Global existence for some kinds of systems of semilinear hyperbolic equations

(ある種の半線形双曲型方程式系に対する大域解の存在)

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Chapter 1

Introduction

Throughout this thesis, we write $\partial_0 := \partial_t = \partial/\partial t$, $\partial_k := \partial/\partial x_k$ for $1 \le k \le d$, and the d'Alembertian as $\Box_c := \partial_t^2 - c^2\Delta$ with a positive constant c. This constant c is called the propagation speed. We simply write $\Box_1$ as $\Box$.

The (free) wave equation $\Box_c u = 0$ and the (free) Klein-Gordon equation $(\Box_c + m^2)u = 0$ with $m > 0$ are two typical examples of hyperbolic partial differential equations. In this thesis, we consider two types of systems of semilinear hyperbolic equations involving the wave equations.

The first one is a coupled system of nonlinear wave and Klein-Gordon equations. To give details, for any $N > 0$, we assume

$$m_j > 0 \text{ for } 1 \le j \le N_0 \quad \text{and} \quad m_j = 0 \text{ for } N_0 + 1 \le j \le N \tag{1.1}$$

with some $N_0 \in \{0, 1, \ldots, N\}$. Let $u = (u_j)_{1 \le j \le N}$ be an $\mathbb{R}^N$-valued unknown function of $(t, x) \in (0, \infty) \times \mathbb{R}^d$. We write $u = (v, w)$ with

$$v = (v_j)_{1 \le j \le N_0} = (u_j)_{1 \le j \le N_0}, \ w = (w_j)_{N_0+1 \le j \le N} = (u_j)_{N_0+1 \le j \le N}. \tag{1.2}$$

When $N_0 = 0$ (resp. $N_0 = N$), (1.1) and (1.2) are understood as $m_j = 0$ and $w_j = u_j$ (resp. $m_j > 0$ and $v_j = u_j$) for $1 \le j \le N$. We call v_j the Klein-Gordon component, and w_j as the wave component. Now we would like to present the first coupled system:

$$(\Box + m_j^2)u_j = F_j(v, \partial u) \quad \text{in } (0, \infty) \times \mathbb{R}^d, 1 \le j \le N, \tag{1.3}$$

where $\partial u = (\partial_a u)_{0 \le a \le d}$. The reason why F_j depends on $(v, \partial v)$ and ∂w but not on w itself will be described later in this chapter. We always suppose that $F = (F_j)_{1 \le j \le N}$, and each F_j is a homogeneous polynomial of degree p in its arguments.

The second one is a system of semilinear wave equations with multiple propagation speeds:

$$\Box_{c_I} u^I = F^I(\partial u) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, 1 \le I \le P, \tag{1.4}$$

where $u = (u^I)_{1 \le I \le P}, u^I = (u^I_j)_{1 \le j \le N^I}, N^1 + \cdots + N^P = N$, and

$$0 < c_1 < c_2 < \cdots < c_P. \tag{1.5}$$

We put $F = (F^I)_{1 \le I \le P}$, and each component of F^I is supposed to be a homogeneous polynomial of degree p in its arguments.

We are interested in the Cauchy problem of the above two types of systems with small, smooth and compactly supported initial data. Therefore, we prescribe the initial condition by

$$u(0,x) = \varepsilon f(x), \quad \partial_t u(0,x) = \varepsilon g(x), \qquad x \in \mathbb{R}^d, \tag{1.6}$$

where f, g are C_0^∞-functions and ε is a small positive parameter.

The system (1.3) with $N_0 = 0$, and the system (1.4) with $P = 1$ and $c_1 = 1$ are none other than a system of semilinear wave equations with a single propagation speed 1, that is

$$\Box u = F(\partial u), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{1.7}$$

where $u = (u_j)_{1 \le j \le N}$ is an $\mathbb{R}^N$-valued unknown, and each component of F is a homogeneous polynomial of degree p in its arguments. We firstly summarize the known results for this simpler system. It is well known that this system has a unique local solution in $C^\infty([0,T) \times \mathbb{R}^d)$ with some $T > 0$ for each C_0^∞-data. The *lifespan* is the supremum of such existence time T for the solution u. The solution u is called the *global solution* if its lifespan is equal to ∞. If we consider the initial condition (1.6), then it is known that the lifespan becomes longer as ε becomes smaller. However, no matter how small ε is, the lifespan for the case $F(\partial u) = (\partial_t u)^{(d+1)/(d-1)}$ stays finite for $d = 2, 3$ unless $f = g = 0$ (see [11, 22]). Now a natural question arises: When does a global solution exist for small initial data? The properties of the solutions to the free wave equation $\Box u^+ = 0$ provide a clue to solving this question. A solution u^+ to $\Box u^+ = 0$ is called a *free solution* in the sequel. A free solution u^+ with compactly supported data enjoys a decay estimate

$$|\partial^\alpha u^+(t,x)| \le C\varepsilon(1 + t + |x|)^{-(d-1)/2}\big(1 + \big|t - |x|\big|\big)^{-|\alpha|-(d-1)/2}$$

for any multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$, where $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$. Furthermore, by the energy identity, we also have

$$\|\partial u^+(t)\|_{L^2} = \|\partial u^+(0)\|_{L^2} < \infty,$$

where

$$\|\phi(t)\|_{L^2} = \left(\int_{\mathbb{R}^d} |\phi(t,x)|^2\, dx\right)^{1/2}$$

for a function $\phi = \phi(t, x)$. Another clue is the energy inequality for the inhomogeneous wave equation $\Box u(t, x) = \Phi(t, x)$, that is

$$\|\partial u(t)\|_{L^2} \leq \|\partial u(0)\|_{L^2} + \int_0^t \|\Phi(\tau)\|_{L^2}\, d\tau, \quad t \in [0, T).$$

If we consider the nonlinear case (1.7), the integrability of $\|F(\partial u)(t)\|_{L^2}$ on $(0, \infty)$ is an important factor. Indeed, when the initial data is small enough, we may expect the solution u of (1.7) to behave similarly to the free solution u^+. Then, we can get

$$\|F(\partial u)(t)\|_{L^2} \leq C \sup_{x \in \mathbb{R}^d} |\partial u(t, x)|^{p-1} \|\partial u(t)\|_{L^2} \leq C(1+t)^{-(p-1)(d-1)/2}.$$

Hence we may expect that there exists a unique global solution to (1.7) if $(p-1)(d-1)/2 > 1$. This expectation was resolved by Klainerman [42] in the following way: If $d \geq 2$ and p is an integer with

$$(p-1)(d-1)/2 > 1, \tag{1.8}$$

then the *small data global existence* (which we refer as SDGE in what follows) holds for (1.7) with (1.6); namely, for any $f, g \in C_0^\infty(\mathbb{R}^d)$, there is a positive constant ε_0 such that the Cauchy problem (1.7) with (1.6) admits a unique classical global solution $u \in C^\infty\big([0, \infty) \times \mathbb{R}^d\big)$ for any $\varepsilon \in (0, \varepsilon_0]$. Note that (1.8) is satisfied for any $p \geq 2$ when $d \geq 4$. (1.8) is also satisfied for $p \geq 3$ when $d = 3$, and for $p \geq 4$ when $d = 2$; however it is violated for $(d, p) = (3, 2)$ and $(d, p) = (2, 3)$ as we have $(p-1)(d-1)/2 = 1$. These are critical cases, as $(p-1)(d-1)/2 = 1$ is equivalent to $p = (d+1)/(d-1)$, and SDGE fails for $F = (\partial_t u)^{(d+1)/(d-1)}$ for $d = 2, 3$, as we have mentioned before.

We are interested in these critical cases, and we always assume

$$(d, p) = (3, 2) \text{ or } (2, 3) \tag{1.9}$$

from now on. Concerning SDGE for the critical cases, Klainerman showed SDGE for $(d, p) = (3, 2)$ under a certain condition called the *null condition* in [44] (see also Christodoulou [9]). Its counterpart for the case $(d, p) = (2, 3)$ was developed by Godin [11] and Hoshiga [16]. We do not go into details, but the null condition for the subcritical case $(d, p) = (2, 2)$ was also studied by Alinhac [3]. To explain the null condition precisely, we write

$$F(\partial u) = F(\partial_t u, \partial_1 u, \ldots, \partial_d u),$$

and define the *reduced nonlinearity* $F^{\rm red}$ by

$$F^{\rm red}(\omega, Y) := F(-Y, \omega_1 Y, \ldots, \omega_d Y)$$

for $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{S}^{d-1}$ and $Y = (Y_j)_{1 \le j \le N} \in \mathbb{R}^N$, where $\mathbb{S}^{d-1}$ denotes the unit $(d-1)$-sphere. In this notation, the null condition in [11, 44] is given by

$$F^{\rm red}(\omega, Y) = 0, \qquad \omega \in \mathbb{S}^{d-1}, Y \in \mathbb{R}^N. \tag{1.10}$$

Roughly speaking, the null condition guarantees that the critical nonlinear terms with the slowest decay disappear. It is also known that global solutions with small data under the null condition are *asymptotically free*; namely, for a global solution u, there is a free solution u^+ such that we have

$$\lim_{t \to \infty} \|\partial u(t) - \partial u^+(t)\|_{L^2} = 0.$$

We can also see that the difference between the initial data for u^+ and that for u is of order ε^p for small ε.

On the other hand, the following simple example

$$\begin{cases} \Box u_1 = -(\partial_t u_2)^3, \\ \Box u_2 = 0, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^2,$$

shows that the null condition is not a sufficient condition for SDGE, since SDGE apparently holds for this system, but the null condition is violated. Therefore, in order to be able to understand such a system, Lindblad-Rodnianski [52] introduced the *weak null condition* for the three space dimensional case (but we can formulate a corresponding condition also for the two space dimensional case in a similar fashion), and conjectured that SDGE would be established under the weak null condition (see Chapter 2 for details). However, this conjecture has not yet been proved. Therefore, sufficient conditions for SDGE, being related to but stronger than the weak null condition, and yet weaker than the null condition, are widely studied. Such conditions were introduced by Alinhac [4], Katayama [29], Katayama-Matoba-Sunagawa [37] in three space dimensions, and by Hoshiga [19], Katayama [27], Katayama-Murotani-Sunagawa [39], Katayama-Matsumura-Sunagawa [38] and Kubo [48] in two space dimensions.

Here we present the conditions introduced in [37] and [38]. These conditions can be unified to one condition, which we call the KMS condition after the initials of the authors: There is $\mathcal{H} = \mathcal{H}(\omega) \in C(\mathbb{S}^{d-1}; \mathcal{S}^N_+)$ such that

$$\langle Y, \mathcal{H}(\omega) F^{\rm red}(\omega, Y) \rangle_{\mathbb{R}^N} \ge 0, \qquad \omega \in \mathbb{S}^{d-1}, Y \in \mathbb{R}^N,$$

where $\mathcal{S}^N_+$ is the set of real symmetric positive-definite matrices of size $N \times N$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$ denotes the standard inner product in $\mathbb{R}^N$. It is trivial to see that the null condition implies the KMS condition. The KMS condition is a generalization of the Agemi condition, which was proposed by Agemi in the 1990s for single wave equations in two space dimensions, to three space dimensions or systems. The KMS condition implies SDGE, and it is known that there are various asymptotic behaviors to occur, differently from the case of the null condition: Some solutions are asymptotically free, and the others are not. Even if the solution is asymptotically free, the asymptotic data can be away from the original data, and decay of the energy may also occur for some nonlinearity. See Katayama [32, 33], Nishii-Sunagawa [54] and Nishii-Sunagawa-Terashita [55] for the details.

Now we turn our attention to (1.3). Before proceeding to the general case, we would like to mention another special casse of $N_0 = N$; in this case, (1.3) is exactly a system of semilinear Klein-Gordon equations, and we would like to recall some known results. Thanks to the mass term m_j^2, solutions to the free Klein-Gordon equation have faster decay than those to the free wave equation in a neighborhood of the light cone $|x| = t$. In fact, the uniform decay rate for the solutions of the Klein-Gordon equation is $(1+t)^{-d/2}$; however, differently from the case of wave equations, there is no gain of decay away from the light cone. Moreover, the mass term enables the energy inequality to control the L^2-norm of the solution as well as that of its derivatives (this is the reason why we allow the nonlinearity F to depend on v itself but not on w itself in (1.3)). Indeed, if we define

$$\|u(t)\|_{E,m} := \left(\frac{m^2}{2} \|u(t)\|_{L^2}^2 + \frac{1}{2} \|\partial u(t)\|_{L^2}^2 \right)^{1/2},$$

then the energy inequality for the linear Klein-Gordon equation

$$(\Box + m^2)u(t,x) = \Phi(t,x)$$

can be written in the form

$$\|u(t)\|_{E,m} \le \|u(0)\|_{E,m} + \int_0^t \|\Phi(\tau)\|_{L^2}\, d\tau.$$

The critical pairs $(d,p) = (3,2)$ and $(d,p) = (2,3)$ for the wave equations are super-critical for the Klein-Gordon equations because the uniform decay rate is $-d/2$ and we have $(p-1)d/2 > 1$ for these pairs. Indeed, it is known that SDGE holds for these cases without any further restriction. See Klainerman [43] and Shatah [57] for the three space dimensional case, and Ozawa-Tsutaya-Tsutsumi [56] and Simon-Taflin [59] for the two space dimensional case.

Now, we would like to recall the previous research about the case where the system (1.3) is actually a coupled system of nonlinear wave and Klein-Gordon equations (namely, the case $1 \le N_0 \le N-1$). Since this coupled system contains the wave equations, some additional condition like the null condition is of course necessary to obtain SDGE when $(d,p) = (3,2)$ or $(d,p) = (2,3)$. The difficulty in treating the null condition for this coupled system is that fewer vector fields can be used than the case where the system involves the wave equations alone. More precisely, Klainerman [44] proved SDGE for systems of nonlinear wave equations satisfying the null condition by the so-called *vector field method*. To be more precise, Klainerman used the vector fields

$$S = t\partial_t + x \cdot \nabla,\ L_k = x_k\partial_t - t\partial_k,\ \Omega_{kl} = x_k\partial_l - x_l\partial_k$$

for $1 \le k, l \le d$, as well as the standard differentiation ∂_a for $0 \le a \le d$. S is called the *scaling operator*, and L_k is called the *Lorentz boost*. Let $[A, B] = AB - BA$ be the commutator for operators A and B. An important point is that we have a commutative property $[\Box, L_k] = [\Box, \Omega_{kl}] = [\Box, \partial_a] = 0$, and that S is "almost" commutable with $\Box$ in the sense that $[\Box, S] = 2\Box$, so that we have $\Box(Su) = (S+2)\Box u$, on whose right-hand side S is replaced by $S+2$, but this causes no serious problem. Klainerman used these vector fields to obtain a weighted L^1-L^∞ decay estimate for the wave equations, and to obtain an enhanced decay estimate for the nonlinearity satisfying the null condition. For the Klein-Gordon equations, the vector fields L_k, Ω_{kl} and ∂_a still commute with $\Box + m^2$ for $m > 0$, but the scaling operator S is not "almost" commutable with $\Box + m^2$, as $[\Box + m^2, S] = 2\Box$. Klainerman [43] developed the vector field method without S for the Klein-Gordon equations. Therefore, in order to treat the coupled system of wave and Klein-Gordon equations, our task is to establish a vector field method without S for the wave equations.

For the three space dimensional case, a pioneering work was done by Georgiev [12]. He introduced the *strong null condition*

$$F^{\mathrm{red}}(\omega, Y) = 0, \qquad \omega \in \mathbb{R}^3, Y \in \mathbb{R}^N$$

(notice that $\omega \in \mathbb{S}^2$ in the null condition for $d = 3$ is replaced by $\omega \in \mathbb{R}^3$ here), and showed SDGE under the strong null condition by developing a weighted L^2-L^∞ decay estimate for wave equations which only requires L_k, Ω_{kl}, and ∂_a, as well as a corresponding estimate for the Klein-Gordon equations, and an enhanced decay estimate without S for the nonlinearity satisfying the strong null condition. Note that the result in [12] is quite weaker than the previous results for the wave and Klein-Gordon equations if we put $N_0 = 0$ or $N_0 = N$. Later, the required condition was extremely relaxed by Katayama [28]. To explain his condition, we introduce some notations: Let $F_j^{(\mathrm{w})}(\partial w)$ be the nonlinear terms consisting of all the terms depending only on ∂w in F_j, and we set $F_{\mathrm{W}}^{(\mathrm{w})} := (F_j^{(\mathrm{w})})_{N_0+1 \le j \le N}$. $F_{\mathrm{W}}^{(\mathrm{w})}$ stands for the interaction between wave components in the wave equations above. Writing

$$F_{\mathrm{W}}^{(\mathrm{w})}(\partial w) = F_{\mathrm{W}}^{(\mathrm{w})}(\partial_t w, \partial_1 w, \ldots, \partial_d w),$$

we define the reduced nonlinearity for this system by

$$F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y) := F_{\mathrm{W}}^{(\mathrm{w})}(-Y, \omega_1 Y, \ldots, \omega_d Y) \tag{1.11}$$

for $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{S}^{d-1}$ and $Y \in \mathbb{R}^{N_1}$ with $N_1 = N - N_0$. In [28], SDGE for $(d, p) = (3, 2)$ was proved by only assuming the null condition for $F_{\mathrm{W}}^{(\mathrm{w})}$, that is

$$F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y) = 0, \quad \omega \in \mathbb{S}^{d-1}, Y \in \mathbb{R}^{N_1} \tag{1.12}$$

(if $N_1 = N$, (1.12) coincides with (1.10)). Katayama used the weighted L^∞-L^∞ estimates for the wave equations, which only require Ω_{kl} and ∂_a; he also obtained an enhanced decay estimate for $F_{\mathrm{W}}^{(\mathrm{w})}$ satisfying the null condition without S by considering $(t - |x|)\partial_a$ instead of S (this idea comes from the multiple-speed case which will be explained below); in order to treat $F_j^{(\mathrm{w})}$ for $1 \le j \le N_0$, the interaction between the wave components in the nonlinearity for the Klein-Gordon components, some kind of transformation was used to eliminate them. We also refer the reader to LeFloch-Ma [51] for an alternative proof, where the hyperboloidal foliation method is used. The counterpart of [28] for $(d, p) = (2, 3)$ was obtained by Aiguchi [2].

We finally review what is known about the systems (1.4) of semilinear wave equations with multiple propagation speeds. In this case, as above, fewer vector fields are available than the single speed case (1.7). This time, the Lorentz boosts L_k are troublesome. Ω_{kl} and ∂_a commute with $\Box_c$ for any $c > 0$, and the scaling operator S is "almost" commutable with $\Box_c$ in the sense that $[\Box_c, S] = 2\Box_c$; however L_k commutes with $\Box_c$ only when $c = 1$. More precisely, a suitable Lorentz boost for $\Box_c$ is $x_k\partial_t + ct\partial_k$, but it does not commute with $\Box_{c'}$ if $c' \neq c$. Therefore, when considering this system, we have to exclude the Lorentz boosts L_k.

The multiple-speed version of the null condition in two and three space dimensions were introduced by Hoshiga-Kubo [20] and Yokoyama [62], respectively, and they proved SDGE under this null condition. Roughly speaking, the interactions between components with different speeds are relatively small, as they propagate in different speeds and each component decays faster in a region away from the corresponding light cone. Hence the multiple-speed version of the null condition is a restriction on interactions between the components with the same speed: Let ${}^*F^I(\partial u^I)$ be the nonlinear terms consisting of all the terms depending only on ∂u^I in F^I, and we write

$${}^*F^I(\partial u^I) = {}^*F^I(\partial_t u^I, \partial_1 u^I, \dots, \partial_d u^I).$$

${}^*F^I$ stands for the interactions between the components with the same speed. We also define the reduced nonlinearity $F^{I,\mathrm{red}}$ by

$$F^{I,\mathrm{red}}(\omega, Y^I) := {}^*F^I(-c_I Y^I, \omega_1 Y^I, \dots, \omega_d Y^I) \tag{1.13}$$

for $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{S}^{d-1}$ and $Y^I = (Y_j^I)_{1 \le j \le N^I} \in \mathbb{R}^{N^I}$. In above notations, the multiple-speed version of the null condition can be expressed in the form

$$F^{I,\mathrm{red}}(\omega, Y^I) = 0, \quad \omega \in \mathbb{S}^{d-1}, Y^I \in \mathbb{R}^{N^I}, 1 \le I \le P \tag{1.14}$$

(observe that if $P = 1$ and $c_1 = 1$, then (1.14) is nothing but the null condition(1.10)). They proved SDGE by developing weighted L^∞-L^∞ decay estimates which only require Ω_{kl} and ∂_a; they also succeeded to obtain an enhanced decay estimate without the Lorentz boosts for ${}^*F^I$ satisfying the null condition, by considering $(t-|x|)\partial_a$ instead of the Lorentz boosts (these vector field method without L_k motivates that without S in [28]). See Sideris-Tu [58], Sogge [60] and Hidano [14] for the alternative proof for $(d,p) = (3,2)$, where different kinds of the decay estimates are used; see Katayama [30] for the asymptotic behavior of global solutions.

