



Title	Studies on the Sarkisov program and minimal model theory
Author(s)	宮本, 恵介
Citation	大阪大学, 2023, 博士論文
Version Type	VoR
URL	<a href="https://doi.org/10.18910/92153">https://doi.org/10.18910/92153</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

**Studies on the Sarkisov program and minimal  
model theory**

Keisuke Miyamoto



# Contents

Chapter 1. Introduction	5
1.1. Preliminaries	9
1.2. Minimal model theory	11
1.3. Toric varieties	14
Chapter 2. The Sarkisov program	17
2.1. Geography of (log) models	18
2.2. Proof of Theorem 2.0.3	24
2.3. Examples	24
Chapter 3. Results related to minimal model theory 1	27
3.1. Theory of quasi-log schemes	27
3.2. Fujita's Theory of $\Delta$ -genus	29
3.3. Proof of Theorem 1.0.8	30
Chapter 4. Results related to minimal model theory 2	35
4.1. Augmented base locus of $\mathbb{R}$ -Cartier divisors	35
4.2. Proof of Theorem 4.0.1	36
4.3. Proof of Theorem 1.0.11	38
4.4. Proof of Theorem 1.0.12	40
4.5. The generalization of Theorem 1.0.11 to algebraic spaces	41
Acknowledgement	43
Bibliography	45



## CHAPTER 1

# Introduction

In this thesis, the author provides three results related to minimal model theory. Throughout this thesis, we will work over an arbitrary algebraically closed field  $k$  of any characteristic unless otherwise mentioned. In this thesis, a *scheme* means a separated scheme of finite type over  $k$ . We call such a scheme a *variety* if it is reduced and irreducible.

The aim of minimal model theory is to establish the following statement.

**CONJECTURE 1.0.1.** *Let  $(Z, B)$  be a  $\mathbb{Q}$ -factorial projective log canonical pair. Then there is a birational map  $f : Z \dashrightarrow X$  such that either*

- (i) (Minimal model)  $K_X + B_X = f_*(K_Z + B)$  is nef, or
- (ii) (Mori fiber space) for a suitable morphism  $\phi : X \rightarrow S$ ,  $-(K_X + B_X)$  is  $\phi$ -ample,  $\rho(X/S) = 1$  and  $\dim X > \dim S$ .

Conjecture 1.0.1 was proved in [3] when  $(Z, B)$  is a kawamata log terminal pair such that  $K_Z + B$  is pseudo-effective with either  $B$  or  $K_Z + B$  big, and that  $K_Z + B$  is not pseudo-effective.

We are interested in the structure of birational map between above models. A birational map between two minimal models is isomorphic in codimension one (for example, minimal models for surfaces are uniquely determined). Moreover, two minimal models of any kawamata log terminal pair with any big boundary divisor are connected by finitely many flops ([3, Corollary 1.1.3]). On the other hand, Kawamata [24] proved that a birational map between two terminal pairs, whose log canonical divisors, that is, the sum of the canonical divisor and a boundary divisor, are nef, is the composition of a sequence of flops.

In contrast, Sarkisov [41] introduced a birational map between 3-fold Mori fiber spaces called an *elementary link* and announced a proof that every birational map between 3-fold Mori fiber spaces is connected by finitely many elementary links, which is called the Sarkisov program. Reid reviewed the Sarkisov program and outlined some key ideas involved in the proof in [40]. Unfortunately, the author could not get their original papers and he found these information in [7]. Eventually, the Sarkisov program for 3-folds with  $\mathbb{Q}$ -factorial terminal singularities was completed by Corti [7]. Matsuki established the toric Sarkisov program for 3-folds with  $\mathbb{Q}$ -factorial terminal singularities ([33, Chapter 14] and [44]). This proof is based on the original idea by Sarkisov (cf. [7]). On the other hand, Hacon–M<sup>c</sup>Kernan [21] proved the Sarkisov program for  $n$ -folds with  $\mathbb{Q}$ -factorial terminal singularities by using a brand new method. Roughly speaking, in the original proof, we keep track of three invariants, called the Sarkisov degree, associated with the singularities and we need to prove that the Sarkisov degree satisfies the ascending chain condition. This proof heavily depends on a detailed study of the singularities. On the other hand, Hacon–M<sup>c</sup>Kernan’s approach is quite different

from this. Hacon–M<sup>c</sup>Kernan used “the geography of (log) models” established in [3] and [6], instead of the Sarkisov degree, and proved the following slightly weak Sarkisov program, called the log Sarkisov program, for kawamata log terminal pairs of any dimensional.

**THEOREM 1.0.2** (Log Sarkisov program for kawamata log terminal pairs, cf. [21, Theorem 1.4]). *Let  $(Z, B)$  be a projective  $\mathbb{Q}$ -factorial kawamata log terminal pair. Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $(K_X + B)$ -MMP. Then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many elementary links.*

As a result, by combining Theorem 1.0.2 and the following easy result, Hacon–M<sup>c</sup>Kernan completed the Sarkisov program for  $n$ -folds with  $\mathbb{Q}$ -factorial terminal singularities.

**PROPOSITION 1.0.3** ([4]). *Let  $\phi : X \rightarrow S$  and  $\psi : X \rightarrow Y$  be two Mori fiber spaces with  $\mathbb{Q}$ -factorial terminal singularities. If  $X$  and  $Y$  are birational, then there is a smooth projective variety  $Z$  such that  $\phi$  and  $\psi$  are outputs of the  $K_Z$ -MMP.*

The log Sarkisov program was first introduced by Bruno–Matsuki [4], and they called the above relation between two Mori fiber spaces, that is, there is a smooth projective variety  $Z$  such that these are outputs of the  $K_Z$ -MMP, the Sarkisov relation. Unfortunately, the log Sarkisov program for log canonical pairs is still open.

**CONJECTURE 1.0.4** (Log Sarkisov program). *Let  $(Z, B)$  be a projective log canonical pair. Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $(K_X + B)$ -MMP. Then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many elementary links.*

Conjecture 1.0.4 for  $Z$  (not necessarily  $\mathbb{Q}$ -factorial) surface was proved in [34].

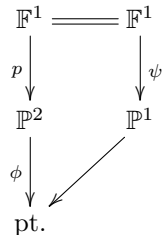
The first purpose of the present thesis is to establish the generalization of Conjecture 1.0.4 for toric varieties as follows. We remark that this result was established in [35].

**THEOREM 1.0.5** ([35, Theorem 1.2]). *Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety and let  $D_Z$  be an  $\mathbb{R}$ -divisor on  $Z$ . Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $D_Z$ -MMP. Then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many Sarkisov links.*

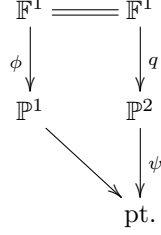
We note that  $D_Z$  is not the log canonical divisor but arbitrary (not necessarily effective)  $\mathbb{R}$ -divisor and there are no assumptions about singularities except  $\mathbb{Q}$ -factoriality.

Elementary links in [41] are known as Sarkisov links. The Sarkisov links for smooth surfaces are very simple as follows:

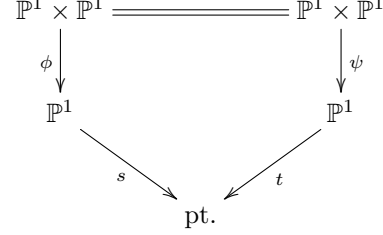
Type (I)



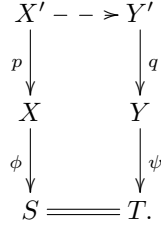
Type (III)



Type (IV)



Type (II)



In a link of Type (I) (resp. (III)),  $p : \mathbb{F}^1 \rightarrow \mathbb{P}^2$  (resp.  $q : \mathbb{F}^1 \rightarrow \mathbb{P}^2$ ) is a blow-up of a smooth point of  $\mathbb{P}^2$ . A link of Type (II) is an elementary transformation of  $\mathbb{P}^1$ -bundles over smooth curves. In a link of Type (IV),  $\phi$  and  $\psi$  are projections to  $\mathbb{P}^1$ . We remark that Sarkisov links for 4 or higher-dimensional Mori fiber spaces are slightly different from the above links. More precisely, links of Type (VI) are separated into two types (see Definition 2.0.1).

The second one is to give a new characterization of projective spaces. Now there are various characterizations of projective spaces. Using Kodaira vanishing theorem, Kobayashi-Ochiai established the characterization of projective spaces from the viewpoint of complete Ricci-flat Kähler metrics as follows.

**THEOREM 1.0.6** ([27, Theorem 1.1]). *Let  $X$  be an  $n$ -dimensional complex projective manifold and let  $H$  be an ample Cartier divisor. Assume that  $-K_X \equiv (n+1)H$ . Then  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ .*

Fujita generalized Kobayashi-Ochiai's characterization by using his own theory, that is, the theory of  $\Delta$ -genus (see Section 3.2 for definition of  $\Delta$ -genus).

**THEOREM 1.0.7** ([19, Theorem 1]). *Let  $X$  be an  $n$ -dimensional complex normal projective variety with rational Gorenstein singularities and let  $H$  be an ample Cartier divisor. Assume that  $K_X + nH$  is not nef. Then  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ .*

We establish a new characterization of projective spaces from the viewpoint of minimal model theory by using the theory of quasi-log schemes (see Section 3.1 for definition of quasi-log schemes) and the theory of  $\Delta$ -genus. We remark that this result was established in [15].

**THEOREM 1.0.8** ([15, Theorem 1.4]). *Let  $[X, \omega]$  be a projective quasi-log canonical pair where  $X$  is defined over  $\mathbb{C}$  and connected. Assume that  $\omega$  is not nef and that  $\omega \equiv rD$  for some Cartier divisor  $D$  on  $X$  with  $r > n = \dim X$ . Then  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$ . Moreover, there are no qlc centers of  $[X, \omega]$ .*



For the reader's convenience, we give Theorem 1.0.8 for log canonical pairs.

**THEOREM 1.0.9.** *Let  $(X, B)$  be a projective log canonical pair where  $X$  is defined over  $\mathbb{C}$ . Assume that  $K_X + B$  is not nef and that  $-(K_X + B) \equiv rD$  for some Cartier divisor  $D$  on  $X$  with  $r > n = \dim X$ . Then  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ . Moreover, there are no lc centers of  $(X, B)$ , that is,  $(X, B)$  is kawamata log terminal.*

This theorem directly follows from Theorem 1.0.8 because arbitrary log canonical pair  $(X, B)$  has a natural quasi-log canonical structure  $[X, K_X + B]$  (cf. Example 3.1.3). This is joint work with his supervisor Osamu Fujino.

The third one is to give the generalization of the following ampleness criterion, known as the Nakai–Moishezon criterion.

**THEOREM 1.0.10** ([22, Theorem 5.1]). *Let  $X$  be a complete scheme and let  $\mathcal{L}$  be a line bundle on  $X$ . Then  $\mathcal{L}$  is ample if and only if  $\mathcal{L}^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed integral subscheme  $Z \subset X$ .*

Theorem 1.0.10 was proved for smooth surfaces by Nakai [37] and for smooth varieties of any dimensional by Moishezon [36]. After that, Kleiman clarified these proofs and gave the simple proof of Theorem 1.0.10 in [26]. Theorem 1.0.10 was generalized to  $\mathbb{R}$ -line bundles on projective schemes by Campana–Petrernell [5] by using Kleiman's method.

Our purpose is to relax the projectivity in Campana–Petrernell's result as follows. We remark that this result was established in [16].

**THEOREM 1.0.11** (Nakai–Moishezon criterion for real line bundles on complete schemes, [16, Theorem 1.3]). *Let  $X$  be a complete scheme and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Then  $\mathcal{L}$  is ample if and only if  $\mathcal{L}^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed integral subscheme  $Z \subset X$ .*

Unfortunately, Campana–Petrernell's arguments do not work for complete nonprojective schemes because they need an ample line bundle, equivalently, the projectivity of schemes. Moreover, Kleiman's ampleness criterion does not always hold for complete nonprojective schemes (see [26, Section 3] and [13, Example 12.1]). We can explicitly construct a complete nonprojective toric threefold  $X$  and a line bundle  $\mathcal{L}$  on  $X$  such that  $\mathcal{L}$  is positive on  $\overline{\text{NE}}(X) \setminus \{0\}$ , where  $\overline{\text{NE}}(X)$  is the Kleiman–Mori cone of  $X$  (see [26, Section 3]). Thus, we need some new idea to generalize to  $\mathbb{R}$ -line bundles on complete schemes.

For the proof of Theorem 1.0.11, we use the characterization of the augmented base loci for  $\mathbb{R}$ -line bundles on projective schemes established by Birkar [2]. Hence, our approach is quite different from those of [26] and [5]. Although we can not directly apply geometric arguments to  $\mathbb{R}$ -line bundles, we can generalize Theorem 1.0.11 for proper morphisms.

**THEOREM 1.0.12** (Relative Nakai–Moishezon criterion for real line bundles on complete algebraic spaces, [16, Theorem 1.5]). *Let  $\pi : X \rightarrow S$  be a proper morphism between schemes and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Then  $\mathcal{L}$  is  $\pi$ -ample if and only if  $\mathcal{L}^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed integral subscheme  $Z \subset X$  such that  $\pi(Z)$  is a point.*

In Section 4.5, we generalize Theorem 1.0.11 to algebraic spaces. It plays a crucial role in Kollár's projectivity criterion for moduli spaces. For details, see [13] and [28]. This is also joint work with his supervisor Osamu Fujino.

The contents of this thesis mainly combine results obtained in [35], [15] and [16].

### 1.1. Preliminaries

In this section, we quickly review the basic definition and results of minimal model theory.

**1.1.1. Fundamentals of algebraic geometry.** Let us start with the definition of  $\mathbb{R}$ -Weil divisors,  $\mathbb{R}$ -Cartier divisors and  $\mathbb{R}$ -line bundles.

**DEFINITION 1.1.1** ( $\mathbb{R}$ -Weil divisors). Let  $X$  be a complete scheme. An  $\mathbb{R}$ -Weil divisor (resp.  $\mathbb{Q}$ -Weil divisor)  $D$  is an element of  $\text{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  (resp.  $\text{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ ), where  $\text{WDiv}(X)$  denotes the group of Weil divisors on  $X$ . For simplicity,  $\mathbb{R}$ -Weil divisors are usually called by  $\mathbb{R}$ -divisors.

**DEFINITION 1.1.2** ( $\mathbb{R}$ -Cartier divisors). Let  $X$  be a complete scheme. An  $\mathbb{R}$ -Cartier divisor (resp. a  $\mathbb{Q}$ -Cartier divisor)  $D$  is an element of  $\text{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  (resp.  $\text{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ ), where  $\text{CDiv}(X)$  denotes the group of Cartier divisors on  $X$ .

**DEFINITION 1.1.3** ( $\mathbb{R}$ -line bundles). Let  $X$  be a complete scheme. An  $\mathbb{R}$ -line bundle (resp. a  $\mathbb{Q}$ -line bundle)  $\mathcal{L}$  is an element of  $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  (resp.  $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ ), where  $\text{Pic}(X)$  is the Picard group of  $X$ .

**REMARK 1.1.4.** In minimal model theory, we usually confuse  $\mathbb{R}$ -Cartier divisors with  $\mathbb{R}$ -line bundles. This is because the following natural homomorphism

$$\psi : \text{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

is surjective if a complete scheme  $X$  is reduced.

On the other hand, in Chapter 4, we discuss the ampleness for  $\mathbb{R}$ -line bundles on a (not necessarily reduced) complete scheme. Thus, we carefully define the nefness, ampleness, and semi-ampleness for  $\mathbb{R}$ -line bundles as follows.

For simplicity of notation, we write the group law of  $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  additively.

**DEFINITION 1.1.5.** Let  $\pi : X \rightarrow U$  be a proper morphism between complete schemes.

An  $\mathbb{R}$ -line bundle  $\mathcal{L}$  on  $X$  is *nef over  $U$*  (or  $\pi$ -nef) if  $\mathcal{L} \cdot C > 0$  for any curve  $C$  contained in a fiber of  $\pi$ .

An  $\mathbb{R}$ -line bundle  $\mathcal{L}$  on  $X$  is *ample over  $U$*  (or  $\pi$ -ample) if we can write  $\mathcal{L} = \sum_i l_i \mathcal{L}_i$  such that  $l_i$  is a positive real number and  $\mathcal{L}_i$  is an ample line bundle on  $X$  over  $U$  for every  $i$ .

An  $\mathbb{R}$ -line bundle  $\mathcal{L}$  on  $X$  is *semi-ample over  $U$*  (or  $\pi$ -semi-ample) if we can write  $\mathcal{L} = \sum_i l_i \mathcal{L}_i$  such that  $l_i$  is a positive real number and  $\mathcal{L}_i$  is a semi-ample line bundle on  $X$  over  $U$  for every  $i$ , that is, the natural homomorphism  $\pi^* \pi_* \mathcal{L} \rightarrow \mathcal{L}$  is surjective.

