternative proof and generalized his result with few or no combinatorial arguments.

THEOREM 1.3.6 ([**14**, Section 4]). *Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D be an $\mathbb{R}$-divisor on Z.*

Then we can run the D-minimal model program (D-MMP, for short). In other words, there is a sequence of divisorial contractions and flips

$$Z = Z_0 \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_n = X$$

such that either

(i) *if D is pseudo-effective, then X is a minimal model of D_X, or*
(ii) *if D is not pseudo-effective, then there is a morphism $X \to S$ which is a Mori fiber space of D_X,*

where D_X is the strict transform of D.

We note that if (Z, B) is a $\mathbb{Q}$-factorial log canonical pair where Z is toric, then $K_Z + B$ is not pseudo-effective and so $(K_Z + B)$-MMP always ends up with a Mori fiber space.

In the rest of this subsection, we assume that variety is $\mathbb{Q}$-factorial. Then definition of weak log canonical models and semi-ample models are coincides by Theorem 1.3.3. Thus we can reformulate Lemma 1.2.7 as follows.

LEMMA 1.3.7. *Let X be a projective toric variety and let D be an $\mathbb{R}$-divisor on X.*

(i) *If $g_i : X \dashrightarrow X_i$ are two ample models of D_i ($i = 1, 2$), then there is an isomorphism $\chi : X_1 \to X_2$ such that $g_2 = \chi \circ g_1$.*
(ii) *If $f : X \dashrightarrow Y$ is a weak log canonical model of D, then there is the ample model $g : X \to Z$ of D and a contraction morphism $h : Y \to Z$ such that $g = h \circ f$ with $f_*D \sim_{\mathbb{R}} h^*H$, where H is an ample $\mathbb{R}$-divisor corresponding to g.*
(iii) *If $f : X \dashrightarrow Y$ is a birational map, then f is the ample model of D if and only if f is a weak log canonical model of D such that f_*D is ample.*

CHAPTER 2

The Sarkisov program

In this chapter, we closely follow [**35**]. Our purpose in this chapter is to establish the log Sarkisov program for $\mathbb{Q}$-factorial and toric varieties:

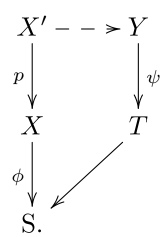
Theorem 1.0.5. Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be an $\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP. Then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many Sarkisov links.

Let us start with the definition of Sarkisov links for toric varieties.

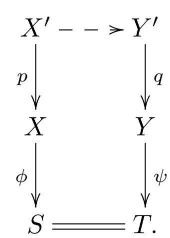
Definition 2.0.1. Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be an $\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP.

The induced birational map $\sigma : X \dashrightarrow Y$ between ϕ and ψ is called a *Sarkisov link* if it is one of the following four types:

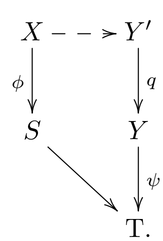
Type (I)



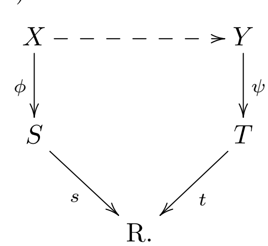
Type (II)



Type (III)



Type (IV)



In the above commutative diagram, the vertical arrows p and q are divisorial contractions and the horizontal dotted arrows are compositions of finitely many flops for the D'_Z-MMP, where D'_Z is an $\mathbb{R}$-divisor on the top left space, that is, X' or X. The spaces X', Y' and R are realized as the ample models of $\mathbb{R}$-divisors on Z, which are the results of the D_Z-MMP. In links of Type (I)-(III), R is $\mathbb{Q}$-factorial. Moreover, links of Type (IV) are separated into two types: (IV$_m$) and (IV$_s$). In a link of Type (IV_m), s and t have both Mori fiber structures and R is $\mathbb{Q}$-factorial. In a link of Type (IV_s), s and t are small birational contractions and R is not $\mathbb{Q}$-factorial. We note that links of Type (IV_s) do not appear for $\dim Z \leq 3$.

Example 2.0.2. For toric 3-folds with terminal singularities, links of Type (I), (II), (III) and (IV_m) are completely classified by shramov and we can find various examples in [**43**].

Let us construction an easy example of links of Type (IV_s). Let $S \to R \leftarrow T$ be a flop for a divisor and we put $X = S \times \mathbb{P}^1$ and $Y = T \times \mathbb{P}^1$. Then here is a link of Type (IV_s).

Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D be a Weil divisor on Z. Then there exists a positive integer m such that mD is linearly equivalent to a torus-invariant Cartier divisor on Z. Hence any $\mathbb{R}$-divisor on Z is $\mathbb{R}$-linearly equivalent to a torus-invariant $\mathbb{R}$-Cartier divisor. Therefore, it is sufficient to prove the following statement for Theorem 1.0.5.

Theorem 2.0.3. *Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be a torus-invariant $\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP. Then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many Sarkisov links.*

By combining Proposition 1.0.3 and Theorem 1.0.5, we obtain the Sarkisov program when Z is $\mathbb{Q}$-factorial and toric.

Corollary 2.0.4. *Let $\phi : X \to S$ and $\psi : X \to Y$ be two toric Mori fiber spaces with $\mathbb{Q}$-factorial terminal singularities. If X and Y are birational, then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many Sarkisov links.*

Moreover, Conjecture 1.0.4 for toric varieties, directly follows from Proposition 1.3.5.

Corollary 2.0.5 (Log Sarkisov program for toric varieties). *Let (Z, B) be a projective log canonical pair, where X is $\mathbb{Q}$-factorial and toric. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the $(K_X + B)$-MMP. Then the induced birational map $\sigma : X \dashrightarrow Y$ is connected by finitely many elementary links.*

2.1. Geography of (log) models

In this section, we explain "geography of (log) models".

Let Z be a $\mathbb{Q}$-factorial projective toric variety. Then the real vector space generated by all torus-invariant prime divisors on Z is denoted by $\mathcal{V}(Z)$.

Let $\mathcal{B}$ be a convex polytope in $\mathcal{V}(Z)$. Let $f : Z \dashrightarrow X$ be a birational contraction to a normal projective variety X and let $g : Z \dashrightarrow Y$ a rational contraction to a normal projective variety Y. Let

$$\begin{aligned}
\mathcal{E}(\mathcal{B}) &= \{D_Z \in \mathcal{B} \mid D_Z \text{ is pseudo-effective}\},\\
\mathcal{M}_f(\mathcal{B}) &= \{D_Z \in \mathcal{E}(\mathcal{B}) \mid f \text{ is a minimal model of } D_Z\},\\
\mathcal{A}_g(\mathcal{B}) &= \{D_Z \in \mathcal{E}(\mathcal{B}) \mid g \text{ is the ample model of } D_Z\},\\
\mathcal{N}(\mathcal{B}) &= \{D_Z \in \mathcal{E}(\mathcal{B}) \mid D_Z \text{ is nef}\}
\end{aligned}$$

and we denote the closure of $\mathcal{A}_g(\mathcal{B})$ by $\mathcal{C}_g(\mathcal{B})$. We simply write $\mathcal{A}_g$ to denote $\mathcal{A}_g(\mathcal{B})$ if there is no risk of confusion.

In the rest of this subsection and next one, we fix the following notation unless otherwise mentioned.

- Z is a $\mathbb{Q}$-factorial projective toric variety,
- $\mathcal{B}$ is a convex polytope of $\mathcal{V}(Z)$, which is defined over $\mathbb{Q}$.

Proposition 2.1.1. *There are only finitely many rational contractions $g_i : Z \dashrightarrow X_i$ $(1 \le i \le l)$ such that*

$$\mathcal{E}(\mathcal{B}) = \bigcup_{i=1}^{l} \mathcal{A}_{g_i},$$

where $\mathcal{A}_{g_i} \neq \mathcal{A}_{g_j}$ for $i \neq j$.

Proof. It follows from the finiteness of minimal models (see [**8**, Theorem 15.5.15]) and Lemma 1.3.7. □

The following two statements come from [**21**, Theorem 3.3] and these are easy consequences of minimal model theory. Thus we give simple proofs. For the details, see [**21**, Theorem 3.3 (2), (3)].

Proposition 2.1.2. *With notation as in Proposition 2.1.1. If $\mathcal{A}_{g_j} \cap \mathcal{C}_{g_i} \neq \emptyset$ for $1 \le i, j \le l$, then there is a contraction morphism $g_{i,j} : X_i \to X_j$ such that $g_j = g_{i,j} \circ g_i$.*

Sketch of Proof. We take $D_Z \in \mathcal{A}_{g_i}$. Running the D_Z-MMP, we end up with a minimal model $f : Z \dashrightarrow X$ of D_Z. Then there is a contraction morphism $g : X \to X_i$ such that $g_i = g \circ f$. Using this morphism g, we can construct a semi-ample divisor on X_i associated to the contraction morphism satisfying the desired property. □

Proposition 2.1.3. *With notation as in Proposition 2.1.1. Assume that $\mathcal{B}$ spans $N^1(Z)$. For any $1 \le i \le l$, the following are equivalent:*

- *there is a rational polytope $\mathcal{C}$ contained in $\mathcal{C}_{g_i}$ which intersects the interior of $\mathcal{B}$ and spans $\mathcal{B}$.*
- *g_i is birational and X_i is $\mathbb{Q}$-factorial.*

Sketch of Proof. Suppose that $\mathcal{C}$ span $\mathcal{B}$. We take D_Z belonging to the relative interior of $\mathcal{C} \cap \mathcal{A}_{g_i}$ belonging to the interior of $\mathcal{B}$. Running the D_Z-MMP, we end up with a minimal model $f : Z \dashrightarrow X$ of D_Z. Then there is certain index $1 \le j \le l$ such that $f = g_j$. Since D_Z belonging to the relative interior of $\mathcal{A}_{g_i}$, we see that $i = j$. Thus $g_i = f$ is birational and $X_i = X$ is $\mathbb{Q}$-factorial. It is easy to see the converse. □

The following proposition is the key result of this section.