Our aim in this thesis is to relax the sufficient conditions for SDGE in the cases of (1.3) and (1.4), in a similar manner to the case of the wave equations where weaker sufficient conditions than the null condition are known. To be more precise, we will introduce conditions for the system (1.3) and (1.4), which correspond to the KMS condition for (1.7), and prove SDGE under these conditions. This is non-trivial because the scaling operator and the Lorentz boosts are essentially used in [37, 38] to treat the KMS condition. This challenge is successful for the two space dimensional case, but only partially resolved for the three space dimensional case due to technical problems. In other words, additional conditions are needed for SDGE to be valid in the three space dimensional case. We will also obtain results on the asymptotic behavior under our conditions, subject to certain additional conditions.

Remark 1. In this chapter, we have assumed that the nonlinear terms are homogeneous polynomials of degree p, but in most cases, we can easily add nonlinear terms of higher order than p. We also assumed that the nonlinear terms do not depend on the unknown functions of the wave equation itself (u for (1.7), w for (1.3), and each u^I for (1.4)), but there are many results that allow them to do so. We refer readers to Katayama [23, 24, 32] and Zha [63] for the wave equations with a single speed, Katayama [25, 26], Katayama-Yokoyama [40], Kubota-Yokoyama [50] and Metcalfe-Nakamura-Sogge [53] for the multiple-speed case, and Katayama [31] for the wave-Klein-Gordon case.

Remark 2. The question naturally arises as to what happens to systems of wave and Klein-Gordon equations with multiple speeds; however, even systems of Klein-Gordon equations with multiple speeds are quite challenging problems, and only few results are known. See Germain [10] for example.

Chapter 2

Main Results

In this chapter, we introduce the KMS conditions for the wave-Klein-Gordon case (1.3) and multiple-speed case (1.4) in two and three space dimensions, and present the main results under these conditions.

Recall that (d,p) is equal to $(2,3)$ or $(3,2)$, where d is the space dimension, and p is the degree of the nonlinear terms. Recall also the $\mathcal{S}_N^+$ is the set of real symmetric positive-definite matrices of size $N \times N$, and $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$ denotes the standard inner product in $\mathbb{R}^N$.

The SDGE part.

We firstly consider the wave-Klein-Gordon case (1.3). For this system, we always suppose (1.1), and each $F_j = F_j(v, \partial u)$ is assumed to be a homogeneous polynomial of degree p in its arguments. The following is our first condition.

Definition 2.1 (The KMS condition for the wave-Klein-Gordon case)**.** We say that the KMS condition for (1.3) is satisfied if there is $\mathcal{H} = \mathcal{H}(\omega) \in C(\mathbb{S}^{d-1}; \mathcal{S}_+^{N_1})$ such that

$$\langle Y, \mathcal{H}(\omega) F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y) \rangle_{\mathbb{R}^{N_1}} \geq 0, \qquad \omega \in \mathbb{S}^{d-1}, Y \in \mathbb{R}^{N_1}, \tag{2.1}$$

where the reduced nonlinearity $F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y)$ is given by (1.11).

Observe that this condition is weaker than the null condition (1.12) for $F_{\mathrm{W}}^{(\mathrm{w})}$. Now we are in a position to state our main results for (1.3). The first was obtained for the two space dimensional case.

Theorem 2.1. *Let* $(d,p) = (2,3)$*. We suppose that the KMS condition for* (1.3) *is satisfied. Then, for any* $f, g \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^N)$*, we can take a positive constant* ε_0 *such that there is a global smooth solution* $u = (v, w)$ *in* $[0, \infty) \times \mathbb{R}^2$ *to the Cauchy problem* (1.3) *with* (1.6) *for any* $\varepsilon \in (0, \varepsilon_0]$.

Example 1. For $u = (v, w)$ with $N = 2$ and $N_0 = N_1 = 1$, we consider a system

$$\begin{cases} (\Box + m^2)v = F_1(v, \partial u), \\ \Box w = F_2(v, \partial u), \end{cases} \tag{2.2}$$

in $(0, \infty) \times \mathbb{R}^2$, where $m > 0$, F_1 and F_2 are homogeneous polynomials of degree 3 in their arguments, and

$$F_2^{(\mathrm{w})}(\partial w) = -c(\partial_a w)^2(\partial_t w) + (\partial_t w)\{(\partial_t w)^2 - (\partial_1 w)^2 - (\partial_2 w)^2\}$$

for some $c \geq 0$ and $a = 0, 1, 2$. As $F_{\mathrm{W}}^{(\mathrm{w})}(\partial w) = F_2^{(\mathrm{w})}(\partial w)$ and $Y = Y_2$, we have

$$F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y) = c\omega_a^2 Y^3.$$

The null condition (1.12) for $F_2^{(\mathrm{w})}$ is violated unless $c = 0$; however the KMS condition is satisfied with $\mathcal{H}(\omega)$ being the identity matrix, as we have

$$\langle Y, F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}} = c\omega_a^2 Y^4 \geq 0, \quad \omega \in \mathbb{S}^1, \ Y(= Y_2) \in \mathbb{R}$$

for any $c \geq 0$.

The next theorem is the three space dimensional version of the above theorem; however, as mentioned in the introduction, this case has not been completely resolved, and at the present time an additional condition is needed. This is due to a lack of sufficient decay of the solution around the t-axis. See Subsection 5.1 below for discussions.

Theorem 2.2. *Let* $(d, p) = (3, 2)$. *We suppose that the KMS condition for* (1.3)*, and the following condition are fulfilled:*

(A1) $F = F(v, \partial u)$ *is independent of* $\partial_t w$*; in other words,* $F = F(v, \partial v, \partial_x w)$ *with* $\partial_x w = (\partial_k w)_{1 \leq k \leq 3}$.

Then, for any $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$*, we can take a positive constant* ε_0 *such that there is a global smooth solution* $u = (v, w)$ *in* $[0, \infty) \times \mathbb{R}^3$ *to the Cauchy problem* (1.3) *with* (1.6) *for any* $\varepsilon \in (0, \varepsilon_0]$.

Remark 3. (1) In three space dimension, (2.1) is equivalent to

$$\langle Y, \mathcal{H}(\omega)F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}^{N_1}} = 0, \qquad \omega \in \mathbb{S}^2, Y \in \mathbb{R}^{N_1}. \tag{2.3}$$

Indeed, since $p = 2$, the left-hand side of (2.1) is cubic in Y, and we get

$$-\langle Y, \mathcal{H}(\omega)F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}^{N_1}} = \langle -Y, \mathcal{H}(\omega)F_{\mathrm{W}}^{\mathrm{red}}(\omega, -Y)\rangle_{\mathbb{R}^{N_1}} \geq 0,$$

which, together with (2.1), implies (2.3).
(2) Investigating the proof of the Theorem 2.2, we can add any nonlinearity of order 3 that depends not only on $v, \partial v, \partial_x w$, but also on $\partial_t w$ (see Remark 11 below).

Because of the additional assumption (A1), the above theorem is not an extension of the previous result in [28]; however the following example shows that Theorem 2.2 can cover some systems to which the previous result in [28] is not applicable.

Example 2. Let $v(= v_1), w_2, w_3$ be a real valued unknowns, and we consider a system

$$\begin{cases} (\Box + 1)v = F_1(v, \partial v, \partial_x w), \\ \Box w_2 = (\partial_1 w_2)(\partial_2 w_3) + \displaystyle\sum_{k,l=1}^{3} Q_{kl}(w_2, w_3) + vG_2(v, \partial v, \partial_x w), \\ \Box w_3 = -(\partial_1 w_2)(\partial_2 w_2) + vG_3(v, \partial v, \partial_x w) \end{cases} \tag{2.4}$$

in $(0, \infty) \times \mathbb{R}^3$, where $w = (w_2, w_3)$, F_1 is a homogeneous polynomial of degree 2 in its argument, and G_1, G_2 are homogeneous polynomials of degree 1, while $Q_{kl}(\phi, \psi)$ is given by

$$Q_{kl}(\phi, \psi) = (\partial_k \phi)(\partial_l \psi) - (\partial_l \phi)(\partial_k \psi). \tag{2.5}$$

In this case, $F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y) = (\omega_1\omega_2 Y_2 Y_3, -\omega_1\omega_2 Y_2^2)$ for $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$ and $Y = (Y_2, Y_3) \in \mathbb{R}^2$. (A1) is apparently satisfied. Since

$$\langle Y, F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}^2} = Y_2(\omega_1\omega_2 Y_2 Y_3) - Y_3(\omega_1\omega_2 Y_2^2) = 0,$$

the KMS condition is satisfied with $\mathcal{H}$ being the identity matrix. However, the null condition is not satisfied as $F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y) \not\equiv (0, 0)$.

Next, we consider the multiple-speed case (1.4). We always assume (1.5), and each component of $F^I(\partial u)$ is supposed to be a homogeneous polynomial of degree p in its arguments. The following is our second condition.

Definition 2.2 (The KMS condition for the multiple-speed case)**.** We say that the KMS condition for (1.4) is satisfied if, for each $1 \le I \le P$, there is $\mathcal{H}^I = \mathcal{H}^I(\omega) \in C(\mathbb{S}^{d-1}; \mathcal{S}_+^{N^I})$ such that

$$\langle Y^I, \mathcal{H}^I(\omega) F^{I,\mathrm{red}}(\omega, Y^I)\rangle_{\mathbb{R}^{N^I}} \ge 0, \qquad \omega \in \mathbb{S}^{d-1}, Y^I \in \mathbb{R}^{N^I}, \tag{2.6}$$

where the reduced nonlinearity $F^{I,\mathrm{red}}(\omega, Y^I)$ is defined by (1.13).

Observe that this condition is weaker than (1.14), the multiple-speed version of the null condition.

As above, we first present the result for the two space dimensional case.

Theorem 2.3. *Let $(d, p) = (2, 3)$. If the KMS condition for (1.4) is satisfied, then for any $f, g \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^N)$, there is a positive constant ε_0 such that the Cauchy problem (1.4) with (1.6) admits a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^2; \mathbb{R}^N)$, provided that $\varepsilon \in (0, \varepsilon_0]$.*

Example 3. We write $\phi = \sum'_{(J,K,L)\neq(I,I,I)} (\partial_a u_j^J)(\partial_b u_k^K)(\partial_c u_l^L)$, if there are some constants C_{abc}^{JjKkLl} such that

$$\phi = \sum_{(J,K,L)\neq(I,I,I)} \sum_{j,k,l} \sum_{a,b,c} C_{abc}^{JjKkLl} (\partial_a u_j^J)(\partial_b u_k^K)(\partial_c u_l^L).$$

For the sake of readability, we write u^I as $u^{(I)}$ here. Let c_1 and c_2 be different positive constants. Let $u = (u^{(1)}, u^{(2)})$ with an $\mathbb{R}^2$-valued unknown $u^{(1)} = (u_1^{(1)}, u_2^{(1)})$ and a real-valued unknown $u^{(2)} = u_1^{(2)}$. We consider a semilinear system

$$\begin{cases} \Box_{c_1} u_1^{(1)} &= \sum_{a,b} C_{ab} (\partial_a u_1^{(1)})^2 (\partial_b u_2^{(1)}) + \sum'_{(J,K,L)\neq(1,1,1)} (\partial_a u_j^J)(\partial_b u_k^K)(\partial_c u_l^L), \\ \Box_{c_1} u_2^{(1)} &= -\sum_{a,b} C_{ab} (\partial_a u_1^{(1)})^2 (\partial_b u_1^{(1)}) + \sum'_{(J,K,L)\neq(1,1,1)} (\partial_a u_j^J)(\partial_b u_k^K)(\partial_c u_l^L), \\ \Box_{c_2} u^{(2)} &= -(\partial_t u^{(2)})^3 + (\partial_t u^{(2)}) Q_0^{c_2}(u^{(2)}, u^{(2)}) \\ &\qquad + \sum'_{(J,K,L)\neq(2,2,2)} (\partial_a u_j^J)(\partial_b u_k^K)(\partial_c u_l^L), \end{cases}$$

in $(0,\infty)\times\mathbb{R}^2$, where C_{ab} are constants and $Q_0^c(\phi,\psi)$ is given by

$$Q_0^c(\phi,\psi) = (\partial_t\phi)(\partial_t\psi) - c^2 \sum_{k=1}^{d} (\partial_k\phi)(\partial_k\psi). \tag{2.7}$$

We suppose that

$$\sum_{a,b} C_{ab}\omega_a^2\omega_b \not\equiv 0, \quad \omega = (\omega_1,\omega_2) \in \mathbb{S}^1 \text{ with } \omega_0 = -c_1. \tag{2.8}$$

Then, all the assumptions in Theorem 2.3 are fulfilled for this system, but the null condition fails to hold, since we have

$$F^{1,\mathrm{red}}\big(\omega, Y^{(1)}\big) = F^{1,\mathrm{red}}\big(\omega, Y_1^{(1)}, Y_2^{(1)}\big) = \sum_{a,b} C_{ab}\omega_a^2\omega_b \begin{pmatrix} (Y_1^{(1)})^2 Y_2^{(1)} \\ -(Y_1^{(1)})^3 \end{pmatrix}$$

with $\omega_0 = -c_1$, and $F^{2,\mathrm{red}}\big(\omega, Y^{(2)}\big) = \big(c_2 Y^{(2)}\big)^3$. Notice that we have $\langle Y^{(1)}, F^{1,\mathrm{red}}\rangle_{\mathbb{R}^2}$ and $\langle Y^{(2)}, F^{2,\mathrm{red}}\rangle_{\mathbb{R}} = (c_2)^3 (Y^{(2)})^4 \geq 0$.

The theorem below was obtained for the three space dimensional case; as in Theorem 2.2, it requires an additional condition at present time, because of a lack of sufficient decay around the t-axis.

Theorem 2.4. *Let $(d,p) = (3,2)$. If the KMS condition for (1.4) and*

(A2) *F depends only on $\partial_t u$, namely $F = F(\partial_t u)$*

are satisfied, then for any $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$, there is a positive constant ε_0 such that the Cauchy problem (1.4) *with* (1.6) *admits a unique global solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3; \mathbb{R}^N)$, provided that $\varepsilon \in (0, \varepsilon_0]$.*

Remark 4. (1) As before, in three space dimensions, (2.6) is equivalent to

$$\langle Y^I, \mathcal{H}^I(\omega) F^{I,\mathrm{red}}(\omega, Y^I)\rangle_{\mathbb{R}^{N^I}} = 0, \qquad \omega \in \mathbb{S}^2, Y \in \mathbb{R}^{N^I}. \tag{2.9}$$

(2) We can add any nonlinearity of order 3 that depends not only on $\partial_t u$, but also on $\partial_x u = (\partial_k u)_{1 \le k \le 3}$.

Because of the additional condition (A2), Theorem 2.4 is not an extension of the previous result in [62], but, as the next example shows, it can cover a certain system which cannot be treated in the previous result.

Example 4. As before, we write $u^{(I)}$ for u^I with $I = 1, 2$ here. Let c_1 and c_2 be different positive constants. Let $u^{(1)}$ be a real-valued unknown, and $u^{(2)} = (u_1^{(2)}, u_2^{(2)})$ be an $\mathbb{R}^2$-valued unknown. We consider a system

$$\begin{cases} \Box_{c_1} u^{(1)} = (\partial_t u^{(1)})(A_1 \partial_t u_1^{(2)} + A_2 \partial_t u_2^{(2)}) + G^{(1)}(\partial_t u^{(2)}), \\ \Box_{c_2} u_1^{(2)} = (\partial_t u_1^{(2)})(\partial_t u_2^{(2)}) + G_1^{(2)}(\partial_t u^{(1)}), \\ \Box_{c_2} u_2^{(2)} = -(\partial_t u_1^{(2)})^2 + G_2^{(2)}(\partial_t u^{(1)}) \end{cases} \tag{2.10}$$

in $(0,\infty) \times \mathbb{R}^3$, where $u = (u^{(1)}, u^{(2)}) = (u^{(1)}, u_1^{(2)}, u_2^{(2)})$, A_1, A_2 are constants, and $G^{(1)}, G_1^{(2)}, G_2^{(2)}$ are homogeneous polynomials of degree 2 in their arguments. For this system, (A2) is trivially satisfied. As $F^{1,\mathrm{red}}(\omega, Y^{(1)}) = 0, F^{2,\mathrm{red}}(\omega, Y^{(2)}) = \left(Y_1^{(2)} Y_2^{(2)}, -\left(Y_1^{(2)}\right)^2\right)$, we have

$$Y^{(1)} F^{1,\mathrm{red}} = \langle Y^{(2)}, F^{2,\mathrm{red}}\rangle_{\mathbb{R}^2} = 0,$$

and the KMS condition is satisfied; however, the null condition is violated as $F^{2,\mathrm{red}} \not\equiv 0$.

The asymptotic behavior part.

For $z \in \mathbb{R}^n$ with a natural number n, we use the notation $\langle z \rangle = \sqrt{1 + |z|^2}$. $\langle z \rangle$ is equivalent to $1 + |z|$, that is to say $C^{-1}\langle z \rangle \le 1 + |z| \le C\langle z \rangle$ for $z \in \mathbb{R}^n$ with some positive constant C. We use the following $\mathcal{O}$-notation in the sequel: Let E be a set, and $\phi = \phi(z), \psi = \psi(z)$, and $\eta = \eta(z)$ be functions of $z \in E$. We write

$$\phi(z) = \psi(z) + \mathcal{O}(\eta(z)), \qquad z \in E,$$

if there is a universal positive constant C such that we have

$$|\phi(z)-\psi(z)| \le C|\eta(z)|, \qquad z \in E.$$

We refer to the above constant C as the constant associated with $\mathcal{O}$.