**DEFINITION 1.1.6.** Let  $\pi : X \rightarrow U$  be a proper morphism between complete schemes. Let  $\psi$  be a natural homomorphism as in Remark 1.1.4. We note that  $\psi$  is not necessarily surjective. Let  $D$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . If the image of  $D$  by  $\psi$  is  $\pi$ -nef,  $\pi$ -ample, and  $\pi$ -semi-ample, then  $D$  is said to be  $\pi$ -nef,  $\pi$ -ample, and  $\pi$ -semi-ample, respectively.

DEFINITION 1.1.7. Let  $\pi : X \rightarrow U$  be a proper morphism between normal quasi-projective varieties.

Two  $\mathbb{R}$ -divisors  $D$  and  $D'$  on  $X$  are  *$\mathbb{R}$ -linearly equivalent over  $U$*  (denoted by  $D \sim_{\mathbb{R}, U} D'$ ) if there is an  $\mathbb{R}$ -Cartier divisor  $B$  on  $U$  such that  $D - D' \sim_{\mathbb{R}} \pi_* B$ .

Two  $\mathbb{R}$ -divisors  $D$  and  $D'$  on  $X$  are *numerically equivalent over  $U$*  (denoted by  $D \equiv_U D'$ ) if  $D - D'$  is an  $\mathbb{R}$ -Cartier divisor and  $(D - D') \cdot C = 0$  for any curve  $C$  contained in a fiber of  $\pi$ .

An  $\mathbb{R}$ -line bundle  $\mathcal{L}$  on  $X$  is *nef over  $U$*  (or  *$\pi$ -nef*) if  $\mathcal{L} \cdot C > 0$  for any curve  $C$  contained in a fiber of  $\pi$ .

An  $\mathbb{R}$ -Cartier divisor  $D$  on  $X$  is *ample over  $U$*  (or  *$\pi$ -ample*) if we can write  $D \sim_{\mathbb{R}, U} \sum_i d_i D_i$  such that  $d_i$  is a positive real number and  $D_i$  is an ample Cartier divisor on  $X$  over  $U$  for every  $i$ .

An  $\mathbb{R}$ -Cartier divisor  $D$  on  $X$  is *semi-ample over  $U$*  (or  *$\pi$ -semi-ample*) if we can write  $D \sim_{\mathbb{R}, U} \sum_i d_i D_i$  such that  $d_i$  is a positive real number and  $D_i$  is a semi-ample Cartier divisor on  $X$  over  $U$  for every  $i$ .

An  $\mathbb{R}$ -divisor  $D$  on  $X$  is *big over  $U$*  (or  *$\pi$ -big*) if there are an ample  $\mathbb{R}$ -Cartier divisor  $A$  and an  $\mathbb{R}$ -divisor  $B \geq 0$  such that  $D \sim_{\mathbb{R}} A + B$ .

An  $\mathbb{R}$ -divisor  $D$  on  $X$  is *pseudo-effective over  $U$*  (or  *$\pi$ -pseudo-effective*) if its numerical class belongs to the closure of the cone of big divisors over  $U$ .

We make similar definition for  $\mathbb{Q}$ -divisors.

Next, let us recall the basic operations and notations for  $\mathbb{R}$ -divisors.

DEFINITION 1.1.8 (Operation of  $\mathbb{R}$ -divisors, [15, Definition 3.2]). Let  $V$  be an equidimensional reduced scheme. Let  $D$  be an  $\mathbb{R}$ -divisor on  $V$  and let  $D = \sum d_i D_i$  be its prime decomposition, that is, each  $D_i$  is an irreducible reduced closed subscheme of  $V$  of pure codimension one with  $D_i \neq D_j$  for  $i \neq j$  and  $d_i$  is a real number for every  $i$ . Then we put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, D^{=1} = \sum_{d_i = 1} d_i D_i \text{ and } D^{>1} = \sum_{d_i > 1} d_i D_i,$$

and

$$\lceil D \rceil = \sum \lceil d_i \rceil D_i \text{ and } \lfloor D \rfloor = -\lceil -D \rceil,$$

where  $\lceil x \rceil$  is the integer defined by  $x \leq \lceil x \rceil < x + 1$ .

For any subset  $S \subset \mathbb{R}$ , we denote  $D \in S$  if  $d_i \in S$  for every  $i$ .

DEFINITION 1.1.9 (Relative Picard numbers). Let  $f : X \rightarrow Y$  be a proper morphism between schemes. We define

$$N^1(X/Y) = \{\text{Pic}(X)/\equiv_Y\} \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$N_1(X/Y) = \{Z_1(X/Y)/\equiv_Y\} \otimes_{\mathbb{Z}} \mathbb{R},$$

where  $Z_1(X/Y)$  is the free abelian group of 1-cycles of  $X$  over  $Y$ . These are inducing the following non-degenerate bilinear pairing:

$$N^1(X/Y) \times N_1(X/Y) \rightarrow \mathbb{R}.$$

It is well-known that  $N^1(X/Y)$  and  $N_1(X/Y)$  are finite-dimensional  $\mathbb{R}$ -vector spaces. We write

$$\rho(X/Y) = \dim_{\mathbb{R}} N^1(X/Y) = \dim_{\mathbb{R}} N_1(X/Y)$$

and call it the *relative Picard number* of  $X$  over  $Y$ . We write  $\rho(X) = \rho(X/Y)$ ,  $N^1(X) = N^1(X/Y)$  and  $N_1(X) = N_1(X/Y)$  when  $Y$  is a point. We simply call  $\rho(X)$  the *Picard number* of  $X$ .

**DEFINITION 1.1.10** (Kleiman-Mori cones). Let  $f : X \rightarrow Y$  be a proper morphism between schemes. The *Kleiman-Mori cone*  $\overline{\text{NE}}(X/Y)$  is defined as the convex cone in  $N_1(X/Y)$  generated by 1-cycles of  $X$  over  $Y$ .

The nef cone  $\text{Nef}(X/Y)$  is defined as the convex cone of all nef  $\mathbb{R}$ -Cartier divisors over  $Y$ . We write  $\overline{\text{NE}}(X) = \overline{\text{NE}}(X/Y)$  and  $\text{Nef}(X) = \text{Nef}(X/Y)$  when  $Y$  is a point. It is well-known that  $\overline{\text{NE}}(X/Y)$  is dual to  $\text{Nef}(X/Y)$ .

**DEFINITION 1.1.11** (Linear systems). Let  $\pi : X \rightarrow U$  be a projective morphism between normal varieties. Let  $D$  be an  $\mathbb{R}$ -divisor on  $X$ . The *real linear system* associated to  $D$  over  $U$  is defined as

$$|D/U|_{\mathbb{R}} = \{B \geq 0 \mid B \sim_{\mathbb{R},U} D\}.$$

The *stable base locus* of an  $\mathbb{R}$ -divisor  $D$  over  $U$  is defined as

$$\mathbf{B}(D/U) = \bigcap_{D' \in |D/U|_{\mathbb{R}}} D'.$$

We consider that  $\mathbf{B}(D/U)$  with the reduced scheme structure.

## 1.2. Minimal model theory

In this section, we collect some basic definition and results of minimal model theory.

**1.2.1. Singularities of pairs.** First of all, we review singularities of pairs in minimal model theory.

**DEFINITION 1.2.1.** Let  $X$  be a normal variety and let  $D$  be an effective  $\mathbb{R}$ -divisor on  $X$ . We say that  $(X, D)$  is a *pair* if  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier, where  $K_X$  is the canonical divisor of  $X$ .

Let  $(X, D)$  be a pair. Let  $f : Y \rightarrow X$  be a proper birational morphism from a normal variety  $Y$ . Then we can write

$$K_Y = f^*(K_X + D) + \sum a_i E_i.$$

Then we say that  $(X, D)$  is

- (i) *kawamata log terminal* (*klt*, for short) if  $a_i > -1$  for any  $f$  and  $i$ , or
- (ii) *log canonical* (*lc*, for short) if  $a_i \geq -1$  for any  $f$  and  $i$ .

We say that  $X$  is  $\mathbb{Q}$ -factorial if every Weil divisor on  $X$  is  $\mathbb{Q}$ -Cartier. In addition, we say that a pair  $(X, D)$  is  $\mathbb{Q}$ -factorial if so is  $X$ .

**DEFINITION 1.2.2** (cf. [3, Definition 3.6.1]). Let  $f : X \dashrightarrow Y$  be a proper birational contraction of normal varieties and let  $D$  be an  $\mathbb{R}$ -Cartier divisor on  $X$  such that  $f_*D$  is also  $\mathbb{R}$ -Cartier. Then we say that  $f$  is *D-non-positive* (resp. *D-negative*) if there is a common resolution  $p : W \rightarrow X$  and  $q : W \rightarrow Y$  such that

$$p^*D = q^*f_*D + E,$$

where  $E \geq 0$  is  $q$ -exceptional (resp.  $E \geq 0$  is  $q$ -exceptional and whose support contains the strict transform of the  $f$ -exceptional divisors).

Next, let us define some models of minimal model theory.

DEFINITION 1.2.3. Let  $f : X \dashrightarrow Y$  be a birational contraction of normal projective varieties and let  $D$  be an  $\mathbb{R}$ -Cartier divisor on  $X$  such that  $f_*D$  is also  $\mathbb{R}$ -Cartier.

We say that  $f$  is a *weak log canonical model* of  $D$  if

- $f$  is  $D$ -non-positive and
- $f_*D$  is nef.

We say that  $f$  is a *minimal model* of  $D$  if

- $Y$  is  $\mathbb{Q}$ -factorial,
- $f$  is  $D$ -negative and
- $f_*D$  is nef.

Let  $\phi : X \rightarrow S$  be a contraction morphism of normal projective varieties.

We say that  $\phi$  is a *Mori fiber space* of  $D$  if

- $X$  is  $\mathbb{Q}$ -factorial,
- $-D$  is  $\phi$ -ample,
- $\rho(X/S) = \rho(X) - \rho(S) = 1$  and
- $\dim S < \dim X$ .

We say that  $\phi$  has a *Mori fiber structure* if  $\phi$  is a Mori fiber space of some  $\mathbb{R}$ -Cartier divisor.

We say that  $f$  is the *output* of the  $D$ -MMP if it is a minimal model of  $D$  or a Mori fiber space of  $D$ . On the other hand, we say that  $f$  is the *result* of running the  $D$ -MMP if it is any sequence of steps (i.e. divisorial contractions and flips) for the  $D$ -MMP. We emphasize that the result of running the  $D$ -MMP is not necessarily a minimal model of  $D$  or a Mori fiber space of  $D$ .

DEFINITION 1.2.4 (Semi-ample models and Ample models, [3, Definition 3.6.5]).

Let  $X$  be a normal projective variety and let  $D$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ .

We say that a birational contraction  $f : X \dashrightarrow Y$  is a *semi-ample model* of  $D$  if

- $f$  is  $D$ -non-positive,
- $Y$  is normal and projective, and
- $f_*D$  is semi-ample.

We say that a rational contraction  $g : X \dashrightarrow Y$  is the *ample model* of  $D$  if

- $Y$  is normal and projective and
- there is an ample  $\mathbb{R}$ -Cartier divisor  $H$  such that if  $p : W \rightarrow X$  and  $q : W \rightarrow Y$  are a common resolution, then we can write  $p^*D \sim_{\mathbb{R}} q^*H + E$ , where  $E \geq 0$ , then  $B \geq E$  for any  $B \in |p^*D|_{\mathbb{R}}$ .

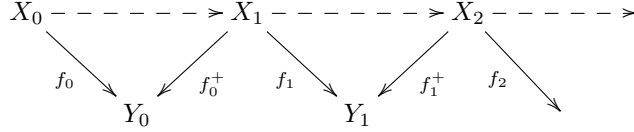
We close this section with explaining a special kind of the MMP with scaling called a *2-ray game*. For details, see [30, Chapter 6].

DEFINITION 1.2.5 (2-ray games). Let  $(X = X_0, B = B_0)$  be a  $\mathbb{Q}$ -factorial log canonical pair and  $\pi : X \rightarrow U$  be a projective morphism of normal varieties. Assume that  $\rho(X/U) = 2$ . This means that the Klaiman-Mori cone  $\overline{\text{NE}}(X_0/U)$  is a 2-dimensional closed cone in  $N_1(X/S)$  and so it is generated by two extremal rays  $R_0$  and  $R'_0$ . If  $K_X + B$  is not nef over  $U$ , then we may assume that  $(K_{X_0} + B_0) \cdot R_0 < 0$ . If the  $(K_{X_0} + B_0)$ -negative extremal contraction  $f_0 : X_0 \rightarrow Y_0$  corresponding to  $R_0$  is divisorial, then we finish.

Assume that  $f_0$  is small and let  $f_0^+ : X_1 \rightarrow Y_0$  be its flip. Then  $\rho(X_1/Y_0) = \rho(X_0/Y_0) = 2$  and so  $\overline{\text{NE}}(X/S)$  is also generated by two extremal rays  $R_1$  and

$R'_1$ . Let  $R'_1$  be the extremal ray consisted of the curves contracted by  $f_0^+$ . Then  $(K_{X_1} + B_1) \cdot R'_1 > 0$ . If  $K_{X_1} + B_1$  is not nef, then the  $(K_{X_1} + B_1)$ -negative extremal ray is uniquely determined and it is  $R_1$ , where  $B_1$  is the strict transform of  $B_0$ . If the  $(K_{X_1} + B_1)$ -negative extremal contraction  $f_1 : X_1 \rightarrow Y_1$  corresponding to  $R_1$  is divisorial, then we finish.

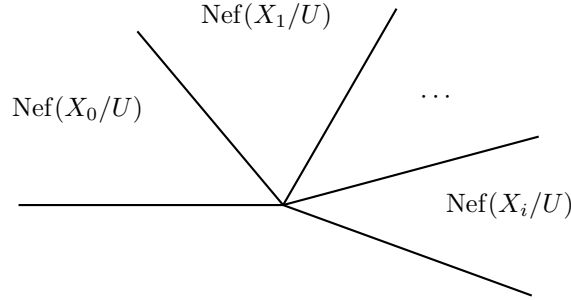
Assume that  $f_1$  is small and let  $f_1^+ : X_2 \rightarrow Y_1$  be its flip. By repeating this discussion, we uniquely obtain the following sequence of steps of the  $(K_{X_0} + B_0)$ -MMP:



The 2-ray game is also explained from the viewpoint of cones of divisors. The induced birational map  $X_i \dashrightarrow X_{i+1}$  is isomorphic in codimension one. Thus there are the following natural isomorphisms:

$$N^1(X_0/U) \cong N^1(X_1/U) \cong \cdots N^1(X_i/U) \cong \cdots$$

The nef cone  $\text{Nef}(X_i/U)$  is a dual cone of  $\overline{\text{NE}}(X_i/U)$  and so it is generated by two extremal rays  $L_i$  and  $L'_i$ . We may assume that  $L_i$  corresponds to  $f_i^+$ , that is,  $L_i \cdot R_i = 0$ , where  $R_i$  is the extremal ray corresponding to  $f_i^+$ . Then  $L_i$  is the pullback of  $\text{Nef}(Y_i/U)$ . Similarly,  $L'_i$  is also the pullback of  $\text{Nef}(Y_{i-1}/U)$ . Thus  $\text{Nef}(X_i/U)$  and  $\text{Nef}(X_{i+1}/U)$  have the same edge coming from the pullback of  $\text{Nef}(Y_i/U)$ .



**1.2.2. Basic results of the minimal model program.** The following lemma is well-known.

**LEMMA 1.2.6.** *Let  $(X, B)$  be a log canonical pair. Let  $D_1, \dots, D_k \geq 0$  be Cartier divisors passing through a closed point  $x \in X$ . If  $(X, B + \sum D_i)$  is log canonical, then  $k \geq \dim X$ .*

**PROOF.** We proceed by induction on  $\dim X$ . If  $\dim X = 1$ , then it is clear. Let  $\nu : Y \rightarrow D_k$  be the normalization of  $D_k$  and we put

$$K_Y + B_Y = \nu^*(K_X + B + D_k).$$

Since  $(X, B + \sum_{i=1}^k D_i)$  is log canonical,  $(Y, B_Y)$  is also log canonical by adjunction and  $D_k$  has no common components with  $D_i$  for  $1 \leq i \leq k-1$ . Thus by adjunction again,  $(Y, B_Y + \sum_{i=1}^{k-1} \nu^* D_i)$  is log canonical. We take a closed point  $y \in \nu^{-1}(x)$ .

Since  $x \in \text{Supp } \nu^* D_i$  for  $1 \leq i \leq k-1$ , it follows from the induction hypothesis that

$$k-1 \leq \dim Y = \dim X - 1.$$

This means that  $k \leq \dim X$ .  $\square$

The next one is the basic properties of ample models.