Proposition 2.1.4 ([**21**, Theorem 3.3 (4)]). *With notation as in Proposition 2.1.1. Assume that $\mathcal{B}$ spans $N^1(Z)$. If $\mathcal{C}_{g_i}$ spans $\mathcal{B}$ and D_Z is a general point of $\mathcal{A}_{g_j} \cap \mathcal{C}_{g_i}$, which is also a point of the interior of $\mathcal{B}$ for $1 \le i, j \le l$, then $\rho(X_i/X_j) = \dim \mathcal{C}_{g_i} - \dim \mathcal{C}_{g_j} \cap \mathcal{C}_{g_i}$.*

Proof. Putting $X = X_i$ and $f = g_i$, by Proposition 2.1.3, X is $\mathbb{Q}$-factorial and f is birational. Let $E_1, \ldots, E_k$ be all f-exceptional prime divisors. Since $\mathcal{B}$ spans $N^1(Z)$, we can take $B_i \in \mathcal{V}(Z)$, which are linear combinations of the elements of $\mathcal{B}$, such that $B_i \equiv E_i$ and put $B_0 = \sum B_i$ and $E_0 = \sum E_i$. Since D_Z is contained in the interior of $\mathcal{B}$, there is a sufficiently small rational number $\delta > 0$ such that $D_Z + \delta B_0 \in \mathcal{B}$. Then f is $(D_Z + \delta E_0)$-negative and so it is a minimal model of $D_Z + \delta E_0$ and g_j is the ample model of $D_Z + \delta E_0$. Thus $D_Z + \delta B_0 \in \mathcal{M}_f(\mathcal{B})$ and $D_Z + \delta B_0 \in \mathcal{A}_{g_j}$. In particular, $D_Z + \delta B_0 \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$. Since D_Z is general in $\mathcal{A}_{g_j} \cap \mathcal{C}_f$, $D_Z \in \mathcal{M}_f(\mathcal{B})$ and so f is D_Z-negative.

We fix a sufficiently small rational number $\epsilon > 0$ such that if $D'_Z \in \mathcal{B}$ with $||D'_Z - D_Z|| < \epsilon$, then $D'_Z \in \mathcal{B}$ and f is D_Z-negative. Then $D'_Z \in \mathcal{C}_f$ if and only if $D'_X = f_* D'_Z$ is nef.

For any $(a_1, \dots, a_k) \in \mathbb{R}^k$, we put $E = \sum a_i E_i$ and $B = \sum a_i B_i$. We put $\mathcal{B}_X = \{D'_X = f_* D'_Z \mid D'_Z \in \mathcal{B}\} \subset \mathcal{V}(X)$. Then $D'_X \in \mathcal{N}(\mathcal{B}_X)$ if and only if $D'_X + f_* B \in \mathcal{N}(\mathcal{B}_X)$ as $D'_Z + B$ is numerically equivalent to $D'_Z + E$. This means that

$$\mathcal{C}_f \simeq \mathcal{N}(\mathcal{B}_X) \times \mathbb{R}^k$$

in a neighbourhood of D_Z.

By the above argument and Lemma 1.3.7, $D'_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$ if and only if it holds that $D'_X = f_* D'_Z \in \mathcal{N}(\mathcal{B}_X)$ and there is an ample $\mathbb{R}$-Cartier divisor H on X_j such that $f_* D'_Z = (g_{i,j})^* H$, where $g_{i,j} : X \to X_j$ is a contraction morphism. Since $D_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$, there is an ample $\mathbb{R}$-Cartier divisor H_0 on X_j such that $f_* D_Z = (g_{i,j})^* H_0$ and so there are ample $\mathbb{R}$-Cartier divisors $H_1, \dots, H_{\rho(X_j)}$, whose images on $N^1(X_j)$ are linearly independent, such that $f_* D'_Z = (g_{i,j})^*(H_0 + \sum b_i H_i)$ for any $(b_1, \dots, b_{\rho(X_j)}) \in \mathbb{R}^{\rho(X_j)}$ with $H_0 + \sum b_i H_i$ is ample. Thus

$$\dim \mathcal{C}_{g_j} \cap \mathcal{C}_f = k + \rho(X_j).$$

Therefore we obtain

$$\begin{aligned} \rho(X_i/X_j) &= \rho(X_i) - \rho(X_j) \\ &= \dim \mathcal{N}(\mathcal{B}_X) - \rho(X_j) \\ &= \dim \mathcal{C}_f - \dim \mathcal{C}_{g_j} \cap \mathcal{C}_f. \end{aligned}$$

□

We recall the following Bertini-type statement for the reader's convenience.

LEMMA 2.1.5 (cf. [**21**, Corollary 3.4]). *Let $\mathcal{P}$ be a convex polytope in $\mathcal{V}(Z)$ which spans $N^1(Z)$. Then for any general affine subspace $\mathcal{H} \subset \mathcal{V}(Z)$, the intersection $\mathcal{P} \cap \mathcal{H}$ of $\mathcal{P}$ and $\mathcal{H}$ satisfies the conclusions of Proposition 2.1.3 and 2.1.4.*

LEMMA 2.1.6 (cf. [**21**, Lemma 3.5]). *Assume that $\mathcal{B}$ satisfies the conclusion of Propositions 2.1.3 and 2.1.4 and that $\dim \mathcal{B} = 2$. Let $f : Z \dashrightarrow X$ and $g : Z \dashrightarrow Y$ be two rational contractions such that $\dim \mathcal{C}_f = 2$ and $\dim \mathcal{O} = 1$, where $\mathcal{O} = \mathcal{C}_f \cap \mathcal{C}_g$. Assume that $\rho(X) \geq \rho(Y)$ and that $\mathcal{O}$ is not contained in the boundary of $\mathcal{B}$. Let D_Z be a point in the relative interior of $\mathcal{O}$ and $D_X = f_* D_Z$.*

Then there is a rational map $\pi : X \dashrightarrow Y$ such that $g = \pi \circ f$ and either

- (1) *$\rho(X) = \rho(Y) + 1$ and π is D_X-trivial,*
  - (a) *π is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}(\mathcal{B})$,*
    - (i) *π is divisorial and $\mathcal{O} \neq \mathcal{C}_g$,*
    - (ii) *π is small and $\mathcal{O} = \mathcal{C}_g$,*
  - (b) *π has a Mori fiber structure and $\mathcal{O} = \mathcal{C}_g$ is contained in the boundary of $\mathcal{E}(\mathcal{B})$,*
- (2) *$\rho(X) = \rho(Y)$, π is a D_X-flop and $\mathcal{O} \neq \mathcal{C}_g$ is not contained in the boundary of $\mathcal{E}(\mathcal{B})$.*

PROOF. By Proposition 2.1.3, f is birational and X is $\mathbb{Q}$-factorial.

If $\mathcal{O}$ is contained in the boundary of $\mathcal{E}(\mathcal{B})$, then $\dim \mathcal{C}_g = 1$ and $\mathcal{O} = \mathcal{C}_g$. By Proposition 2.1.4, there is a contraction $\pi : X \to Y$ which has a Mori fiber structure. This is (1, b).

In the rest of proof, we may assume that $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}(\mathcal{B})$. If $\dim \mathcal{C}_g = 1$, then $\mathcal{O} = \mathcal{C}_g$. By Proposition 2.1.4, there is a contraction $\pi : X \to Y$ with $\rho(X/Y) = 1$. Since D_Z is not contained in the boundary of $\mathcal{E}(\mathcal{B})$, D_Z is big and so π is birational. Thus by Proposition 2.1.3, Y is not $\mathbb{Q}$-factorial and so π is small. This is (1, a, ii).

We assume that $\dim \mathcal{C}_g = 2$. Then g is birational and Y is $\mathbb{Q}$-factorial. Let $h : Z \to W$ be the ample model of D_Z. By Proposition 2.1.4, there are two contractions $p : X \to W$ and $q : Y \to W$ with $\rho(X/W), \rho(Y/W) \leq 1$. Then we can explicitly calculate the Picard numbers of X and Y and there are only two cases below:

(1) $\rho(X) = \rho(Y) + 1$ or
(2) $\rho(X) = \rho(Y)$.

In (1), $h = g$ and we put $\pi = p$. Then π is divisorial and this is (1,a, i).

In (2), $\rho(X/W) = \rho(Y/W) = 1$. Then $\dim \mathcal{C}_h = 1$ since $\dim \mathcal{O} = 1$. By Theorem 2.1.3, W is not $\mathbb{Q}$-factorial. Thus p and q are small and so π is D_X-flop. □

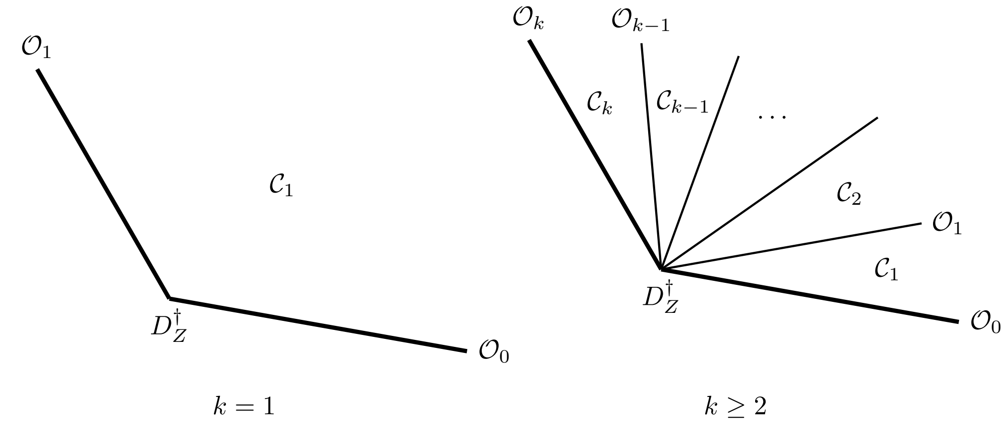
LEMMA 2.1.7 (cf. [**21**, Lemma 3.6]). *Let $f : Z \dashrightarrow X$ be a birational contraction between $\mathbb{Q}$-factorial projective toric varieties. Let D_Z and D'_Z be two torus-invariant $\mathbb{R}$-divisors on Z. If f is the ample model of D'_Z and $D'_Z - D_Z$ is ample, then f is the result of running the D_Z-MMP.*

PROOF. We take an ample divisor H on Z such that $D'_Z + H$ is ample and $tH \sim_{\mathbb{R}} D_Z - D'_Z$ for some positive real number $0 < t < 1$. Then f is the ample model of $D'_Z + tH$. We take any $s < t$ sufficiently close to t. Since f is $(D'_Z + tH)$-non-positive and H is ample, f is $(D'_Z + sH)$-negative. Then f is the unique minimal model of $D'_Z + sH$. In particular, if we run the $(D'_Z + sH)$-MMP with scaling of H and the value of the scalar is s, then the induced morphism is f. □

Next we will see that a certain point contained in the boundary of $\mathcal{E}(V)$ corresponds to a Sarkisov link. Before that we introduce the following additional notation.