To describe the asymptotic behavior of solutions to (1.3) and (1.4) in a unified way, we introduce some notations: Let n be an integer, $\phi=\phi(t,x)$ be an $\mathbb{R}^n$-valued function of $(t,x) \in (0,\infty)\times\mathbb{R}^d$, and let $\Psi=\Psi(\partial u)$ be an $\mathbb{R}^n$-valued function whose components are homogeneous polynomials of degree p in its arguments. Let $c>0$. Writing $\Psi(\partial\phi)=\Psi(\partial_t\phi,\partial_1\phi,\dots,\partial_d\phi)$, we define

$$\Psi^{\mathrm{red}}(\omega,Y)=\Psi^{\mathrm{red}}(\omega,Y;c)=\Psi(-cY,\omega_1 Y,\dots,\omega_d Y), \quad \omega\in\mathbb{S}^{d-1}, Y\in\mathbb{R}^n.$$

We suppose that there is $\mathcal{H}=\mathcal{H}(\omega)\in C(\mathbb{S}^{d-1};\mathcal{S}_n^+)$ such that

$$\langle Y,\mathcal{H}(\omega)\Psi^{\mathrm{red}}(\omega,Y)\rangle_{\mathbb{R}^n}\ge 0, \qquad \omega\in\mathbb{S}^{d-1}, Y\in\mathbb{R}^n. \tag{2.11}$$

Given an $\mathbb{R}^n$-valued function $\psi=\psi(\sigma,\omega)$, we write $A=A[c,\Psi;\psi](t,\sigma,\omega)$ for the unique solution to the Cauchy problem

$$\partial_t A(t,\sigma,\omega)=-\frac{1}{2c^2 t}\Psi^{\mathrm{red}}\big(\omega,A(t,\sigma,\omega);c\big) \tag{2.12}$$

for $(t,\sigma,\omega)\in(1,\infty)\times\mathbb{R}\times\mathbb{S}^{d-1}$ with the initial condition

$$A(1,\sigma,\omega)=\psi(\sigma,\omega), \qquad (\sigma,\omega)\in\mathbb{R}\times\mathbb{S}^{d-1}. \tag{2.13}$$

Under the condition (2.11), the Cauchy problem (2.12)–(2.13), which can be viewed as an ODE system with parameters (σ,ω), admits a unique global solution. Indeed, we have

$$\partial_t\langle A,\mathcal{H}A\rangle_{\mathbb{R}^n}=2\langle A,\mathcal{H}\partial_t A\rangle_{\mathbb{R}^n}=-\frac{1}{c^2 t}\langle A,\mathcal{H}\Psi^{\mathrm{red}}(\omega,A)\rangle_{\mathbb{R}^n}\le 0,$$

which leads to an *a priori* estimate $|A(t,\sigma,\omega)|^2\le C|\psi(\sigma,\omega)|^2$, because there is a positive constant C_0 such that

$$\frac{1}{C_0}|Y|^2\le\langle Y,\mathcal{H}(\omega)Y\rangle_{\mathbb{R}^n}\le C_0|Y|^2, \quad \omega\in\mathbb{S}^{d-1}, Y\in\mathbb{R}^n.$$

When we consider (1.3), the system (2.12) with

$$c=1,\ n=N_1,\ \phi=w,\ \Psi(\partial w)=F_{\mathrm{W}}^{(\mathrm{w})}(\partial w)$$

(and thus $\Psi^{\mathrm{red}}(\omega,Y)=F_{\mathrm{W}}^{\mathrm{red}}(\omega,Y)$) is called the *reduced system* associated with (1.3), and we define

$$A[\psi](t,\sigma,\omega)=A[1,F_{\mathrm{W}}^{(\mathrm{w})};\psi](t,\sigma,\omega). \tag{2.14}$$

On the other hand, when (1.4) is considered, the systems (2.12) with

$$c = c_I,\ n = N^I,\ \phi = u^I,\ \Psi(\partial u^I) = {}^*F^I(\partial u^I)$$

(and thus $\Psi^{\mathrm{red}}(\omega, Y) = F^{I,\mathrm{red}}(\omega, Y)$) for $1 \le I \le P$ are called the *reduced systems* associated with (1.4), and we put

$$A^I[\psi](t, \sigma, \omega) = A[c_I, {}^*F^I; \psi](t, \sigma, \omega), \quad 1 \le I \le P. \tag{2.15}$$

Observe that, in both cases, the KMS condition ensures (2.11). As we will see later, the reduced system plays an important role in the derivation of the asymptotic behavior, as well as in the proof of SDGE.

Remark 5. As for the system (1.7) of semilinear wave equations with the single speed 1, its reduced system is (2.12) with $c = 1, n = N, \phi = u$ and $\Psi(\partial u) = F(\partial u)$. We say that the weak null condition is satisfied for (1.7) if there is a global solution to the reduced system (2.12) for small initial data ψ decaying sufficiently fast as $|\sigma| \to \infty$. The above observation shows the the KMS condition implies the weak null condition.

Using the above notation, we first state a result for (1.3).

Theorem 2.5. *Let* (d, p) *is equal to* $(2, 3)$ *or* $(3, 2)$*, and suppose that the KMS condition for* (1.3) *holds; we also assume (A1) when* $(d, p) = (3, 2)$*. Let* κ *be a small positive number and* $u = (v, w)$ *be the global solution to* (1.3) *with* (1.6)*. Let* $f, g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^N)$*, and* ε *be sufficiently small. Then we have the following:*

(1) *The Klein-Gordon component* v *is asymptotically free; namely there is the asymptotic data*

$$(\varphi^+, \psi^+) = \Big((\varphi_j^+), (\psi_j^+)\Big) \in H^1(\mathbb{R}^d; \mathbb{R}^{N_0}) \times L^2(\mathbb{R}^d; \mathbb{R}^{N_0})$$

such that

$$\lim_{t\to\infty} \Big(\|v(t) - v^+(t)\|_{H^1(\mathbb{R}^d)} + \|\partial_t v(t) - \partial_t v^+(t)\|_{L^2(\mathbb{R}^d)}\Big) = 0,$$

where $v^+ = (v_j^+)_{1\le j\le N_0}$ *satisfies* $(\Box + m_j^2)v_j^+ = 0$ *with* $(v_j^+, \partial_t v_j^+)(0) = (\varphi_j^+, \psi_j^+)$ *for* $1 \le j \le N_0$*. Moreover, we have*

$$(\varphi_j^+, \psi_j^+) = \varepsilon(f_j, g_j) + \mathcal{O}(\varepsilon^p) \qquad \text{in } H^1 \times L^2.$$

(2) *When* $(d, p) = (2, 3)$*, we assume either of the following conditions* (B1) *or* (B2) *in addition:*

(B1) $\langle Y, \mathcal{H}(\omega) F_{\mathrm{W}}^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}^{N_1}} = 0, \quad (\omega, Y) \in \mathbb{S}^1 \times \mathbb{R}^{N_1}.$

(B2) $N_1 = 1,$

where $\mathcal{H}$ *is from the KMS condition. Then there is a function*

$$\psi \in L^\infty(\mathbb{R} \times \mathbb{S}^{d-1}; \mathbb{R}^{N_1}) \cap L^2(\mathbb{R} \times \mathbb{S}^{d-1}; \mathbb{R}^{N_1}),$$

which depends on f, g, ε*, but is independent of the choice of* κ*, such that the wave component* w *enjoys*

$$r^{\frac{d-1}{2}} \partial_a w(t,x) = \omega_a A[\psi](t, r-t, \omega) + \mathcal{O}\Big(\varepsilon \langle t+r\rangle^{\frac{\kappa-1}{2}} \langle t-r\rangle^{\frac{\kappa-1}{2}}\Big) \qquad (2.16)$$

for $0 \le a \le d$*, where* $A[\psi](t,\sigma,\omega)$ *is given by* (2.14)*,* $r = |x|$*,* $\omega = x/|x|$*, and* $\omega_0 = -1$*. Moreover, we have*

$$\psi(\sigma,\omega) = \mathcal{O}(\varepsilon\langle\sigma\rangle^{\kappa-1}).$$

The asymptotic behavior for the system (1.4) is quite similar to that for the wave component w in (1.3) above.

Theorem 2.6. *Let* (d,p) *is equal to* $(2,3)$ *or* $(3,2)$*. We assume the KMS condition for* (1.4)*;* (A2) *is also assumed when* $(d,p) = (3,2)$*. Let* κ *be a small positive number. When* $(d,p) = (2,3)$*, we assume one of the following conditions* (B1') *or* (B2') *for each fixed* $I = 1, \dots, P$ *in addition:*

(B1') *We have*

$$\langle Y^I, \mathcal{H}^I(\omega) F^{I,\mathrm{red}}(\omega, Y^I)\rangle_{\mathbb{R}^{N^I}} = 0, \quad (\omega, Y^I) \in \mathbb{S}^1 \times \mathbb{R}^{N^I},$$

where $\mathcal{H}^I$ *is from* (2.6)*.*

(B2') *The size* N^I *of* u^I *is equal to* 1*; in other words, we have* $u^I(t,x) \in \mathbb{R}$*.*

Then, for any $f, g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^N)$ *and sufficiently small* ε*, there is a function* $\psi = (\psi^1, \dots, \psi^P) \in L^\infty(\mathbb{R} \times \mathbb{S}^{d-1}; \mathbb{R}^N) \cap L^2(\mathbb{R} \times \mathbb{S}^{d-1}; \mathbb{R}^N)$*, which is independent of the choice of* κ*, such that we have*

$$\begin{aligned} r^{\frac{d-1}{2}} \partial_a u^I(t,x) &= \omega_a A^I[\psi^I](t, r - c_I t, \omega) \\ &\quad + \mathcal{O}\Big(\varepsilon \langle t+r\rangle^{\frac{\kappa-1}{2}} \langle r - c_I t\rangle^{\frac{\kappa-1}{2}}\Big) \end{aligned} \qquad (2.17)$$

for $0 \le a \le d$*, where* $A^I[\psi](t,\sigma,\omega)$ *is given by* (2.15)*,* $r = |x|, \omega = x/|x|$*, and* $\omega_0 = -c_I$*. Moreover, we have*

$$\psi^I(\sigma,\omega) = \mathcal{O}\big(\varepsilon\langle\sigma\rangle^{\kappa-1}\big).$$

Remark 6. (1) In two theorems above, the conditions (A1) and (A2) for $(d,p) = (3,2)$ is not directly used in their proof; but we need them to ensure the existence of global solutions with the desired decay estimates.

(2) The reason why (B1) or (B2) (resp. (B1') or (B2')) are not necessary for $(d,p) = (3,2)$ in Theorem 2.5 (resp. Theorem 2.6) is that the KMS condition for (1.3) (resp. (1.4)) implies (B1) (resp. (B1')), as mentioned before. We believe that these additional assumptions for $(d,p) = (2,3)$ can be removed, but this is an open problem.

As we will see in the examples below, a key feature here is that the main part A (resp. A^I) of the asymptotic behavior for w in (1.3) (resp. u^I in (1.4)) can be determined by a separated system (2.12), and the influence of v (resp. u^J with $J \neq I$) appears only in the initial condition (2.13). This enables us to apply the arguments in the previous works like [32] for analysis of further properties of the asymptotic behavior. For each component u_i^I of the solution u to (1.4), other kind of the asymptotic behavior can occur, but the following two are typical:

(I) The component u_i^I is asymptotically free, and its asymptotic data is close to the original data for small ε. To be more precise, there is a free solutionϕ^+ to $\Box_{c_I}\Phi^+ = 0$ such that

$$\lim_{t\to\infty} \|\partial u_i^I(t) - \partial\phi^+(t)\|_{L^2} = 0,$$

and, for the asymptotic data $\big(\phi^+(0), \partial_t\phi^+(0)\big)$, we have

$$\big(\phi^+(0), \partial_t\phi^+(0)\big) = \big(u_i^I(0), \partial_t u_i^I(0)\big) + \mathcal{O}(\varepsilon^p) \qquad \text{in } H^1 \times L^2.$$

(II) The energy for u_i^I vanishes as $t \to \infty$, namely $\|\partial u_i^I(t)\|_{L^2} \to 0$ as $t \to \infty$, despite the choice of the initial data.

Remark 7. Type (II)-behavior can be also understood that u_i^I is asymptotically free, but the asymptotic data is always $(0,0)$. Therefore it is a special case of another situation: u_i^I is asymptotically free, but the asymptotic data is far from the original data in the sense that their difference is not of $o(\varepsilon)$ as $\varepsilon \to 0$, where o denotes Landau's little-o.

We can similarly formulate the cases (I) and (II) for a component w_i of the solution $u = (v, w)$ with replacing c_I by 1.

Example 5. We start with the case which involves the two typical situations. Theorem 2.6 can be applied to the system in Example 3. Indeed, (B1) is satisfied for $I = 1$, while we have (B2) for $I = 2$. Hence there is $\psi = (\psi^1, \psi^2)$ such that the main part of $\partial_a u^I(t,x)$ is given by $\omega_a A^I[\psi^I](t, r - c_I t, \omega)$ with $\omega_0 = -c_I$. As before, u^1, u^2, A^1, A^2 are written as $u^{(1)}, u^{(2)}, A^{(1)}, A^{(2)}$, respectively. Suppose that we have (2.8). The reduced system for $A^{(1)} = (A_1^{(1)}, A_2^{(1)})$ is

$$\partial_t \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix} = -\frac{1}{2c_1^2 t} \sum_{a,b} C_{ab}\omega_a^2\omega_b \begin{pmatrix} \big(A_1^{(1)}\big)^2 A_2^{(1)} \\ -\big(A_1^{(1)}\big)^3 \end{pmatrix}.$$

Observe that the right-hand side vanishes when $A_1^{(1)} = 0$. Therefore, for ω satisfying $\sum_{a,b} C_{ab}\omega_a^2\omega_b \neq 0$, we can show that $A_1^{(1)}(t,\sigma,\omega) \to 0$ as $t \to \infty$, while $A_2^{(1)}(t,\sigma,\omega)$ converge to some function of (σ,ω) as $t \to \infty$ by investigating the reduced system above. A system of (single-speed) semilinear wave equations with just the same reduced system was considered in [33], and applying the argument there, we can show that $u_1^{(1)}$ has Type (II)-behavior, while $u_2^{(1)}$ has Type (I)-behavior.

The reduced equation for $A^{(2)}$ is

$$\partial_t A^{(2)} = -\frac{1}{2c_2^2 t}\left(c_2 A^{(2)}\right)^3.$$

This can be explicitly solved, and we see that $A^{(2)}(t,\sigma,\omega) \to 0$ as $t \to \infty$: Following the argument in [39], we can show that $u^{(2)}$ has Type (II)-behavior.

Theorem 2.6 can be also applied to the system in Example 4 in three space dimensions. It is easy to see that $A^{(1)}$ is independent of t, as $F^{1,\mathrm{red}} \equiv 0$, and we can show that $u^{(1)}$ has Type (I)-behavior. As for $u^{(2)}$, similarly to the above, we can show that $A_1^{(2)}(t,\sigma,\omega) \to 0$ as $t \to \infty$, and $A_2^{(2)}(t,\sigma,\omega)$ tends to a function of (σ,ω) as $t \to \infty$. A similar reduced system can be found in [37], and using the argument there, we see that $u_1^{(2)}$ and $u_2^{(2)}$ has Type (II)- and Type (I)-behavior, respectively.

Example 6. We turn our attention to the examples of (1.3). Theorem 2.5 can be applicable to systems in Examples 1 and 2. In both cases, the Klein-Gordon component v is asymptotically free, and nothing interesting happens. Hence we restrict our attention to the wave component w. For Example 1, the reduced system is

$$\partial_t A = -\frac{\omega_a^2}{2t} A^3.$$

As before, unless $\omega_a = 0$, we see that $A \to 0$ as $t \to \infty$, and we can show that this w has Type (II)-behavior. We do not go into details, but we can also apply the argument in [55] to obtain the decay rate of the energy. As for Example 2, the reduced system is

$$\partial_t \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = -\frac{\omega_1\omega_2}{2t}\begin{pmatrix} A_2 A_3 \\ -A_2^2 \end{pmatrix},$$

and using the argument in [37], we see that the wave components w_2 and w_3 have Type (II)- and Type (I)-behavior, respectively.

The rest of this thesis is organized as follows; In Chapter 3, we give some preliminaries. We also discuss the profile system related to the KMS condition, and show that the asymptotic behavior is given by the reduced system; we also introduce a technical transformation for the Klein-Gordon components. The key estimates for the wave equations and the proof of Theorems 2.1 and 2.3, as well as the two space dimensional parts of Theorems 2.5 and

2.6 will be given in Chapter 4. We will show the three space dimensional results in Chapter 5. At there, we also discuss different versions of decay estimates of solutions to wave equations in connection with the additional assumptions (A1) and (A2).

Chapter 3

Preliminaries

In this chapter, unless otherwise stated, for $x \in \mathbb{R}^d$, we write $r = |x|$ and $\omega = x/|x|$, so that $x = r\omega$. In the polar coordinates (r, ω), we have

$$\partial_r = \sum_{k=1}^{d} \omega_k \partial_k.$$

3.1 The vector field method

Following Klainerman [44], we introduce vector fields

$$\begin{aligned} S &= t\partial_t + x \cdot \nabla_x = t\partial_t + \sum_{j=1}^{d} x_j \partial_j, \\ L_k &= t\partial_k + x_k \partial_t, \qquad\qquad 1 \leq k \leq d, \\ \Omega_{kl} &= x_k \partial_l - x_l \partial_k, \qquad\qquad 1 \leq k, l \leq d. \end{aligned}$$

We put $L = (L_k)_{1 \leq k \leq d}$ and $\Omega = (\Omega_{kl})_{1 \leq k < l \leq d}$. Let $[A, B] = AB - BA$ for linear operators A and B. Recall that we have $[\Box + m^2, L_k] = [\Box + m^2, \Omega_{kl}] = [\Box + m^2, \partial_a] = 0$ for $1 \leq k, l \leq d$, $0 \leq a \leq d$, and $m \geq 0$. Hence these vector fields L_k, Ω_{kl}, and ∂_a are compatible with wave and Klein-Gordon equations since $[\Box + m^2, S] = 2\Box$, as mentioned in the introduction. We put

$${}^L\Gamma = ({}^L\Gamma_a)_{1 \leq a \leq d_0} = ((L_k)_{1 \leq k \leq d}, (\Omega_{kl})_{1 \leq k < l \leq d}, (\partial_a)_{0 \leq a \leq d}),$$

where $d_0 = (d^2 + 3d + 2)/2$.

Similarly, we have $[\Box_c, S] = 2\Box_c, [\Box_c, \Omega_{kl}] = [\Box_c, \partial_a] = 0$ for $1 \leq k, l \leq d, 0 \leq a \leq d$, and $c > 0$. Since these vector fields S, Ω_{kl}, and ∂_a are commutable or "almost" commutable with $\Box_c$, they are compatible with the multiple-speed case. On the other hand, $[\Box_c, L_k] = 2(1 - c^2)\partial_t \partial_k$ for $c \neq 1$ has no good property. We set

$${}^S\Gamma = ({}^S\Gamma_a)_{1 \leq a \leq d_1} = (S, (\Omega_{kl})_{1 \leq k < l \leq d}, (\partial_a)_{0 \leq a \leq d}),$$

where $d_1 = (d^2 + d + 4)/2$.

For two sets Γ, Ξ of vector fields, we symbolically write $[\Gamma, \Xi] = \Xi$, if $[A, B]$ can be written as a linear combination of the vector fields in Ξ for any $A \in \Gamma$ and $B \in \Xi$. Then one can check that

$$[{}^L\Gamma, {}^L\Gamma] = {}^L\Gamma,\ [{}^L\Gamma, \partial] = \partial,\ [{}^S\Gamma, {}^S\Gamma] = {}^S\Gamma,\ [{}^S\Gamma, \partial] = \partial. \tag{3.1}$$

For a set $\Gamma = (\Gamma_1, \dots, \Gamma_m)$ of vector fields and a multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$, we put

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_m^{\alpha_m}.$$

For a smooth function $\phi = \phi(t, x)$ and a non-negative integer s, we set

$$|\phi(t,x)|_{\Gamma,s} = \sum_{|\alpha| \le s} |\Gamma^\alpha \phi(t,x)|,\ \|\phi(t)\|_{\Gamma,s} = \| |\phi(t,\cdot)|_{\Gamma,s} \|_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

For $c > 0$ and a positive integer s, we put

$$[\phi(t,x)]_{\Gamma,c,s} = |\phi(t,x)|_{\Gamma,s} + \langle ct - r \rangle |\partial \phi(t,x)|_{\Gamma,s-1}. \tag{3.3}$$

As we will see below, this quantity plays an important role in the modification of the arguments in the previous works.

The following Sobolev type inequality will be used to combine decay estimates with the energy estimates (see Klainerman [45] for the proof):

Lemma 3.1. *There exists a positive constant C such that*

$$\sup_{x \in \mathbb{R}^d} \langle x \rangle^{\frac{d-1}{2}} |\varphi(x)| \le C \sum_{|\alpha| + |\beta| \le [d/2]+1} \|\partial_x^\alpha \Omega^\beta \varphi\|_{L^2(\mathbb{R}^d)} \tag{3.4}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^d)$.

In the previous works [37, 38], the full set of vector fields, including the scaling operator S and the Lorentz boosts L_k, was used to take advantage of the KMS condition. We must modify the arguments because only the restricted sets ${}^L\Gamma$ or ${}^S\Gamma$ can be used for our systems (1.3) and (1.4). We start with some notations and basic estimates.

Let $0 < T \le \infty$ and $R > 0$. Given $0 < b_1 \le b_2$, we define

$$\Lambda_{T,R}^{b_1,b_2} := \left\{ (t,x) \in [0,T) \times \mathbb{R}^d; b_1 \le \frac{b_1 t}{2} \le r \le b_2 t + R \right\}. \tag{3.5}$$

Note that $t, \langle t \rangle, r, \langle r \rangle, t + r$, and $\langle t + r \rangle$ are equivalent to each other in $\Lambda_{T,R}^{b_1,b_2}$, where we say that two functions $f(t,r)$ and $g(t,r)$ are equivalent if we have $C^{-1} f(t,r) \le g(t,r) \le C f(t,r)$ with some universal positive constant C. Throughout this section, we suppose that

$$b_1 \le c \le b_2$$

and either of the following two holds:

(a) $\Gamma = {}^L\Gamma$ and $c = 1$,

or

(b) $\Gamma = {}^S\Gamma$.