LEMMA 1.2.7 ([3, Lemma 3.6.6]). *Let  $X$  be a projective variety and let  $D$  be an  $\mathbb{R}$ -divisor on  $X$ .*

- (i) *If  $g_i : X \dashrightarrow X_i$  are two ample models of  $D$  ( $i = 1, 2$ ), then there is an isomorphism  $\chi : X_1 \rightarrow X_2$  such that  $g_2 = \chi \circ g_1$ .*
- (ii) *If  $f : X \dashrightarrow Y$  is a semi-ample model of  $D$ , then there is the ample model  $g : X \rightarrow Z$  of  $D$  and a contraction morphism  $h : Y \rightarrow Z$  such that  $g = h \circ f$  with  $f_* D \sim_{\mathbb{R}} h^* H$ , where  $H$  is an ample  $\mathbb{R}$ -Cartier divisor corresponding to  $g$ .*
- (iii) *If  $f : X \dashrightarrow Y$  is a birational map, then  $f$  is the ample model of  $D$  if and only if  $f$  is a semi-ample model of  $D$  such that  $f_* D$  is ample.*

### 1.3. Toric varieties

In this section, we quickly review minimal model theory for toric varieties.

**1.3.1. Toric varieties.** First of all, let us recall the definition of toric varieties. Let  $N \simeq \mathbb{Z}^n$  be a lattice of rank  $n$  and let  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  be its dual lattice.

DEFINITION 1.3.1 (cf. [14, 2.1]). A *toric variety*  $X(\Delta)$  is associated to a fan  $\Delta = \{\sigma\}$ , a finite collection of convex cones  $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  satisfying:

- (i) Each convex cone  $\sigma$  is rational polyhedral in the sense that there are finitely many  $v_1, \dots, v_s \in N \subset N_{\mathbb{R}}$  such that

$$\sigma = \{r_1 v_1, \dots, r_s v_s \mid r_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}$$

and it is strongly convex in the sense

$$\sigma \cap -\sigma = \{0\}.$$

- (ii) Each face  $\tau$  of a convex cone  $\sigma \in \Delta$  is again contained in  $\Delta$ .
- (iii) The intersection of two cones in  $\Delta$  is a face of each.

DEFINITION 1.3.2. The *dimension*  $\dim \sigma$  of a polyhedral cone  $\sigma$  is the dimension of the span of  $\sigma$ , that is, the smallest subspace of  $N_{\mathbb{R}}$  containing  $\sigma$ .

THEOREM 1.3.3 ([8, Theorem 6.3.12]). *Let  $D$  be  $\mathbb{R}$ -Cartier divisor on a normal toric variety  $X$ . The following are equivalent:*

- (i)  *$D$  is semi-ample.*
- (ii)  *$D$  is nef.*
- (iii)  *$D \cdot C \geq 0$  for all torus-invariant irreducible complete curve  $C \subset X$ .*

The following lemma is the combinatorial characterization of  $\mathbb{Q}$ -factoriality.

LEMMA 1.3.4. [8, Proposition 4.2.7] *Let  $X = X(\Delta)$  be a toric variety. Then  $X$  is  $\mathbb{Q}$ -factorial if and only if each of  $\sigma \in \Delta$  is simplicial, that is, there are  $v_1, \dots, v_s \in \sigma \cap N$  such that  $\sigma = \{\sum r_i v_i \mid r_i \geq 0\}$  where  $s = \dim \sigma$ .*

The following lemma is a well-known criteria for singularities of toric pairs.

LEMMA 1.3.5 ([8, Proposition 11.4.24]). *Let  $(X, D)$  be a pair where  $X$  is toric variety. If  $D \in [0, 1]$ , then  $(X, D)$  is log canonical. Moreover if  $D \in [0, 1)$ , then it is kawamata log terminal.*

**1.3.2. The minimal model theory for toric varieties.** In this subsection, we collect the basic results of the toric minimal model theory without proof.

The toric minimal model theory was first established by Reid [39] for projective toric varieties with  $\mathbb{Q}$ -factorial terminal singularities. His proof depends on the combinatorics for toric varieties. On the other hand, Fujino–Sato [14] gave an alternative proof and generalized his result with few or no combinatorial arguments.

THEOREM 1.3.6 ([14, Section 4]). *Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety and let  $D$  be an  $\mathbb{R}$ -divisor on  $Z$ .*

*Then we can run the  $D$ -minimal model program ( $D$ -MMP, for short). In other words, there is a sequence of divisorial contractions and flips*

$$Z = Z_0 \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_n = X$$

*such that either*

- (i) *if  $D$  is pseudo-effective, then  $X$  is a minimal model of  $D_X$ , or*
- (ii) *if  $D$  is not pseudo-effective, then there is a morphism  $X \rightarrow S$  which is a Mori fiber space of  $D_X$ ,*

*where  $D_X$  is the strict transform of  $D$ .*

We note that if  $(Z, B)$  is a  $\mathbb{Q}$ -factorial log canonical pair where  $Z$  is toric, then  $K_Z + B$  is not pseudo-effective and so  $(K_Z + B)$ -MMP always ends up with a Mori fiber space.

In the rest of this subsection, we assume that variety is  $\mathbb{Q}$ -factorial. Then definition of weak log canonical models and semi-ample models are coincides by Theorem 1.3.3. Thus we can reformulate Lemma 1.2.7 as follows.

LEMMA 1.3.7. *Let  $X$  be a projective toric variety and let  $D$  be an  $\mathbb{R}$ -divisor on  $X$ .*

- (i) *If  $g_i : X \dashrightarrow X_i$  are two ample models of  $D_i$  ( $i = 1, 2$ ), then there is an isomorphism  $\chi : X_1 \rightarrow X_2$  such that  $g_2 = \chi \circ g_1$ .*
- (ii) *If  $f : X \dashrightarrow Y$  is a weak log canonical model of  $D$ , then there is the ample model  $g : X \rightarrow Z$  of  $D$  and a contraction morphism  $h : Y \rightarrow Z$  such that  $g = h \circ f$  with  $f_*D \sim_{\mathbb{R}} h^*H$ , where  $H$  is an ample  $\mathbb{R}$ -divisor corresponding to  $g$ .*
- (iii) *If  $f : X \dashrightarrow Y$  is a birational map, then  $f$  is the ample model of  $D$  if and only if  $f$  is a weak log canonical model of  $D$  such that  $f_*D$  is ample.*





## CHAPTER 2

### The Sarkisov program

In this chapter, we closely follow [35]. Our purpose in this chapter is to establish the log Sarkisov program for  $\mathbb{Q}$ -factorial and toric varieties:

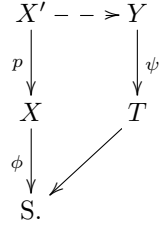
**Theorem 1.0.5.** Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety and let  $D_Z$  be an  $\mathbb{R}$ -divisor on  $Z$ . Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $D_Z$ -MMP. Then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many Sarkisov links.

Let us start with the definition of Sarkisov links for toric varieties.

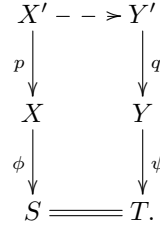
**DEFINITION 2.0.1.** Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety and let  $D_Z$  be an  $\mathbb{R}$ -divisor on  $Z$ . Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $D_Z$ -MMP.

The induced birational map  $\sigma : X \dashrightarrow Y$  between  $\phi$  and  $\psi$  is called a *Sarkisov link* if it is one of the following four types:

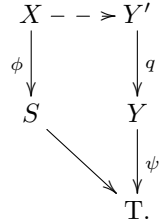
Type (I)



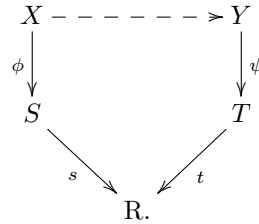
Type (II)



Type (III)



Type (IV)



In the above commutative diagram, the vertical arrows  $p$  and  $q$  are divisorial contractions and the horizontal dotted arrows are compositions of finitely many flops for the  $D'_Z$ -MMP, where  $D'_Z$  is an  $\mathbb{R}$ -divisor on the top left space, that is,  $X'$  or  $X$ . The spaces  $X'$ ,  $Y'$  and  $R$  are realized as the ample models of  $\mathbb{R}$ -divisors on  $Z$ , which are the results of the  $D_Z$ -MMP. In links of Type (I)-(III),  $R$  is  $\mathbb{Q}$ -factorial. Moreover, links of Type (IV) are separated into two types:  $(IV_m)$  and  $(IV_s)$ . In a

link of Type  $(IV_m)$ ,  $s$  and  $t$  have both Mori fiber structures and  $R$  is  $\mathbb{Q}$ -factorial. In a link of Type  $(IV_s)$ ,  $s$  and  $t$  are small birational contractions and  $R$  is not  $\mathbb{Q}$ -factorial. We note that links of Type  $(IV_s)$  do not appear for  $\dim Z \leq 3$ .

EXAMPLE 2.0.2. For toric 3-folds with terminal singularities, links of Type (I), (II), (III) and  $(IV_m)$  are completely classified by Shramov and we can find various examples in [43].

Let us construct an easy example of links of Type  $(IV_s)$ . Let  $S \rightarrow R \leftarrow T$  be a flop for a divisor and we put  $X = S \times \mathbb{P}^1$  and  $Y = T \times \mathbb{P}^1$ . Then here is a link of Type  $(IV_s)$ .

Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety and let  $D$  be a Weil divisor on  $Z$ . Then there exists a positive integer  $m$  such that  $mD$  is linearly equivalent to a torus-invariant Cartier divisor on  $Z$ . Hence any  $\mathbb{R}$ -divisor on  $Z$  is  $\mathbb{R}$ -linearly equivalent to a torus-invariant  $\mathbb{R}$ -Cartier divisor. Therefore, it is sufficient to prove the following statement for Theorem 1.0.5.

THEOREM 2.0.3. *Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety and let  $D_Z$  be a torus-invariant  $\mathbb{R}$ -divisor on  $Z$ . Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $D_Z$ -MMP. Then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many Sarkisov links.*

By combining Proposition 1.0.3 and Theorem 1.0.5, we obtain the Sarkisov program when  $Z$  is  $\mathbb{Q}$ -factorial and toric.

COROLLARY 2.0.4. *Let  $\phi : X \rightarrow S$  and  $\psi : X \rightarrow Y$  be two toric Mori fiber spaces with  $\mathbb{Q}$ -factorial terminal singularities. If  $X$  and  $Y$  are birational, then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many Sarkisov links.*

Moreover, Conjecture 1.0.4 for toric varieties, directly follows from Proposition 1.3.5.

COROLLARY 2.0.5 (Log Sarkisov program for toric varieties). *Let  $(Z, B)$  be a projective log canonical pair, where  $X$  is  $\mathbb{Q}$ -factorial and toric. Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces, which are outputs of the  $(K_X + B)$ -MMP. Then the induced birational map  $\sigma : X \dashrightarrow Y$  is connected by finitely many elementary links.*

## 2.1. Geography of (log) models

In this section, we explain “geography of (log) models”.

Let  $Z$  be a  $\mathbb{Q}$ -factorial projective toric variety. Then the real vector space generated by all torus-invariant prime divisors on  $Z$  is denoted by  $\mathcal{V}(Z)$ .

Let  $\mathcal{B}$  be a convex polytope in  $\mathcal{V}(Z)$ . Let  $f : Z \dashrightarrow X$  be a birational contraction to a normal projective variety  $X$  and let  $g : Z \dashrightarrow Y$  a rational contraction to a normal projective variety  $Y$ . Let

$$\begin{aligned} \mathcal{E}(\mathcal{B}) &= \{D_Z \in \mathcal{B} \mid D_Z \text{ is pseudo-effective}\}, \\ \mathcal{M}_f(\mathcal{B}) &= \{D_Z \in \mathcal{E}(\mathcal{B}) \mid f \text{ is a minimal model of } D_Z\}, \\ \mathcal{A}_g(\mathcal{B}) &= \{D_Z \in \mathcal{E}(\mathcal{B}) \mid g \text{ is the ample model of } D_Z\}, \\ \mathcal{N}(\mathcal{B}) &= \{D_Z \in \mathcal{E}(\mathcal{B}) \mid D_Z \text{ is nef}\} \end{aligned}$$

and we denote the closure of  $\mathcal{A}_g(\mathcal{B})$  by  $\mathcal{C}_g(\mathcal{B})$ . We simply write  $\mathcal{A}_g$  to denote  $\mathcal{A}_g(\mathcal{B})$  if there is no risk of confusion.

In the rest of this subsection and next one, we fix the following notation unless otherwise mentioned.

- $Z$  is a  $\mathbb{Q}$ -factorial projective toric variety,
- $\mathcal{B}$  is a convex polytope of  $\mathcal{V}(Z)$ , which is defined over  $\mathbb{Q}$ .

PROPOSITION 2.1.1. *There are only finitely many rational contractions  $g_i : Z \dashrightarrow X_i$  ( $1 \leq i \leq l$ ) such that*

$$\mathcal{E}(\mathcal{B}) = \bigcup_{i=1}^l \mathcal{A}_{g_i},$$

where  $\mathcal{A}_{g_i} \neq \mathcal{A}_{g_j}$  for  $i \neq j$ .

PROOF. It follows from the finiteness of minimal models (see [8, Theorem 15.5.15]) and Lemma 1.3.7.  $\square$

The following two statements come from [21, Theorem 3.3] and these are easy consequences of minimal model theory. Thus we give simple proofs. For the details, see [21, Theorem 3.3 (2), (3)].

PROPOSITION 2.1.2. *With notation as in Proposition 2.1.1. If  $\mathcal{A}_{g_j} \cap \mathcal{C}_{g_i} \neq \emptyset$  for  $1 \leq i, j \leq l$ , then there is a contraction morphism  $g_{i,j} : X_i \rightarrow X_j$  such that  $g_j = g_{i,j} \circ g_i$ .*

SKETCH OF PROOF. We take  $D_Z \in \mathcal{A}_{g_i}$ . Running the  $D_Z$ -MMP, we end up with a minimal model  $f : Z \dashrightarrow X$  of  $D_Z$ . Then there is a contraction morphism  $g : X \rightarrow X_i$  such that  $g_i = g \circ f$ . Using this morphism  $g$ , we can construct a semi-ample divisor on  $X_i$  associated to the contraction morphism satisfying the desired property.  $\square$

PROPOSITION 2.1.3. *With notation as in Proposition 2.1.1. Assume that  $\mathcal{B}$  spans  $N^1(Z)$ . For any  $1 \leq i \leq l$ , the following are equivalent:*

- *there is a rational polytope  $\mathcal{C}$  contained in  $\mathcal{C}_{g_i}$  which intersects the interior of  $\mathcal{B}$  and spans  $\mathcal{B}$ .*
- *$g_i$  is birational and  $X_i$  is  $\mathbb{Q}$ -factorial.*

SKETCH OF PROOF. Suppose that  $\mathcal{C}$  span  $\mathcal{B}$ . We take  $D_Z$  belonging to the relative interior of  $\mathcal{C} \cap \mathcal{A}_{g_i}$  belonging to the interior of  $\mathcal{B}$ . Running the  $D_Z$ -MMP, we end up with a minimal model  $f : Z \dashrightarrow X$  of  $D_Z$ . Then there is certain index  $1 \leq j \leq l$  such that  $f = g_j$ . Since  $D_Z$  belonging to the relative interior of  $\mathcal{A}_{g_i}$ , we see that  $i = j$ . Thus  $g_i = f$  is birational and  $X_i = X$  is  $\mathbb{Q}$ -factorial. It is easy to see the converse.  $\square$

The following proposition is the key result of this section.

PROPOSITION 2.1.4 ([21, Theorem 3.3 (4)]). *With notation as in Proposition 2.1.1. Assume that  $\mathcal{B}$  spans  $N^1(Z)$ . If  $\mathcal{C}_{g_i}$  spans  $\mathcal{B}$  and  $D_Z$  is a general point of  $\mathcal{A}_{g_j} \cap \mathcal{C}_{g_i}$ , which is also a point of the interior of  $\mathcal{B}$  for  $1 \leq i, j \leq l$ , then  $\rho(X_i/X_j) = \dim \mathcal{C}_{g_i} - \dim \mathcal{C}_{g_j} \cap \mathcal{C}_{g_i}$ .*

PROOF. Putting  $X = X_i$  and  $f = g_i$ , by Proposition 2.1.3,  $X$  is  $\mathbb{Q}$ -factorial and  $f$  is birational. Let  $E_1, \dots, E_k$  be all  $f$ -exceptional prime divisors. Since  $\mathcal{B}$  spans  $N^1(Z)$ , we can take  $B_i \in \mathcal{V}(Z)$ , which are linear combinations of the elements of  $\mathcal{B}$ , such that  $B_i \equiv E_i$  and put  $B_0 = \sum B_i$  and  $E_0 = \sum E_i$ . Since  $D_Z$  is contained

in the interior of  $\mathcal{B}$ , there is a sufficiently small rational number  $\delta > 0$  such that  $D_Z + \delta B_0 \in \mathcal{B}$ . Then  $f$  is  $(D_Z + \delta E_0)$ -negative and so it is a minimal model of  $D_Z + \delta E_0$  and  $g_j$  is the ample model of  $D_Z + \delta E_0$ . Thus  $D_Z + \delta B_0 \in \mathcal{M}_f(\mathcal{B})$  and  $D_Z + \delta B_0 \in \mathcal{A}_{g_j}$ . In particular,  $D_Z + \delta B_0 \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$ . Since  $D_Z$  is general in  $\mathcal{A}_{g_j} \cap \mathcal{C}_f$ ,  $D_Z \in \mathcal{M}_f(\mathcal{B})$  and so  $f$  is  $D_Z$ -negative.