NOTATION 2.1.8. Assume that $\mathcal{B}$ satisfies the conclusion of Propositions 2.1.3 and 2.1.4 and that $\dim \mathcal{B} = 2$. Let $D^{\dagger}_Z$ be a point contained in the boundary of $\mathcal{E}(\mathcal{B})$ and the interior of $\mathcal{B}$. If $D^{\dagger}_Z$ is contained in only one polytope of the form $\mathcal{C}_{\bullet}$ of two-dimensional, then we assume that it is a vertex of $\mathcal{E}(\mathcal{B})$.

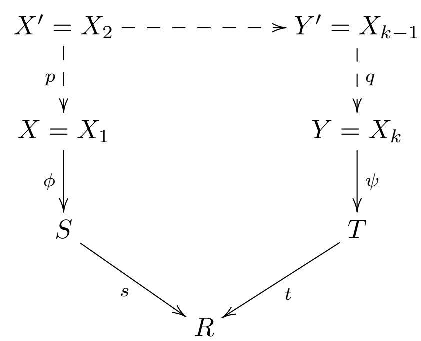
Let $\mathcal{C}_{f_1}, \ldots, \mathcal{C}_{f_k}$ be all two-dimensional rational polytopes containing $D^{\dagger}_Z$, where $f_i : Z \dashrightarrow X_i$ are rational maps. Note that f_i is birational and X_i is $\mathbb{Q}$-factorial by Proposition 2.1.3. Renumbering $\mathcal{C}_{f_i}$ to $\mathcal{C}_i$, let $\mathcal{O}_0$ (resp. $\mathcal{O}_k$) be the intersection of $\mathcal{C}_1$ (resp. $\mathcal{C}_k$) with the boundary of $\mathcal{E}(\mathcal{B})$, and let $\mathcal{O}_i := \mathcal{C}_i \cap \mathcal{C}_{i+1}$ ($1 \leq i \leq k-1$). Then, we may assume that $\mathcal{O}_i$ is one-dimensional for any i. Let $g_i : Z \dashrightarrow S_i$ be the rational contractions associated to $\mathcal{O}_i$. We put $f = f_1 : Z \dashrightarrow X = X_1$, $g = f_k : Z \dashrightarrow Y = X_k$, $X' = X_2$ and $Y' = X_{k-1}$. Then, by Proposition 2.1.2, there are contraction morphisms $\phi : X \to S = S_0$ and $\psi : Y \to T = S_k$. Let $h : Z \dashrightarrow R$ be the ample model of $D^{\dagger}_Z$.



Theorem 2.1.9 (cf. [**21**, Theorem 3.7]). *Let $\mathcal{B}$ and $D_Z^\dagger$ be notation as above. Let D_Z be an $\mathbb{R}$-divisor on Z with $D_Z^\dagger - D_Z$ is ample.*

Then ϕ and ψ are Mori fiber spaces, which are outputs of the D_Z-MMP, and f_i are the result of running the D_Z-MMP. Moreover, if $D_Z^\dagger$ is contained in more than two polytopes, then ϕ and ψ are connected by a Sarkisov link.

Proof. By Lemma 2.1.6, we have the following commutative diagram:



where p, q and the horizontal arrow $X' \dashrightarrow Y'$ are birational and ϕ and ψ have Mori fiber structures. Since $D_Z^\dagger - D_Z$ is ample, for any i we can take $D_i \in \mathcal{C}_i$ such that $D_i - D_Z$ is ample. By Lemma 2.1.7, f_i is the result of running the D_Z-MMP. By Proposition 2.1.4, there is a contraction $X_i \to R$ with $\rho(X_i/R) \leq 2$. If $\rho(X_i/R) = 0$, then $f_i = h$ and this case does not happen. If $\rho(X_i/R) = 1$, then $X_i \to R$ gives a Mori fiber structure. By Lemma 2.1.6, $\dim \mathcal{C}_h = 1$ and there is a facet of $\mathcal{C}_i$ contained in the boundary of $\mathcal{E}(V)$ and so $i = 1$ or k. Therefore if $k \geq 3$, then $\rho(X_i/R) = 2$ for any $1 < i < k$ and $X' \dashrightarrow Y'$ is connected by flops by Lemma 2.1.6 again. Moreover since $\rho(X'/R) = 2$, p is divisorial and s is the identity, or p is flop and s is not the identity. For q and t, similar conditions follow and there are only 7 possibilities below:

(1) $k = 1$,
(2) $k = 2$, $\rho(X/R) = 1$ and $\rho(Y/R) = 2$,
(3) $k = 2$, $\rho(X/R) = 2$ and $\rho(Y/R) = 1$,
(4) $k \geq 3$, p and q are divisorial, and s and t are the identities,
(5) $k \geq 3$, p divisorial, q is flop, s is the identity and t is not the identity,

(6) $k \geq 3$, p is flop, q is divisorial, s is not the identity and t is the identity,
(7) $k \geq 3$, p and q are flops, and s and t are not the identities.

In (1), $X = Y$ and this is a link of Type (IV). In (2), s is the identity and $\rho(Y) \geq \rho(X)$. Then by Lemma 2.1.6, there is a divisorial contraction $X' = Y \to X$. Thus this is a special case of a link of Type (I). In (3), this is similar to (2) and we obtain a special case of a link of Type (III). In (4), this is a link of Type (II). In (5), this is a link of Type (I). In (6), this is a link of Type (III). In (7), this is a link of Type (IV).

The rest of the proof is that a link of Type (IV) is splitting into two types (IV_m) and (IV_s) in (1) and (7).

We assume that s is a divisorial contraction. Then there is a prime divisor F on S which is contracted by s. Since $\rho(X/S) = 1$, there is a prime divisor E on X such that $mE = \phi^*F$ for some $m \in \mathbb{Z}_{\geq 0}$. Since $D_X^\dagger = f_*D_Z^\dagger$ is numerically trivial over R, $\mathbf{B}(D_X^\dagger + E/R) = E$. Since $\rho(X/R) = 2$, by the 2-ray game, there are birational contractions $X \dashrightarrow V \xrightarrow{f'} W \xrightarrow{g} U$ such that f' is a divisorial contraction and g has a Mori fiber space. In (1), this is a contradiction since $X = Y$, ϕ and ψ are Mori fiber spaces and $\rho(X/R) = 2$. In (7), $W = Y$ and $U = T$ and so this gives a link of Type (III) and this is a contradiction. Similarly, t is not divisorial. Thus s and t are not divisorial.

If s has a Mori fiber structure, then R is $\mathbb{Q}$-factorial and so t has also a Mori fiber structure. Thus this is a link of Type (IV_m).

If s is a small contraction, then R is not $\mathbb{Q}$-factorial and so t is also small. Thus this is a link of Type (IV_s). □

Lemma 2.1.10 (cf. [**21**, Lemma 4.1]). *Let D_Z be a torus-invariant $\mathbb{R}$-divisor on Z. Let $f : Z \dashrightarrow X$ and $g : Z \dashrightarrow Y$ be the results of the D_Z-MMP. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces which are outputs of the D_Z-MMP.*

Then we can find a two-dimensional convex polytope $\mathcal{B} \subset \mathcal{V}(Z)$, which is defined over $\mathbb{Q}$, with the following properties:

(1) *$D'_Z - D_Z$ is ample for any $D'_Z \in \mathcal{E}(\mathcal{B})$,*
(2) *$\mathcal{A}_{\phi\circ f}$ and $\mathcal{A}_{\psi\circ g}$ are not contained in the boundary of $\mathcal{B}$,*
(3) *$\mathcal{C}_f$ and $\mathcal{C}_g$ are two-dimensional,*
(4) *$\mathcal{C}_{\phi\circ f}$ and $\mathcal{C}_{\psi\circ g}$ are one-dimensional,*
(5) *$L = \{D'_Z \in \mathcal{E}(\mathcal{B}) \mid D'_Z$ is not big$\}$ is connected.*

Proof. We take ample torus-invariant divisors $H_1, \ldots, H_r \geq 0$, which generate $N^1(Z)$, and we put $H = H_1 + \cdots + H_r$. By assumption, there are ample divisors C on S and D on T, respectively, such that

$$-f_*D_Z + \phi^*C \text{ and } -g_*D_Z + \psi^*D$$

are both ample. Let $\epsilon > 0$ be a sufficiently small rational number. Then

$$-f_*D_Z + \epsilon f_*H + \phi^*C \text{ and } -g_*D_Z + \epsilon g_*H + \psi^*D$$

are both ample and f and g are both $(D_Z + \epsilon H)$-negative. Replacing H by ϵH, we may assume that $\epsilon = 1$. We take a torus-invariant $\mathbb{Q}$-divisor $\widetilde{D}_Z$ on X sufficiently close to D_Z. Then

$$-f_*\widetilde{D}_Z + f_*H + \phi^*C \text{ and } -g_*\widetilde{D}_Z + g_*H + \psi^*D$$

are both ample and f and g are both $(\widetilde{D}_Z + H)$-negative. We take torus-invariant $\mathbb{Q}$-divisors H'_{r+1} and H'_{r+2} on X and Y, respectively, such that

$$H'_{r+1} \in |-f_*\widetilde{D}_Z + f_*H + \phi^*C|_{\mathbb{Q}} \text{ and } H'_{r+2} \in |-g_*\widetilde{D}_Z + g_*H + \phi^*D|_{\mathbb{Q}}.$$

There are torus-invariant $\mathbb{Q}$-divisors H_{r+1} and H_{r+2} on Z such that

$$H_{r+1} \sim_{\mathbb{Q}} f^*H'_{r+1} \text{ and } H_{r+2} \sim_{\mathbb{Q}} g^*H'_{r+2}.$$

Let $a > 0$ be a sufficiently large rational number and we put a rational convex polytope

$$\mathcal{B}_0 = \left\{ \widetilde{D}_Z + a\sum_{i=1}^{r+2} t_i H_i \,\middle|\, \sum_{i=1}^{r+2} t_i \leq 1, t_i \geq 0 \right\}.$$

Possibly replacing H_i by suitable ones, we may assume that (2) holds for $\mathcal{B}_0$.