If we put $\sigma_c = r - ct$, then $(t, r\omega) \in \Lambda_{T,R}^{b_1,b_2}$ is equivalent to $(t, \sigma_c, \omega) \in \mathcal{L}_{T,R}^c$, where

$$\begin{aligned}\mathcal{L}_{T,R}^c = \mathcal{L}_{T,R}^{c;b_1,b_2} := \Big\{&(t,\sigma,\omega) \in [2,T) \times \mathbb{R} \times \mathbb{S}^{d-1};\\ &-\left(c - \frac{b_1}{2}\right) t \le \sigma \le (b_2 - c)t + R\Big\}.\end{aligned} \tag{3.6}$$

$\mathcal{L}_{T,R}^c$ can be also written as

$$\mathcal{L}_{T,R}^c = \{(t,\sigma,\omega) \in [2,T) \times \Sigma^c \times \mathbb{S}^{d-1}; t \ge t_0^c(\sigma)\}, \tag{3.7}$$

where $\Sigma^c = \mathbb{R}$ for $b_1 \le c < b_2$, $\Sigma^{b_2} = (-\infty, R]$, and

$$\begin{aligned}t_0^c(\sigma) &= t_0^c(\sigma; b_1, b_2, R)\\ &:= \begin{cases} -\dfrac{2}{2c - b_1}\sigma, & \sigma < -(2c - b_1),\\ 2, & -(2c - b_1) \le \sigma \le 2(b_2 - c) + R,\\ \dfrac{\sigma - R}{b_2 - c}, & \sigma > 2(b_2 - c) + R.\end{cases}\end{aligned} \tag{3.8}$$

Observe that, by (3.8), there is a positive constant C such that

$$C^{-1}\langle\sigma\rangle \le t_0^c(\sigma) \le \langle\sigma\rangle, \qquad \sigma \in \Sigma^c. \tag{3.9}$$

We define

$$\partial_{\pm,c} = \partial_t \pm c\partial_r, \quad \mathcal{D}_c = -\frac{1}{2c}\partial_{-,c} = \frac{1}{2}\left(\partial_r - \frac{1}{c}\partial_t\right). \tag{3.10}$$

We write $\partial_\pm = \partial_{\pm,1} = \partial_t \pm \partial_r$ and $\mathcal{D} = \mathcal{D}_1 = (\partial_r - \partial_t)/2$.

Lemma 3.2. *There is a positive constant C such that*

$$|\partial_{+,c}\phi(t,x)| \le C\langle t + r\rangle^{-1}[\phi(t,x)]_{\Gamma,c,1}, \quad (t,x) \in \Lambda_{T,R}^{b_1,b_2}$$

for a smooth function ϕ on $[0,T) \times \mathbb{R}^d$, where $[\phi(t,x)]_{\Gamma,c,1}$ is given by (3.3).

Proof. The following identities show the desired result immediately, as $t+r$ is equivalent to $\langle t + r\rangle$ in $\Lambda_{T,R}^{b_1,b_2}$;

$$\partial_+ = \frac{1}{t + r}\left(2\sum_{k=1}^d \omega_k L_k + (t - r)(\partial_t - \partial_r)\right), \tag{3.11}$$

$$\partial_{+,c} = \frac{(c+1)S + (r - ct)(\partial_t - \partial_r)}{t + r}. \tag{3.12}$$

Observe that L_k and S appear only in (3.11) and (3.12), respectively. Hence we can use (3.11) when (a) above is assumed, and (3.12) when (b) above is assumed. □

Remark 8. Compared to previous works [37, 38], we need $\langle r-ct\rangle|\partial\phi|$ to compensate the lack of the scaling operator or the Lorentz boosts.

Lemma 3.3. *For a smooth function ϕ, we have*

$$\partial_a\phi(t,x) = \omega_a\mathcal{D}_c\phi(t,x) + \mathcal{O}\left(\langle t+r\rangle^{-1}[\phi(t,x)]_{\Gamma,c,1}\right), \tag{3.13}$$

$$r^{\frac{d-1}{2}}\partial_a\phi(t,x) = \omega_a\mathcal{D}_c\left(r^{\frac{d-1}{2}}\phi(t,x)\right) + \mathcal{O}\left(\langle t+r\rangle^{\frac{d-3}{2}}[\phi(t,x)]_{\Gamma,c,1}\right) \tag{3.14}$$

for $(t,x)\in\Lambda_{T,R}^{b_1,b_2}$ and $0\le a\le d$, where $\omega_0=-c$, and a constant associated with $\mathcal{O}$ is independent of T.

Proof. Let $(t,x)\in\Lambda_{T,R}^{b_1,b_2}$. In polar coordinates, we have

$$\partial_k\phi = \omega_k\partial_r\phi - \sum_{l=1}^{d}\frac{\omega_l}{r}\Omega_{kl}\phi = \omega_k\partial_r\phi + \mathcal{O}\left(\langle t+r\rangle^{-1}|\Gamma\phi|\right) \tag{3.15}$$

for $1\le k\le d$. By Lemma 3.2, we get

$$\partial_t\phi = \frac{1}{2}(\partial_{+,c}\phi+\partial_{-,c}\phi) = -c\mathcal{D}_c\phi + \mathcal{O}\left(\langle t+r\rangle^{-1}[\phi]_{\Gamma,c,1}\right),$$
$$\partial_r\phi = \frac{1}{2c}(\partial_{+,c}\phi-\partial_{-,c}\phi) = \mathcal{D}_c\phi + \mathcal{O}\left(\langle t+r\rangle^{-1}[\phi]_{\Gamma,c,1}\right).$$

To sum up, we obtain (3.13). Observing that

$$r^{\frac{d-1}{2}}\mathcal{D}_c\phi = \mathcal{D}_c(r^{\frac{d-1}{2}}\phi) + \mathcal{O}(r^{\frac{d-3}{2}}|\phi|),$$

we get (3.14) from (3.13). □

3.2 Profile systems

To describe the arguments for the wave component w of (1.3), and for each u^I of (1.4) in a unified way, we consider the following system of wave equations

$$\Box_c\phi = \Psi(\partial\phi) + \Theta(t,x), \qquad (t,x)\in(0,T)\times\mathbb{R}^d, \tag{3.16}$$

where ϕ is an $\mathbb{R}^n$-valued function, each component of Ψ is a homogeneous polynomial of degree p in its argument, and Θ is a function which is supposed to decay faster than $\Psi(\partial\phi)$. We assume that (d,p) is equal to $(3,2)$ or $(2,3)$ in the sequel. Recall that we have $(p-1)(d-1)/2=1$. We consider one of the following settings:

(WKG) $\phi = w$, $\Psi = F_{\rm W}^{(\rm w)}$, $\Theta = F_{\rm W}(v, \partial u) - F_{\rm W}^{(\rm w)}(\partial w)$, $n = N_1 (= N - N_0)$, $c = b_1 = b_2 = 1$, $\Gamma = {}^L\Gamma$

with $F_{\rm W}^{(\rm w)}(\partial w)$ being given in the introduction and

$$F_{\rm W}(v, \partial u) = \big(F_j(v, \partial u)\big)_{N_0+1 \le j \le N},$$

or

(MSW) $\phi = u^I$, $\Psi = {}^*F^I$, $\Theta = F^I(\partial u) - {}^*F^I(\partial u^I)$, $n = N^I$, $c = c_I$, $b_1 = c_1$, $b_2 = c_P$, $\Gamma = {}^S\Gamma$

with $1 \le I \le P$ and ${}^*F^I(\partial u^I)$ being given in the introduction.

Writing $\Psi(\partial\phi) = \Psi(\partial_t\phi, \partial_1\phi, \dots, \partial_d\phi)$, we define

$$\Psi^{\rm red}(\omega, Y) = \Psi(-cY, \omega_1 Y, \dots, \omega_d Y).$$

Observe that $\Psi^{\rm red}(\omega, Y) = F_{\rm W}^{\rm red}(\omega, Y)$ under (WKG), and $\Psi^{\rm red}(\omega, Y) = F^{I,\rm red}(\omega, Y)$ under (MSW), where $F_{\rm W}^{\rm red}$ and $F^{I,\rm red}$ are defined by (1.11) and (1.13), respectively. Therefore the KMS conditions for (1.3) and (1.4) can be expressed in the following way: There is $\mathcal{H} \in C(\mathbb{S}^{d-1}; \mathcal{S}_n^+)$ such that

$$\langle Y, \mathcal{H}(\omega)\Psi^{\rm red}(\omega, Y)\rangle_{\mathbb{R}^n} \ge 0, \quad \omega \in \mathbb{S}^{d-1}, Y \in \mathbb{R}^n. \tag{3.17}$$

We refer to this condition as the KMS condition for (3.16).

Now we start our discussion for (3.16). Writing the d'Alembertian in the polar coordinate, we have

$$r^{\frac{d-1}{2}}\Box_c\phi = \partial_{+,c}\partial_{-,c}\big(r^{\frac{d-1}{2}}\phi\big) - c^2 r^{\frac{d-5}{2}}\left(\Delta_\omega\phi - \frac{(d-1)(d-3)}{4}\phi\right), \tag{3.18}$$

where $\Delta_\omega = \sum_{1\le k<l\le d}\Omega_{kl}^2$, and $\partial_{\pm,c}$ is defined in (3.10).

For a solution ϕ to (3.16) in $[0, T) \times \mathbb{R}^d$, we define

$$\Phi(t, x) = \mathcal{D}_c\big(r^{\frac{d-1}{2}}\phi(t, x)\big), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \tag{3.19}$$

where $\mathcal{D}_c$ is defined by (3.10). Recall the definitions (3.3) and (3.5) for $[\cdot]_{\Gamma,c,s}$ and $\Lambda_{T,R}^{b_1,b_2}$, respectively. Since $\langle t + r\rangle$ and t are equivalent to each other in $\Lambda_{T,R}^{b_1,b_2}$, (3.14) can be written as

$$r^{\frac{d-1}{2}}\partial_a\phi = \omega_a\Phi + \mathcal{O}\Big(t^{\frac{d-3}{2}}[\phi]_{\Gamma,c,1}\Big) \qquad \text{in } \Lambda_{T,R}^{b_1,b_2} \tag{3.20}$$

for $0 \le a \le d$, where $\omega_0 = -c$. By (3.19), it is easy to see that there is a positive constant C, independent of T, such that

$$|\Phi(t, x)| \le C t^{\frac{d-1}{2}}\langle r - ct\rangle^{-1}[\phi]_{\Gamma,c,1}, \quad (t, x) \in \Lambda_{T,R}^{b_1,b_2}. \tag{3.21}$$

Recall the definition (3.6) or (3.7) for $\mathcal{L}^c_{T,R} = \mathcal{L}^{c;b_1,b_2}_{T,R}$. For a smooth function $\psi = \psi(t,x)$, we define $\mathscr{T}_c[\psi]$ by

$$\mathscr{T}_c[\psi](t,\sigma,\omega) = \psi(t,r\omega)|_{r=ct+\sigma}, \quad (t,\sigma,\omega) \in \mathcal{L}^c_{T,R},$$

so that we have

$$\partial_t \mathscr{T}_c[\psi](t,\sigma,\omega) = \mathscr{T}_c[\partial_{+,c}\psi](t,\sigma,\omega). \tag{3.22}$$

It is apparent that we have

$$\psi(t,x) = \psi(t,r\omega) = \mathscr{T}_c[\psi](t,r-ct,\omega), \quad (t,x) \in \Lambda^{b_1,b_2}_{T,R}.$$

For simplicity of exposition, we sometimes use the following $*$-notation for $\mathscr{T}_c$:

$$\psi^* = \psi^{*_c}(t,\sigma,\omega) = \mathscr{T}_c[\psi](t,\sigma,\omega), \quad (t,\sigma,\omega) \in \mathcal{L}^c_{T,R} \tag{3.23}$$

for a function $\psi = \psi(t,x)$ in $\Lambda^{b_1,b_2}_{T,R}$, where we use ψ^* when there is no fear of confusion. In what follows, for functions $\mathscr{F}, \mathscr{G}$ defined in $\mathcal{L}^c_{T,R}$ and a function $h = h(t,x)$ in $\Lambda^{b_1,b_2}_{T,R}$, we write

$$\mathscr{F} = \mathscr{G} + \mathcal{O}_c(h)$$

if there is a positive constant C such that

$$|(\mathscr{F} - \mathscr{G})(t,\sigma,\omega)| \leq C|\mathscr{T}_c[h](t,\sigma,\omega)|, \quad (t,\sigma,\omega) \in \mathcal{L}^c_{T,R}.$$

By (3.18), (3.19) and (3.22), we immediately obtain the following lemma for $\Phi^* = \mathscr{T}_c[\Phi]$ (recall that $\mathcal{D}_c = -(2c)^{-1}\partial_{-,c}$).

Lemma 3.4. *It holds that*

$$\partial_t \Phi^* = -\frac{1}{2c}\mathscr{T}_c[r^{\frac{d-1}{2}}\Box_c \phi] + \mathcal{O}_c(t^{\frac{d-5}{2}}|\phi|_{\Gamma,2}).$$

The next lemma reveals the meaning of the reduced nonlinearity Ψ^{red}.

Lemma 3.5.

$$\mathscr{T}_c\Big[r^{\frac{d-1}{2}}\Psi(\partial\phi)\Big] = \frac{1}{ct}\Psi^{\mathrm{red}}(\omega,\Phi^*) + \mathcal{O}_c\Big(t^{\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi(t,x)]^p_{\Gamma,c,1}\Big).$$

Proof. Let $(t,x) \in \Lambda^{b_1,b_2}_{T,R}$. Since Ψ is a homogeneous function of degree p and $(p-1)(d-1)/2 = 1$, we get

$$r^{\frac{d-1}{2}}\Psi(\partial\phi) = r^{\frac{(d-1)(1-p)}{2}}\Psi(r^{\frac{d-1}{2}}\partial\phi) = \frac{1}{r}\Psi(r^{\frac{d-1}{2}}\partial\phi).$$

It is easy to see that

$$\begin{aligned}\frac{1}{r}\Psi(r^{\frac{d-1}{2}}\partial\phi) &= \Big(\frac{1}{ct}+\frac{ct-r}{ctr}\Big)\Psi(r^{\frac{d-1}{2}}\partial\phi)\\ &= \frac{1}{ct}\Psi(r^{\frac{d-1}{2}}\partial\phi)+\mathcal{O}\Big(t^{-2+\frac{p(d-1)}{2}}\langle ct-r\rangle|\partial\phi|^p\Big)\\ &= \frac{1}{ct}\Psi(r^{\frac{d-1}{2}}\partial\phi)+\mathcal{O}\Big(t^{\frac{d-3}{2}}\langle ct-r\rangle^{1-p}[\phi]^p_{\Gamma,c,1}\Big).\end{aligned}$$

Recalling the definition of Ψ^{red}, and using the mean value theorem, we obtain

$$\begin{aligned}&\left|\frac{1}{ct}\Psi(r^{\frac{d-1}{2}}\partial\phi)-\frac{1}{ct}\Psi^{\mathrm{red}}(\omega,\Phi)\right|\\ &\quad= \frac{1}{ct}\left|\Psi(r^{\frac{d-1}{2}}\partial_0\phi,\dots,r^{\frac{d-1}{2}}\partial_d\phi)-\Psi(\omega_0\Phi,\dots,\omega_d\Phi)\right|\\ &\quad\le Ct^{-1}(|r^{\frac{d-1}{2}}\partial\phi|+|\phi|)^{p-1}\sum_{a=0}^{n}|r^{\frac{d-1}{2}}\partial_a\phi-\omega_a\Phi|.\end{aligned}$$

By (3.20) and (3.21), we obtain

$$\begin{aligned}\frac{1}{ct}\Psi(r^{\frac{d-1}{2}}\partial\phi) &= \frac{1}{ct}\Psi^{\mathrm{red}}(\omega,\Phi)+\mathcal{O}\Big(t^{-1+\frac{(d-1)(p-1)}{2}+\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi]^p_{\Gamma,c,1}\Big)\\ &= \frac{1}{ct}\Psi^{\mathrm{red}}(\omega,\Phi)+\mathcal{O}\Big(t^{\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi]^p_{\Gamma,c,1}\Big).\end{aligned}$$

Gathering the above estimates, we obtain the desired result, since

$$\mathscr{T}_c[\Psi^{\mathrm{red}}(\omega,\Phi)] = \Psi^{\mathrm{red}}(\omega,\Phi^*).$$

This completes the proof. □

By (3.16), the next lemma is an immediate consequence of the lemma above.

Lemma 3.6. *We have*

$$\mathscr{T}_c[r^{\frac{d-1}{2}}\Box_c\phi] = \frac{1}{ct}\Psi^{\mathrm{red}}(\omega,\Phi^*)+\mathcal{O}_c\left(\widetilde{\mathcal{R}}\right),$$

where

$$\widetilde{\mathcal{R}} = t^{\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi(t,x)]^p_{\Gamma,c,1}+t^{\frac{d-1}{2}}|\Theta(t,x)|.$$

By Lemmas 3.4 and 3.6, we find

$$\partial_t\Phi^* = -\frac{1}{2c^2t}\Psi^{\mathrm{red}}(\omega,\Phi^*)+\mathcal{O}_c(\mathcal{R}), \tag{3.24}$$

where

$$\mathcal{R} = t^{\frac{d-5}{2}}|\phi(t,x)|_{\Gamma,2}+t^{\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi(t,x)]^p_{\Gamma,c,1}+t^{\frac{d-1}{2}}|\Theta(t,x)|. \tag{3.25}$$

We call (3.24) the *profile system*, and it plays an important role to obtain *a priori* estimates for $\partial\phi$. If we neglect the error term $\mathcal{R}$, and replace Φ^* with A, we obtain the reduced system

$$\partial_t A = -\frac{1}{2c^2 t}\Psi^{\mathrm{red}}(\omega, A). \tag{3.26}$$

As was seen in Theorems 2.5 and 2.6, the reduced system is useful to describe the asymptotic behavior of global solution ϕ (see the next section for the detailed discussion).

The following lemma shows that how the KMS condition and (3.24) are used to estimate $\partial\phi$.

Lemma 3.7. *Suppose that the KMS condition for* (3.16) *is satisfied. Then there is a positive constant C, which is independent of T, such that*

$$\begin{aligned} r^{\frac{d-1}{2}}|\partial\phi(t,x)| \leq & C\left(|\Phi^*(t_0^c(\sigma_c),\sigma_c,\omega)| + \int_{t_0^c(\sigma_c)}^t \mathscr{T}_c[\mathcal{R}](\tau,\sigma_c,\omega)d\tau\right) \\ & + Ct^{\frac{d-3}{2}}[\phi(t,x)]_{\Gamma,c,1} \end{aligned}$$

in $\Lambda_{T,R}^{b_1,b_2}$, where $\sigma_c = r - ct$.

Proof. Because of (3.14), we only need to estimate $\Phi^*(t,\sigma_c,\omega)$. Let $\mathcal{H}$ be from (3.17). As $\mathcal{H}$ is positive-definite and continuous on a compact set $\mathbb{S}^{d-1}$, there is a positive constant C such that

$$\frac{1}{C}|Y|^2 \leq \langle Y, \mathcal{H}(\omega)Y\rangle_{\mathbb{R}^n} \leq C|Y|^2, \qquad Y \in \mathbb{R}^n. \tag{3.27}$$

Since $\mathcal{H}$ is real-symmetric, it follows from (3.24) that

$$\begin{aligned} \partial_t\langle\Phi^*, \mathcal{H}(\omega)\Phi^*\rangle_{\mathbb{R}^n} &= 2\langle\Phi^*, \mathcal{H}(\omega)\partial_t\Phi^*\rangle_{\mathbb{R}^n} \\ &= -\frac{1}{c^2 t}\langle\Phi^*, \mathcal{H}(\omega)\Psi^{\mathrm{red}}(\omega,\Phi^*)\rangle_{\mathbb{R}^n} + \mathcal{O}(\mathscr{T}_c[\mathcal{R}]|\Phi^*|) \end{aligned}$$

for $(t,\sigma,\omega) \in \mathcal{L}_{T,R}^c$. Now the KMS condition implies

$$\partial_t\langle\Phi^*, \mathcal{H}\Phi^*\rangle_{\mathbb{R}^n} \leq C\mathscr{T}_c[\mathcal{R}]|\Phi^*| \quad \text{in } \mathcal{L}_{T,R}^c.$$

Using (3.27), we obtain

$$|\Phi^*(t,\sigma_c,\omega)| \leq C|\Phi^*(t_0^c(\sigma_c),\sigma_c,\omega)| + C\int_{t_0^c(\sigma_c)}^t \mathscr{T}_c[\mathcal{R}](\tau,\sigma_c,\omega)d\tau.$$

This completes the proof. □

We are going to obtain a similar expression to (3.24) for generalized derivatives of higher order. We can only obtain rather rough estimates, as the structure in the KMS condition is almost broken by differentiation.