We fix a sufficiently small rational number  $\epsilon > 0$  such that if  $D'_Z \in \mathcal{B}$  with  $\|D'_Z - D_Z\| < \epsilon$ , then  $D'_Z \in \mathcal{B}$  and  $f$  is  $D_Z$ -negative. Then  $D'_Z \in \mathcal{C}_f$  if and only if  $D'_X = f_*D'_Z$  is nef.

For any  $(a_1, \dots, a_k) \in \mathbb{R}^k$ , we put  $E = \sum a_i E_i$  and  $B = \sum a_i B_i$ . We put  $\mathcal{B}_X = \{D'_X = f_*D'_Z \mid D'_Z \in \mathcal{B}\} \subset \mathcal{V}(X)$ . Then  $D'_X \in \mathcal{N}(\mathcal{B}_X)$  if and only if  $D'_X + f_*B \in \mathcal{N}(\mathcal{B}_X)$  as  $D'_Z + B$  is numerically equivalent to  $D'_Z + E$ . This means that

$$\mathcal{C}_f \simeq \mathcal{N}(\mathcal{B}_X) \times \mathbb{R}^k$$

in a neighbourhood of  $D_Z$ .

By the above argument and Lemma 1.3.7,  $D'_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$  if and only if it holds that  $D'_X = f_*D'_Z \in \mathcal{N}(\mathcal{B}_X)$  and there is an ample  $\mathbb{R}$ -Cartier divisor  $H$  on  $X_j$  such that  $f_*D'_Z = (g_{i,j})^*H$ , where  $g_{i,j} : X \rightarrow X_j$  is a contraction morphism. Since  $D_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$ , there is an ample  $\mathbb{R}$ -Cartier divisor  $H_0$  on  $X_j$  such that  $f_*D_Z = (g_{i,j})^*H_0$  and so there are ample  $\mathbb{R}$ -Cartier divisors  $H_1, \dots, H_{\rho(X_j)}$ , whose images on  $N^1(X_j)$  are linearly independent, such that  $f_*D'_Z = (g_{i,j})^*(H_0 + \sum b_i H_i)$  for any  $(b_1, \dots, b_{\rho(X_j)}) \in \mathbb{R}^{\rho(X_j)}$  with  $H_0 + \sum b_i H_i$  is ample. Thus

$$\dim \mathcal{C}_{g_j} \cap \mathcal{C}_f = k + \rho(X_j).$$

Therefore we obtain

$$\begin{aligned} \rho(X_i/X_j) &= \rho(X_i) - \rho(X_j) \\ &= \dim \mathcal{N}(\mathcal{B}_X) - \rho(X_j) \\ &= \dim \mathcal{C}_f - \dim \mathcal{C}_{g_j} \cap \mathcal{C}_f. \end{aligned}$$

□

We recall the following Bertini-type statement for the reader's convenience.

**LEMMA 2.1.5** (cf. [21, Corollary 3.4]). *Let  $\mathcal{P}$  be a convex polytope in  $\mathcal{V}(Z)$  which spans  $N^1(Z)$ . Then for any general affine subspace  $\mathcal{H} \subset \mathcal{V}(Z)$ , the intersection  $\mathcal{P} \cap \mathcal{H}$  of  $\mathcal{P}$  and  $\mathcal{H}$  satisfies the conclusions of Proposition 2.1.3 and 2.1.4.*

**LEMMA 2.1.6** (cf. [21, Lemma 3.5]). *Assume that  $\mathcal{B}$  satisfies the conclusion of Propositions 2.1.3 and 2.1.4 and that  $\dim \mathcal{B} = 2$ . Let  $f : Z \dashrightarrow X$  and  $g : Z \dashrightarrow Y$  be two rational contractions such that  $\dim \mathcal{C}_f = 2$  and  $\dim \mathcal{O} = 1$ , where  $\mathcal{O} = \mathcal{C}_f \cap \mathcal{C}_g$ . Assume that  $\rho(X) \geq \rho(Y)$  and that  $\mathcal{O}$  is not contained in the boundary of  $\mathcal{B}$ . Let  $D_Z$  be a point in the relative interior of  $\mathcal{O}$  and  $D_X = f_*D_Z$ .*

*Then there is a rational map  $\pi : X \dashrightarrow Y$  such that  $g = \pi \circ f$  and either*

- (1)  $\rho(X) = \rho(Y) + 1$  and  $\pi$  is  $D_X$ -trivial,
  - (a)  $\pi$  is birational and  $\mathcal{O}$  is not contained in the boundary of  $\mathcal{E}(\mathcal{B})$ ,
    - (i)  $\pi$  is divisorial and  $\mathcal{O} \neq \mathcal{C}_g$ ,
    - (ii)  $\pi$  is small and  $\mathcal{O} = \mathcal{C}_g$ ,
  - (b)  $\pi$  has a Mori fiber structure and  $\mathcal{O} = \mathcal{C}_g$  is contained in the boundary of  $\mathcal{E}(\mathcal{B})$ ,
- (2)  $\rho(X) = \rho(Y)$ ,  $\pi$  is a  $D_X$ -flop and  $\mathcal{O} \neq \mathcal{C}_g$  is not contained in the boundary of  $\mathcal{E}(\mathcal{B})$ .

PROOF. By Proposition 2.1.3,  $f$  is birational and  $X$  is  $\mathbb{Q}$ -factorial.

If  $\mathcal{O}$  is contained in the boundary of  $\mathcal{E}(\mathcal{B})$ , then  $\dim \mathcal{C}_g = 1$  and  $\mathcal{O} = \mathcal{C}_g$ . By Proposition 2.1.4, there is a contraction  $\pi : X \rightarrow Y$  which has a Mori fiber structure. This is (1, b).

In the rest of proof, we may assume that  $\mathcal{O}$  is not contained in the boundary of  $\mathcal{E}(\mathcal{B})$ . If  $\dim \mathcal{C}_g = 1$ , then  $\mathcal{O} = \mathcal{C}_g$ . By Proposition 2.1.4, there is a contraction  $\pi : X \rightarrow Y$  with  $\rho(X/Y) = 1$ . Since  $D_Z$  is not contained in the boundary of  $\mathcal{E}(\mathcal{B})$ ,  $D_Z$  is big and so  $\pi$  is birational. Thus by Proposition 2.1.3,  $Y$  is not  $\mathbb{Q}$ -factorial and so  $\pi$  is small. This is (1, a, ii).

We assume that  $\dim \mathcal{C}_g = 2$ . Then  $g$  is birational and  $Y$  is  $\mathbb{Q}$ -factorial. Let  $h : Z \rightarrow W$  be the ample model of  $D_Z$ . By Proposition 2.1.4, there are two contractions  $p : X \rightarrow W$  and  $q : Y \rightarrow W$  with  $\rho(X/W), \rho(Y/W) \leq 1$ . Then we can explicitly calculate the Picard numbers of  $X$  and  $Y$  and there are only two cases below:

- (1)  $\rho(X) = \rho(Y) + 1$  or
- (2)  $\rho(X) = \rho(Y)$ .

In (1),  $h = g$  and we put  $\pi = p$ . Then  $\pi$  is divisorial and this is (1, a, i).

In (2),  $\rho(X/W) = \rho(Y/W) = 1$ . Then  $\dim \mathcal{C}_h = 1$  since  $\dim \mathcal{O} = 1$ . By Theorem 2.1.3,  $W$  is not  $\mathbb{Q}$ -factorial. Thus  $p$  and  $q$  are small and so  $\pi$  is  $D_X$ -flop.  $\square$

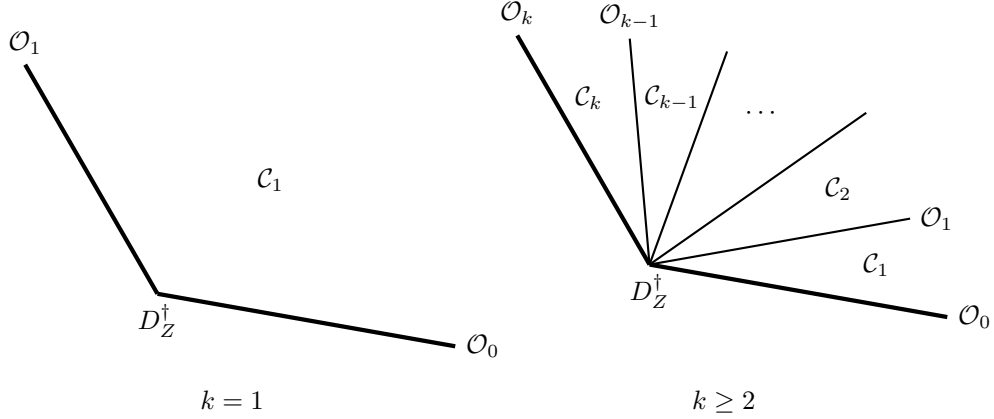
LEMMA 2.1.7 (cf. [21, Lemma 3.6]). *Let  $f : Z \dashrightarrow X$  be a birational contraction between  $\mathbb{Q}$ -factorial projective toric varieties. Let  $D_Z$  and  $D'_Z$  be two torus-invariant  $\mathbb{R}$ -divisors on  $Z$ . If  $f$  is the ample model of  $D'_Z$  and  $D'_Z - D_Z$  is ample, then  $f$  is the result of running the  $D_Z$ -MMP.*

PROOF. We take an ample divisor  $H$  on  $Z$  such that  $D'_Z + H$  is ample and  $tH \sim_{\mathbb{R}} D_Z - D'_Z$  for some positive real number  $0 < t < 1$ . Then  $f$  is the ample model of  $D'_Z + tH$ . We take any  $s < t$  sufficiently close to  $t$ . Since  $f$  is  $(D'_Z + tH)$ -non-positive and  $H$  is ample,  $f$  is  $(D'_Z + sH)$ -negative. Then  $f$  is the unique minimal model of  $D'_Z + sH$ . In particular, if we run the  $(D'_Z + sH)$ -MMP with scaling of  $H$  and the value of the scalar is  $s$ , then the induced morphism is  $f$ .  $\square$

Next we will see that a certain point contained in the boundary of  $\mathcal{E}(V)$  corresponds to a Sarkisov link. Before that we introduce the following additional notation.

NOTATION 2.1.8. Assume that  $\mathcal{B}$  satisfies the conclusion of Propositions 2.1.3 and 2.1.4 and that  $\dim \mathcal{B} = 2$ . Let  $D_Z^\dagger$  be a point contained in the boundary of  $\mathcal{E}(\mathcal{B})$  and the interior of  $\mathcal{B}$ . If  $D_Z^\dagger$  is contained in only one polytope of the form  $\mathcal{C}_\bullet$  of two-dimensional, then we assume that it is a vertex of  $\mathcal{E}(\mathcal{B})$ .

Let  $\mathcal{C}_{f_1}, \dots, \mathcal{C}_{f_k}$  be all two-dimensional rational polytopes containing  $D_Z^\dagger$ , where  $f_i : Z \dashrightarrow X_i$  are rational maps. Note that  $f_i$  is birational and  $X_i$  is  $\mathbb{Q}$ -factorial by Proposition 2.1.3. Renumbering  $\mathcal{C}_{f_i}$  to  $\mathcal{C}_i$ , let  $\mathcal{O}_0$  (resp.  $\mathcal{O}_k$ ) be the intersection of  $\mathcal{C}_1$  (resp.  $\mathcal{C}_k$ ) with the boundary of  $\mathcal{E}(\mathcal{B})$ , and let  $\mathcal{O}_i := \mathcal{C}_i \cap \mathcal{C}_{i+1}$  ( $1 \leq i \leq k-1$ ). Then, we may assume that  $\mathcal{O}_i$  is one-dimensional for any  $i$ . Let  $g_i : Z \dashrightarrow S_i$  be the rational contractions associated to  $\mathcal{O}_i$ . We put  $f = f_1 : Z \dashrightarrow X = X_1$ ,  $g = f_k : Z \dashrightarrow Y = X_k$ ,  $X' = X_2$  and  $Y' = X_{k-1}$ . Then, by Proposition 2.1.2, there are contraction morphisms  $\phi : X \rightarrow S = S_0$  and  $\psi : Y \rightarrow T = S_k$ . Let  $h : Z \dashrightarrow R$  be the ample model of  $D_Z^\dagger$ .



**THEOREM 2.1.9** (cf. [21, Theorem 3.7]). *Let  $\mathcal{B}$  and  $D_Z^\dagger$  be notation as above. Let  $D_Z$  be an  $\mathbb{R}$ -divisor on  $Z$  with  $D_Z^\dagger - D_Z$  is ample.*

*Then  $\phi$  and  $\psi$  are Mori fiber spaces, which are outputs of the  $D_Z$ -MMP, and  $f_i$  are the result of running the  $D_Z$ -MMP. Moreover, if  $D_Z^\dagger$  is contained in more than two polytopes, then  $\phi$  and  $\psi$  are connected by a Sarkisov link.*

**PROOF.** By Lemma 2.1.6, we have the following commutative diagram:

$$\begin{array}{ccc}
 X' = X_2 & \dashrightarrow & Y' = X_{k-1} \\
 \downarrow p & & \downarrow q \\
 X = X_1 & & Y = X_k \\
 \downarrow \phi & & \downarrow \psi \\
 S & & T \\
 \searrow s & & \swarrow t \\
 & R &
 \end{array}$$

where  $p$ ,  $q$  and the horizontal arrow  $X' \dashrightarrow Y'$  are birational and  $\phi$  and  $\psi$  have Mori fiber structures. Since  $D_Z^\dagger - D_Z$  is ample, for any  $i$  we can take  $D_i \in \mathcal{C}_i$  such that  $D_i - D_Z$  is ample. By Lemma 2.1.7,  $f_i$  is the result of running the  $D_Z$ -MMP. By Proposition 2.1.4, there is a contraction  $X_i \rightarrow R$  with  $\rho(X_i/R) \leq 2$ . If  $\rho(X_i/R) = 0$ , then  $f_i = h$  and this case does not happen. If  $\rho(X_i/R) = 1$ , then  $X_i \rightarrow R$  gives a Mori fiber structure. By Lemma 2.1.6,  $\dim \mathcal{C}_h = 1$  and there is a facet of  $\mathcal{C}_i$  contained in the boundary of  $\mathcal{E}(V)$  and so  $i = 1$  or  $k$ . Therefore if  $k \geq 3$ , then  $\rho(X_i/R) = 2$  for any  $1 < i < k$  and  $X' \dashrightarrow Y'$  is connected by flops by Lemma 2.1.6 again. Moreover since  $\rho(X'/R) = 2$ ,  $p$  is divisorial and  $s$  is the identity, or  $p$  is flop and  $s$  is not the identity. For  $q$  and  $t$ , similar conditions follow and there are only 7 possibilities below:

- (1)  $k = 1$ ,
- (2)  $k = 2$ ,  $\rho(X/R) = 1$  and  $\rho(Y/R) = 2$ ,
- (3)  $k = 2$ ,  $\rho(X/R) = 2$  and  $\rho(Y/R) = 1$ ,
- (4)  $k \geq 3$ ,  $p$  and  $q$  are divisorial, and  $s$  and  $t$  are the identities,
- (5)  $k \geq 3$ ,  $p$  divisorial,  $q$  is flop,  $s$  is the identity and  $t$  is not the identity,

- (6)  $k \geq 3$ ,  $p$  is flop,  $q$  is divisorial,  $s$  is not the identity and  $t$  is the identity,
- (7)  $k \geq 3$ ,  $p$  and  $q$  are flops, and  $s$  and  $t$  are not the identities.

In (1),  $X = Y$  and this is a link of Type (IV). In (2),  $s$  is the identity and  $\rho(Y) \geq \rho(X)$ . Then by Lemma 2.1.6, there is a divisorial contraction  $X' = Y \rightarrow X$ . Thus this is a special case of a link of Type (I). In (3), this is similar to (2) and we obtain a special case of a link of Type (III). In (4), this is a link of Type (II). In (5), this is a link of Type (I). In (6), this is a link of Type (III). In (7), this is a link of Type (IV).

The rest of the proof is that a link of Type (IV) is splitting into two types ( $IV_m$ ) and ( $IV_s$ ) in (1) and (7).