On the other hand, since f is $(\widetilde{D}_Z + H + H_{r+1})$-negative and $(\phi \circ f)_*(\widetilde{D}_Z + H + H_{r+1})$ is ample, $\widetilde{D}_Z + H + H_{r+1} \in \mathcal{A}_{\phi\circ f}(\mathcal{B}_0)$. Similarly, $\widetilde{D}_Z + H + H_{r+2} \in \mathcal{A}_{\psi\circ g}(\mathcal{B}_0)$ and f and g are weak log canonical models of $\widetilde{D}_Z + H + H_{r+1}$ and $\widetilde{D}_Z + H + H_{r+2}$, respectively. Thus $\widetilde{D}_Z + H + H_{r+1} \in \mathcal{C}_f(\mathcal{B}_0)$ and $\widetilde{D}_Z + H + H_{r+2} \in \mathcal{C}_g(\mathcal{B}_0)$.

Let $\mathcal{H}_0$ be the translation by $\widetilde{D}_Z$ of the affine subspace generated by $H + H_{r+1}$ and $H + H_{r+2}$ and let $\mathcal{H}$ be a small perturbation of $\mathcal{H}_0$, which is defined over $\mathbb{Q}$. Putting $\mathcal{B} = \mathcal{B}_0 \cap \mathcal{H}$, $\mathcal{B}$ satisfies (1) and (2). Since $\mathcal{B}_0$ spans $N^1(Z)$, $\mathcal{C}_{\phi\circ f}(\mathcal{B}_0)$ spans $\mathcal{B}_0$ and so by Lemma 2.1.5, $\mathcal{B}$ satisfies (3). By Proposition 2.1.4, $\dim \mathcal{C}_{\phi\circ f}(\mathcal{B}) = \dim \mathcal{C}_{\psi\circ g}(\mathcal{B}) = 1$ and so $\mathcal{B}$ satisfies (4).

Finally we see that we can take $\mathcal{B}$ satisfying (5). Since ϕ and ψ are Mori fiber spaces, we may assume that $\rho(Z) \geq 2$. There is a surjective linear map from $\mathcal{V}(Z)$ to $N^1(Z)$. Then the pullback of the pseudo-effective cone $\overline{\mathrm{Eff}}(Z) \subset N^1(Z)$ via this map is the convex polyhedron $\mathcal{P}$ containing a $(\dim Z)$-dimensional vector subspace V since $\dim \mathcal{V}(Z) = \rho(Z) + \dim Z$. Then possibly replacing H_i by suitable ones, we can take a two-dimensional rational convex polytope $\mathcal{B}$, which does not contan V, since $\operatorname{codim} V = \rho(Z) \geq 2$. Thus $\mathcal{B}$ satisfies (5) as $\mathcal{E}(\mathcal{B}) = \mathcal{B} \cap \mathcal{P}$. □

2.2. Proof of Theorem 2.0.3

Proof of Theorem 2.0.3. We take a two-dimensional rational convex polytope $\mathcal{B} \subset \mathcal{V}(Z)$ given by Lemma 2.1.10. We take $D_0 \in \mathcal{A}_{\phi\circ f}$ and $D_1 \in \mathcal{A}_{\psi\circ g}$ belonging to the interior of $\mathcal{B}$. As $\mathcal{B}$ is two-dimensional, removing two points D_0 and D_1, the boundary of $\mathcal{E}(\mathcal{B})$ separates into two parts. Then one of the two parts of the boundary of $\mathcal{E}(\mathcal{B})$ is contained in L by Lemma 2.1.10 (5). Tracing this part from D_0 to D_1, we obtain finitely many points D_i $(2 \leq i \leq k)$ which are contained in rational polytopes of two-dimensional. By Theorem 2.1.9, each of D_i gives a Sarkisov link and σ is connected by these links. □

2.3. Examples

In this section, we give some examples of toric Sarkisov links, which are $\mathbb{Q}$-factorial surfaces with singularities.

Example 2.3.1 ([**34**]). We fix a lattice $N = \mathbb{Z}^2$ and take lattice points

$$v_1 = (1, 0), v_2 = (1, n), v_3 = (-1, 0), v_4 = (-1, -n),$$

where n is an integer with $n \geq 2$.

We consider the following fan

$$\Delta = \{\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2, \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3, \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4,\\ \mathbb{R}_{\geq 0}v_4 + \mathbb{R}_{\geq 0}v_1 \text{ and their faces}\}.$$

Then the associated toric variety $X = X(\Delta)$ is a projective $\mathbb{Q}$-factorial toric surface. We consider $\mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto y$ and $\mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto nx - y$.

Then, we obtain two different toric morphisms $p_1, p_2 : X \to \mathbb{P}^1$. This gives a Sarkisov link of Type (IV).

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{P}^1 & & \mathbb{P}^1 \\ & \searrow \quad \swarrow & \\ & \text{pt.} & \end{array}$$

We note that $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.3.2. In Example 2.3.1, we put

$$v_1 = (1, 0), v_2 = (1, n), v_3 = (0, 1), v_4 = (-1, -1),$$

where n is an integer with $n \geq 2$.

Then the associated toric variety $X = X(\Delta)$ is a projective $\mathbb{Q}$-factorial toric surface with $\rho(X) = 2$. By removing v_2, we obtain a toric birational morphism $X \to \mathbb{P}^2$. By removing v_3, we get $X \to \mathbb{P}(1, n-1, n)$.

Thus we have a Sarkisov link of Type (II).

$$\begin{array}{ccc} X & = & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & & \mathbb{P}(1, n-1, n) \\ \downarrow & & \downarrow \\ \text{pt} & = & \text{pt.} \end{array}$$

Example 2.3.3. In Example 2.3.1, we put

$$v_1 = (1, 0), v_2 = (1, n), v_3 = (0, 1), v_4 = (-1, -n),$$

where n is an integer with $n \geq 2$.

We consider the associated toric variety $X = X(\Delta)$. By removing v_2, we have a proper birational morhism $X \to \mathbb{P}(1, 1, n)$. By considering $\mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto nx - y$, we can construct $X \to \mathbb{P}^1$.

Thus we obtain the following commutative diagram:

$$
\begin{array}{ccc}
X & = & X \\
\downarrow & & \downarrow \\
\mathbb{P}(1,1,n) & & \mathbb{P}^1 \\
\downarrow & \swarrow & \\
\text{pt.} & &
\end{array}
$$

So we have Sarkisov links of Types (I) and (III).

CHAPTER 3

Results related to minimal model theory 1

In this chapter, the author provides the results related to minimal model theory joint work with his supervisor Osamu Fujino and we closely follow [**15**]. In this chapter, we will work over the complex number field $\mathbb{C}$ unless otherwise mentioned and our purpose is to give the proof of Theorem 1.0.8.

Let us start with the definition of semi-log canonical pairs to give the corollary of Theorem 1.0.8.

Definition 3.0.1 (Semi-log canonical pairs). Let X be a reduced equidimensional scheme which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let B be an effective $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier.

Let $X = \bigcup X_i$ be the irreducible decomposition of X and let $\nu : X^\nu := \coprod X_i^\nu \to X = \bigcup X_i$ be the normalization, where $\nu|_{X_i^\nu} : X_i^\nu \to X_i$ is the usual normalization for any i. Let B^ν be an $\mathbb{R}$-divisor on X_i such that $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$ and let $B_i^\nu = B^\nu|_{X_i^\nu}$.

Then we say that (X, B) is a *semi-log canonical pair* (an *slc pair*, for short) if each pair (X^ν, B_i^ν) is log canonical.

We introduce some easy examples of semi-log canonical pairs

Example 3.0.2. Let (X, B) be a log canonical pair. Then (X, B) is a semi-log canonical pair.

Example 3.0.3. Let (X, B) be a semi-log canonical pair where X is normal. Then (X, B) is a log canonical pair.

We directly obtain the following result by combining Theorem 1.0.8 and Theorem 3.1.5.

Corollary 3.0.4 ([**16**, Corollary 1.3]). *Let (X, B) be a projective semi-log canonical pair where X is connected. Assume that $K_X + B$ is not nef and that $K_X + \Delta \equiv rB$ for some Cartier divisor D on X with $r > n = \dim X$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(B) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$ and (X, B) is kawamata log terminal.*

3.1. Theory of quasi-log schemes

In this section, we collect some basic definition and results of the theory of quasi-log schemes. The notion of quasi-log schemes was first introduced by Ambro [**1**] as quasi-log varieties. First of all, we recall Fujino's definition of quasi-log schemes, which is slightly different from Ambro's original one. For details, see [**11**, Chapter 6]. Let us define globally embedded simple normal crossing pairs to define quasi-log schemes.

Definition 3.1.1 ([**11**, Definition 6.2.1]). Let Y be a simple normal crossing divisor on a smooth variety M. Let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$

is a simple normal crossing divisor on M and that D and Y have no common irreducible components. Putting $B_Y = D|Y$, we consider the pair (Y, B_Y). Then we say that (Y, B_Y) is a *globally embedded simple normal crossing pair* and M is the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is a log canonical center of $(M, Y + D)$ that is contained in Y.

Let us define quasi-log schemes.

Definition 3.1.2 ([**11**, Definition 6.2.2]). A *quasi-log scheme* $(X, \omega, f : (Y, B_Y) \to X)$ is the data which is X is a scheme, ω is an $\mathbb{R}$-Cartier divisor on X, $X_{-\infty} \subset X$ is a proper closed subscheme, C is a finite collection of reduced and irreducible subschemes of X, and $f : (Y, B_Y) \to X$ is a proper morphism from a globally embedded simple normal crossing pair satisfying the following properties:

(i) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(ii) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\sim} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}} \subset \mathcal{O}_X$ is the ideal sheaf defining a subscheme $X_{-\infty}$.
(iii) The collection of reduced and irreducible subschemes C coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote $(X, \omega, f : (Y, B_Y) \to X)$ if there is no risk of confusion. We say that each member of C is a *qlc stratum* of $[X, \omega]$, $X_{-\infty}$ is the *non-qlc locus* of $[X, \omega]$, and $f : (Y, B_Y) \to X$ is a *quasi-log resolution* of $[X, \omega]$. A qlc stratum C is a *qlc center* of $[X, \omega]$ if it is not an irreducible component of X. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$.