For a multi-index α of an arbitrary size, we set $\widetilde{\Gamma}^\alpha = {}^L\Gamma^\alpha$ when $\Gamma = {}^L\Gamma$, and $\widetilde{\Gamma}^\alpha = ({}^S\Gamma_1 + 2)^{\alpha_1}\,{}^S\Gamma_2^{\alpha_2} \cdots {}^S\Gamma_{d_1}^{\alpha_{d_1}}$ when $\Gamma = {}^S\Gamma$, so that we have

$$\Box_c(\Gamma^\alpha \phi) = \widetilde{\Gamma}^\alpha\big(\Psi(\partial\phi)\big) + \widetilde{\Gamma}^\alpha \Theta$$

(recall the ${}^S\Gamma_1 = S$ and $[\Box_c, S] = 2\Box_c$). For a multi-index α, we put

$$\Phi^{(\alpha)}(t,x) = \mathcal{D}_c\big(r^{\frac{d-1}{2}}\Gamma^\alpha\phi(t,x)\big), \quad (t,x) \in (0,T) \times \mathbb{R}^d.$$

Let $(t,x) \in \Lambda_{T,R}^{b_1,b_2}$. Then similarly to (3.20), we have

$$r^{\frac{d-1}{2}}\partial_a\Gamma^\alpha\phi = \omega_a\Phi^{(\alpha)} + \mathcal{O}\big(t^{\frac{d-3}{2}}[\phi]_{\Gamma,c,|\alpha|+1}\big), \tag{3.28}$$

as well as

$$\left|\Phi^{(\alpha)}\right| \le C\big(t^{\frac{d-1}{2}}\langle r - ct\rangle^{-1}[\phi]_{\Gamma,c,|\alpha|+1}\big). \tag{3.29}$$

Recall the $*$-notation in (3.23), and let $\Phi^{(\alpha)*} = \mathscr{T}_c[\Phi^{(\alpha)}]$. Similarly to Lemma 3.4, we have

$$\partial_t\Phi^{(\alpha)*} = -\frac{1}{2c}\mathscr{T}_c[r^{\frac{d-1}{2}}\Box_c\Gamma^\alpha\phi] + \mathcal{O}_c(t^{\frac{d-5}{2}}|\phi|_{\Gamma,|\alpha|+2}).$$

We suppose that $|\alpha| \ge 1$ from now on. It is easy to see that

$$r^{\frac{d-1}{2}}\Box_c\Gamma^\alpha\phi = r^{\frac{d-1}{2}}\widetilde{\Gamma}^\alpha(\Box_c\phi) = r^{\frac{d-1}{2}}\widetilde{\Gamma}^\alpha\{\Psi(\partial\phi)\} + \mathcal{O}\big(\widetilde{\mathcal{R}}^{(\alpha)}\big) \tag{3.30}$$

with

$$\widetilde{\mathcal{R}}^{(\alpha)} = t^{\frac{d-1}{2}}|\Theta(t,x)|_{\Gamma,|\alpha|}.$$

By the commutative relationship (3.1), (3.13) and (3.14), we get

$$\begin{aligned}
&ctr^{\frac12}\widetilde{\Gamma}^\alpha\{(\partial_a\phi_j)(\partial_b\phi_k)(\partial_c\phi_l)\}\\
&\quad= ctr^{\frac12}(\partial_a\Gamma^\alpha\phi_j)(\partial_b\phi_k)(\partial_c\phi_l) + ctr^{\frac12}(\partial_a\phi_j)(\partial_b\Gamma^\alpha\phi_k)(\partial_c\phi_l)\\
&\qquad+ ctr^{\frac12}(\partial_a\phi_j)(\partial_b\phi_k)(\partial_c\Gamma^\alpha\phi_l) + \mathcal{O}(tr^{\frac12}|\partial\phi|^3_{\Gamma,|\alpha|-1})\\
&\quad= (r^{\frac12}\partial_a\Gamma^\alpha\phi_j)(r^{\frac12}\partial_b\phi_k)(r^{\frac12}\partial_c\phi_l) + (r^{\frac12}\partial_a\phi_j)(r^{\frac12}\partial_b\Gamma^\alpha\phi_k)(r^{\frac12}\partial_c\phi_l)\\
&\qquad+ (r^{\frac12}\partial_a\phi_j)(r^{\frac12}\partial_b\phi_k)(r^{\frac12}\partial_c\Gamma^\alpha\phi_l)\\
&\qquad+ \mathcal{O}\big(r^{\frac12}\langle ct - r\rangle^{-2}[\phi]^3_{\Gamma,c,|\alpha|+1} + tr^{\frac12}|\partial\phi|^3_{\Gamma,|\alpha|-1}\big)
\end{aligned}$$

for $d=2$. It follows from (3.20), (3.21), (3.28) and (3.29) that

$$
\begin{aligned}
&\left|(r^{\frac12}\partial_a\Gamma^\alpha\phi_j)(r^{\frac12}\partial_b\phi_k)(r^{\frac12}\partial_c\phi_l)-\omega_a\omega_b\omega_c\Phi_j^{(\alpha)}\Phi_k\Phi_l\right|\\
&\quad\le\left(|r^{\frac12}\partial\phi|+|r^{\frac12}\partial\Gamma^\alpha\phi|+|\Phi|+|\Phi^{(\alpha)}|\right)^2\\
&\qquad\times\left(|r^{\frac12}\partial_a\Gamma^\alpha\phi_j-\omega_a\Phi_j^{(\alpha)}|+|r^{\frac12}\partial_b\phi_k-\omega_b\Phi_k|+|r^{\frac12}\partial_c\phi_l-\omega_b\Phi_l|\right)\\
&\quad\le Ct^{\frac12}\langle r-ct\rangle^{-2}[\phi]^3_{\Gamma,c,|\alpha|+1}.
\end{aligned}
$$

Going a similar way, we end up with

$$
\begin{aligned}
&ctr^{\frac12}\widetilde{\Gamma}^\alpha\{(\partial_a\phi_j)(\partial_b\phi_k)(\partial_c\phi_l)\}\\
&\quad=\omega_a\omega_b\omega_c\left(\Phi_j^{(\alpha)}\Phi_k\Phi_l+\Phi_j\Phi_k^{(\alpha)}\Phi_l+\Phi_j\Phi_k\Phi_l^{(\alpha)}\right)\\
&\qquad+\mathcal{O}(t^{\frac12}\langle ct-r\rangle^{-2}[\phi]^3_{\Gamma,c,|\alpha|+1}+t^{\frac32}|\partial\phi|^3_{\Gamma,|\alpha|-1}),
\end{aligned}
$$

For $d=3$, just in the same manner, we obtain

$$
\begin{aligned}
&ctr\widetilde{\Gamma}^\alpha\{(\partial_a\phi_j)(\partial_b\phi_k)\}\\
&\quad=ctr(\partial_a\Gamma^\alpha\phi_j)(\partial_b\phi_k)+ctr(\partial_a\phi_j)(\partial_b\Gamma^\alpha\phi_k)+\mathcal{O}(tr|\partial\phi|^2_{\Gamma,|\alpha|-1})\\
&\quad=(r\partial_a\Gamma^\alpha\phi_j)(r\partial_b\phi_k)+(r\partial_a\phi_j)(r\partial_b\Gamma^\alpha\phi_k)\\
&\qquad+\mathcal{O}\big(r\langle ct-r\rangle^{-1}[\phi]^2_{\Gamma,c,|\alpha|+1}+tr|\partial\phi|^2_{\Gamma,|\alpha|-1}\big)\\
&\quad=\omega_a\omega_b\left(\Phi_j^{(\alpha)}\Phi_k+\Phi_j\Phi_k^{(\alpha)}\right)\\
&\qquad+\mathcal{O}\big(t\langle ct-r\rangle^{-1}[\phi]^2_{\Gamma,c,|\alpha|+1}+t^2|\partial\phi|^2_{\Gamma,|\alpha|-1}\big)
\end{aligned}
$$

Since Ψ is a linear combination of $(\partial_a\phi_j)(\partial_b\phi_k)(\partial_c\phi_l)$ (resp. $(\partial_a\phi_j)(\partial_b\phi_k)$) when $(d,p)=(2,3)$ (resp. $(d,p)=(3,2)$), we obtain

$$
\begin{aligned}
\mathscr{T}_c[r^{\frac{d-1}{2}}\widetilde{\Gamma}^\alpha\{\Psi(\partial\phi)\}]=&\frac{\mathcal{G}(\omega,\Phi^*)}{ct}\Phi^{(\alpha)*}\\
&+\mathcal{O}_c(t^{\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi]^p_{\Gamma,c,|\alpha|+1}+t^{\frac{d-1}{2}}|\partial\phi|^p_{\Gamma,|\alpha|-1}),
\end{aligned}
\tag{3.31}
$$

where

$$
\mathcal{G}(\omega,Y)=(\partial_{Y_k}\Psi_j^{\mathrm{red}}(\omega,Y))_{1\le j,k\le n}. \tag{3.32}
$$

Finally, for $1\le|\alpha|\le s$, we obtain the following by (3.30) and (3.31):

$$
\partial_t\Phi^{(\alpha)*}=-\frac{1}{2c^2t}\mathcal{G}(\omega,\Phi^*)\Phi^{(\alpha)*}+\mathcal{O}_c\big(\mathcal{R}^{(\alpha)}\big) \tag{3.33}
$$

where

$$
\begin{aligned}
\mathcal{R}^{(\alpha)}=&t^{\frac{d-1}{2}}|\partial\phi|^p_{\Gamma,|\alpha|-1}+t^{\frac{d-5}{2}}|\phi|_{\Gamma,|\alpha|+2}\\
&+t^{\frac{d-3}{2}}\langle ct-r\rangle^{-(p-1)}[\phi]^p_{\Gamma,c,|\alpha|+1}+t^{\frac{d-1}{2}}|\Theta|_{\Gamma,|\alpha|}.
\end{aligned}
\tag{3.34}
$$

We put $\mathcal{R}^{(\alpha)} = \mathcal{R}$ when $|\alpha| = 0$. For a real matrix $\mathcal{M}$ of size $n \times n$, we define

$$\|\mathcal{M}\| = \sup_{Y \in \mathbb{R}^n, |Y|=1} |\mathcal{M}Y|.$$

Lemma 3.8. *Supposed that the KMS condition for* Ψ *is satisfied. If we have*

$$\|\mathcal{G}\big(\omega, \Phi^*(t,\sigma,\omega)\big)\| \le 2c^2 C_0 \varepsilon^{p-1}$$

for $(t,\sigma,\omega) \in \mathcal{L}^c_{T,R}$ *with a positive constant* C_0*, then we have*

$$\begin{aligned} r^{\frac{d-1}{2}} |\partial\phi(t,x)|_{\Gamma,s} \le\, & C \sum_{|\alpha|\le s} \left(\frac{t}{t_0^c(\sigma_c)}\right)^{C_0\varepsilon^{p-1}} |\Phi^{(\alpha)*}(t_0^c(\sigma_c),\sigma_c,\omega)| \\ & + C \sum_{|\alpha|\le s} \int_{t_0^c(\sigma_c)}^{t} \left(\frac{t}{\tau}\right)^{C_0\varepsilon^{p-1}} |\mathscr{T}_c[\mathcal{R}^{(\alpha)}](\tau,\sigma_c,\omega)| d\tau \\ & + C t^{\frac{d-3}{2}} [\phi(t,x)]_{\Gamma,c,s+1} \end{aligned}$$

for $s \ge 1$ *in* $\Lambda^{b_1,b_2}_{T,R}$*, where* $\sigma_c = r - ct$*.*

Proof. In view of (3.14) and Lemma 3.7, our task is to estimate

$$\Phi^{(\alpha)*}(t,\sigma_c,\omega)$$

for $1 \le |\alpha| \le s$. It follows from (3.33) that

$$\begin{aligned} \partial_t |\Phi^{(\alpha)*}|^2 &= 2\langle \Phi^{(\alpha)*}, \partial_t \Phi^{(\alpha)*}\rangle_{\mathbb{R}^n} \\ &\le \frac{1}{c^2 t}\|\mathcal{G}(\omega,\Phi^*)\|\,|\Phi^{(\alpha)*}|^2 + C\,|\Phi^{(\alpha)*}|\,|\mathcal{R}^{(\alpha)*}| \\ &\le \frac{2C_0\varepsilon^{p-1}}{t}|\Phi^{(\alpha)*}|^2 + C|\Phi^{(\alpha)*}|\,|\mathcal{R}^{(\alpha)*}|. \end{aligned}$$

Therefore we get

$$\partial_t \left\{ t^{-2C_0\varepsilon^{p-1}} \big|\Phi^{(\alpha)*}\big|^2 \right\} \le C t^{-2C_0\varepsilon^{p-1}} \big|\Phi^{(\alpha)*}\big|\,\big|\mathcal{R}^{(\alpha)*}\big|, \quad (t,\sigma,\omega) \in \mathcal{L}^c_{T,R},$$

from which we easily see that $\Phi^{(\alpha)*}$ has the desired bound. □

3.3 ODE lemmas for the asymptotic behavior

We continue our investigation of (3.16) satisfying the KMS condition, and would like to obtain the asymptotic behavior of $r^{\frac{d-1}{2}}\partial_a\phi$ in $\Lambda^{b_1,b_2}_{T,R}$, especially when $T = \infty$. In view of (3.14), our task is to investigate the asymptotic behavior of the solution Φ^* to the profile system (3.24). For this purpose, we would like to approximate Φ^* for large t by a solution A to the reduced system (3.26) with appropriately chosen data, under some additional assumptions if necessary.

To continue our discussion, we repeat our aim again with equations being explicitly given. Let Φ^* satisfy

$$\partial_t \Phi^* = -\frac{1}{2c^2 t}\Psi^{\mathrm{red}}(\omega, \Phi^*) + \eta \quad \text{in } \mathcal{L}^c_{\infty,R} \tag{3.35}$$

with $\eta(t,\sigma,\omega) = \mathcal{O}_c(\mathcal{R})$, where $\mathcal{L}^c_{\infty,R} = \mathcal{L}^{c;b_1,b_2}_{\infty,R}$ be given by (3.6) or (3.7), and $\mathcal{R}$ is given by (3.25). We suppose that the KMS condition is satisfied: There is $\mathcal{H} \in C(\mathbb{S}^{d-1}; \mathcal{S}_n^+)$ such that

$$\langle Y, \mathcal{H}(\omega)\Psi^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}^n} \ge 0, \quad \omega \in \mathbb{S}^{d-1},\ Y \in \mathbb{R}^n. \tag{3.36}$$

Let $A = A(t,\sigma,\omega)$ satisfies

$$\partial_t A = -\frac{1}{2c^2 t}\Psi^{\mathrm{red}}(\omega, Y), \quad \text{in } (1,\infty)\times\Sigma^c\times\mathbb{S}^{d-1} \tag{3.37}$$

with

$$A(1,\sigma,\omega) = \psi(\sigma,\omega), \quad (\sigma,\omega) \in \Sigma^c \times \mathbb{S}^{d-1}, \tag{3.38}$$

where $\Sigma^c = \mathbb{R}$ when $b_1 \le c < b_2$, and $\Sigma^c = (-\infty, R]$ when $c = b_2$. Assuming certain assumption on $\mathcal{R}$ and some additional condition on Ψ^{red} if necessary, we would like to find ψ such that A approximates Φ^* as $t \to \infty$. In correspondence to the conditions (B1) and (B2) in Theorem 2.5, or (B1') and (B2') in Theorem 2.6, we will consider two situations.

Firstly we assume that (3.36) is replaced by

$$\langle Y, \mathcal{H}(\omega)\Psi^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}^n} = 0, \quad \omega \in \mathbb{S}^{d-1},\ Y \in \mathbb{R}^n. \tag{3.39}$$

As we have seen, this is nothing but the KMS condition when $(d,p) = (3,2)$; however this is a stronger restriction when $(d,p) = (2,3)$, and corresponds to (B1) or (B1'). Since $\mathcal{H} = \mathcal{H}(\omega)$ is real-symmetric and positive-definite for each $\omega \in \mathbb{S}^{d-1}$, there are a real-symmetric and positive-definite matrix $\sqrt{\mathcal{H}(\omega)}$ and its inverse. We write $\mathcal{H}^{\frac12}(\omega)$ and $\mathcal{H}^{-\frac12}(\omega)$ for $\sqrt{\mathcal{H}(\omega)}$ and its inverse, respectively. Then it turns out that $\mathcal{H}^{\pm\frac12} \in C(\mathbb{S}^{d-1}; \mathcal{S}_n^+)$, and there is a positive constant C such that

$$\left\| \mathcal{H}^{\pm\frac12}(\omega) \right\| \le C, \quad \omega \in \mathbb{S}^{d-1}. \tag{3.40}$$

We put

$$\begin{aligned} \Phi^\star(t,\sigma,\omega) &= \mathcal{H}^{\frac12}(\omega)\Phi^*(t,\sigma,\omega), \\ \Psi^\star(\omega, Y) &= \mathcal{H}^{\frac12}(\omega)\Psi^{\mathrm{red}}\big(\omega, \mathcal{H}^{-\frac12}(\omega)Y\big). \end{aligned}$$

Then we have

$$\partial_t \Phi^\star = -\frac{1}{2c^2 t}\Psi^\star(\omega, \Phi^\star) + \mathcal{H}^{\frac{1}{2}}(\omega)\eta. \tag{3.41}$$

Note that we have $\mathcal{H}^{\frac{1}{2}}\eta = \mathcal{O}_c(\mathcal{R})$ because of (3.40). Moreover, (3.39) implies

$$\begin{aligned}\langle Y, \Psi^\star(\omega, Y)\rangle_{\mathbb{R}^n} &= \langle Y, \mathcal{H}^{\frac{1}{2}}(\omega)\Psi^{\mathrm{red}}\big(\omega, \mathcal{H}^{-\frac{1}{2}}(\omega)Y\big)\rangle_{\mathbb{R}^n} \\ &= \langle \mathcal{H}^{-\frac{1}{2}}(\omega)Y, \mathcal{H}(\omega)\Psi^{\mathrm{red}}\big(\omega, \mathcal{H}^{-\frac{1}{2}}(\omega)Y\rangle_{\mathbb{R}^n} = 0\end{aligned}$$

for $\omega \in \mathbb{S}^{d-1}$ and $Y \in \mathbb{R}^n$. In other words, (3.39) holds for (3.41) with the corresponding $\mathcal{H}$ being the identity matrix. Consider the reduced system for (3.41):

$$\partial_t A^\star = -\frac{1}{2c^2 t}\Psi^\star(\omega, A^\star) \tag{3.42}$$

with

$$A^\star(1, \sigma, \omega) = \psi^\star(\sigma, \omega).$$

If we put $\psi(\sigma, \omega) = \mathcal{H}^{-\frac{1}{2}}(\omega)\psi^\star(\sigma, \omega)$, and $A = \mathcal{H}^{-\frac{1}{2}}A^\star$, then it is easy to see that A solves (3.37)–(3.38). Moreover, by (3.40), we obtain

$$\begin{aligned}|\Phi^*(t, \sigma, \omega) - A(t, \sigma, \omega)| &= \left|\mathcal{H}^{-\frac{1}{2}}(\omega)\big(\Phi^\star(t, \sigma, \omega) - A^\star(t, \sigma, \omega)\big)\right| \\ &\le C\left|\Phi^\star(t, \sigma, \omega) - A^\star(t, \sigma, \omega)\right|.\end{aligned}$$

Therefore, If $A^\star$ approximates $\Phi^\star$ for large t, then A approximates Φ^*. In conclusion, we found that we may assume $\mathcal{H}(\omega)$ is the identity matrix for our aim. Hence we can use the following:

Lemma 3.9 ([32, Lemma 10.22])**.** *Let $P(\omega, Y)$ be an $\mathbb{R}^n$-valued function whose components are homogeneous polynomials of degree p with bounded coefficients depending on $\omega \in \mathbb{S}^{d-1}$. If we have*

$$\langle Y, P(\omega, Y)\rangle_{\mathbb{R}^n} = 0, \quad (\omega, Y) \in \mathbb{S}^{d-1} \times \mathbb{R}^n,$$

then there is an $n \times n$-matrix-valued function $\mathcal{Q} = \mathcal{Q}(\omega, Y)$, whose components are homogeneous polynomials of degree $p-1$ in Y with bounded coefficients depending on ω, such that

$${}^{\mathrm{t}}\mathcal{Q}(\omega, Y) = -\mathcal{Q}(\omega, Y)$$

and

$$P(\omega, Y) = \mathcal{Q}(\omega, Y)Y, \quad (\omega, Y) \in \mathbb{S}^{d-1} \times \mathbb{R}^n,$$

where ${}^{\mathrm{t}}\mathcal{Q}$ is the transpose of $\mathcal{Q}$.