We assume that  $s$  is a divisorial contraction. Then there is a prime divisor  $F$  on  $S$  which is contracted by  $s$ . Since  $\rho(X/S) = 1$ , there is a prime divisor  $E$  on  $X$  such that  $mE = \phi^*F$  for some  $m \in \mathbb{Z}_{\geq 0}$ . Since  $D_X^\dagger = f_*D_Z^\dagger$  is numerically trivial over  $R$ ,  $\mathbf{B}(D_X^\dagger + E/R) = E$ . Since  $\rho(X/R) = 2$ , by the 2-ray game, there are birational contractions  $X \dashrightarrow V \xrightarrow{f'} W \xrightarrow{g} U$  such that  $f'$  is a divisorial contraction and  $g$  has a Mori fiber space. In (1), this is a contradiction since  $X = Y$ ,  $\phi$  and  $\psi$  are Mori fiber spaces and  $\rho(X/R) = 2$ . In (7),  $W = Y$  and  $U = T$  and so this gives a link of Type (III) and this is a contradiction. Similarly,  $t$  is not divisorial. Thus  $s$  and  $t$  are not divisorial.

If  $s$  has a Mori fiber structure, then  $R$  is  $\mathbb{Q}$ -factorial and so  $t$  has also a Mori fiber structure. Thus this is a link of Type ( $IV_m$ ).

If  $s$  is a small contraction, then  $R$  is not  $\mathbb{Q}$ -factorial and so  $t$  is also small. Thus this is a link of Type ( $IV_s$ ).  $\square$

LEMMA 2.1.10 (cf. [21, Lemma 4.1]). *Let  $D_Z$  be a torus-invariant  $\mathbb{R}$ -divisor on  $Z$ . Let  $f : Z \dashrightarrow X$  and  $g : Z \dashrightarrow Y$  be the results of the  $D_Z$ -MMP. Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two Mori fiber spaces which are outputs of the  $D_Z$ -MMP.*

*Then we can find a two-dimensional convex polytope  $\mathcal{B} \subset \mathcal{V}(Z)$ , which is defined over  $\mathbb{Q}$ , with the following properties:*

- (1)  $D'_Z - D_Z$  is ample for any  $D'_Z \in \mathcal{E}(\mathcal{B})$ ,
- (2)  $\mathcal{A}_{\phi \circ f}$  and  $\mathcal{A}_{\psi \circ g}$  are not contained in the boundary of  $\mathcal{B}$ ,
- (3)  $\mathcal{C}_f$  and  $\mathcal{C}_g$  are two-dimensional,
- (4)  $\mathcal{C}_{\phi \circ f}$  and  $\mathcal{C}_{\psi \circ g}$  are one-dimensional,
- (5)  $L = \{D'_Z \in \mathcal{E}(\mathcal{B}) \mid D'_Z \text{ is not big}\}$  is connected.

PROOF. We take ample torus-invariant divisors  $H_1, \dots, H_r \geq 0$ , which generate  $N^1(Z)$ , and we put  $H = H_1 + \dots + H_r$ . By assumption, there are ample divisors  $C$  on  $S$  and  $D$  on  $T$ , respectively, such that

$$-f_*D_Z + \phi^*C \text{ and } -g_*D_Z + \psi^*D$$

are both ample. Let  $\epsilon > 0$  be a sufficiently small rational number. Then

$$-f_*D_Z + \epsilon f_*H + \phi^*C \text{ and } -g_*D_Z + \epsilon g_*H + \psi^*D$$

are both ample and  $f$  and  $g$  are both  $(D_Z + \epsilon H)$ -negative. Replacing  $H$  by  $\epsilon H$ , we may assume that  $\epsilon = 1$ . We take a torus-invariant  $\mathbb{Q}$ -divisor  $\tilde{D}_Z$  on  $X$  sufficiently close to  $D_Z$ . Then

$$-f_*\tilde{D}_Z + f_*H + \phi^*C \text{ and } -g_*\tilde{D}_Z + g_*H + \psi^*D$$



are both ample and  $f$  and  $g$  are both  $(\tilde{D}_Z + H)$ -negative. We take torus-invariant  $\mathbb{Q}$ -divisors  $H'_{r+1}$  and  $H'_{r+2}$  on  $X$  and  $Y$ , respectively, such that

$$H'_{r+1} \in |-f_*\tilde{D}_Z + f_*H + \phi^*C|_{\mathbb{Q}} \text{ and } H'_{r+2} \in |-g_*\tilde{D}_Z + g_*H + \phi^*D|_{\mathbb{Q}}.$$

There are torus-invariant  $\mathbb{Q}$ -divisors  $H_{r+1}$  and  $H_{r+2}$  on  $Z$  such that

$$H_{r+1} \sim_{\mathbb{Q}} f^*H'_{r+1} \text{ and } H_{r+2} \sim_{\mathbb{Q}} g^*H'_{r+2}.$$

Let  $a > 0$  be a sufficiently large rational number and we put a rational convex polytope

$$\mathcal{B}_0 = \left\{ \tilde{D}_Z + a \sum_{i=1}^{r+2} t_i H_i \mid \sum_{i=1}^{r+2} t_i \leq 1, t_i \geq 0 \right\}.$$

Possibly replacing  $H_i$  by suitable ones, we may assume that (2) holds for  $\mathcal{B}_0$ .

On the other hand, since  $f$  is  $(\tilde{D}_Z + H + H_{r+1})$ -negative and  $(\phi \circ f)_*(\tilde{D}_Z + H + H_{r+1})$  is ample,  $\tilde{D}_Z + H + H_{r+1} \in \mathcal{A}_{\phi \circ f}(\mathcal{B}_0)$ . Similarly,  $\tilde{D}_Z + H + H_{r+2} \in \mathcal{A}_{\psi \circ g}(\mathcal{B}_0)$  and  $f$  and  $g$  are weak log canonical models of  $\tilde{D}_Z + H + H_{r+1}$  and  $\tilde{D}_Z + H + H_{r+2}$ , respectively. Thus  $\tilde{D}_Z + H + H_{r+1} \in \mathcal{C}_f(\mathcal{B}_0)$  and  $\tilde{D}_Z + H + H_{r+2} \in \mathcal{C}_g(\mathcal{B}_0)$ .

Let  $\mathcal{H}_0$  be the translation by  $\tilde{D}_Z$  of the affine subspace generated by  $H + H_{r+1}$  and  $H + H_{r+2}$  and let  $\mathcal{H}$  be a small perturbation of  $\mathcal{H}_0$ , which is defined over  $\mathbb{Q}$ . Putting  $\mathcal{B} = \mathcal{B}_0 \cap \mathcal{H}$ ,  $\mathcal{B}$  satisfies (1) and (2). Since  $\mathcal{B}_0$  spans  $N^1(Z)$ ,  $\mathcal{C}_{\phi \circ f}(\mathcal{B}_0)$  spans  $\mathcal{B}_0$  and so by Lemma 2.1.5,  $\mathcal{B}$  satisfies (3). By Proposition 2.1.4,  $\dim \mathcal{C}_{\phi \circ f}(\mathcal{B}) = \dim \mathcal{C}_{\psi \circ g}(\mathcal{B}) = 1$  and so  $\mathcal{B}$  satisfies (4).

Finally we see that we can take  $\mathcal{B}$  satisfying (5). Since  $\phi$  and  $\psi$  are Mori fiber spaces, we may assume that  $\rho(Z) \geq 2$ . There is a surjective linear map from  $\mathcal{V}(Z)$  to  $N^1(Z)$ . Then the pullback of the pseudo-effective cone  $\overline{\text{Eff}}(Z) \subset N^1(Z)$  via this map is the convex polyhedron  $\mathcal{P}$  containing a  $(\dim Z)$ -dimensional vector subspace  $V$  since  $\dim \mathcal{V}(Z) = \rho(Z) + \dim Z$ . Then possibly replacing  $H_i$  by suitable ones, we can take a two-dimensional rational convex polytope  $\mathcal{B}$ , which does not contain  $V$ , since  $\text{codim } V = \rho(Z) \geq 2$ . Thus  $\mathcal{B}$  satisfies (5) as  $\mathcal{E}(\mathcal{B}) = \mathcal{B} \cap \mathcal{P}$ .  $\square$

## 2.2. Proof of Theorem 2.0.3

PROOF OF THEOREM 2.0.3. We take a two-dimensional rational convex polytope  $\mathcal{B} \subset \mathcal{V}(Z)$  given by Lemma 2.1.10. We take  $D_0 \in \mathcal{A}_{\phi \circ f}$  and  $D_1 \in \mathcal{A}_{\psi \circ g}$  belonging to the interior of  $\mathcal{B}$ . As  $\mathcal{B}$  is two-dimensional, removing two points  $D_0$  and  $D_1$ , the boundary of  $\mathcal{E}(\mathcal{B})$  separates into two parts. Then one of the two parts of the boundary of  $\mathcal{E}(\mathcal{B})$  is contained in  $L$  by Lemma 2.1.10 (5). Tracing this part from  $D_0$  to  $D_1$ , we obtain finitely many points  $D_i$  ( $2 \leq i \leq k$ ) which are contained in rational polytopes of two-dimensional. By Theorem 2.1.9, each of  $D_i$  gives a Sarkisov link and  $\sigma$  is connected by these links.  $\square$

## 2.3. Examples

In this section, we give some examples of toric Sarkisov links, which are  $\mathbb{Q}$ -factorial surfaces with singularities.

EXAMPLE 2.3.1 ([34]). We fix a lattice  $N = \mathbb{Z}^2$  and take lattice points

$$v_1 = (1, 0), v_2 = (1, n), v_3 = (-1, 0), v_4 = (-1, -n),$$

where  $n$  is an integer with  $n \geq 2$ .

We consider the following fan

$$\Delta = \{\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2, \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3, \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4, \\ \mathbb{R}_{\geq 0}v_4 + \mathbb{R}_{\geq 0}v_1 \text{ and their faces}\}.$$

Then the associated toric variety  $X = X(\Delta)$  is a projective  $\mathbb{Q}$ -factorial toric surface. We consider  $\mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $(x, y) \mapsto y$  and  $\mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $(x, y) \mapsto nx - y$ .

Then, we obtain two different toric morphisms  $p_1, p_2 : X \rightarrow \mathbb{P}^1$ . This gives a Sarkisov link of Type (IV).

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ p_1 \downarrow & & \downarrow p_2 \\ \mathbb{P}^1 & & \mathbb{P}^1 \\ & \searrow & \swarrow \\ & \text{pt.} & \end{array}$$

We note that  $X \not\cong \mathbb{P}^1 \times \mathbb{P}^1$ .

EXAMPLE 2.3.2. In Example 2.3.1, we put

$$v_1 = (1, 0), v_2 = (1, n), v_3 = (0, 1), v_4 = (-1, -1),$$

where  $n$  is an integer with  $n \geq 2$ .

Then the associated toric variety  $X = X(\Delta)$  is a projective  $\mathbb{Q}$ -factorial toric surface with  $\rho(X) = 2$ . By removing  $v_2$ , we obtain a toric birational morphism  $X \rightarrow \mathbb{P}^2$ . By removing  $v_3$ , we get  $X \rightarrow \mathbb{P}(1, n-1, n)$ .

Thus we have a Sarkisov link of Type (II).

$$\begin{array}{ccc} X & \xlongequal{\quad} & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & & \mathbb{P}(1, n-1, n) \\ \downarrow & & \downarrow \\ \text{pt.} & \xlongequal{\quad} & \text{pt.} \end{array}$$

EXAMPLE 2.3.3. In Example 2.3.1, we put

$$v_1 = (1, 0), v_2 = (1, n), v_3 = (0, 1), v_4 = (-1, -n),$$

where  $n$  is an integer with  $n \geq 2$ .

We consider the associated toric variety  $X = X(\Delta)$ . By removing  $v_2$ , we have a proper birational morphism  $X \rightarrow \mathbb{P}(1, 1, n)$ . By considering  $\mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $(x, y) \mapsto nx - y$ , we can construct  $X \rightarrow \mathbb{P}^1$ .

Thus we obtain the following commutative diagram:

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \downarrow & & \downarrow \\
 \mathbb{P}(1, 1, n) & & \mathbb{P}^1 \\
 \downarrow & \swarrow & \\
 \text{pt.} & & 
 \end{array}$$

So we have Sarkisov links of Types (I) and (III).

## CHAPTER 3

### Results related to minimal model theory 1

In this chapter, the author provides the results related to minimal model theory joint work with his supervisor Osamu Fujino and we closely follow [15]. In this chapter, we will work over the complex number field  $\mathbb{C}$  unless otherwise mentioned and our purpose is to give the proof of Theorem 1.0.8.

Let us start with the definition of semi-log canonical pairs to give the corollary of Theorem 1.0.8.

**DEFINITION 3.0.1** (Semi-log canonical pairs). Let  $X$  be a reduced equidimensional scheme which satisfies Serre's  $S_2$  condition and is normal crossing in codimension one. Let  $B$  be an effective  $\mathbb{R}$ -divisor such that  $K_X + B$  is  $\mathbb{R}$ -Cartier.

Let  $X = \bigcup X_i$  be the irreducible decomposition of  $X$  and let  $\nu : X^\nu := \coprod X_i^\nu \rightarrow X = \bigcup X_i$  be the normalization, where  $\nu|_{X_i^\nu} : X_i^\nu \rightarrow X_i$  is the usual normalization for any  $i$ . Let  $B^\nu$  be an  $\mathbb{R}$ -divisor on  $X_i$  such that  $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$  and let  $B_i^\nu = B^\nu|_{X_i^\nu}$ .

Then we say that  $(X, B)$  is a *semi-log canonical pair* (an *slc pair*, for short) if each pair  $(X^\nu, B_i^\nu)$  is log canonical.

We introduce some easy examples of semi-log canonical pairs

**EXAMPLE 3.0.2.** Let  $(X, B)$  be a log canonical pair. Then  $(X, B)$  is a semi-log canonical pair.

**EXAMPLE 3.0.3.** Let  $(X, B)$  be a semi-log canonical pair where  $X$  is normal. Then  $(X, B)$  is a log canonical pair.

We directly obtain the following result by combining Theorem 1.0.8 and Theorem 3.1.5.

**COROLLARY 3.0.4** ([16, Corollary 1.3]). *Let  $(X, B)$  be a projective semi-log canonical pair where  $X$  is connected. Assume that  $K_X + B$  is not nef and that  $K_X + \Delta \equiv rB$  for some Cartier divisor  $D$  on  $X$  with  $r > n = \dim X$ . Then  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(B) \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$  and  $(X, B)$  is kawamata log terminal.*

#### 3.1. Theory of quasi-log schemes

In this section, we collect some basic definition and results of the theory of quasi-log schemes. The notion of quasi-log schemes was first introduced by Ambro [1] as quasi-log varieties. First of all, we recall Fujino's definition of quasi-log schemes, which is slightly different from Ambro's original one. For details, see [11, Chapter 6]. Let us define globally embedded simple normal crossing pairs to define quasi-log schemes.

**DEFINITION 3.1.1** ([11, Definition 6.2.1]). Let  $Y$  be a simple normal crossing divisor on a smooth variety  $M$ . Let  $D$  be an  $\mathbb{R}$ -divisor on  $M$  such that  $\text{Supp}(D + Y)$

is a simple normal crossing divisor on  $M$  and that  $D$  and  $Y$  have no common irreducible components. Putting  $B_Y = D|_Y$ , we consider the pair  $(Y, B_Y)$ . Then we say that  $(Y, B_Y)$  is a *globally embedded simple normal crossing pair* and  $M$  is the *ambient space* of  $(Y, B_Y)$ . A *stratum* of  $(Y, B_Y)$  is a log canonical center of  $(M, Y + D)$  that is contained in  $Y$ .

Let us define quasi-log schemes.

DEFINITION 3.1.2 ([11, Definition 6.2.2]). A *quasi-log scheme*  $(X, \omega, f : (Y, B_Y) \rightarrow X)$  is the data which is  $X$  is a scheme,  $\omega$  is an  $\mathbb{R}$ -Cartier divisor on  $X$ ,  $X_{-\infty} \subset X$  is a proper closed subscheme,  $C$  is a finite collection of reduced and irreducible subschemes of  $X$ , and  $f : (Y, B_Y) \rightarrow X$  is a proper morphism from a globally embedded simple normal crossing pair satisfying the following properties:

- (i)  $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$ .
- (ii) The natural map  $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y([- (B_Y^{<1})])$  induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\sim} f_*\mathcal{O}_Y([- (B_Y^{<1})] - \lfloor B_Y^{>1} \rfloor),$$

where  $\mathcal{I}_{X_{-\infty}} \subset \mathcal{O}_X$  is the ideal sheaf defining a subscheme  $X_{-\infty}$ .

- (iii) The collection of reduced and irreducible subschemes  $C$  coincides with the images of  $(Y, B_Y)$ -strata that are not included in  $X_{-\infty}$ .