A quasi-log scheme $[X, \omega]$ is a *quasi-log canonical pair* if $X_{-\infty} = \emptyset$.

We give a very important example of quasi-log canonical pairs, i.e. quasi-log schemes. Thanks to this example, we can consider log canonical pairs as quasi-log canonical pairs.

Example 3.1.3 ([**11**, 6.4.1]). Let (X, B) be a pair. Let $f : X \to Y$ be a log resolution of (X, B), that is, $\mathrm{Exc}(f)$ is a divisor and $\mathrm{Supp}\, f_*^{-1}B \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor on Y. We put

$$K_Y + B_Y = f^*(K_X + B).$$

We put $\omega = K_X + B$. Then $(X, \omega, f : (Y, B_Y) \to X)$ becomes a quasi-log scheme. By construction, (X, B) is log canonical if and only if $[X, \omega]$ is quasi-log canonical. We note that C is a log canonical center of (X, B) if and only if C is a qlc center of $[X, \omega]$.

The following theorem is the adjunction and the vanishing theorem for quasi-log schemes.

Theorem 3.1.4 ([**11**, Theorem 6.3.5]). *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty} = \mathrm{Nqlc}(X, \omega)$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then the following properties hold:*

(i) (Adjunction theorem) *Assume that $X' \neq X_\infty$. Then X' has a quasi-log structure with $\omega' = \omega|_X'$ and $\mathrm{Nqlc}(X', \omega') = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are the qlc strata of $[X, \omega]$ that are included in X'.*

(ii) (Vanishing theorem) *Let* $\pi : X \to S$ *be a proper morphism of schemes. Let* L *be a Cartier divisor on* X *such that* $L - \omega$ *is nef and big over* S*. Then* $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ *for every* $i > 0$*, where* $\mathcal{I}_{X'} \subset \mathcal{O}_X$ *is the ideal sheaf defining* X'*.*

Thanks to the following theorem, semi-log canonical pairs consider as quasi-log canonical pairs.

THEOREM 3.1.5 ([**10**, Theorem 1.1]). *Let* (X, B) *be a quasi-projective semi-log canonical pair. Then* $[X, K_X + B]$ *has a quasi-log structure with only qlc singularities. Moreover, the set of slc strata of* (X, B) *gives the set of qlc centers of* $[X, K_X + B]$*.*

The following theorem is well-known for log canonical pairs.

THEOREM 3.1.6 ([**11**, Theorem 6.3.11]). *Let* $[X, \omega]$ *be a quasi-log scheme with* $X_{-\infty} = \emptyset$*. Then two intersection of two qlc strata is a union of qlc strata.*

The following lemma is the generalization of Lemma 1.2.6 to quasi-log canonical pairs.

LEMMA 3.1.7 (([**11**, Lemma 6.3.13]). *Let* $[X, \omega]$ *be a qlc pair. Let* $D_1, \cdots, D_k \geq 0$ *be Cartier divisors passing through a closed point* $x \in X$*. If* $[X, \omega + \sum D_i]$ *is qlc, then* $k \leq \dim X$*.*

We closed this section with very useful proposition of quasi-log schemes.

PROPOSITION 3.1.8 ([**11**, Proposition 6.3.1]). *Let* $f : V \to W$ *be a proper birational morphism between smooth varieties and let* B_W *be an* $\mathbb{R}$*-divisor on* W *such that* $\operatorname{Supp} B_W$ *is a simple normal crossing divisor on* W*. Assume that*

$$K_V + B_V = f^*(K_W + B_W)$$

and that $\operatorname{Supp} B_V$ *is a simple normal crossing divisor on* V*. Then we have*

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor).$$

Furthermore, let S *be a simple normal crossing divisor on* W *such that* $S \subset \operatorname{Supp} B_W^{=1}$*. Let* T *be the union of the irreducible components of* $B_V^{=1}$ *that are mapped into* S *by* f*. Assume that* $\operatorname{Supp} f_*B_W \cup \operatorname{Exc}(f)$ *is a simple normal crossing divisor on* V*. Then we have*

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1}) \rceil - \lfloor B_S^{>1} \rfloor),$$

where $(K_V + B_V)|_T = K_T + B_T$ *and* $(K_W + B_W)|_S = K_S + B_S$*.*

3.2. Fujita's Theory of Δ-genus

In this section, we quickly review Fujita's Δ-genus. For details, see, for example, [**17**], [**18**], [**20**], and [**23**]. Throughout this section, we will work over an algebraically closed field of characteristic zero. Let us start with the definition of Δ-genus.

DEFINITION 3.2.1. Let X be a variety and let D be a Cartier divisor on X. Then Δ-genus $\Delta(X, D)$ is defined to be $\dim X + \deg B - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H))$.

The following theorem is the characterization of projective spaces by Δ-genus.

THEOREM 3.2.2 ([**20**, Theorem 1.1], [**27**, Theorem 2.1]). *Let* (X, D) *be a* n*-dimensional projective polarized variety, that is,* X *is a variety and* D *is an ample Cartier divisor on* X*. Then following are equivalent:*

(i) $D^n = 1$ *and* $\Delta(X, D) = 0$.
(ii) X *is isomorphic to* $\mathbb{P}^n$ *with* $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.

3.3. Proof of Theorem 1.0.8

Before we prove Theorem 1.0.8, we give a sketch of proof of Theorem 1.0.9, that is, Theorem 1.0.8 for log canonical pairs.

Sketch of proof of Theorem 1.0.9. Since $K_X + B$ is not nef, there is a $(K_X + B)$-negative extremal contraction $\phi : X \to W$ by the cone and contraction theorem for log canonical pairs (see [**9**, Thorem1.1]). Let us divide into several cases.

Case 1 ($\dim W \geq 1$). By Lemma 1.2.6, we can take an effective $\mathbb{R}$-Cartier divisor B on W with the following properties:

(i) $(X, B + \phi^* B)$ is log canonical outside finitely many points, and
(ii) there exists a log canonical center C of $(X, B + \phi^* B)$ such that $\phi(C)$ is a point with $\dim C \geq 1$.

In this situation, we obtain that

$$-(K_X + B + \phi^* B)|_C \equiv rD|_C$$

and $D|_C$ is ample since $\phi(C)$ is a point. Therefore, by the vanishing theorem for quasi-log schemes (see Lemma 3.3.1 below), we obtain

$$\chi(C, \mathcal{O}_C(tD)) \equiv 0.$$

This is a contradiction since $H|_C$ is ample. This means that $\dim W \geq 1$ does not happen.

Case 2 ($\dim W = 1$). Since $\phi : X \to W$ is a $(K_X + B)$-negative extremal contraction, we see that H is ample. We can explicitly determine

$$\chi(X, \mathcal{O}_X(tD))$$

by $-(K_X + B) \equiv rD$ with $r > n$ and the vanishing theorem for log canonical pairs (see [**9**, Theorem 8.1]). Then we get $H^n = 1$ and

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = n + 1.$$

Therefore,

$$\Delta(X, D) = n + H^n - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = 0$$

holds. By Theorem proj-space, this implies that $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.

□

Lemma 3.3.1. *Let* $[X, \omega]$ *be a projective quasi-log canonical pair such that* X *is irreducible with* $\dim X = n \geq 1$. *Let* H *be an ample Cartier divisor on* X*, Assume that* $-\omega \equiv rH$ *for some* $r > n$. *Then* $X \simeq \mathbb{P}^n$ *with* $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$, $r \leq n + 1$, *and there are no qlc centers of* $[X, \omega]$.

Proof. We divide the proof into several steps.

STEP 1. In this step, we will see that $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. We consider the Hilbert polynomial of the polarized pair (X, H):

$$\chi(X, \mathcal{O}_X(tH)) = \sum_{i=0}^{n} a_n(-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(tH)).$$

Since H is ample, it is a nontrivial polynomial of degree n. Since

$$tH - \omega \equiv (t + r)H \tag{3.3.1}$$

is ample for $t \geq -n$ by assumption, we obtain that

$$H^i(X, \mathcal{O}_X(tH) = 0$$

for any $i > 0$ and $t \geq -n$ by Theorem 3.1.4 (ii). Since

$$H^0(X, \mathcal{O}_X(tH)) = 0$$

for $t < 0$ and

$$\chi(X, \mathcal{O}_X) = \dim(X, \mathcal{O}_X) = 1,$$

we have

$$\chi(X, \mathcal{O}_X(tH)) = \frac{1}{n}(t + 1) \cdots (t + n). \tag{3.3.2}$$

Thus we obtain that $H^n = 1$ and

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H)) = n + 1.$$

Therefore, we have

$$\Delta(X, H) = n + 1 - (n + 1) = 0$$

and so by Theorem 3.2.2, $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$.

STEP 2. In this step, we will see that $r \leq n+1$. Assume that $r > n+1$. Then by (3.3.1) and Theorem 3.1.4 (ii), we have

$$H^i(X, \mathcal{O}_X(-(n + 1)) = 0$$

for $i > 0$. Thus we obtain that

$$\chi(X, \mathcal{O}_X(-(n + 1)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(-(n + 1)H) = 0.$$

On the other hand, by (3.3.2), we have

$$\chi(X, \mathcal{O}_X(-(n + 1)) = (-1)^n \neq 0.$$

This is a contradiction. Therefore, we see that $r \leq n + 1$.

STEP 3. In this step, we will see that $[X, \omega]$ has no qlc centers. Assume that there is a zero-dimensional qlc center P of $[X, \omega]$. Then the evaluation map

$$H^0(X, \mathcal{O}_X(-H)) \to \mathbb{C}(P)$$

is surjective since

$$H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(-H)) = 0$$

by Theorem 3.1.4 (ii), where $\mathcal{I}_P \subset \mathcal{O}_X$ is the ideal sheaf defining P. Thuswe obtain that

$$H^0(X, \mathcal{O}_X(-H)) \neq 0.$$

This is a contradiction since H is ample. Therefore, there are no zero-dimensional qlc centers of $[X, \omega]$.