Motivated by this structure, we consider the following system for an $\mathbb{R}^n$-valued unknown w, since σ and ω in (3.35) can be considered as parameters:

$$\begin{cases} w'(t) = \dfrac{1}{t}\mathcal{Q}\big(w(t)\big)w(t) + J(t), \quad t > t_0, \\ w(t_0) = \xi \end{cases} \tag{3.43}$$

with some $t_0 \geq 1$, where $\mathcal{Q} = \mathcal{Q}(Y)$ is an $n \times n$-matrix-valued function, whose components are homogeneous polynomials of degree $p-1$ in Y, and satisfies ${}^t\mathcal{Q}(Y) = -\mathcal{Q}(Y)$, while $J \in C([t_0,\infty);\mathbb{R}^n) \cap L^1([t_0,\infty);\mathbb{R}^n)$.

We write $w^+[\xi^+]$ for a unique global solution w^+ to

$$\begin{cases} (w^+)'(t) = \dfrac{1}{t}\mathcal{Q}\big(w^+(t)\big)w^+(t), \quad t > 1, \\ w^+(1) = \xi^+. \end{cases} \tag{3.44}$$

To obtain the asymptotic behavior for (3.43), we use the following lemma, which is a slight modification of [32, Proposition 10.21].

Lemma 3.10. *Let $\rho > 0$. Suppose that*

$$\int_t^\infty |J(\tau)|d\tau \leq \frac{C_0}{\rho}t^{-\rho}, \quad t \geq t_0, \tag{3.45}$$

$$\|\mathcal{Q}(Y) - \mathcal{Q}(Z)\| \leq D_0\left(\frac{|Y|+|Z|}{2}\right)^{p-2}|Y-Z|, \quad Y, Z \in \mathbb{R}^n$$

with some positive constants C_0 and D_0. Suppose also that

$$|\xi| + \frac{C_0 t_0^{-\rho}}{\rho} \leq E_0.$$

If $D_0E_0^{p-1} < \rho$, then there is $\xi^+ \in \mathbb{R}^n$ with $|\xi^+| \leq E_0$ such that

$$|w(t) - w^+[\xi^+](t)| \leq \frac{C_0}{\rho - D_0E_0^{p-1}}t^{-\rho}, \quad t \geq t_0.$$

Proof. This lemma with (3.45) replaced by

$$|J(t)| \leq C_0t^{-1-\rho}, \quad t \geq t_0 \tag{3.46}$$

is proved in Proposition 10.21 of [32]; however, as (3.46) is only used to obtain (3.45) in its proof, this lemma can be proved by the same proof. □

Secondly we consider the case $n = 1$. Since the general KMS condition for $(d,p) = (3,2)$ is covered by the previous case, we may assume $(d,p) = (2,3)$ here. Then this case corresponds to (B2) or (B2'). Since $n = 1$ and $p = 3$, Ψ^{red} has the form

$$\Psi^{\mathrm{red}}(\omega, Y) = P(\omega)Y^3,$$

where P is a polynomial in ω of at most degree 3. Then the KMS condition can be written as

$$\langle Y, \mathcal{H}(\omega)\Psi^{\mathrm{red}}(\omega, Y)\rangle_{\mathbb{R}} = \mathcal{H}(\omega)P(\omega)Y^4 \geq 0, \quad \omega \in \mathbb{S}^1,\ Y \in \mathbb{R}.$$

As $\mathcal{H}$ is a positive and continuous function on the compact set $\mathbb{S}^1$, this is equivalent to

$$P(\omega) \geq 0, \quad \omega \in \mathbb{S}^1.$$

Motivated by this, we consider

$$\begin{cases} z'(t) = -\dfrac{K}{2t} z(t)^3 + J(t), & t \geq t_0, \\ z(t_0) = z_0, \end{cases} \tag{3.47}$$

where z is a real-valued unknown, $K \geq 0$, and $J \in C\big([t_0, \infty)\big) \cap L^1\big([t_0, \infty)\big)$. Given $z_0^+ \in \mathbb{R}$, we write $z^+[z_0^+]$ for a unique global solution z^+ to

$$\begin{cases} (z^+)'(t) = -\dfrac{K}{2t} z^+(t)^3, & t \geq 1, \\ z^+(1) = z_0^+. \end{cases} \tag{3.48}$$

We will use the following modification of [32, Lemma 10.25] (see also [39]).

Lemma 3.11. *Let $0 \leq K < K_0$ with some positive constant K_0, and let $\rho > 0$. Suppose that we can take positive constants ν, c_0, and E_0, as well as parameters $\sigma \in \mathbb{R}$ and $\varepsilon > 0$, such that*

$$\int_t^\infty \tau^\theta |J(\tau)| d\tau \leq \frac{E_0 \varepsilon \langle\sigma\rangle^{-\nu}}{\rho - \theta} t^{-\rho+\theta}, \quad t \geq t_0,\ 0 < \theta \ll 1, \tag{3.49}$$

$$|z_0| \leq E_0 \varepsilon \langle\sigma\rangle^{-\rho-\nu}, \tag{3.50}$$

$$c_0^{-1}\langle\sigma\rangle < t_0 < c_0\langle\sigma\rangle. \tag{3.51}$$

Then, for $0 < \theta \ll 1$, there are positive constants $\varepsilon_1 = \varepsilon_1(K_0, E_0, c_0, \rho)$ and $C_0 = C_0(K_0, E_0, c_0, \rho, \theta)$ such that, for any $\varepsilon \in (0, \varepsilon_1]$, there is $z_0^+ \in \mathbb{R}$ being independent of θ and satisfying

$$|z(t) - z^+[z_0^+](t)| \leq C_0 \varepsilon t^{-\rho+\theta} \langle\sigma\rangle^{-\nu-\theta}, \quad t \geq t_0,$$

$$|z_0^+| \leq C_0 \varepsilon \langle\sigma\rangle^{-\rho-\nu}.$$

Proof. Similarly to the above, this lemma with (3.49) replaced by

$$|J(\tau)| \leq E_0 \varepsilon \langle\sigma\rangle^{-\delta} t^{-1-\rho} \tag{3.52}$$

was proved in [32, Lemma 10.25], where (3.52) is only used to obtain (3.49). Hence this lemma is also valid by the same proof. □

Finally, using Lemmas 3.10 and 3.11, we obtain the following:

Lemma 3.12. *Let ϕ be a solution to* (3.16) *satisfying*

$$\phi(t,x)=0, \quad t>0,\ |x|\geq b_2t+R,$$

and $\mathcal{R}$ be given by (3.25)*. We suppose that we have*

$$r^{\frac{d-1}{2}}[\phi(t,x)]_{\Gamma,c,1}\leq C\langle t+r\rangle^{\lambda}\langle ct-r\rangle^{-\mu}, \quad (t,x)\in[0,\infty)\times\mathbb{R}^d,$$

where $\lambda>0$ and $\mu\geq 0$. We also suppose that

$$\int_t^\infty \tau^\theta\,|\mathscr{T}_c[\mathcal{R}](\tau,\sigma,\omega)|\,d\tau\leq C\varepsilon t^{-\rho+\theta}\langle\sigma\rangle^{-\nu}, \quad (t,\sigma,\omega)\in\mathcal{L}^{c;b_1,b_2}_{\infty,R}$$

for $0\leq\theta\ll 1$, where ν and ρ are positive numbers.

We assume that either of the following two conditions holds:

(i) (3.39) *is satisfied.*

(ii) $(d,p)=(2,3)$ *and* $n=1$*.*

If $\lambda-1\leq-\rho$ and $\lambda-1-\mu\leq-\rho-\nu$, then, for $0<\theta\ll 1$ and sufficiently small ε, there is

$$\psi\in L^\infty(\mathbb{R}\times\mathbb{S}^{d-1};\mathbb{R}^n)\cap L^2(\mathbb{R}\times\mathbb{S}^{d-1};\mathbb{R}^n),$$

which is independent of θ, such that

$$r^{\frac{d-1}{2}}\partial_a\phi(t,x)=\omega_aA[\psi](t,r-ct,\omega)+\mathcal{O}\left(\varepsilon\langle t+r\rangle^{-\rho+\theta}\langle ct-r\rangle^{-\nu-\theta}\right)$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^d$ with $|x|\leq b_2t+R$, where $\omega_0=-c$, $A[\psi]$ is the solution to (3.37)–(3.38)*. Moreover*

$$\psi(\sigma,\omega)=\mathcal{O}\big(\langle\sigma\rangle^{-\rho-\nu}\big), \quad \sigma\in\mathbb{R},\ \omega\in\mathbb{S}^{d-1}.$$

If (i) *is satisfied, we can also choose $\theta=0$.*

Proof. For a while, we suppose that $(t,x)\in\Lambda^{b_1,b_2}_{\infty,R}$. Note that t, r, $\langle t+r\rangle$ are equivalent to each other. Since $\langle r-ct\rangle\leq C\langle t+r\rangle$, by (3.20) and the definition of Φ^*, we get

$$\begin{aligned}\left|r^{\frac{d-1}{2}}\partial_a\phi(t,x)-\omega_a\Phi^*(t,r-ct,\omega)\right|&\leq Ct^{\frac{d-3}{2}}[\phi(t,x)]_{\Gamma,c,1}\\&\leq C\langle t+r\rangle^{\lambda-1}\langle ct-r\rangle^{-\mu}\\&\leq C\langle t+r\rangle^{-\rho+\theta}\langle ct-r\rangle^{-\nu-\theta}.\end{aligned}$$

Hence our task is to show that there is some ψ such that

$$\Phi^*(t,\sigma,\omega)=A[\psi](t,\sigma,\omega)+\mathcal{O}\left(\varepsilon t^{-\rho+\theta}\langle\sigma\rangle^{-\nu-\theta}\right) \quad \text{in } \mathcal{L}^c_{\infty,R}.$$

Let $t_0^c(\sigma)$ be given by (3.8). In view of (3.9) and (3.21), we get

$$\left|\Phi^*\big(t_0^c(\sigma),\sigma,\omega\big)\right| \le C\varepsilon\langle\sigma\rangle^{\lambda-\mu-1} \le C\varepsilon\langle\sigma\rangle^{-\rho-\nu}. \tag{3.53}$$

Firstly we consider the case of (i). As we have observed, we may assume that $\mathcal{H}(\omega)$ is the identity matrix, and applying Lemma 3.9, we see that there is an anti-symmetric-matrix-valued function $\mathcal{Q}=\mathcal{Q}(\omega,Y)$ such that $\Psi^{\mathrm{red}}(\omega,Y)=\mathcal{Q}(\omega,Y)Y$, and each component of $\mathcal{Q}$ is a homogeneous polynomial of degree $p-1$ with bounded coefficients depending on ω. Therefore, there is a positive constant C such that

$$\|\mathcal{Q}(\omega,Y)-\mathcal{Q}(\omega,Z)\| \le C\left(\frac{|Y|+|Z|}{2}\right)^{p-2}|Y-Z|, \quad Y,Z\in\mathbb{R}^n.$$

By the assumption, we have

$$\int_t^\infty \mathscr{T}_c[\mathcal{R}](\tau,\sigma,\omega)d\tau \le \frac{C\varepsilon\langle\sigma\rangle^{-\nu}}{\rho}t^{-\rho}, \quad t\ge t_0^c(\sigma).$$

By (3.53) and (3.9), we get

$$\left|\Phi^*\big(t_0^c(\sigma),\sigma,\omega\big)\right| + \frac{C\varepsilon\langle\sigma\rangle^{-\nu}}{\rho}\big(t_0^c(\sigma)\big)^{-\rho} \le C\varepsilon\langle\sigma\rangle^{-\rho-\nu}.$$

Hence, if ε is sufficiently small, then for arbitrarily fixed (σ,ω), we can apply Lemma 3.10 with $w(t)=\Phi^*(t,\sigma,\omega), J(t)=\eta(t,\sigma,\omega)=\mathcal{O}_c\big(\mathcal{R}\big), \mathcal{Q}(Y)=\mathcal{Q}(\omega,Y), \xi=\Phi^*\big(t_0^c(\sigma),\sigma,\omega\big)$ and $t_0=t_0^c(\sigma)$ to conclude that there is $\psi=\psi(\sigma,\omega)$ and a positive constant C, which is independent of (σ,ω), such that

$$|\Phi^*(t,\sigma,\omega)-[\psi](\sigma,\omega)| \le C\varepsilon\langle\sigma\rangle^{-\nu}t^{-\rho} \le C\varepsilon t^{-\rho+\theta}\langle\sigma\rangle^{-\nu-\theta}$$

for $(t,\sigma,\omega)\in\mathcal{L}^c_{\infty,R}$. This is the desired result. We also have

$$\psi(\sigma,\omega)=\mathcal{O}(\varepsilon\langle\sigma\rangle^{-\rho-\nu}), \quad (\sigma,\omega)\in\Sigma_c\times\mathbb{S}^{d-1}.$$

Recall that $\Sigma^c=\mathbb{R}$ for $b_1\le c<b_2$ and $\Sigma^c=(-\infty,R]$ for $c=b_2$. Hence, when $c=b_2$, we define

$$\psi(\sigma,\omega)=0, \quad (\sigma,\omega)\in(R,\infty)\times\mathbb{S}^{d-1},$$

so that ψ is defined on $\mathbb{R}\times\mathbb{S}^{d-1}$.

Secondly, we consider the case of (ii). In this case, as we have observed, $\Psi^{\mathrm{red}}(\omega,Y)=P(\omega)Y^3$. As P is a polynomial of at most degree 3, there is a positive constant C such that

$$|P(\omega)|\le C, \quad \omega\in\mathbb{S}^{d-1}.$$

In view of (3.9), (3.53) and the assumption on $\mathcal{R}$, we can apply Lemma 3.11 with $w(t)=\Phi^*(t,\sigma,\omega), S=P(\omega)/c, J(t)=\eta(t,\sigma,\omega), t_0=t_0^c(\sigma)$ and $z_0=$

$\Phi^*\big(t_0^c(\sigma),\sigma,\omega\big)$, for arbitrarily fixed (σ,ω) and sufficiently small ε, to show the existence of $\psi=\psi(\sigma,\omega)$ such that

$$|\Phi^*(t,\sigma,\omega)-A[\psi](t,\sigma,\omega)|\le C\varepsilon t^{-\rho+\theta}\langle\sigma\rangle^{-\nu-\theta},\quad (t,\sigma,\omega)\in\mathcal{L}^c_{\infty,R},$$
$$\psi(\sigma,\omega)=\mathcal{O}\big(\varepsilon\langle\sigma\rangle^{-\rho-\nu}\big),\qquad (\sigma,\omega)\in\Sigma^c\times\mathbb{S}^{d-1},$$

which are the desired results. As above, we define $\psi(\sigma,\omega)=0$ for $(\sigma,\omega)\in(R,\infty)\times\mathbb{S}^{d-1}$ when $c=b_2$.

Now we assume that $(t,x)\in\big([0,\infty)\times\mathbb{R}^d\big)\setminus\Lambda^{b_1,b_2}_{\infty,R}$. Then we have $t<2$, or $r<b_1t/2$ or $r\ge b_2t+R$. As $b_1\le c\le b_2$,

Let $t<2$ or $r<b_1t/2$. Then we have $|ct-r|\ge C\langle t+r\rangle$. Hence we get

$$\begin{aligned}
r^{\frac{d-1}{2}}|\partial_a u(t,x)| &\le C\varepsilon r^{\frac{d-1}{2}}\langle ct-r\rangle^{-1}[\phi(t,x)]_{\Gamma,c,1}\le C\varepsilon\langle t+r\rangle^{\lambda-\mu-1}\\
&\le C\varepsilon\langle t+r\rangle^{-\rho-\nu}\le C\varepsilon\langle t+r\rangle^{-\rho+\theta}\langle ct-r\rangle^{-\nu-\theta},\\
|A[\psi](t,r-ct,\omega)| &\le C|\psi(r-ct,\omega)|\le C\varepsilon\langle r-ct\rangle^{-\rho-\nu}\\
&\le C\varepsilon\langle t+r\rangle^{-\rho-\nu}\le C\varepsilon\langle t+r\rangle^{-\rho+\theta}\langle ct-r\rangle^{-\nu-\theta},
\end{aligned}$$

which yield the desired result.

Finally, let $r>b_2t+R$. Then $\partial_a\phi(t,x)=0$ by the assumption. If $c=b_2$, then $\psi(r-ct,\omega)=0$ by the definition, which leads to $A(t,\sigma,\omega)=0$. On the other hand, if $b_1\le c<b_2$, we have $\langle ct-r\rangle\ge C\langle t+r\rangle$, and going the same way as above, we get $|A[\psi](t,\sigma,\omega)|\le C\varepsilon\langle t+r\rangle^{-\rho+\theta}\langle ct-r\rangle^{-\nu-\theta}$. This completes the proof. □

3.4 Decay estimates and a transformation for the Klein-Gordon equations

In this section, we consider a decay estimate and some transformation for the Klein-Gordon equations. We always assume $\Gamma={}^L\Gamma$ and $c=1$ in this section. Therefore we abbreviate $|\cdot|_{\Gamma,s}$, $\|\cdot\|_{\Gamma,s}$, and $[\cdot]_{\Gamma,c,s}$ as $|\cdot|_s$, $\|\cdot\|_s$, and $[\cdot]_s$, respectively.

First we describe a known decay result for solutions to linear Klein-Gordon equations

$$(\Box+m^2)v(t,x)=\Phi(t,x),\qquad (t,x)\in(0,\infty)\times\mathbb{R}^d \tag{3.54}$$

with $m>0$. We use the decay estimate in Georgiev [13]; however, since we are working in the situation of compactly supported data, the statement can be simplified as follows.

Lemma 3.13. *Suppose that $0<T\le\infty$ and $R>0$. Let v be a smooth solution to (3.54). We assume that $v(0,\cdot)$ is supported on a ball*

$$\{x\in\mathbb{R}^d;|x|\le R\},$$

and that $\Phi(t,x)=0$ *for all* $(t,x)\in[0,T)\times\mathbb{R}^d$ *satisfying* $|x|\ge t+R$*. For any positive constant* $\rho\ne 1$*, there exists a positive constant* $C=C(m,\rho,R)$*, being independent of* T*, such that*

$$\langle t+|x|\rangle^{\frac{d}{2}}|v(t,x)|\le C\|v(0)\|_5+Ct^{(1-\rho)_+}\sup_{\tau\in[0,t]}\langle\tau\rangle^{\rho}\|\Phi(\tau)\|_4 \tag{3.55}$$

for $(t,x)\in(0,T)\times\mathbb{R}^d$*, where* $a_+=\max\{a,0\}$ *for* $a\in\mathbb{R}$*.*

Proof. Let $\{\chi_j\}_{j\in\mathbb{Z}_{\ge 0}}\big(\subset C_0^\infty(\mathbb{R})\big)$ be a partition of unity on $[0,\infty)$ satisfying $\operatorname{supp}\chi_j\subset[2^{j-1},2^{j+1}]$ for $j\ge 1$ and $\operatorname{supp}\chi_0\cap[0,\infty)\subset[0,2)$. Then it is proved in [13] that

$$\begin{aligned}\langle t+|x|\rangle|v(t,x)|^{\frac{d}{2}}\le C&\sum_{j=0}^{\infty}\sum_{|\alpha|\le 5}\left\|\langle\cdot\rangle^{\frac{d}{2}}\chi_j(|\cdot|)\Gamma^\alpha v(0,\cdot)\right\|_{L^2}\\&+C\sum_{j=0}^{\infty}\sum_{|\alpha|\le 4}\sup_{\tau\in[0,t]}\chi_j(\tau)\big\|\langle\tau+|\cdot|\rangle\Gamma^\alpha\Phi(\tau,\cdot)\big\|_{L^2}\end{aligned}$$

without any support condition on $v(0)$ and Φ. It is easy to see that the first term on the right-hand side is bounded by $C\|v(0)\|_5$ if $v(0)$ is supported on the ball of radius R. Let $\|\Phi(t)\|_4\le M_0\langle t\rangle^{-\rho}$ with $M_0>0$. Then, in view of the support condition for Φ, we see that the second term is bounded by

$$CM_0\sum_{j=0}^{\infty}\sup_{\tau\in[0,t]}\chi_j(\tau)\langle\tau\rangle^{1-\rho}\le CM_0\langle t\rangle^{(1-\rho)_+}$$

because we have $\chi_j(\tau)\langle\tau\rangle^{1-\rho}\le C2^{j(1-\rho)}$. □

Among the nonlinear terms in $F_{\rm K}=(F_j)_{1\le j\le N_0}$ for the Klein-Gordon components, $F_{\rm K}^{(\rm w)}=(F_j^{(\rm w)})_{1\le j\le N_0}$ has the slowest decay. To treat it in the decay estimate for the Klein-Gordon components, we use the transformation in Tsutsumi [61]. The idea of this transformation can go back to Kosecki [46]. We would like to summarize the argument here.