We simply write  $[X, \omega]$  to denote  $(X, \omega, f : (Y, B_Y) \rightarrow X)$  if there is no risk of confusion. We say that each member of  $C$  is a *qlc stratum* of  $[X, \omega]$ ,  $X_{-\infty}$  is the *non-qlc locus* of  $[X, \omega]$ , and  $f : (Y, B_Y) \rightarrow X$  is a *quasi-log resolution* of  $[X, \omega]$ . A qlc stratum  $C$  is a *qlc center* of  $[X, \omega]$  if it is not an irreducible component of  $X$ . We sometimes use  $\text{Nqlc}(X, \omega)$  to denote  $X_{-\infty}$ .

A quasi-log scheme  $[X, \omega]$  is a *quasi-log canonical pair* if  $X_{-\infty} = \emptyset$ .

We give a very important example of quasi-log canonical pairs, i.e. quasi-log schemes. Thanks to this example, we can consider log canonical pairs as quasi-log canonical pairs.

EXAMPLE 3.1.3 ([11, 6.4.1]). Let  $(X, B)$  be a pair. Let  $f : X \rightarrow Y$  be a log resolution of  $(X, B)$ , that is,  $\text{Exc}(f)$  is a divisor and  $\text{Supp } f_*^{-1}B \cup \text{Exc}(f)$  is a simple normal crossing divisor on  $Y$ . We put

$$K_Y + B_Y = f^*(K_X + B).$$

We put  $\omega = K_X + B$ . Then  $(X, \omega, f : (Y, B_Y) \rightarrow X)$  becomes a quasi-log scheme. By construction,  $(X, B)$  is log canonical if and only if  $[X, \omega]$  is quasi-log canonical. We note that  $C$  is a log canonical center of  $(X, B)$  if and only if  $C$  is a qlc center of  $[X, \omega]$ .

The following theorem is the adjunction and the vanishing theorem for quasi-log schemes.

THEOREM 3.1.4 ([11, Theorem 6.3.5]). *Let  $[X, \omega]$  be a quasi-log scheme and let  $X'$  be the union of  $X_{-\infty} = \text{Nqlc}(X, \omega)$  with a (possibly empty) union of some qlc strata of  $[X, \omega]$ . Then the following properties hold:*

- (i) (Adjunction theorem) *Assume that  $X' \neq X_{-\infty}$ . Then  $X'$  has a quasi-log structure with  $\omega' = \omega|_{X'}$  and  $\text{Nqlc}(X', \omega') = X_{-\infty}$ . Moreover, the qlc strata of  $[X', \omega']$  are the qlc strata of  $[X, \omega]$  that are included in  $X'$ .*

- (ii) (Vanishing theorem) *Let  $\pi : X \rightarrow S$  be a proper morphism of schemes. Let  $L$  be a Cartier divisor on  $X$  such that  $L - \omega$  is nef and big over  $S$ . Then  $R^i \pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$  for every  $i > 0$ , where  $\mathcal{I}_{X'} \subset \mathcal{O}_X$  is the ideal sheaf defining  $X'$ .*

Thanks to the following theorem, semi-log canonical pairs consider as quasi-log canonical pairs.

**THEOREM 3.1.5** ([10, Theorem 1.1]). *Let  $(X, B)$  be a quasi-projective semi-log canonical pair. Then  $[X, K_X + B]$  has a quasi-log structure with only qlc singularities. Moreover, the set of slc strata of  $(X, B)$  gives the set of qlc centers of  $[X, K_X + B]$ .*

The following theorem is well-known for log canonical pairs.

**THEOREM 3.1.6** ([11, Theorem 6.3.11]). *Let  $[X, \omega]$  be a quasi-log scheme with  $X_{-\infty} = \emptyset$ . Then two intersection of two qlc strata is a union of qlc strata.*

The following lemma is the generalization of Lemma 1.2.6 to quasi-log canonical pairs.

**LEMMA 3.1.7** ([11, Lemma 6.3.13]). *Let  $[X, \omega]$  be a qlc pair. Let  $D_1, \dots, D_k \geq 0$  be Cartier divisors passing through a closed point  $x \in X$ . If  $[X, \omega + \sum D_i]$  is qlc, then  $k \leq \dim X$ .*

We closed this section with very useful proposition of quasi-log schemes.

**PROPOSITION 3.1.8** ([11, Proposition 6.3.1]). *Let  $f : V \rightarrow W$  be a proper birational morphism between smooth varieties and let  $B_W$  be an  $\mathbb{R}$ -divisor on  $W$  such that  $\text{Supp } B_W$  is a simple normal crossing divisor on  $W$ . Assume that*

$$K_V + B_V = f^*(K_W + B_W)$$

*and that  $\text{Supp } B_V$  is a simple normal crossing divisor on  $V$ . Then we have*

$$f_* \mathcal{O}_V(\lceil -(B_V^{<1}) \rceil - \lfloor B_V^{>1} \rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1}) \rceil - \lfloor B_W^{>1} \rfloor).$$

*Furthermore, let  $S$  be a simple normal crossing divisor on  $W$  such that  $S \subset \text{Supp } B_W^{<1}$ . Let  $T$  be the union of the irreducible components of  $B_V^{<1}$  that are mapped into  $S$  by  $f$ . Assume that  $\text{Supp } f_* B_W \cup \text{Exc}(f)$  is a simple normal crossing divisor on  $V$ . Then we have*

$$f_* \mathcal{O}_T(\lceil -(B_T^{<1}) \rceil - \lfloor B_T^{>1} \rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1}) \rceil - \lfloor B_S^{>1} \rfloor),$$

*where  $(K_V + B_V)|_T = K_T + B_T$  and  $(K_W + B_W)|_S = K_S + B_S$ .*

### 3.2. Fujita's Theory of $\Delta$ -genus

In this section, we quickly review Fujita's  $\Delta$ -genus. For details, see, for example, [17], [18], [20], and [23]. Throughout this section, we will work over an algebraically closed field of characteristic zero. Let us start with the definition of  $\Delta$ -genus.

**DEFINITION 3.2.1.** Let  $X$  be a variety and let  $D$  be a Cartier divisor on  $X$ . Then  $\Delta$ -genus  $\Delta(X, D)$  is defined to be  $\dim X + \deg B - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H))$ .

The following theorem is the characterization of projective spaces by  $\Delta$ -genus.

**THEOREM 3.2.2** ([20, Theorem 1.1], [27, Theorem 2.1]). *Let  $(X, D)$  be a  $n$ -dimensional projective polarized variety, that is,  $X$  is a variety and  $D$  is an ample Cartier divisor on  $X$ . Then following are equivalent:*

- (i)  $D^n = 1$  and  $\Delta(X, D) = 0$ .
- (ii)  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ .

### 3.3. Proof of Theorem 1.0.8

Before we prove Theorem 1.0.8, we give a sketch of proof of Theorem 1.0.9, that is, Theorem 1.0.8 for log canonical pairs.

SKETCH OF PROOF OF THEOREM 1.0.9. Since  $K_X + B$  is not nef, there is a  $(K_X + B)$ -negative extremal contraction  $\phi : X \rightarrow W$  by the cone and contraction theorem for log canonical pairs (see [9, Theorem 1.1]). Let us divide into several cases.

CASE 1 ( $\dim W \geq 1$ ). By Lemma 1.2.6, we can take an effective  $\mathbb{R}$ -Cartier divisor  $B$  on  $W$  with the following properties:

- (i)  $(X, B + \phi^*B)$  is log canonical outside finitely many points, and
- (ii) there exists a log canonical center  $C$  of  $(X, B + \phi^*B)$  such that  $\phi(C)$  is a point with  $\dim C \geq 1$ .

In this situation, we obtain that

$$-(K_X + B + \phi^*B)|_C \equiv rD|_C$$

and  $D|_C$  is ample since  $\phi(C)$  is a point. Therefore, by the vanishing theorem for quasi-log schemes (see Lemma 3.3.1 below), we obtain

$$\chi(C, \mathcal{O}_C(tD)) \equiv 0.$$

This is a contradiction since  $H|_C$  is ample. This means that  $\dim W \geq 1$  does not happen.

CASE 2 ( $\dim W = 1$ ). Since  $\phi : X \rightarrow W$  is a  $(K_X + B)$ -negative extremal contraction, we see that  $H$  is ample. We can explicitly determine

$$\chi(X, \mathcal{O}_X(tD))$$

by  $-(K_X + B) \equiv rD$  with  $r > n$  and the vanishing theorem for log canonical pairs (see [9, Theorem 8.1]). Then we get  $H^n = 1$  and

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = n + 1.$$

Therefore,

$$\Delta(X, D) = n + H^n - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = 0$$

holds. By Theorem proj-space, this implies that  $X \simeq \mathbb{P}^n$  with  $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ . □

LEMMA 3.3.1. *Let  $[X, \omega]$  be a projective quasi-log canonical pair such that  $X$  is irreducible with  $\dim X = n \geq 1$ . Let  $H$  be an ample Cartier divisor on  $X$ . Assume that  $-\omega \equiv rH$  for some  $r > n$ . Then  $X \simeq \mathbb{P}^n$  with  $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ ,  $r \leq n + 1$ , and there are no qlc centers of  $[X, \omega]$ .*

PROOF. We divide the proof into several steps.

STEP 1. In this step, we will see that  $X \simeq \mathbb{P}^n$  with  $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ . We consider the Hilbert polynomial of the polarized pair  $(X, H)$ :

$$\chi(X, \mathcal{O}_X(tH)) = \sum_{i=0}^n a_n(-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(tH)).$$

Since  $H$  is ample, it is a nontrivial polynomial of degree  $n$ . Since

$$(3.3.1) \quad tH - \omega \equiv (t+r)H$$

is ample for  $t \geq -n$  by assumption, we obtain that

$$H^i(X, \mathcal{O}_X(tH)) = 0$$

for any  $i > 0$  and  $t \geq -n$  by Theorem 3.1.4 (ii). Since

$$H^0(X, \mathcal{O}_X(tH)) = 0$$

for  $t < 0$  and

$$\chi(X, \mathcal{O}_X) = \dim(X, \mathcal{O}_X) = 1,$$

we have

$$(3.3.2) \quad \chi(X, \mathcal{O}_X(tH)) = \frac{1}{n}(t+1) \cdots (t+n).$$

Thus we obtain that  $H^n = 1$  and

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(H)) = n+1.$$

Therefore, we have

$$\Delta(X, H) = n+1 - (n+1) = 0$$

and so by Theorem 3.2.2,  $X \simeq \mathbb{P}^n$  with  $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ .

STEP 2. In this step, we will see that  $r \leq n+1$ . Assume that  $r > n+1$ . Then by (3.3.1) and Theorem 3.1.4 (ii), we have

$$H^i(X, \mathcal{O}_X(-(n+1))) = 0$$

for  $i > 0$ . Thus we obtain that

$$\chi(X, \mathcal{O}_X(-(n+1))) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(-(n+1)H)) = 0.$$

On the other hand, by (3.3.2), we have

$$\chi(X, \mathcal{O}_X(-(n+1))) = (-1)^n \neq 0.$$

This is a contradiction. Therefore, we see that  $r \leq n+1$ .

STEP 3. In this step, we will see that  $[X, \omega]$  has no qlc centers. Assume that there is a zero-dimensional qlc center  $P$  of  $[X, \omega]$ . Then the evaluation map

$$H^0(X, \mathcal{O}_X(-H)) \rightarrow \mathbb{C}(P)$$

is surjective since

$$H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(-H)) = 0$$

by Theorem 3.1.4 (ii), where  $\mathcal{I}_P \subset \mathcal{O}_X$  is the ideal sheaf defining  $P$ . Thus we obtain that

$$H^0(X, \mathcal{O}_X(-H)) \neq 0.$$

This is a contradiction since  $H$  is ample. Therefore, there are no zero-dimensional qlc centers of  $[X, \omega]$ .



Assume that there is a qlc center  $C$  of  $[X, \omega]$  with  $\dim X \geq 1$ . By Theorem 3.1.4 (i),  $[C, \omega|_C]$  is also a quasi-log canonical pair with  $\dim C < \dim X$ . Since

$$-\omega \equiv rH$$

with  $r > n$ , we have

$$-\omega|_C \equiv rH|_C$$

with  $r > n \geq \dim C + 1$ . This is a contradiction with Step 2. Thus there are no qlc centers of  $[X, \omega]$ .

We finish the proof of Lemma 3.3.1. □

The following lemma is an easy consequence of Theorem 3.1.4.

LEMMA 3.3.2. *Let  $[X, \omega]$  be a projective quasi-log scheme with  $\dim X_{-\infty} = 0$  or  $X_{-\infty} = \emptyset$ , where  $X_{-\infty} = \text{Nqlc}(X, \omega)$ . Let  $L$  be a Cartier divisor on  $X$  such that  $L - \omega$  is ample. Then*

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for  $i > 0$ .

PROOF. It follows from Theorem 3.1.4 if  $X_{-\infty} = \emptyset$ . Thus we may assume that  $\dim X_{-\infty} = 0$ . We consider the following short exact sequence:

$$0 \rightarrow \mathcal{I}_{X_{-\infty}} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_{-\infty}} \rightarrow 0.$$

Then the above sequence induces the following long exact sequence:

$$(3.3.3) \quad \cdots \rightarrow H^i(X, \mathcal{I}_{X_{-\infty}} \otimes \mathcal{O}_X(L)) \rightarrow H^i(X, \mathcal{O}_X(L)) \rightarrow H^i(X, \mathcal{I}_{X_{-\infty}}(L)) \rightarrow \cdots.$$

By Theorem 3.1.4 (ii), we obtain that

$$H^i(X, \mathcal{I}_{X_{-\infty}} \otimes \mathcal{O}_X(L)) = 0$$

for  $i > 0$ . Since  $\dim X_{-\infty} = 0$  by assumption, we have

$$H^i(X, \mathcal{I}_{X_{-\infty}}(L)) = 0$$

for  $i > 0$ . Therefore, by (3.3.3), we see that

$$H^i(X, \mathcal{O}_X(L)) = 0$$

holds true for  $i > 0$ . □

LEMMA 3.3.3. *Let  $[X, \omega]$  be a quasi-log canonical pair such that  $X$  is irreducible. Let  $\phi : X \rightarrow W$  be a proper surjective morphism to a quasi-projective variety  $W$  with  $\dim W \geq 1$ . Let  $P \in W$  be a closed point such that  $\dim \phi^{-1}(P) \geq 1$ . Then there is an  $\mathbb{R}$ -Cartier divisor  $B \geq 0$  on  $W$  such that  $[W, \omega + \phi^*B]$  is a quasi-log scheme with the following properties:*

- (i)  $[X, \omega + \phi^*B]$  is quasi-log canonical outside finitely many points, and
- (ii) there is a qlc center  $C$  of  $[X, \omega + \phi^*B]$  such that  $\phi(C) = P$  with  $\dim C \geq 1$ .

PROOF.

CASE 1. In this case, we assume that there are no qlc centers of  $[X, \omega]$  in  $\phi^{-1}(P)$ .

Let  $f : (Y, B_Y) \rightarrow X$  be a quasi-log resolution of  $[X, \omega]$ . We take general very ample Cartier divisors  $B_1, \dots, B_{n+1}$  on  $W$  such that  $P \in \text{Supp } B_i$  for any  $i$ . By 3.1.8, we may further assume that

$$\left( Y, \sum_{i=1}^{n+1} (\phi \circ f)^* B_i + \text{Supp } B_Y \right)$$

is a globally embedded simple normal crossing pair. By Lemma 3.1.7, we can take  $0 < \epsilon < 1$  with the following properties:

- (i)  $\left( B_Y + c \sum_{i=1}^{n+1} (\phi \circ f)^* B_i \right)^{>1} = 0$  or  $\dim f \left( \text{Supp} \left( B_Y + c \sum_{i=1}^{n+1} (\phi \circ f)^* B_i \right)^{>1} \right) = 0$ , and
- (ii) there is an irreducible component  $G$  of  $\left( B_Y + c \sum_{i=1}^{n+1} (\phi \circ f)^* B_i \right)^{=1}$  such that  $\dim f(G) \geq 1$ .

We put  $B = c \sum_{i=1}^{n+1} B_i$ . Then

$$f : (Y, B_Y + (\phi \circ f)^* B) \rightarrow [X, \omega + \phi^* B]$$

gives a desired quasi-log structure on  $[X, \omega + \phi^* B]$  by construction.

CASE 2. In this case, we assume that there is a qlc center  $C$  of  $[X, \omega]$  in  $\phi^{-1}(P)$  with  $\dim C \geq 1$ ,

It is clear when we put  $B = 0$ .

CASE 3. In this case, we assume that every qlc center of  $[X, \omega]$  contained in  $\phi^{-1}(P)$  is zero-dimensional.