Assume that there is a qlc center C of $[X, \omega]$ with $\dim X \geq 1$. By Theorem 3.1.4 (i), $[C, \omega|_C]$ is also a quasi-log canonical pair with $\dim C < \dim X$. Since

$$-\omega \equiv rH$$

with $r > n$, we have

$$-\omega|_C \equiv rH|_C$$

with $r > n \geq \dim C + 1$. This is a contradiction with Step 2. Thus there are no qlc centers of $[X, \omega]$.

We finish the proof of Lemma 3.3.1.

□

The following lemma is an easy consequence of Theorem 3.1.4.

LEMMA 3.3.2. *Let $[X, \omega]$ be a projective quasi-log scheme with $\dim X_{-\infty} = 0$ or $X_{-\infty} = \emptyset$, where $X_{-\infty} = \mathrm{Nqlc}(X, \omega)$. Let L be a Cartier divisor on X such that $L - \omega$ is ample. Then*

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for $i > 0$.

PROOF. It follows from Theorem 3.1.4 if $X_{-\infty} = 0$. Thus we may assume that $\dim X_{-\infty} = 0$. We consider the following short exact sequence:

$$0 \to \mathcal{I}_{X_\infty} \to \mathcal{O}_X \to \mathcal{O}_{X_\infty} \to 0.$$

Then the above sequence induces the following long exact sequence:

$$(3.3.3) \quad \cdots \to H^i(X, \mathcal{I}_{X_\infty} \otimes \mathcal{O}_X(L)) \to H^i(X, \mathcal{O}_X(L)) \to H^i(X, \mathcal{I}_{X_\infty}(L)) \to \cdots .$$

By Theorem 3.1.4 (ii), we obtain that

$$H^i(X, \mathcal{I}_{X_\infty} \otimes \mathcal{O}_X(L)) = 0$$

for$i > 0$. Since $\dim X_\infty = 0$ by assumption, we have

$$H^i(X, \mathcal{I}_{X_\infty}(L)) = 0$$

for $i > 0$. Therefore, by (3.3.3), we see that

$$H^i(X, \mathcal{O}_X(L)) = 0$$

holds true for $i > 0$. □

LEMMA 3.3.3. *Let $[X, \omega]$ be a quasi-log canonical pair such that X is irreducible. Let $\phi : X \to W$ be a proper surjective morphism to a quasi-projective variety W with $\dim W \geq 1$. Let $P \in W$ be a closed point such that $\dim \phi^{-1}(P) \geq 1$. Then there is an $\mathbb{R}$-Cartier divisor $B \geq$ on W such that $[W, \omega + \phi^* B]$ is a quasi-log scheme with the following properties:*

(i) *$[X, \omega + \phi^* B]$ is quasi-log canonical outside finitely many points, and*
(ii) *there is a qlc center C of $[X, \omega + \phi^* B]$ such that $\phi(C) = P$ with $\dim C \geq 1$.*

PROOF.

Case 1. In this case, we assume that there are no qlc centers of $[X,\omega]$ in $\phi^{-1}(P)$.

Let $f:(Y,B_Y)\to X$ be a quasi-log resolution of $[X,\omega]$. We take general very ample Cartier divisors $B_1,\dots B_{n+1}$ on W such that $P\in \operatorname{Supp} B_i$ for any i. By 3.1.8, we may further assume that

$$\left(Y,\sum_{i=1}^{n+1}(\phi\circ f)^*B_i+\operatorname{Supp} B_Y\right)$$

is a globally embedded simple normal crossing pair. By Lemma 3.1.7, we can take $0<\epsilon<1$ with the following properties:

(i) $\left(B_y+c\sum_{l=1}^{n+1}(\phi\circ f)^*B_i\right)^{>1}=0$ or $\dim f\left(\operatorname{Supp}\left(B_Y+c\sum_{l=1}^{n+1}(\phi\circ f)^*B_i\right)^{>1}\right)=0$, and

(ii) there is an irreducible component G of $\left(B_Y+c\sum_{l=1}^{n+1}(\phi\circ f)^*B_i\right)^{=1}$ such that $\dim f(G)\geq 1$.

We put $B=c\sum_{l=1}^{n+1}B_i$. Then

$$f:(Y,B_Y+(\phi\circ f)^*B)\to[X,\omega+\phi^*B]$$

gives a desired quasi-log structure on $[X,\omega+\phi^*B]$ by construction.

Case 2. In this case, we assume that there is a qlc center C of $[X,\omega]$ in $\phi^{-1}(P)$ with $\dim C\geq 1$,

It is clear when we put $B=0$.

Case 3. In this case, we assume that every qlc center of $[X,\omega]$ contained in $\phi^{-1}(P)$ is zero-dimensional.

Let $f:(Y,B_Y)\to X$ be a quasi-log resolution of $[X,\omega]$. We take general very ample Cartier divisors $B_1,\dots,B_{n+1}$ on W such that $P\in\operatorname{Supp} B_i$ for any i as in Case 1. Let X' be the union of all qlc centers contained in $\phi^{-1}(P)$. By 3.1.8, we may assume that the union of all strata of (Y,B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y''=Y-Y'$, $K_{Y''}+B_{Y''}=(K_Y+B_Y)|_{Y''}$, and $f^{''}=f|_{f''}$. We may further assume that

$$\left(Y'',\sum_{i=1}^{n+1}(\phi\circ f'')^*B_i+\operatorname{Supp} B_{Y''}\right)$$

is a globally embedded simple normal crossing pair in the same way as in Case 1. Then we note that

$$\mathcal{I}_{X'}=f^{''}_*\mathcal{O}_{Y''}(\lfloor-(B^{<1}_{Y''}\rfloor-Y'|_{Y''})$$

holds, where $\mathcal{I}_{X'}$ is the ideal sheaf defining X'. We also note that $B^{''}_Y\geq Y'|^{''}_Y$ by construction. By Lemma 3.1.7, we can take $0<\epsilon<1$ with the following properties:

(i) $\dim f^{''}\left(\operatorname{Supp}\left(B_{Y''}+c\sum_{l=1}^{n+1}(\phi\circ f^{''})^*B_i\right)^{>1}\right)=0$, and

(ii) there is an irreducible component G of $\left(B_{Y''}+c\sum_{l=1}^{n+1}(\phi\circ f^{''})^*B_i\right)^{=1}$ such that $\dim f^{''}(G)\geq 1$.

We put $B=c\sum_{l=1}^{n+1}B_i$.

$$f^{''}:(Y^{''},B_{Y''}+(\phi\circ f^{''})^*B)\to[X,\omega+\phi^*B]$$

gives a desired quasi-log structure on $[X, \omega + \phi^* B]$ by construction.

We finish the proof of Lemma 3.3.3.

□

Proof of Theorem 1.0.8. We put $H = -D$.

Case 1. In this case, we assume that X is irreducible. Since ω is not nef, there is an ω-negative extremal contraction $\phi : X \to W$ by the cone and contraction theorem for quasi-log schemes([**11**, Theorem 6.7.3 and Theorem 6.7.4]). If $\dim W \geq 1$, then we can take an $\mathbb{R}$-Cartier divisor D on W satisfying the properties in Lemma 3.3.3. Let C be a qlc center of $[X, \omega + \phi^* B]$ as in Lemma 3.3.3. We put

$$C' = C \cup \mathrm{Nqlc}(X, \omega + \phi^* B).$$

By Theorem 3.1.4 (i), $[C', \omega|'_C]$ is a quasi-log scheme. There is the following short exact sequence:

$$0 \longrightarrow \mathrm{Ker}\,\alpha \longrightarrow \mathcal{O}'_C \xrightarrow{\alpha} \mathcal{O}_C \longrightarrow 0. \tag{3.3.4}$$

such that $\mathrm{Ker}\,\alpha = 0$ or the support of $\mathrm{Ker}\,\alpha$ is zero-dimensional. Since $\dim \phi(C') = 0$, we have

$$-(\omega + \phi^* B)|'_C \equiv rH|'_C.$$

Thus by assumption,

$$(tH - (\omega + \phi^* B))|'_C \equiv (t + r)H|'_C$$

is ample for $t \geq -n$. Therefore, by Lemma 3.3.3 and (3.3.4),

$$H^i(C, \mathcal{O}_C(tH)) = H^i(C', \mathcal{O}'_C(tH)) = 0$$

for $i > 0$ and $t \geq -n$. Since $H|_C$ is ample, we have

$$H^0(C, \mathcal{O}_C(tH)) = 0$$

for $t < 0$. Thus we have

$$\chi(C, \mathcal{O}_C(tH)) = 0$$

for $t = 1, \ldots, -n$ and so we get that

$$\chi(C, \mathcal{O}_C(tH)) \equiv 0$$

since $n \geq \dim C + 1$. This is a contradiction since $H|_C$ is ample. This means that $W = 0$ and so H is ample. By Lemma 3.3.3, $X \simeq \mathbb{P}^n$ with $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$, and there are no qlc centers of $[X, \omega]$.

Case 2. We assume that X is not irreducible and we take an irreducible component X' of X such that $\omega' = \omega|'_X$ is not nef. Then by Theorem 3.1.4 (i), $[X', \omega']$ is a quasi-log canonical pair such that $\omega' \equiv rD|'_X$ with $r > n \geq \dim X'$. By the above argument, we see that $H|'_X$ is ample. Thus by Lemma 3.3.1, $X' \simeq \mathbb{P}^n$ with $\mathcal{O}_{X'}(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. On the other hand, if $X \neq X'$, then $[X', \omega']$ has a qlc center since X is connected by Theorem 3.1.6 and Theorem 3.1.4 (i). This is a contradiction and this case does not happen.

□

CHAPTER 4

Results related to minimal model theory 2

In this chapter, the author provides the results related to minimal model theory joint work with his supervisor Osamu Fujino and we closely follow [**16**]. In this chapter, our purpose is to establish the generalization of the classical result, the Nakai–Moishezon ampleness criterion, to $\mathbb{R}$-line bundles on complete schemes. By the standard reduction argument, it is sufficient to treat the case where X is a complete normal variety. Therefore, all we have to do is to establish the following theorem.