$Q_0(\phi,\psi)=Q_0^1(\phi,\psi)=(\partial_t\phi)(\partial_t\psi)-(\nabla_x\phi)\cdot(\nabla_x\psi)$ (cf. (2.7)) is one of the *null forms* introduced in [44] to characterize the null condition. A key feature of the null forms is their faster decay. Indeed, we have the following:

Lemma 3.14. *For any non-negative integer* s*, there is a positive constant* C *such that*

$$\begin{aligned}|Q_0(\lambda_j,\lambda_k)(t,x)|_s\le C\langle t+r\rangle^{-1}\big([\lambda(t,x)]_{[s/2]+1}&|\partial\lambda(t,x)|_s\\&+|\partial\lambda(t,x)|_{[s/2]}[\lambda(t,x)]_{s+1}\big)\end{aligned}$$

for $t>0$*,* $x\in\mathbb{R}^d$*, a smooth function* $\lambda=(\lambda_j)_{1\le j\le n}$ *and* $1\le j,k\le n$*.*

Proof. This estimate in three space dimensions was proved in [28], but the proof is also valid in two space dimensions with an apparent modification. Here we give an outline of the proof. Because $\Gamma^\alpha Q_0(\lambda_j,\lambda_k)$ can be written as a linear combination of $Q_0(\Gamma^\beta\lambda_j,\Gamma^\gamma\lambda_k)$ with $|\beta|+|\gamma|\le|\alpha|$ (see [32] for instance), it suffices to prove

$$|Q_0(\phi,\psi)|\le\langle t+r\rangle^{-1}\big([\phi]_1|\partial\psi|+|\partial\phi|[\psi]_1\big).$$

If $0\le t\le 2$ or $r\le t/2$, then we have $\langle t+r\rangle\le C\langle t-r\rangle$. Hence we get

$$|Q_0(\phi,\psi)|\le C|\partial\phi||\partial\psi|\le C\langle t+r\rangle^{-1}\langle t-r\rangle|\partial\phi||\partial\psi|\le C\langle t+r\rangle^{-1}[\phi]_1|\partial\psi|.$$

On the other hand, when $1\le t/2\le r$, (3.13) implies

$$|(\partial_a\phi)(\partial_b\psi)-(\omega_a\mathcal{D}\phi)(\omega_b\mathcal{D}\psi)|\le C\langle t+r\rangle^{-1}\big([\phi]_1|\partial\psi|+|\partial\phi|[\psi]_1\big),$$

where $\mathcal{D}=\mathcal{D}_1=(\partial_r-\partial_t)/2$, and $\omega_0=-1$. From this we find

$$\begin{aligned}Q_0(\phi,\psi)&=(\omega_0^2-|\omega|^2)(\mathcal{D}\phi)(\mathcal{D}\psi)+\mathcal{O}\big(\langle t+r\rangle^{-1}([\phi]_1|\partial\psi|+|\partial\phi|[\psi]_1)\big)\\&=\mathcal{O}\big(\langle t+r\rangle^{-1}([\phi]_1|\partial\psi|+|\partial\phi|[\psi]_1)\big),\end{aligned}$$

since $\omega_0^2-|\omega|^2=0$. This completes the proof. □

Now we return our attention to the transformation. For $N_0+1\le k,l,m\le N$, direct calculations yield

$$\begin{aligned}&\Box\big((\partial_a w_k)(\partial_b w_l)(\partial_c w_m)\big)\\&\quad=(\partial_a\Box w_k)(\partial_b w_l)(\partial_c w_m)+(\partial_a w_k)(\partial_b\Box w_l)(\partial_c w_m)+(\partial_a w_k)(\partial_b w_l)(\partial_c\Box w_m)\\&\qquad+2(\partial_a w_k)Q_0(\partial_b w_l,\partial_c w_m)+2(\partial_b w_l)Q_0(\partial_c w_m,\partial_a w_k)\\&\qquad+2(\partial_c w_m)Q_0(\partial_a w_k,\partial_b w_l),\end{aligned}$$

and

$$\Box\big((\partial_a w_k)(\partial_b w_l)\big)=(\partial_a\Box w_k)(\partial_b w_l)+(\partial_a w_k)(\partial_b\Box w_l)+2Q_0(\partial_a w_k,\partial_b w_l).$$

Using that $\partial_a\Box w_k=\partial_a\big(F_k(v,\partial u)\big)$, and also applying Lemma 3.14 to estimate Q_0, we obtain

$$\begin{aligned}\big|\Box\big((\partial_a w_k)(\partial_b w_l)(\partial_c w_m)\big)\big|_s\le{}&C|(v,\partial u)|^4_{[(s+1)/2]}|(v,\partial u)|_{s+1}\\&+C\langle t+r\rangle^{-1}[w]_{s+1}|\partial w|_s^2\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}\big|\Box\big((\partial_a w_k)(\partial_b w_l)\big)\big|_s\le{}&C|(v,\partial u)|^2_{[(s+1)/2]}|(v,\partial u)|_{s+1}\\&+C\langle t+r\rangle^{-1}[w]_{s+1}|\partial w|_s\end{aligned}\tag{3.57}$$

for non-negative integer s. Recalling that $|\partial w|_s\le\langle t-r\rangle^{-1}[w]_{s+1}$, (3.56) for $(d,p)=(2,3)$ and (3.57) for $(d,p)=(3,2)$ yields

$$\begin{aligned}\big|\Box F_j^{(\mathrm{w})}(\partial w)\big|_s\le{}&C\big|(v,\partial u)\big|^{2p-2}_{[(s+1)/2]}|(v,\partial u)|_{s+1}\\&+C\langle t+r\rangle^{-1}\langle t-r\rangle^{-(p-2)}[w]^{p-1}_{s+1}|\partial w|_s.\end{aligned}\tag{3.58}$$

Lemma 3.15. *Let* $1 \le j \le N_0$. *We put*

$$\widetilde{v}_j := v_j - m_j^{-2} F_j^{(\mathrm{w})}(\partial w).$$

Then, for any non-negative integer s, we have

$$\begin{aligned} &|(\Box + m_j^2)\widetilde{v}_j - \big(F_j(v, \partial u) - F_j^{(\mathrm{w})}(\partial w)\big)|_s \\ &\quad \le C|(v, \partial u)|_{[(s+1)/2]}^{2p-2} |(v, \partial u)|_{s+1} + C\langle t + r\rangle^{-1}\langle t - r\rangle^{-(p-2)}[w]_{s+1}^{p-1}|\partial w|_s. \end{aligned}$$

Proof. As the definition of $\widetilde{v}_j$ yields

$$(\Box + m_j^2)\widetilde{v}_j = \big(F_j(v, \partial u) - F_j^{(\mathrm{w})}(\partial w)\big) - m_j^{-2}\Box\big(F_j^{(\mathrm{w})}\big),$$

this lemma is an immediate consequence of (3.58). $\Box$

For the asymptotic behavior of v_j, we have the following.

Lemma 3.16. *Let* $\widetilde{v}_j$ *be defined as in Lemma* 3.15. *If*

$$\big\|(\Box + m_j^2)\widetilde{v}_j(t)\big\|_{L^2} \le C\varepsilon^p(1 + t)^{-\rho}$$

with some $\rho > 1$, *and*

$$\lim_{t\to\infty} \Big\|F_j^{(\mathrm{w})}(\partial w)(t)\Big\|_1 = 0,$$

then v_j *is asymptotically free, and its asymptotic data* (φ_j^+, ψ_j^+) *satisfies*

$$(\varphi_j^+, \psi_j^+) = \varepsilon(f_j, g_j) + \mathcal{O}(\varepsilon^p) \quad \text{in } H^1 \times L^2.$$

Proof. Let j be fixed. In this proof, we consider

$$\|\phi\|_{H^1}^2 = m_j^2\|\phi\|_{L^2}^2 + \|\nabla_x\phi\|_{L^2}^2,$$

which is an equivalent norm to the standard Sobolev norm. Let $\mathcal{S}(t)$ be the solution operator to the free Klein-Gordon equation

$$(\Box + m_j^2)\phi = 0,$$

that is to say that $\mathcal{S}(t)\big(\phi(0), \partial_t\phi(0)\big) = \big(\phi(t), \partial_t\phi(t)\big)$. Then $\mathcal{S}(t)$ is a unitary operator on $H^1 \times L^2$.

Let $\widetilde{v}_j$ be defined as in Lemma 3.15. We put $\widetilde{\varphi}_j = \widetilde{v}_j(0)$ and $\widetilde{\psi}_j = \partial_t\widetilde{v}_j(0)$. Then we have

$$\big(\widetilde{\varphi}_j, \widetilde{\psi}_j\big) = \varepsilon(f_j, g_j) + \mathcal{O}(\varepsilon^p) \quad \text{in } H^1 \times L^2.$$

Writing $\widetilde{F}_j = (\Box + m_j^2)\widetilde{v}_j$, we obtain

$$\big(\widetilde{v}_j(t), \partial_t\widetilde{v}_j(t)\big) = \mathcal{S}_j(t)\big(\widetilde{\varphi}_j, \widetilde{\psi}_j\big) + \int_0^t \mathcal{S}_j(t - \tau)\big(0, \widetilde{F}_j(\tau)\big)d\tau.$$

By the assumption,

$$(\varphi_j^+, \psi_j^+) = \big(\widetilde{\varphi}_j, \widetilde{\psi}_j\big) + \int_0^\infty \mathcal{S}(-\tau)\big(0, \widetilde{F}_j(\tau)\big)d\tau$$

is well-defined, because the assumption implies

$$\begin{aligned}\int_0^\infty \left\|\mathcal{S}(-\tau)\big(0, \widetilde{F}_j(\tau)\big)\right\|_{H^1\times L^2} d\tau &= \int_0^\infty \left\|\widetilde{F}_j(\tau)\right\|_{L^2} d\tau \\ &\le C\varepsilon^p \int_0^\infty (1+\tau)^{-\rho} d\tau \le C\varepsilon^p.\end{aligned}$$

This also leads to

$$(\varphi_j^+, \psi_j^+) = \varepsilon(f_j, g_j) + \mathcal{O}(\varepsilon^p) \quad \text{in } H^1 \times L^2.$$

It follows that

$$\begin{aligned}&\left\|\big(\widetilde{v}_j(t), \partial_t \widetilde{v}_j(t)\big) - \mathcal{S}(t)\big(\varphi_j^+, \psi_j^+\big)\right\|_{H^1\times L^2} \\ &\quad \le \int_t^\infty \left\|\mathcal{S}(t-\tau)\big(0, \widetilde{F}_j(\tau)\big)\right\|_{H^1\times L^2} d\tau = \int_t^\infty \left\|\widetilde{F}_j(\tau)\right\|_{L^2} d\tau \\ &\quad \le C\varepsilon^p (1+t)^{1-\rho} \to 0, \quad t \to \infty.\end{aligned}$$

By the assumption,

$$\big\|\big(v_j(t), \partial_t v_j(t)\big) - \big(\widetilde{v}_j(t), \partial_t \widetilde{v}_j(t)\big)\big\|_{H^1\times L^2} \le C \left\|F_j^{(\mathrm{w})}(\partial w)(t)\right\|_1 \to 0$$

as $t \to \infty$. To sum up, we obtain

$$\left\|\big(v_j(t), \partial_t v_j(t)\big) - \mathcal{S}(t)\big(\varphi_j^+, \psi_j^+\big)\right\|_{H^1\times L^2} \to 0, \quad t \to \infty.$$

This completes the proof. □

Chapter 4

Two space dimensional case

4.1 Key decay estimates for the wave equations

In this chapter, we will prove our theorems for the two space dimensional case. To begin with, we describe known decay results for solutions to linear wave equations

$$\begin{aligned} &\Box_c w(t,x) = \Psi(t,x), && (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ &w(0,x) = w_0(x),\ (\partial_t w)(0,x) = w_1(x), && x \in \mathbb{R}^2 \end{aligned} \tag{4.1}$$

with $c > 0$. We use the weighted L^∞-L^∞ estimates in Hoshiga-Kubo [21] and Kubo [49]. The first pair consists of decay estimates of solutions and their derivatives to the homogeneous wave equation (namely the case where $\Psi = 0$ in $(0,\infty) \times \mathbb{R}^2$).

Lemma 4.1. *Let w be a smooth solution to* (4.1) *with $\Psi = 0$. Suppose that $\mu > 0$. For any $\bigl(w_0, w_1\bigr) \in C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$, it holds that*

$$\begin{aligned} \langle t+r\rangle^{1/2}\langle r-ct\rangle^{1/2}|w(t,x)| &\le C\mathcal{A}_{2+\mu,0}[w_0,w_1], \\ \langle t+r\rangle^{1/2}\langle r-ct\rangle^{3/2}|\partial w(t,x)| &\le C\mathcal{A}_{3+\mu,1}[w_0,w_1] \end{aligned} \tag{4.2}$$

for $(t,x) \in [0,T) \times \mathbb{R}^2$, where we put

$$\mathcal{A}_{\rho,s}[w_0,w_1] := \sum_{|\alpha|+|\beta|\le s} \left(\sum_{|\gamma|\le 1} \|\langle\cdot\rangle^\rho \partial_x^{\alpha+\gamma}\Omega^\beta w_0\|_{L^\infty(\mathbb{R}^2)} + \|\langle\cdot\rangle^\rho \partial_x^{\alpha}\Omega^\beta w_1\|_{L^\infty(\mathbb{R}^2)} \right).$$

The second pair is for the inhomogeneous wave equation with zero initial data (namely the case where $w_0(x) = w_1(x) = 0$ for all $x \in \mathbb{R}^2$). To state decay estimates, we introduce some notation. Let n be an arbitrary natural number, and $c^1, \ldots, c^n$ be given positive constants. For $\kappa, \mu > 0$ and a non-negative integer s, we define

$$\mathcal{B}_{\kappa,\mu,s}[\Phi](t,x) = \sup_{0\le\tau<t}\sup_{|y-x|\le t-\tau} \langle y\rangle^{1/2}\langle \tau+|y|\rangle^{1+\kappa}\mathcal{W}_-(\tau,|y|)^{1+\mu} \times \sum_{|\alpha|+|\beta|\le s}|\partial^\alpha\Omega^\beta\Phi(\tau,y)|,$$

where

$$\mathcal{W}_-(t,r) = \mathcal{W}_-(t,r;c^1,\dots,c^n) := \min\{\langle r\rangle, \langle r-c^1t\rangle,\dots,\langle r-c^nt\rangle\}. \tag{4.3}$$

Lemma 4.2. *Let w be a smooth solution to (4.1) with initial data $w_0 = w_1 = 0$. Suppose that $\xi \ge 0$, $0<\zeta<1/2$ and $\eta>0$. Then there exists a positive constant $C = C(\xi,\zeta,\eta)$ such that*

$$\langle t+|x|\rangle^{1/2-\xi}\langle r-ct\rangle^{\zeta}|w(t,x)| \le C\mathcal{B}_{\zeta-\xi,\eta,0}[\Psi](t,x), \tag{4.4}$$

$$\langle t+|x|\rangle^{-\xi}\langle x\rangle^{1/2}\langle r-ct\rangle^{1+\zeta}|\partial w(t,x)| \le C\mathcal{B}_{\zeta+\eta-\xi,0,1}[\Psi](t,x) \tag{4.5}$$

for $(t,x)\in[0,T)\times\mathbb{R}^2$.

Remark 9. (1) In [49, Lemmas 4.2 and A.3], the above lemma is proved for the case $\xi = 0$. However, using the fact that $\langle\tau+|y|\rangle^\xi \le \langle t+|x|\rangle^\xi$ for (τ,y) satisfying $0\le\tau<t$ and $|y-x|\le t-\tau$, we can easily show the case where $\xi>0$.

(2) For the multiple-speed case (1.4), let us call the nonlinear terms in $F^I(\partial u) - {}^*F^I(\partial u^I)$ *non-resonant terms.* In [20], additional decay estimates with different weights are used to estimate non-resonant terms like $(\partial_a u_i^J)(\partial_b u_j^J)(\partial_c u_l^J)$ with $J\ne I$ in F^I. In our proof, we do not need any additional decay estimate, and all types of the non-resonant terms are treated in a unified and intuitive way.

To obtain a decay estimate for a general solution to (4.1), we write $w = \hat{w}^0+\hat{w}^1$, where $\hat{w}^0$ is a solution to $\Box\hat{w}^0 = 0$ with $(\hat{w}^0,\partial_t\hat{w}^0)(0) = (w_0,w_1)$, and $\hat{w}^1$ is a solution to $\Box\hat{w}^1 = \Psi$ with $(\hat{w}^1,\partial_t\hat{w}^1)(0) = (0,0)$. Then we can apply Lemmas 4.1 and 4.2 to $\hat{w}^0$ and $\hat{w}^1$, respectively, to obtain a decay estimate for w.

We conclude this section with the following elementary lemma, which will be used in the proof of our theorems.

Lemma 4.3. *Let $\rho\ge 0$ and $\nu>1$. There is a positive constant C such that*

$$\mathcal{I}(t,\sigma) := \int_t^\infty \tau^{-\rho}\langle c\tau-\sigma\rangle^{-\nu}d\tau \le Ct^{-\rho}$$

for $t\ge1$ and $\sigma\in\mathbb{R}$.

Proof. By a simple change of variable, we obtain

$$\mathcal{I}(t,\sigma) \le t^{-\rho}\int_t^\infty\langle c\tau-\sigma\rangle^{-\nu}d\tau \le \frac{1}{c}t^{-\rho}\int_{-\infty}^\infty\langle\tau\rangle^{-\nu}d\tau \le Ct^{-\rho}.$$

This completes the proof. □

4.2 Proof of SDGE

In this chapter, as well as in the next chapter for the three space dimensional case, we always assume

$$f(x) = g(x) = 0, \quad |x| \geq R \tag{4.6}$$

in the initial condition (1.6), where R is a positive constant. Then, by the finite speed of propagation, we have

$$u(t,x) = 0, \quad |x| \geq t + R,\ 0 \leq t < T \tag{4.7}$$

for a solution u to (1.3)–(1.6) on $[0,T) \times \mathbb{R}^d$. Similarly, for a solution $u = (u^I)_{1 \leq I \leq P}$ to (1.4)–(1.6) with (1.5), we have

$$u^I(t,x) = 0, \qquad |x| \geq c_P t + R,\ 0 \leq t < T \tag{4.8}$$

for $1 \leq I \leq P$ (note that (4.8) is not necessary true if $|x| \geq c_I t + R$).

4.2.1 Proof of Theorem 2.1

Recall the definitions in Section 3.1. Throughout this subsection, we put $\Gamma = {}^L\Gamma$, and we simply write $|\cdot|_s, \|\cdot\|_s$, $[\cdot]_s$, and $\mathscr{T}[\cdot]$ for $|\cdot|_{\Gamma,s}, \|\cdot\|_{\Gamma,s}$, $[\cdot]_{\Gamma,1,s}$, and $\mathscr{T}_1[\cdot]$, respectively. We put

$$\Lambda_{T,R} = \Lambda^{1,1}_{T,R}, \quad \mathcal{L}_{T,R} = \mathcal{L}^1_{T,R} = \mathcal{L}^{1;1.1}_{T,R}$$

and

$$t_0(\sigma) = t_0^1(\sigma; 1, 1, R) = \max\{\langle r\rangle, \langle t-r\rangle\}, \quad \sigma \leq R.$$

We also set $\mathcal{W}_-(t,r) = \mathcal{W}_-(t,r;1) = \min\{\langle r\rangle, \langle t-r\rangle\}$.