Let  $f : (Y, B_Y) \rightarrow X$  be a quasi-log resolution of  $[X, \omega]$ . We take general very ample Cartier divisors  $B_1, \dots, B_{n+1}$  on  $W$  such that  $P \in \text{Supp } B_i$  for any  $i$  as in Case 1. Let  $X'$  be the union of all qlc centers contained in  $\phi^{-1}(P)$ . By 3.1.8, we may assume that the union of all strata of  $(Y, B_Y)$  mapped to  $X'$  by  $f$ , which is denoted by  $Y'$ , is a union of some irreducible components of  $Y$ . We put  $Y'' = Y - Y'$ ,  $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$ , and  $f'' = f|_{Y''}$ . We may further assume that

$$\left( Y'', \sum_{i=1}^{n+1} (\phi \circ f'')^* B_i + \text{Supp } B_{Y''} \right)$$

is a globally embedded simple normal crossing pair in the same way as in Case 1. Then we note that

$$\mathcal{I}_{X'} = f''_* \mathcal{O}_{Y''}([-(B_{Y''}^{<1}] - Y'|_{Y''})$$

holds, where  $\mathcal{I}_{X'}$  is the ideal sheaf defining  $X'$ . We also note that  $B_{Y''}^{>1} \geq Y'|_{Y''}$  by construction. By Lemma 3.1.7, we can take  $0 < \epsilon < 1$  with the following properties:

- (i)  $\dim f'' \left( \text{Supp} \left( B_{Y''} + c \sum_{i=1}^{n+1} (\phi \circ f'')^* B_i \right)^{>1} \right) = 0$ , and
- (ii) there is an irreducible component  $G$  of  $\left( B_{Y''} + c \sum_{i=1}^{n+1} (\phi \circ f'')^* B_i \right)^{=1}$  such that  $\dim f''(G) \geq 1$ .

We put  $B = c \sum_{i=1}^{n+1} B_i$ .

$$f'' : (Y'', B_{Y''} + (\phi \circ f'')^* B) \rightarrow [X, \omega + \phi^* B]$$

gives a desired quasi-log structure on  $[X, \omega + \phi^*B]$  by construction.

We finish the proof of Lemma 3.3.3.

□

PROOF OF THEOREM 1.0.8. We put  $H = -D$ .

CASE 1. In this case, we assume that  $X$  is irreducible. Since  $\omega$  is not nef, there is an  $\omega$ -negative extremal contraction  $\phi : X \rightarrow W$  by the cone and contraction theorem for quasi-log schemes ([11, Theorem 6.7.3 and Theorem 6.7.4]). If  $\dim W \geq 1$ , then we can take an  $\mathbb{R}$ -Cartier divisor  $D$  on  $W$  satisfying the properties in Lemma 3.3.3. Let  $C$  be a qlc center of  $[X, \omega + \phi^*B]$  as in Lemma 3.3.3. We put

$$C' = C \cup \text{Nqlc}(X, \omega + \phi^*B).$$

By Theorem 3.1.4 (i),  $[C', \omega|'_C]$  is a quasi-log scheme. There is the following short exact sequence:

$$(3.3.4) \quad 0 \longrightarrow \text{Ker } \alpha \longrightarrow \mathcal{O}'_C \xrightarrow{\alpha} \mathcal{O}_C \longrightarrow 0.$$

such that  $\text{Ker } \alpha = 0$  or the support of  $\text{Ker } \alpha$  is zero-dimensional. Since  $\dim \phi(C') = 0$ , we have

$$-(\omega + \phi^*B)|'_C \equiv rH|'_C.$$

Thus by assumption,

$$(tH - (\omega + \phi^*B))|'_C \equiv (t + r)H|'_C$$

is ample for  $t \geq -n$ . Therefore, by Lemma 3.3.3 and (3.3.4),

$$H^i(C, \mathcal{O}_C(tH)) = H^i(C', \mathcal{O}'_C(tH)) = 0$$

for  $i > 0$  and  $t \geq -n$ . Since  $H|_C$  is ample, we have

$$H^0(C, \mathcal{O}_C(tH)) = 0$$

for  $t < 0$ . Thus we have

$$\chi(C, \mathcal{O}_C(tH)) = 0$$

for  $t = 1, \dots, -n$  and so we get that

$$\chi(C, \mathcal{O}_C(tH)) \equiv 0$$

since  $n \geq \dim C + 1$ . This is a contradiction since  $H|_C$  is ample. This means that  $W = 0$  and so  $H$  is ample. By Lemma 3.3.3,  $X \simeq \mathbb{P}^n$  with  $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ , and there are no qlc centers of  $[X, \omega]$ .

CASE 2. We assume that  $X$  is not irreducible and we take an irreducible component  $X'$  of  $X$  such that  $\omega' = \omega|'_{X'}$  is not nef. Then by Theorem 3.1.4 (i),  $[X', \omega']$  is a quasi-log canonical pair such that  $\omega' \equiv rD|'_{X'}$  with  $r > n \geq \dim X'$ . By the above argument, we see that  $H|'_{X'}$  is ample. Thus by Lemma 3.3.1,  $X' \simeq \mathbb{P}^n$  with  $\mathcal{O}_{X'}(H) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ . On the other hand, if  $X \neq X'$ , then  $[X', \omega']$  has a qlc center since  $X$  is connected by Theorem 3.1.6 and Theorem 3.1.4 (i). This is a contradiction and this case does not happen.

□

## CHAPTER 4

### Results related to minimal model theory 2

In this chapter, the author provides the results related to minimal model theory joint work with his supervisor Osamu Fujino and we closely follow [16]. In this chapter, our purpose is to establish the generalization of the classical result, the Nakai–Moishezon ampleness criterion, to  $\mathbb{R}$ -line bundles on complete schemes. By the standard reduction argument, it is sufficient to treat the case where  $X$  is a complete normal variety. Therefore, all we have to do is to establish the following theorem.

**THEOREM 4.0.1** (Nakai–Moishezon ampleness criterion for real Cartier divisors on complete normal varieties, [16, Theorem 1.4]). *Let  $X$  be a complete normal variety and let  $\mathcal{L}$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . Then  $\mathcal{L}$  is ample if and only if  $\mathcal{L}^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed subvarieties  $Z \subset X$ .*

#### 4.1. Augmented base locus of $\mathbb{R}$ -Cartier divisors

In this section, we explain some properties of augmented base loci of  $\mathbb{R}$ -divisors following [2]. Let us recall the definition of base loci and stable base loci.

**DEFINITION 4.1.1** (Base loci and stable base loci of  $\mathbb{Q}$ -divisors). Let  $X$  be a projective scheme and let  $D$  be a Cartier divisor on  $X$ . The *base locus* of  $D$  is defined as

$$\mathrm{Bs}|D| = \{x \in X \mid \alpha \text{ vanishes at } x \text{ for every } \alpha \in H^0(X, \mathcal{O}_X(D))\}.$$

We consider  $\mathrm{Bs}|D|$  with the reduced scheme structure.

The *stable base locus* of a  $\mathbb{Q}$ -Cartier divisor  $L$  on  $X$  is defined as

$$\mathbf{B}(L) = \bigcap_m \mathrm{Bs}|mL|$$

where  $m$  runs over all positive integers such that  $mL$  is Cartier. Note that  $\mathbf{B}(L)$  is considered with the reduced scheme structure.

**DEFINITION 4.1.2.** Let  $X$  be a projective scheme and let  $L$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . The *augmented base locus* of  $L$  is defined as

$$\mathbf{B}_+(L) = \bigcap_H \mathbf{B}(L - H),$$

where  $H$  runs over all ample  $\mathbb{R}$ -Cartier divisors on  $X$  such that  $L - H$  is  $\mathbb{Q}$ -Cartier. We consider  $\mathbf{B}_+(L)$  with the reduced scheme structure.

**DEFINITION 4.1.3.** Let  $X$  be a projective scheme and let  $L$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . The *exceptional locus* of  $L$  is defined as

$$\mathbb{E}(L) = \bigcup_{\dim V > 0, L|_V \text{ is not big}} V,$$

that is, the union runs over all positive-dimensional subvarieties  $V \subset X$  such that  $L|_V$  is not big.

It is called the *null locus* when  $L$  is nef.

DEFINITION 4.1.4 (Augmented base loci of  $\mathbb{R}$ -divisors). Let  $X$  be a projective scheme and let  $L$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . The *augmented base locus* of  $L$  is defined as

$$\mathbf{B}_+(L) = \bigcap_H \mathbf{B}(L - H)$$

where  $H$  runs over all ample  $\mathbb{R}$ -divisors such that  $L - H$  is  $\mathbb{Q}$ -Cartier. As usual, we consider  $\mathbf{B}_+(L)$  with the reduced scheme structure.

Birkar defined  $\mathbf{B}_+(L)$  differently (see [2, Definition 1.2]). Then he proved that his definition coincides with the usual one (see Definition 4.1.4). For the details, see [2, Lemma 3.1 (3)].

In order to explain Birkar's theorem (see Theorem 4.1.6), it is convenient to introduce the notion of exceptional loci of  $\mathbb{R}$ -divisors.

DEFINITION 4.1.5 (Exceptional loci of  $\mathbb{R}$ -divisors). Let  $X$  be a projective scheme and let  $L$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . The *exceptional locus* of  $L$  is defined as

$$\mathbb{E}(L) = \bigcup_{\substack{\dim V > 0, \\ L|_V \text{ is not big}}} V,$$

that is, the union runs over the positive-dimensional subvarieties  $V \subset X$  such that  $L|_V$  is not big.

Note that  $\mathbb{E}(L)$  is sometimes called the *null locus* of  $L$  when  $L$  is nef.

The following result is a key ingredient of this section.

THEOREM 4.1.6 ([2, Theorem 1.4]). *Let  $X$  be a projective scheme and let  $L$  be a nef  $\mathbb{R}$ -Cartier divisor on  $X$ . Then*

$$\mathbf{B}_+(L) = \mathbb{E}(L)$$

*holds.*

For the details of Theorem 4.1.6, we strongly recommend the reader to see Birkar's original statement in [2, Theorem 1.4]. We will use Theorem 4.1.6 when  $X$  is a normal projective variety in the proof of Theorem 4.0.1.

## 4.2. Proof of Theorem 4.0.1

Let us start with the following lemma.

LEMMA 4.2.1. *Let  $X$  be an  $n$ -dimensional projective variety and let  $L$  be a nef  $\mathbb{R}$ -Cartier divisor on  $X$ . Then  $L$  is big if and only if  $L^n > 0$ .*

Before we prove Lemma 4.2.1, we need the following bigness criterion:

LEMMA 4.2.2 ([31, Theorem 2.2.15]). *Let  $X$  be an  $n$ -dimensional projective variety and let  $L$  and  $A$  be a nef  $\mathbb{Q}$ -divisors on  $X$ . If*

$$L^n > n \cdot (L^{n-1} \cdot A)$$

*holds, then  $L - A$  is big.*

PROOF OF LEMMA 4.2.1. Assume that  $L$  is big, then there are an ample  $\mathbb{R}$ -divisor  $A$  and an effective  $\mathbb{R}$ -Cartier divisor  $E$  on  $X$  such that  $L \sim_{\mathbb{R}} A + E$ . Then

$$L^n = (A + E) \cdot L^{n-1} \geq A \cdot L^{n-1} > \dots > A^n > 0.$$

Conversely, assume that  $L^n > 0$ . We take ample  $\mathbb{R}$ -divisors  $A_1$  and  $A_2$  on  $X$  such that  $L + A_1$  and  $A_1 + A_2$  are  $\mathbb{Q}$ -Cartier divisors on  $X$ . Since  $L^n > 0$ , we may assume that

$$(L + A_1)^n > n((L + A_1)^{n-1} \cdot (A_1 + A_2))$$

by replacing  $A_1$  and  $A_2$  with sufficiently small ones. By Lemma 4.2.2,

$$L - A_2 = (L + A_1) - (A_1 + A_2)$$

is big. Hence  $L$  is also big.  $\square$

PROOF OF THEOREM 4.0.1. Let

$$X = \bigcup_{i=1}^k U_i$$

be a finite affine Zariski open cover of  $X$  and let  $\overline{U}_i$  is the closure of  $U_i$  in  $\mathbb{P}^N$ . By [32, Lemma 2.2], we can take an ideal sheaf  $\mathcal{I}$  on  $U_i$  with  $\text{Supp } \mathcal{O}(U_i)/\mathcal{I} \subset \overline{U}_i \setminus U_i$  such that the blow-up of  $U_i$  along  $\mathcal{I}$  eliminates the indeterminacy of  $U_i \dashrightarrow X$ . Therefore, by taking the normalization of the blow-up of  $U_i$  along  $\mathcal{I}$ , we get a projective birational morphism  $\pi_i : X_i \rightarrow X$  from a normal projective variety  $X_i$  such that  $\pi_i : \pi_i^{-1}(U_i) \rightarrow U_i$  is an isomorphism.

$$\begin{array}{ccc} & X_i & \\ \swarrow & & \searrow \pi_i \\ \overline{U}_i & \text{-----} & X \end{array}$$

Since  $\pi_i^*L$  is nef by assumption, it follows from Theorem 4.1.6 that there is an ample  $\mathbb{R}$ -Cartier divisor  $H_i$  on  $X_i$  such that  $\pi_i^*L - H_i$  is  $\mathbb{Q}$ -Cartier and that

$$\mathbf{B}(\pi_i^*L - H_i) = \mathbf{B}_+(\pi_i^*L) = \mathbb{E}(\pi_i^*L)$$

holds. Thus by Lemma 4.2.1 and the assumption that  $L^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed subvariety  $Z \subset X$ ,

$$\mathbb{E}(\pi_i^*L) = \text{Exc}(\pi_i)$$

holds. Since  $L$  is  $\mathbb{R}$ -Cartier, we can write

$$L = \sum_{j \in J} l_j L_j,$$

where  $l_j \in \mathbb{R}$  and  $L_j$  is Cartier for any  $j \in J$ . There is a real number  $\epsilon > 0$  such that if  $m_j \in \mathbb{Q}$  with  $|l_j - m_j| < \epsilon$  for any  $j \in J$ , then

$$\pi_i^* \left( \sum_{j \in J} m_j L_j \right) - \pi_i^*L + H_i$$

is ample. Then

$$\mathbf{B} \left( \pi_i^* \left( \sum_{j \in J} m_j L_j \right) \right) \subset \mathbf{B}(\pi_i^*L - H_i) = \text{Exc}(\pi_i)$$

holds. This implies that

$$\mathbf{B} \left( \sum_{j \in J} m_j L_j \right) \subset \pi_i(\text{Exc}(\pi_i)) \subset X \setminus U_i$$

holds. Therefore, we have

$$\mathbf{B} \left( \sum_{j \in J} m_j L_j \right) \subset \bigcap_{i=1}^k (X \setminus U_i) = \emptyset.$$

This means that  $\sum m_j L_j$  is semi-ample for any  $m_j \in \mathbb{Q}$  with  $|l_j - m_j| < \epsilon$  and so we can write

$$L = \sum_p r_p M_p,$$

where  $r_p \in \mathbb{R}$  and  $M_p$  is a semi-ample  $\mathbb{Q}$ -divisor for every  $p$ . Thus  $L$  is semi-ample and so there is a morphism  $f : X \rightarrow Y$  onto a normal projective variety  $Y$  with  $f_* \mathcal{O}_X \simeq \mathcal{O}_Y$  and an ample  $\mathbb{R}$ -Cartier divisor  $A$  on  $Y$  such that  $L \sim_{\mathbb{R}} f^* A$ . Since  $L \cdot C > 0$  for any curve  $C \subset X$  by assumption,  $f$  is an isomorphism. Thus  $L$  is ample.  $\square$

#### 4.3. Proof of Theorem 1.0.11

LEMMA 4.3.1. *Let  $X$  be a complete scheme and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Let  $X = \sum_{i=1}^k \bigcup X_i$  be the irreducible decomposition of  $X$ . Then  $\mathcal{L}$  is ample if and only if  $\mathcal{L}|_{(X_i)_{\text{red}}}$  is ample for every  $i$ .*

PROOF. This statement is well known for  $\mathbb{Q}$ -line bundles. Hence, we will freely use this lemma for  $\mathbb{Q}$ -line bundles in this proof. Assume that  $\mathcal{L}$  is ample. Then we can write

$$\mathcal{L} = \sum_j a_j \mathcal{L}_j,$$

where  $a_i$  is a positive real number and  $\mathcal{L}_j$  is an ample line bundle for every  $j$ . Thus  $\mathcal{L}|_{(X_i)_{\text{red}}}$  is also ample for every  $i$ .