THEOREM 4.0.1 (Nakai–Moishezon ampleness criterion for real Cartier divisors on complete normal varieties, [**16**, Theorem 1.4]). *Let X be a complete normal variety and let $\mathcal{L}$ be an $\mathbb{R}$-Cartier divisor on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed subvarieties $Z \subset X$.*

4.1. Augmented base locus of $\mathbb{R}$-Cartier divisors

In this section, we explain some properties of augmented base loci of $\mathbb{R}$-divisors following [**2**]. Let us recall the definition of base loci and stable base loci.

DEFINITION 4.1.1 (Base loci and stable base loci of $\mathbb{Q}$-divisors). Let X be a projective scheme and let D be a Cartier divisor on X. The *base locus* of D is defined as

$$\mathrm{Bs}|D| = \{x \in X \mid \alpha \text{ vanishes at } x \text{ for every } \alpha \in H^0(X, \mathcal{O}_X(D))\}.$$

We consider $\mathrm{Bs}|D|$ with the reduced scheme structure.

The *stable base locus* of a $\mathbb{Q}$-Cartier divisor L on X is defined as

$$\mathbf{B}(L) = \bigcap_m \mathrm{Bs}|mL|$$

where m runs over all positive integers such that mL is Cartier. Note that $\mathbf{B}(L)$ is considered with the reduced scheme structure.

DEFINITION 4.1.2. Let X be a projective scheme and let L be an $\mathbb{R}$-Cartier divisor on X. The *augmented base locus* of L is defined as

$$\mathbf{B}_+(L) = \bigcap_H \mathbf{B}(L - H),$$

where H runs over all ample $\mathbb{R}$-Cartier divisors on X such that $L - H$ is $\mathbb{Q}$-Cartier. We consider $\mathbf{B}_+(L)$ with the reduced scheme structure.

DEFINITION 4.1.3. Let X be a projective scheme and let L be an $\mathbb{R}$-Cartier divisor on X. The *exceptional locus* of L is defined as

$$\mathbb{E}(L) = \bigcup_{\dim V > 0, L|_V \text{is not big}} V,$$

that is, the union runs over all positive-dimensional subvarieties $V \subset X$ such that $L|_V$ is not big.

It is called the *null locus* when L is nef.

DEFINITION 4.1.4 (Augmented base loci of $\mathbb{R}$-divisors). Let X be a projective scheme and let L be an $\mathbb{R}$-Cartier divisor on X. The *augmented base locus* of L is defined as

$$\mathbf{B}_+(L) = \bigcap_H \mathbf{B}(L - H)$$

where H runs over all ample $\mathbb{R}$-divisors such that $L - H$ is $\mathbb{Q}$-Cartier. As usual, we consider $\mathbf{B}_+(L)$ with the reduced scheme structure.

Birkar defined $\mathbf{B}_+(L)$ differently (see [**2**, Definition 1,2]). Then he proved that his definition coincides with the usual one (see Definition 4.1.4). For the details, see [**2**, Lemma 3.1 (3)].

In order to explain Birkar's theorem (see Theorem 4.1.6), it is convenient to introduce the notion of exceptional loci of $\mathbb{R}$-divisors.

DEFINITION 4.1.5 (Exceptional loci of $\mathbb{R}$-divisors). Let X be a projective scheme and let L be an $\mathbb{R}$-Cartier divisor on X. The *exceptional locus* of L is defined as

$$\mathbb{E}(L) = \bigcup_{\dim V > 0,\ L|_V \text{ is not big}} V,$$

that is, the union runs over the positive-dimensional subvarieties $V \subset X$ such that $L|_V$ is not big.

Note that $\mathbb{E}(L)$ is sometimes called the *null locus* of L when L is nef.

The following result is a key ingredient of this section.

THEOREM 4.1.6 ([**2**, Theorem 1.4]). *Let X be a projective scheme and let L be a nef $\mathbb{R}$-Cartier divisor on X. Then*

$$\mathbf{B}_+(L) = \mathbb{E}(L)$$

holds.

For the details of Theorem 4.1.6, we strongly recommend the reader to see Birkar's original statement in [**2**, Theorem 1.4]. We will use Theorem 4.1.6 when X is a normal projective variety in the proof of Theorem 4.0.1.

4.2. Proof of Theorem 4.0.1

Let us start with the following lemma.

LEMMA 4.2.1. *Let X be an n-dimensional projective variety and let L be a nef $\mathbb{R}$-Cartier divisor on X. Then L is big if and only if $L^n > 0$.*

Before we prove Lemma 4.2.1, we need the following bigness criterion:

LEMMA 4.2.2 ([**31**, Theorem 2.2.15]). *Let X be an n-dimensional projective variety and let L and A be a nef $\mathbb{Q}$-divisors on X. If*

$$L^n > n \cdot (L^{n-1} \cdot A)$$

holds, then $L - A$ is big.

Proof of Lemma 4.2.1. Assume that L is big, then there are an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-Cartier divisor E on X such that $L \sim_{\mathbb{R}} A + E$. Then

$$L^n = (A + E) \cdot L^{n-1} \geq A \cdot L^{n-1} > \cdots > A^n > 0.$$

Conversely, assume that $L^n > 0$. We take ample $\mathbb{R}$-divisors A_1 and A_2 on X such that $L + A_1$ and $A_1 + A_2$ are $\mathbb{Q}$-Cartier divisors on X. Since $L^n > 0$, we may assume that

$$(L + A_1)^n > n((L + A_1)^{n-1} \cdot (A_1 + A_2))$$

by replacing A_1 and A_2 with sufficiently small ones. By Lemma 4.2.2,

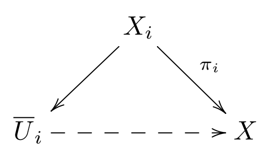
$$L - A_2 = (L + A_1) - (A_1 + A_2)$$

is big. Hence L is also big. $\square$

Proof of Theorem 4.0.1. Let

$$X = \bigcup_{i=1}^{k} U_i$$

be a finite affine Zariski open cover of X and let $\overline{U}_i$ is the closure of U_i in $\mathbb{P}^N$. By [**32**, Lemma 2.2], we can take an ideal sheaf $\mathcal{I}$ on U_i with $\operatorname{Supp} \mathcal{O}(U_i)/\mathcal{I} \subset \overline{U}_i \backslash U_i$ such that the blow-up of U_i along $\mathcal{I}$ eliminates the indeterminacy of $U_i \dashrightarrow X$. Therefore, by taking the normalization of the blow-up of U_i along $\mathcal{I}$, we get a projective birational morphism $\pi_i : X_i \to X$ from a normal projective variety X_i such that $\pi_i : \pi^{-1}(U_i) \to U_i$ is an isomorphism.



Since $\pi_i^* L$ is nef by assumption, it follows from Theorem 4.1.6 that there is an ample $\mathbb{R}$-Cartier divisor H_i on X_i such that $\pi_i^* L - H_i$ is $\mathbb{Q}$-Cartier and that

$$\mathbf{B}(\pi_i^* L - H_i) = \mathbf{B}_+(\pi_i^* L) = \mathbb{E}(\pi_i^* L)$$

holds. Thus by Lemma 4.2.1 and the assumption that $L^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed subvariety $Z \subset X$,

$$\mathbb{E}(\pi_i^* L) = \operatorname{Exc}(\pi_i)$$

holds. Since L is $\mathbb{R}$-Cartier, we can write

$$L = \sum_{j \in J} l_j L_j,$$

where $l_j \in \mathbb{R}$ and L_j is Cartier for any $j \in J$. There is a real number $\epsilon > 0$ such that if $m_j \in \mathbb{Q}$ with $|l_j - m_j| < \epsilon$ for any $j \in J$, then

$$\pi_i^* \left(\sum_{j \in J} m_j L_j \right) - \pi_i^* L + H_i$$

is ample. Then

$$\mathbf{B}\left(\pi_i^* \left(\sum_{j \in J} m_j L_j \right) \right) \subset \mathbf{B}(\pi_i^* L - H_i) = \operatorname{Exc}(\pi_i)$$

holds. This implies that

$$\mathbf{B}\left(\sum_{j\in J} m_jL_j\right)\subset \pi_i(\mathrm{Exc}(\pi_i))\subset X\setminus U_i$$

holds. Therefore, we have

$$\mathbf{B}\left(\sum_{j\in J} m_jL_j\right)\subset \bigcap_{i=1}^{k}(X\setminus U_i)=\emptyset.$$

This means that $\sum m_jL_j$ is semi-ample for any $m_j\in\mathbb{Q}$ with $|l_j-m_j|<\epsilon$ and so we can write

$$L=\sum_p r_pM_p,$$

where $r_p\in\mathbb{R}$ and M_p is a semi-ample $\mathbb{Q}$-divisor for every p. Thus L is semi-ample and so there is a morphism $f: X\to Y$ onto a normal projective variety Y with $f_*\mathcal{O}_X\simeq\mathcal{O}_Y$ and an ample $\mathbb{R}$-Cartier divisor A on Y such that $L\sim_{\mathbb{R}} f^*A$. Since $L\cdot C>0$ for any curve $C\subset X$ by assumption, f is an isomorphism. Thus L is ample. □

4.3. Proof of Theorem 1.0.11

LEMMA 4.3.1. *Let X be a complete scheme and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Let $X=\sum_{i=1}^{k}\bigcup X_i$ be the irreducible decomposition of X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}|_{(X_i)_{\mathrm{red}}}$ is ample for every i.*

PROOF. This statement is well known for $\mathbb{Q}$-line bundles. Hence, we will freely use this lemma for $\mathbb{Q}$-line bundles in this proof. Assume that $\mathcal{L}$ is ample. Then we can write

$$\mathcal{L}=\sum_j a_j\mathcal{L}_j,$$

where a_i is a positive real number and $\mathcal{L}_j$ is an ample line bundle for every j. Thus $\mathcal{L}|_{(X_i)_{\mathrm{red}}}$ is also ample for every i.