Conversely, assume that  $\mathcal{L}|_{(X_i)_{\text{red}}}$  is ample for every  $i$ . Since  $\mathcal{L}$  is an  $\mathbb{R}$ -line bundle, we can write

$$\mathcal{L} = \sum_{j=1}^m l_j \mathcal{L}_j,$$

where  $l_j$  is a real number and  $\mathcal{L}_j$  is a line bundle for every  $j$ . We put

$$V_i = \left\{ (p_1, \dots, p_m) \in \mathbb{R}^m \mid \sum_{j=1}^m p_j \mathcal{L}_j|_{(X_i)_{\text{red}}} \text{ is ample} \right\}$$

for every  $i$ . Then  $V_i$  contains a Euclidean open neighborhood of  $l = (l_1, \dots, l_m)$  for every  $i$  since  $\sum_{j=1}^m p_j \mathcal{L}_j|_{(X_i)_{\text{red}}}$  is ample by assumption. Hence  $V = \bigcap_{i=1}^k V_i$  contains a Euclidean open neighborhood of  $l \in \mathbb{R}^m$ . Thus we can take positive real numbers  $r_1, \dots, r_p$  and

$$v_1 = (v_{11}, \dots, v_{1m}), \dots, v_p = (v_{p1}, \dots, v_{pm}) \in V \cap \mathbb{Q}^m$$

such that  $l = \sum_{\alpha=1}^p r_\alpha v_\alpha$ . Then

$$\mathcal{A}_\alpha := \sum_{j=1}^m v_{\alpha j} \mathcal{L}_j \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is ample for every  $\alpha$  since  $v_\alpha \in V \cap \mathbb{Q}^m$ . Since we can write

$$\mathcal{L} = \sum_{\alpha=1}^p r_\alpha \mathcal{A}_\alpha,$$

$\mathcal{L}$  is ample by definition.  $\square$

LEMMA 4.3.2. *Let  $X$  be a complete scheme and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Let  $\pi : Y \rightarrow X$  be a finite surjective morphism between complete varieties. Then  $\mathcal{L}$  is ample if and only if  $\pi^* \mathcal{L}$  is ample.*

PROOF. This statement is well known for  $\mathbb{Q}$ -line bundles. Hence, we will freely use this lemma for  $\mathbb{Q}$ -line bundles in this proof. Assume that  $\mathcal{L}$  is ample. Then it is obvious that  $\pi^* \mathcal{L}$  is ample.

Conversely, assume that  $\pi^* \mathcal{L}$  is ample. Since  $\mathcal{L}$  is an  $\mathbb{R}$ -line bundle, we can write

$$\mathcal{L} = \sum_{j=1}^m l_j \mathcal{L}_j,$$

where  $l_j$  is a real number and  $\mathcal{L}_j$  is a line bundle for every  $j$ . Since  $\pi^* \mathcal{L}$  is ample, there exists a positive real number  $\epsilon$  such that if  $|l_j - \alpha_j| < \epsilon$  for every  $j$ , then

$$\pi^* \left( \sum_{j=1}^m \alpha_j \mathcal{L}_j \right)$$

is ample. Moreover, if we further assume that  $\alpha_j \in \mathbb{Q}$  for every  $j$ , then

$$\sum_{j=1}^m \alpha_j \mathcal{L}_j$$

is ample since  $\pi$  is a finite surjective morphism. Hence we can write

$$\mathcal{L} = \sum_{j=1}^m r_j \mathcal{A}_i$$

such that  $r_i$  is a positive real number and  $\mathcal{A}_i$  is an ample line bundle for every  $i$ . This means that  $\mathcal{L}$  is ample by definition.  $\square$

PROOF OF THEOREM 1.0.11. By Lemma 4.3.1, we may assume that  $X$  is a variety. Let  $\nu : X^\nu \rightarrow X$  be the normalization of  $X$ . Note that  $\nu$  is a finite surjective morphism. Then, by Lemma 4.3.2, it is sufficient to prove that  $\nu^* \mathcal{L}$  is ample. Hence we further assume that  $X$  is a complete normal variety. In this case, the ampleness of  $\mathcal{L}$  follows from Theorem 4.0.1.  $\square$



#### 4.4. Proof of Theorem 1.0.12

In this section, we prove Theorem 1.0.12. More precisely, we reduce Theorem 1.0.12 to a special case where  $X$  is a normal variety, which is nothing but Theorem 4.0.1. Let us start with the following elementary lemma.

LEMMA 4.4.1. *Let  $\pi : X \rightarrow S$  be a proper surjective morphism between schemes and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Assume that  $\mathcal{L}|_{X_s}$  is ample for every closed point  $s \in S$ , where  $X_s = \pi^{-1}(s)$ . Then  $\mathcal{L}$  is  $\pi$ -ample.*

Before we prove Lemma 4.4.1, we prepare the following lemma, which is also well known for  $\mathbb{Q}$ -line bundles.

LEMMA 4.4.2. *Let  $\pi : X \rightarrow S$  be a proper surjective morphism between schemes and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Assume that  $\mathcal{L}|_{X_{s_0}}$  is ample for some closed point  $s_0 \in S$ , where  $X_{s_0} = \pi^{-1}(s_0)$ . Then there exists a Zariski open neighborhood  $U_{s_0}$  of  $s_0$  such that  $\mathcal{L}|_{\pi^{-1}(U_{s_0})}$  is ample over  $U_{s_0}$ .*

PROOF. This statement is well known for  $\mathbb{Q}$ -line bundles. Hence, we will freely use this lemma for  $\mathbb{Q}$ -line bundles in this proof. For details, see, for example, [31, Theorem 1.7.8].

Since  $\mathcal{L}$  is an  $\mathbb{R}$ -line bundle, there exist line bundles  $\mathcal{M}_j$  for  $1 \leq j \leq k$  such that

$$\mathcal{L} = \sum_{j=1}^k b_j \mathcal{M}_j$$

in  $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ , where  $b_j$  is a real number for every  $j$ . We put

$$\mathcal{A} = \left\{ (c_1, \dots, c_k) \in \mathbb{R}^k \left| \sum_{j=1}^k c_j \mathcal{M}_j|_{X_{s_0}} \text{ is ample} \right. \right\}.$$

Then  $\mathcal{A}$  contains a Euclidean open neighborhood of  $(b_1, \dots, b_k)$ . Hence we can write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i,$$

where  $\mathcal{L}_i$  is a line bundle on  $X$  such that  $a_i$  is a positive real number and  $\mathcal{L}_i|_{X_{s_0}}$  is ample for every  $i$ . Since  $\mathcal{L}_i|_{X_{s_0}}$  is ample for every  $i$ , there exists a Zariski open neighborhood  $U_{s_0}$  of  $s_0$  such that  $\mathcal{L}_i|_{\pi^{-1}(U_{s_0})}$  is ample over  $U_{s_0}$  for every  $i$  (see, for example, [29, Proposition 1.41]). Therefore,  $\mathcal{L}_i|_{\pi^*(U_{s_0})} = \sum_i a_i \mathcal{L}_i|_{\pi^{-1}(U_{s_0})}$  is ample over  $U_{s_0}$ .  $\square$

Let us prove Lemma 4.4.2.

PROOF OF LEMMA 4.4.1. We use the same notation as in the proof of Lemma 4.4.2. Note that  $S$  is Noetherian since it is a separated scheme of finite type over an algebraically closed field. Hence, by Lemma 4.4.2, we can take  $s_1, \dots, s_l \in S$  such that

$$\bigcup_{\alpha=1}^l U_{S_\alpha} = S,$$

where  $U_{S_\alpha}$  is a Zariski open neighborhood of  $S_\alpha$  in  $S$  for every  $\alpha$ , and that  $\mathcal{L}|_{\pi^{-1}(U_{S_\alpha})}$  is ample over  $U_{S_\alpha}$  for every  $\alpha$ . We put

$$\mathcal{A}_\alpha = \left\{ (c_1, \dots, c_k) \in \mathbb{R}^k \left| \sum_{j=1}^k c_j \mathcal{M}_j|_{\pi^{-1}(U_{S_\alpha})} \text{ is } \pi\text{-ample over } U_{S_\alpha} \right. \right\}.$$

Then  $\mathcal{A}_\alpha$  contains a Euclidean open neighborhood of  $(b_1, \dots, b_k)$ . Therefore,  $\bigcap_{\alpha=1}^l \mathcal{A}_\alpha$  also contains a Euclidean open neighborhood of  $(b_1, \dots, b_k)$ . Hence, we can write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i$$

in  $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  such that  $a_i$  is a positive real number and  $\mathcal{L}_i$  is a  $\pi$ -ample line bundle on  $X$  for every  $i$ .  $\square$

Finally, we prove Theorem 1.0.12.

**PROOF OF THEOREM 1.0.12.** If  $\mathcal{L}$  is  $\pi$ -ample, then it is obvious that it satisfies the desired property. Hence, by Lemma 4.4.1, it is sufficient to prove that  $\mathcal{L}|_{X_s}$  is ample for every closed point  $s \in S$ , where  $X_s = \pi^{-1}(s)$ , under the assumption that  $\mathcal{L}^{\dim Z} \cdot Z > 0$ . This follows from the Nakai–Moishezon ampleness criterion for  $\mathbb{R}$ -line bundles on complete schemes (see Theorem 1.0.11).  $\square$

#### 4.5. The generalization of Theorem 1.0.11 to algebraic spaces

In this section, as an application of Theorem 1.0.11, we prove the Nakai–Moishezon ampleness criterion for  $\mathbb{R}$ -line bundles on complete algebraic spaces.

**THEOREM 4.5.1** (Nakai–Moishezon ampleness criterion for real line bundles on complete algebraic spaces, [16, Theorem 1.6]). *Let  $X$  be a complete algebraic space over an algebraically closed field  $k$  and let  $\mathcal{L}$  be an  $\mathbb{R}$ -line bundle on  $X$ . Then  $\mathcal{L}$  is ample if and only if  $\mathcal{L}^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed integral subspace  $Z \subset X$ .*

Before we prove Theorem 4.5.1, let us start with two well-known results of algebraic spaces.

**THEOREM 4.5.2** (Nakai–Moishezon criterion for line bundles on complete algebraic spaces, [28, Theorem 3.11] or [38, Theorem (1.4)]). *Let  $X$  be a complete algebraic space and let  $\mathcal{L}$  be a line bundle on  $X$ . Then  $\mathcal{L}$  is ample if and only if  $\mathcal{L}^{\dim Z} \cdot Z > 0$  for every positive-dimensional closed integral subspace  $Z \subset X$ .*

**LEMMA 4.5.3** ([28, Lemma 2.8]). *Let  $X$  be an algebraic space of finite type. Then there is a scheme  $Y$  and a finite surjective morphism  $f : Y \rightarrow X$ .*

**PROOF OF THEOREM 4.5.1.** By Lemma 4.5.3, there is a finite surjective morphism  $f : Y \rightarrow X$  from a complete scheme  $Y$ . By Theorem 1.0.11,  $f^* \mathcal{L}$  is an ample  $\mathbb{R}$ -line bundle on  $Y$ . We write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i,$$

where  $a_i$  is a positive real number and  $\mathcal{L}_i$  is a line bundle on  $X$  for every  $i$ . We put

$$\mathcal{M} = \sum_i b_i \mathcal{L}_i,$$

where  $b_i$  is a positive real number for every  $i$ . If  $|a_i - b_i| \ll 1$  for every  $i$ , then  $f^*\mathcal{M}$  is an ample  $\mathbb{Q}$ -line bundle on  $Y$  since  $f^*\mathcal{L}$  is ample. Therefore,  $m\mathcal{M}$  is an ample line bundle on  $X$  for some positive integer  $m$  by Theorem 4.5.2. This implies that  $X$  is projective. Thus, by Theorem 1.0.11 again,  $\mathcal{L}$  is an ample  $\mathbb{R}$ -line bundle on  $X$ .  $\square$

## Acknowledgement

The author would like to thank his supervisor Osamu Fujino for various suggestions and warm encouragement. The author also thanks Professor Takehiko Yasuda for their helpful advice and suggestions.



## Bibliography

- [1] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233.
- [2] C. Birkar, The augmented base locus of real divisors over arbitrary fields, Math. Ann. **368** (2017), no. 3–4, 905–921.
- [3] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.
- [4] A. Bruno, K. Matsuki, Log Sarkisov program, Internat. J. Math. **8** (1997), no. 4, 451–494.
- [5] F. Campana, T. Peternell, Algebraicity of the ample cone of projective varieties, J. Reine Angew. Math. **407** (1990), 160–166.
- [6] S. R. Choi, V. V. Shokurov, Geography of log models: theory and applications, Cent. Eur. J. Math. **9** (2011), no. 3, 489–534.
- [7] A. Corti, Factoring birational maps of threefolds after Sarkisov, J. Algebraic Geom. **4** (1995), no. 2, 223–254.
- [8] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, **124**, American Mathematical Society, Providence, RI, 2011.
- [9] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), 727–789.
- [10] O. Fujino, Fundamental theorems for semi log canonical pairs, Algebr. Geom. **1** (2014), 194–228.
- [11] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs **35**, Mathematical Society of Japan, Tokyo, 2017.
- [12] O. Fujino, Semipositivity theorems for moduli problems, Ann. Math. (2) **187**(3) (2018), 639–665.
- [13] O. Fujino, Minimal model theory for log surfaces in Fujiki’s class  $C$ , Nagoya Math. J. **244** (2021), 256–282.
- [14] O. Fujino, H. Sato, Introduction to the toric Mori theory, Michigan Math. J. **52** (2004), no. 3, 649–665.
- [15] O. Fujino, K. Miaymoto, A characterization of projective spaces from the Mori theoretic viewpoint, Osaka J. Math. **58** (2021), no. 4, 827–837.
- [16] O. Fujino, K. Miaymoto, Nakai–Moishezon ampleness criterion for real line bundles, Math. Ann. **385** (2023), no. 1–2, 459–470.
- [17] T. Fujita, On the structure of polarized varieties with  $\Delta$ -genera zero, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 103–115.
- [18] T. Fujita, The structure and classification of polarized varieties, (Japanese) Sūgaku **27** (1975), 316–326.
- [19] T. Fujita, On polarized manifolds whose adjoint bundles are not semipositive, In Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, 167–178. North-Holland, Amsterdam 1987.
- [20] T. Fujita, *Classification theories of polarized varieties*, London Mathematical Society Lecture Note Series **155**, Cambridge University Press, Cambridge, 1990.
- [21] C. D. Hacon, J. McKernan, The Sarkisov program, J. Algebraic Geom. **22** (2013), no. 2, 389–405.
- [22] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
- [23] S. Itaka, Kakankan-ron. (Japanese) [Theory of commutative rings], Second edition, Iwanami Shoten Kiso Sūgaku [Iwanami Lectures on Fundamental Mathematics] 8, Daisū [Algebra], iv. Iwanami Shoten, Tokyo, 1984.

- [24] Y. Kawamata, flops connect minimal models, *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 419–423.
- [25] Y. Kawamata, K. Matsuda, K. Matsuki, Introduction to the minimal model problem, *Algebraic geometry*, Sendai, 1985, 283–360, *Adv. Stud. Pure Math.* **10**, North-Holland, Amsterdam, 1987, 283–360.
- [26] S. Kleiman, Toward a numerical theory of ampleness, *Annals of Math.* **84** (1966), 293–344.
- [27] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, *J. Math. Kyoto Univ.* **13** (1973), 31–47.
- [28] J. Kollár, Projectivity of complete moduli, *J. Differ. Geom.* **32**(1) (1990), 235–268.
- [29] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original, *Cambridge Tracts in Mathematics*, **134**, Cambridge University Press, Cambridge, 1998.
- [30] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, *Astérisque* No. 211 (1992).
- [31] R. Lazarsfeld, *Positivity in algebraic geometry*, I. Classical setting: line bundles and linear series, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **48**. Springer-Verlag, Berlin, 2004.
- [32] W. Lütkebohmert, On compactification of schemes, *Manuscripta Math.* **80** (1993), no. 1, 95–111.
- [33] K. Matsuki, *Introduction to the Mori program*, *Universitext*, Springer-Verlag, New York, 2002.
- [34] K. Miaymoto, The Sarkisov program for log surface, Master’s Thesis, Osaka University, 2020.
- [35] K. Miaymoto, Remarks on birational relations of toric Mori fiber spaces. [arXiv:2207.06171](https://arxiv.org/abs/2207.06171).
- [36] B. G. Moishezon, A criterion for projectivity of complete algebraic abstract varieties, *Amer. Math. Soc. Translations* **63** (1967), 1–50.
- [37] Y. Nakai, A criterion of an ample sheaf on a projective scheme, *Amer. J. Math.* **85** (1963), 14–26.
- [38] P. Pascual, Ampleness criteria for algebraic spaces, *Arch. Math. (Basel)* **45**(3) (1985), 270–274.
- [39] M. Reid, Decomposition of toric morphisms, *Arithmetic and geometry*, vol. II, *Progr. Math.* **36**, pp. 395–418, Birkhäuser, Boston, 1983.
- [40] M. Reid, Birational geometry of 3-folds according to Sarkisov, preprint (1991).
- [41] V. G. Sarkisov, Birational maps of standard  $\mathbb{Q}$ -Fano fiberings, I. B. Kurchatov Institute of Atomic Energy, preprint (1989).
- [42] V. V. Shokurov, 3-fold log models, *Algebraic geometry*, 4. *J. Math. Sci.* **81** (1996), no. 3, 2667–2699.
- [43] K. A. Shramov, Elementary Birational Maps between Mori Toric Fiber 3-Spaces, *Math. Notes*, **78**:1 (2005), 120–127.
- [44] K. A. Shramov, Ascending chain condition for the set of canonical thresholds of toric varieties, *Sb. Math.*, **197**:4 (2006), 623–631.