Conversely, assume that $\mathcal{L}|_{(X_i)_{\mathrm{red}}}$ is ample for every i. Since $\mathcal{L}$ is an $\mathbb{R}$-line bundle, we can write

$$\mathcal{L}=\sum_{j=1}^{m} l_j\mathcal{L}_j,$$

where l_j is a real number and $\mathcal{L}_j$ is a line bundle for every j. We put

$$V_i=\left\{(p_1,\dots,p_m)\in\mathbb{R}^m \;\middle|\; \sum_{j=1}^{m}p_j\mathcal{L}_j|_{(X_i)_{\mathrm{red}}} \text{ is ample}\right\}$$

for every i. Then V_i contains a Euclidean open neighborhood of $l=(l_1,\dots,l_m)$ for every i since $\sum_{j=1}^{m}p_j\mathcal{L}_j|_{(X_i)_{\mathrm{red}}}$ is ample by assumption. Hence $V=\bigcap_{i=1}^{k}V_i$ contains a Euclidean open neighborhood of $l\in\mathbb{R}^m$. Thus we can take positive real numbers $r_1,\dots,r_p$ and

$$v_1=(v_{11},\dots,v_{1m}),\dots,v_p=(v_{p1},\dots,v_{pm})\in V\cap\mathbb{Q}^m$$

such that $l = \sum_{\alpha=1}^{p} r_\alpha v_\alpha$. Then

$$\mathcal{A}_\alpha := \sum_{j=1}^{m} v_{\alpha j} \mathcal{L}_j \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is ample for every α since $v_\alpha \in V \cap \mathbb{Q}^m$. Since we can write

$$\mathcal{L} = \sum_{\alpha=1}^{p} r_\alpha \mathcal{A}_\alpha,$$

$\mathcal{L}$ is ample by definition. □

LEMMA 4.3.2. *Let X be a complete scheme and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Let $\pi : Y \to X$ be a finite surjective morphism between complete varieties. Then $\mathcal{L}$ is ample if and only if $\pi^*\mathcal{L}$ is ample.*

PROOF. This statement is well known for $\mathbb{Q}$-line bundles. Hence, we will freely use this lemma for $\mathbb{Q}$-line bundles in this proof. Assume that $\mathcal{L}$ is ample. Then it is obvious that $\pi^*\mathcal{L}$ is ample.

Conversely, assume that $\pi^*\mathcal{L}$ is ample. Since $\mathcal{L}$ is an $\mathbb{R}$-line bundle, we can write

$$\mathcal{L} = \sum_{j=1}^{m} l_j \mathcal{L}_j,$$

where l_j is a real number and $\mathcal{L}_j$ is a line bundle for every j. Since $\pi^*\mathcal{L}$ is ample, there exists a positive real number ϵ such that if $|l_j - \alpha_j| < \epsilon$ for every j, then

$$\pi^* \left(\sum_{j=1}^{m} \alpha_j \mathcal{L}_j \right)$$

is ample. Moreover, if we further assume that $\alpha_j \in \mathbb{Q}$ for every j, then

$$\sum_{j=1}^{m} \alpha_j \mathcal{L}_j$$

is ample since π is a finite surjective morphism. Hence we can write

$$\mathcal{L} = \sum_{j=1}^{m} r_j \mathcal{A}_i$$

such that r_i is a positive real number and $\mathcal{A}_i$ is an ample line bundle for every i. This means that $\mathcal{L}$ is ample by definition. □

PROOF OF THEOREM 1.0.11. By Lemma 4.3.1, we may assume that X is a variety. Let $\nu : X^\nu \to X$ be the normalization of X. Note that ν is a finite surjective morphism. Then, by Lemma 4.3.2, it is sufficient to prove that $\nu^*\mathcal{L}$ is ample. Hence we further assume that X is a complete normal variety. In this case, the ampleness of $\mathcal{L}$ follows from Theorem 4.0.1. □

4.4. Proof of Theorem 1.0.12

In this section, we prove Theorem 1.0.12. More precisely, we reduce Theorem 1.0.12 to a special case where X is a normal variety, which is nothing but Theorem 4.0.1. Let us start with the following elementary lemma.

LEMMA 4.4.1. *Let $\pi : X \to S$ be a proper surjective morphism between schemes and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Assume that $\mathcal{L}|_{X_s}$ is ample for every closed point $s \in S$, where $X_s = \pi^{-1}(s)$. Then $\mathcal{L}$ is π-ample.*

Before we prove Lemma 4.4.1, we prepare the following lemma, which is also well known for $\mathbb{Q}$-line bundles.

LEMMA 4.4.2. *Let $\pi : X \to S$ be a proper surjective morphism between schemes and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Assume that $\mathcal{L}|_{X_{s_0}}$ is ample for some closed point $s_0 \in S$, where $X_{s_0} = \pi^{-1}(s_0)$. Then there exists a Zariski open neighborhood U_{s_0} of s_0 such that $\mathcal{L}|_{\pi^{-1}(U_{s_0})}$ is ample over U_{s_0}.*

PROOF. This statement is well known for $\mathbb{Q}$-line bundles. Hence, we will freely use this lemma for $\mathbb{Q}$-line bundles in this proof. For details, see, for example, [**31**, Theorem 1.7.8.].

Since $\mathcal{L}$ is an $\mathbb{R}$-line bundle, there exist line bundles $\mathcal{M}_j$ for $1 \leq k \leq k$ such that

$$\mathcal{L} = \sum_{j=1}^{k} b_j \mathcal{M}_j$$

in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where b_j is a real number for every j. We put

$$\mathcal{A} = \left\{ (c_1, \dots, c_k) \in \mathbb{R}^k \;\middle|\; \sum_{j=1}^{k} c_j \mathcal{M}_j|_{X_{s_0}} \text{ is ample} \right\}.$$

Then $\mathcal{A}$ contains a Euclidean open neighborhood of $(b_1, \dots, b_k)$. Hence we can write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i,$$

where $\mathcal{L}_i$ is a line bundle on X such that a_i is a positive real number and $\mathcal{L}_i|_{X_{s_0}}$ is ample for every i. Since $\mathcal{L}_i|_{X_{s_0}}$ is ample for every i, there exists a Zariski open neighborhood U_{s_0} of s_0 such that $\mathcal{L}_i|_{\pi^{-1}(U_{s_0})}$ is ample over U_{s_0} for every i (see, for example, [**29**, Proposition 1.41]). Therefore, $\mathcal{L}_i|_{\pi^*(U_{s_0})} = \sum_i a_i \mathcal{L}_i|_{\pi^{-1}(U_{s_0})}$ is ample over U_{s_0}. □

Let us prove Lemma 4.4.2.

PROOF OF LEMMA 4.4.1. We use the same notation as in the proof of Lemma 4.4.2. Note that S is Noetherian since it is a separated scheme of finite type over an algebraically closed field. Hence, by Lemma 4.4.2, we can take $s_1, \dots, s_l \in S$ such that

$$\bigcup_{\alpha=1}^{l} U_{S_\alpha} = S,$$

where U_{S_α} is a Zariski open neighborhood of S_α in S for every α, and that $\mathcal{L}|_{\pi^{-1}(U_{S_\alpha})}$ is ample over U_{S_α} for every α. We put

$$\mathcal{A}_\alpha = \left\{ (c_1, \ldots, c_k) \in \mathbb{R}^k \;\middle|\; \sum_{j=1}^{k} c_j \mathcal{M}_j|_{\pi^{-1}(U_{S_\alpha})} \text{ is } \pi\text{-ample over } U_{S_\alpha} \right\}.$$

Then $\mathcal{A}_\alpha$ contains a Euclidean open neighborhood of $(b_1, \ldots, b_k)$. Therefore, $\bigcap_{\alpha=1}^{l} \mathcal{A}_\alpha$ also contains a Euclidean open neighborhood of $(b_1, \ldots, b_k)$. Hence, we can write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i$$

in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ such that a_i is a positive real number and $\mathcal{L}_i$ is a π-ample line bundle on X for every i. □

Finally, we prove Theorem 1.0.12.

Proof of Theorem 1.0.12. If $\mathcal{L}$ is π-ample, then it is obvious that it satisfies the desired property. Hence, by Lemma 4.4.1, it is sufficient to prove that $\mathcal{L}|_{X_s}$ is ample for every closed point $s \in S$, where $X_s = \pi^{-1}(s)$, under the assumption that $\mathcal{L}^{\dim Z} \cdot Z > 0$. This follows from the Nakai–Moishezon ampleness criterion for $\mathbb{R}$-line bundles on complete schemes (see Theorem 1.0.11). □

4.5. The generalization of Theorem 1.0.11 to algebraic spaces

In this section, as an application of Theorem 1.0.11, we prove the Nakai–Moishezon ampleness criterion for $\mathbb{R}$-line bundles on complete algebraic spaces.

Theorem 4.5.1 (Nakai–Moishezon ampleness criterion for real line bundles on complete algebraic spaces, [**16**, Theorem 1.6]). *Let X be a complete algebraic space over an algebraically closed field k and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subspace $Z \subset X$.*

Before we prove Theorem 4.5.1, let us start with two well-known results of algebraic spaces.

Theorem 4.5.2 (Nakai–Moishezon criterion for line bundles on complete algebraic spaces, [**28**, Theorem 3.11] or [**38**, Theorem (1.4)]). *Let X be a complete algebraic space and let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subspace $Z \subset X$.*

Lemma 4.5.3 ([**28**, Lemma 2.8]). *Let X be an algebraic space of finite type. Then there is a scheme Y and a finite surjective morphism $f : Y \to X$.*

Proof of Theorem 4.5.1. By Lemma 4.5.3, there is a finite surjective morphism $f : Y \to X$ from a complete scheme Y. By Theorem 1.0.11, $f^*\mathcal{L}$ is an ample $\mathbb{R}$-line bundle on Y. We write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i,$$

where a_i is a positive real number and L_i is a line bundle on X for every i. We put

$$\mathcal{M} = \sum_i b_i \mathcal{L}_i,$$

where b_i is a positive real number for every i. If $|a_i - b_i| << 1$ for every i, then $f^*\mathcal{M}$ is an ample $\mathbb{Q}$-line bundle on Y since $f^*\mathcal{L}$ is ample. Therefore, $m\mathcal{M}$ is an ample line bundle on X for some positive integer m by Theorem 4.5.2. This implies that X is projective. Thus, by Theorem 1.0.11 again, $\mathcal{L}$ is an ample $\mathbb{R}$-line bundle on X. □

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