Notations and the goal

For a smooth solution $u = (v,w)$ to the Cauchy problem (1.3) with (1.6) on $[0,T) \times \mathbb{R}^2$, we define

$$\begin{aligned}\mathcal{E}(T) := &\sup_{(t,x)\in[0,T)\times\mathbb{R}^2} \langle r\rangle^{\frac{1}{2}} \langle t-r\rangle^{1-\kappa}\big(|\partial w(t,x)| + \langle t+r\rangle^{-\rho}|\partial w(t,x)|_K\big)\\ &+ \sup_{(t,x)\in[0,T)\times\mathbb{R}^2} \langle t+r\rangle |v(t,x)|_{K+1},\end{aligned} \tag{4.9}$$

where ρ and κ are small positive constants to be fixed later, and K is a fixed positive integer with $K \geq 9$. ρ is assumed to be so small compared to κ.

For the proof of Theorem 2.1, it suffices to show the following: If we choose appropriate κ and ρ, then for any large number M, there is a positive number $\varepsilon_0 = \varepsilon_0(M)$, being independent of T, such that $\mathcal{E}(T) \leq M\varepsilon$ implies $\mathcal{E}(T) \leq M\varepsilon/2$ for $\varepsilon \in (0, \varepsilon_0]$. Indeed, this property and the bootstrap argument lead to *a priori* estimates of u for sufficiently small ε, and we have the global existence.

We assume $\mathcal{E}(T) \le M\varepsilon$ from now on. In the following arguments, $M(\ge 1)$ is sufficiently large, and ε is supposed to be so small that $M\varepsilon \le M^2\varepsilon \le 1$.

By (4.9), we have

$$|\big(v(t,x), \partial u(t,x)\big)| \le CM\varepsilon\langle t+r\rangle^{-\frac{1}{2}}\mathcal{W}_-(t,r)^{-\frac{1}{2}}, \tag{4.10}$$

$$|\big(v(t,x), \partial u(t,x)\big)|_K \le CM\varepsilon\langle t+r\rangle^{\rho-\frac{1}{2}}\mathcal{W}_-(t,r)^{-\frac{1}{2}-\rho} \tag{4.11}$$

for $(t,x) \in [0,T) \times \mathbb{R}^2$, since there is a positive constant C such that

$$C^{-1}\langle t+r\rangle\mathcal{W}_-(t,r) \le \langle r\rangle\langle t-r\rangle \le C\langle t+r\rangle\mathcal{W}_-(t,r)$$

for $(t,x) \in [0,T) \times \mathbb{R}^2$.

Energy estimates for wave and Klein-Gordon components.

Recall the definitions of $|\phi(t,x)|_s$ and $\|\phi(t)\|_s$ for a smooth function ϕ and a non-negative integer s. For simplicity of exposition, we set $|\phi(t,x)|_{-1}$ and $\|\phi(t)\|_{-1}$ to be zero in what follows. Let $0 \le s \le 2K+1$. By the Leibniz formula, (4.10) and (4.11), we obtain

$$\begin{aligned}|F(v,\partial u)|_s &\le C\Big(|(v,\partial u)|^2|(v,\partial u)|_s + |(v,\partial u)|_{[s/2]}^2|(v,\partial u)|_{s-1}\Big)\\ &\le CM^2\varepsilon^2\left((1+t)^{-1}|(v,\partial u)|_s + (1+t)^{2\rho-1}|(v,\partial u)|_{s-1}\right),\end{aligned} \tag{4.12}$$

since we have $[s/2] \le K$ for $0 \le s \le 2K+1$. Because of (3.1), we get

$$\begin{aligned}\|(v,\partial u)(t)\|_s &\le C\left(\|(v,\partial u)(0)\|_s + \int_0^t \big\|F\big((v,\partial u)(\tau)\big)\big\|_s\,d\tau\right)\\ &\le C\varepsilon + CM^2\varepsilon^2\int_0^t (1+\tau)^{-1}\|(v,\partial u)\|_s\,d\tau\\ &\quad + CM^2\varepsilon^2\int_0^t (1+\tau)^{2\rho-1}\|(v,\partial u)(\tau)\|_{s-1}\,d\tau,\end{aligned} \tag{4.13}$$

where the constant C can be chosen independently of s.

It follows from (4.13) with $s=0$ and the Gronwall lemma that

$$\|(v,\partial u)(t)\|_0 \le C\varepsilon(1+t)^{CM^2\varepsilon^2}.$$

Similarly, applying the Gronwall lemma to (4.13), we can inductively show

$$\|(v,\partial u)(t)\|_s \le C\varepsilon(1+t)^{CM^2\varepsilon^2+2s\rho} \tag{4.14}$$

for $0 \le s \le 2K+1$. Especially, we have

$$\langle t\rangle^{2\rho}\|(v,\partial u)(t)\|_{2K} + \|(v,\partial u)(t)\|_{2K+1} \le C\varepsilon\langle t\rangle^{\delta}, \tag{4.15}$$

where

$$\delta = 4(K+1)\rho,$$

provided that ε is small enough to satisfy $CM^2\varepsilon^2 \leq 2\rho$.

Remark 10. We can also add any nonlinearity of order greater than 4 to the nonlinear term F. Indeed, in order to do so, we need to add

$$|(v, \partial u)|_{[s/2]}^2 |(v, \partial u)|_s$$

in the first line of the estimate (4.12), but its second line stays valid, and the conclusion of this step is true. We can also easily modify the estimates below in the rest of the proof.

Rough decay estimates for wave and Klein-Gordon components.

By (4.10), (4.11), (4.12)T and (4.15), we obtain

$$||F(v, \partial u)(t)||_{2K} \leq CM^2\varepsilon^2\langle t\rangle^{2\rho-1}\|(v, \partial u)\|_{2K} \leq CM^2\varepsilon^2\langle t\rangle^{\delta-1}.$$

It follows from Lemma 3.13 that

$$\langle t+r\rangle|v(t,x)|_{2K-3} \leq C\varepsilon + CM^2\varepsilon^2\langle t\rangle^{\delta} \leq C\varepsilon\langle t\rangle^{\delta}. \tag{4.16}$$

By (4.11) and (4.15), together with Lemma 3.1, we get

$$\begin{aligned}\langle r\rangle^{\frac{1}{2}}|F_{\mathrm{W}}(v, \partial u)|_{2K-1} &\leq C\langle r\rangle^{\frac{1}{2}}|(v, \partial u)|_K^2|(v, \partial u)|_{2K-1}\\ &\leq C|(v, \partial u)|_K^2\|(v, \partial u)\|_{2K+1}\\ &\leq CM^2\varepsilon^3\langle t+r\rangle^{\delta+2\rho-1}\mathcal{W}_-(t,r)^{-2\rho-1},\end{aligned} \tag{4.17}$$

where $F_{\mathrm{W}} = (F_j)_{N_0+1\leq j\leq N}$. As (4.17) implies

$$\sum_{s=0}^{1}\sum_{|\alpha|\leq 2K-1-s}\mathcal{B}_{\rho+\rho-(4\rho+\delta),\rho,s}[\Gamma^\alpha F_{\mathrm{W}}](t,x) \leq CM^2\varepsilon^3 \leq C\varepsilon,$$

it follows from Lemmas 4.1 and 4.2 with $\xi = 4\rho + \delta$ and $\zeta = \eta = \rho$ that

$$|w(t,x)|_{2K-1} \leq C\varepsilon\langle t+r\rangle^{(4\rho+\delta)-\frac{1}{2}}\langle t-r\rangle^{-\rho}, \tag{4.18}$$

$$|\partial w(t,x)|_{2K-2} \leq C\varepsilon\langle t+r\rangle^{4\rho+\delta}\langle r\rangle^{-\frac{1}{2}}\langle t-r\rangle^{-1-\rho}. \tag{4.19}$$

Hence we get

$$[w]_{2K-1} \leq C\varepsilon\langle t+r\rangle^{\delta'}\langle r\rangle^{-\frac{1}{2}}\langle t-r\rangle^{-\rho}, \tag{4.20}$$

where

$$\delta' = 4\rho + \delta = 4(K+2)\rho.$$

Note that (4.16) and (4.19) yield

$$|(v, \partial u)(t,x)|_{2K-4} \le C\varepsilon \langle t+r\rangle^{\delta'} \langle r\rangle^{-\frac{1}{2}} \langle t-r\rangle^{-\frac{1}{2}-4\rho}. \tag{4.21}$$

We set $\Lambda^{\rm c}_{T,R} = \bigl([0,T)\times\mathbb{R}^2\bigr)\backslash\Lambda_{T,R}$. If $r > t+R$, we have $\partial w(t,x) = 0$. On the other hand, when $t < 2$ or $t > 2r$, we have $\langle t+r\rangle \le C\langle t-r\rangle$ with some universal positive constant C. Therefore (4.19) leads to

$$|\partial w(t,x)|_{2K-2} \le C\varepsilon \langle r\rangle^{-1/2} \langle t-r\rangle^{(3\rho+\delta)-1} \le C\varepsilon \langle r\rangle^{-\frac{1}{2}} \langle t-r\rangle^{\kappa-1}$$

for $(t,x)\in\Lambda^{\rm c}_{T,R}$, as ρ is sufficiently small to satisfy $3\rho+\delta = (4K+7)\rho < \kappa$. Therefore we get

$$\sup_{(t,x)\in\Lambda^{\rm c}_{T,R}} \langle r\rangle^{\frac{1}{2}} \langle t-r\rangle^{1-\kappa} |\partial w(t,x)|_{2K-2} \le C\varepsilon. \tag{4.22}$$

Better decay estimates for the wave components.

Let $(t,x)\in\Lambda_{T,R}$, and $\sigma = r-t$ in the rest of this step. Then we have $(t,\sigma,\omega)\in\mathcal{L}_{T,R}$. Recall that t, r and $\langle t+r\rangle$ are equivalent to each other in $\Lambda_{T,R}$. We apply the arguments in Section 3.2 with

$$\phi = w, c = 1, \Psi = F^{(\rm w)}_{\rm W}, \Theta = F_{\rm W}(v,\partial u) - F^{(\rm w)}_{\rm W}(\partial w)$$

in (3.16) (cf. (WKG) in Section 3.2). Then Φ and $\Phi^{(\alpha)}$ in Section 3.2 are

$$\Phi(t,x) = \mathcal{D}\bigl(r^{\frac{1}{2}} w(t,x)\bigr), \quad \Phi^{(\alpha)}(t,x) = \mathcal{D}\bigl(r^{\frac{1}{2}}\Gamma^\alpha w(t,x)\bigr)$$

with $\mathcal{D} = (\partial_r - \partial_t)/2$. Let $|\alpha| = s \le 2K-4$. By (4.18) and (4.20), we obtain

$$\mathscr{T}[t^{-\frac{3}{2}}|w|_{s+2}] = \mathcal{O}(\varepsilon t^{\delta'-2}\langle\sigma\rangle^{-\rho}), \tag{4.23}$$

$$\mathscr{T}[t^{-\frac{1}{2}}[w]^3_{s+1}] = \mathcal{O}(\varepsilon^3 t^{3\delta'-2}\langle\sigma\rangle^{-3\rho}). \tag{4.24}$$

By the definition of $F^{(\rm w)}_j(\partial w)$, each term in $\Theta = F_{\rm W}(v,\partial u) - F^{(\rm w)}_{\rm W}(\partial w)$ has at least one factor like $\partial^\alpha v_k$ with $|\alpha|\le 1$. Therefore, by (4.16) and (4.21), we get

$$\mathscr{T}[t^{\frac{1}{2}}|\Theta|_s] \le C\mathscr{T}\Bigl[t^{\frac{1}{2}}|(v,\partial u)|^2_s |v|_{s+1}\Bigr] \le C\varepsilon^3 t^{3\delta'-\frac{3}{2}}\langle\sigma\rangle^{-8\rho-1}. \tag{4.25}$$

Let $\mathcal{R}$ be given by (3.25). For $0\le\theta\ll 1$, we obtain

$$\begin{aligned}\int_t^\infty \tau^\theta \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\,d\tau &\le C\varepsilon^3 t^{\theta+3\delta'-1}\langle\sigma\rangle^{-3\rho} + C\varepsilon t^{\theta+\delta'-1}\langle\sigma\rangle^{-\rho} \\ &\quad + C\varepsilon^3 t^{\theta+3\delta'-\frac{1}{2}}\langle\sigma\rangle^{-8\rho-1} \\ &\le C\varepsilon t^{\theta+\frac{\kappa}{4}-\frac{1}{2}}\langle\sigma\rangle^{\frac{3\kappa}{4}-\frac{1}{2}},\end{aligned} \tag{4.26}$$

provided that ρ is small enough to satisfy

$$3\delta' = 12(K+2)\rho < \frac{\kappa}{4}.$$

Especially, with the help of (3.9), we have

$$\int_{t_0(\sigma)}^{\infty} \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\, d\tau \le C\varepsilon\langle\sigma\rangle^{\kappa-1}. \tag{4.27}$$

Recall the $*$-notation in (3.23). By (4.20), we get

$$t^{-\frac{1}{2}}[w(t,x)]_{2K-1} \le C\varepsilon t^{\delta'-1}\langle\sigma\rangle^{-\rho} \le C\varepsilon\langle\sigma\rangle^{\kappa-1}. \tag{4.28}$$

By (3.21) and (3.29), we also obtain

$$\begin{aligned}|\Phi^*(t,\sigma,\omega)|, |\Phi^{(\alpha)*}(t,\sigma,\omega)| &\le C\mathscr{T}\big[t^{\frac{1}{2}}\langle t-r\rangle^{-1}[w]_{2K-4}\big]\\ &\le C\varepsilon t^{\delta'}\langle\sigma\rangle^{-\rho-1}.\end{aligned} \tag{4.29}$$

Especially, we have

$$\big|\Phi^*\big(t_0(\sigma),\sigma,\omega\big)\big|, \big|\Phi^{(\alpha)*}\big(t_0(\sigma),\sigma,\omega\big)| \le C\varepsilon\langle\sigma\rangle^{\delta'-1-\rho} \le C\varepsilon\langle\sigma\rangle^{\kappa-1}. \tag{4.30}$$

It follows from Lemma 3.7, (4.27), (4.28) and (4.30) that

$$\sup_{(t,x)\in\Lambda_{T,R}} \langle r\rangle^{\frac{1}{2}}\langle t-r\rangle^{1-\kappa}|\partial w(t,x)| \le C\varepsilon. \tag{4.31}$$

From (4.29), we get $|\Phi^*(t,\sigma,\omega)| \le C\varepsilon$, which implies

$$\big\|\mathcal{G}\big(\omega,\Phi^*(t,\sigma,\omega)\big)\big\| \le 2C_0\varepsilon^2$$

with some positive constant C_0, where $\mathcal{G}(\omega,Y)$ is given by (3.32). Let $\mathcal{R}^{(\alpha)}$ be given by (3.34). Similarly to (4.27) we obtain

$$\begin{aligned}&\sum_{|\alpha|\le s}\int_{t_0(\sigma)}^{t}\left(\frac{t}{\tau}\right)^{C_0\varepsilon^2}|\mathscr{T}[\mathcal{R}^{(\alpha)}](\tau,\sigma,\omega)|d\tau\\ &\quad\le Ct^{C_0\varepsilon^2}\int_{t_0(\sigma)}^{t}\tau^{-C_0\varepsilon^2-1}\left(\tau^{\frac{1}{2}}\mathscr{T}[|\partial w|_{s-1}](t,\sigma,\omega)\right)^3 d\tau\\ &\qquad+C\varepsilon t^{C_0\varepsilon^2}\langle\sigma\rangle^{\kappa-1}.\end{aligned} \tag{4.32}$$

It follows from Lemma 3.8, (4.28), (4.30), and (4.32) that

$$\begin{aligned}t^{\frac{1}{2}}|\partial w(t,x)|_s \le{}& Ct^{C_0\varepsilon^2}\int_{t_0(\sigma)}^{t}\tau^{-C_0\varepsilon^2-1}\left(\tau^{\frac{1}{2}}\mathscr{T}[|\partial w|_{s-1}](t,\sigma,\omega)\right)^3 d\tau\\ &+C\varepsilon t^{C_0\varepsilon^2}\langle\sigma\rangle^{\kappa-1}.\end{aligned}$$

By induction, we can show that

$$t^{\frac{1}{2}}|\partial w(t,x)|_s \le C\varepsilon t^{3^{s-1}C_0\varepsilon^2}\langle t-r\rangle^{\kappa-1}$$

for $1 \le s \le 2K-4$. If ε is sufficiently small to satisfy

$$3^{2K-5}C_0\varepsilon^2 \le \rho,$$

we obtain

$$\sup_{(t,x)\in\Lambda_{T,R}} t^{-\rho}\langle r\rangle^{\frac{1}{2}}\langle t-r\rangle^{1-\kappa}|\partial w(t,x)|_{2K-4} \le C\varepsilon. \tag{4.33}$$

Better decay estimates for the Klein-Gordon components.

We will make use of Lemma 3.15 to improve decay estimates for the Klein-Gordon components. Let $(t,x) \in [0,T)\times\mathbb{R}^2$. For $1 \le j \le N_0$, we put

$$\widetilde{v}_j = v_j - m_j^{-2}F_j^{(\mathrm{w})}(\partial w).$$

Since, as before, each term in $F_j(v,\partial u) - F_j^{(\mathrm{w})}(\partial w)$ has at least one factor of the form $\partial^\alpha v_k$ with $|\alpha| \le 1$, it follows from (4.11) and (4.16) that

$$\begin{aligned}|F_j(v,\partial u) - F_j^{(\mathrm{w})}(\partial w)|_{2K-4} &\le C|v|_{2K-3}|(v,\partial u)|_{K-2}|(v,\partial u)|_{2K-4}\\ &\le CM\varepsilon^2\langle t\rangle^{\delta+\rho-\frac{3}{2}}|(v,\partial u)|_{2K-4}.\end{aligned}$$

Hence, by (4.15), we obtain

$$\begin{aligned}\|F_j(v,\partial u) - F_j^{(\mathrm{w})}(\partial u)\|_{2K-4} &\le CM\varepsilon^2\langle t\rangle^{\delta-\frac{3}{2}}\langle t\rangle^{\rho}\|(v,\partial u)\|_{2K-4}\\ &\le CM\varepsilon^3\langle t\rangle^{2\delta-\frac{3}{2}}.\end{aligned} \tag{4.34}$$

From (4.11) and (4.15), we also have

$$\left\||(v,\partial u)|_{K-2}^4|(v,\partial u)|_{2K-3}\right\|_{L^2} \le CM^4\varepsilon^5\langle t\rangle^{\delta+2\rho-2}.$$

By (4.20) and (4.15), we obtain

$$\begin{aligned}\left\|\langle t\rangle^{-1}\langle t-|\cdot|\rangle^{-1}[w]_{2K-3}^2|\partial w|_{2K-4}\right\|_{L^2} &\le C\varepsilon^2\langle t\rangle^{2\delta'-2}\|\partial w\|_{2K-4}\\ &\le C\varepsilon^3\langle t\rangle^{3\delta'-2}.\end{aligned}$$

Gathering the above estimates, we see from Lemma 3.15 that

$$\|(\Box + m_j^2)\widetilde{v}_j\|_{2K-4} \le CM\varepsilon^3\langle t\rangle^{-1-\rho} \le C\varepsilon\langle t\rangle^{-1-\rho}. \tag{4.35}$$

Lemma 3.13 implies

$$\langle t+r\rangle|\widetilde{v}(t,x)|_{2K-8} \le C\varepsilon. \tag{4.36}$$

By (4.11) and (4.19), we have

$$|v(t,x)-\widetilde{v}(t,x)|_{2K-8} \le C\left|F_{\rm K}^{(\rm w)}(\partial w)(t,x)\right|_{2K-8} \le CM^2\varepsilon^3\langle t+r\rangle^{2\rho+\delta'-\frac{3}{2}}$$
$$\le C\varepsilon\langle t+r\rangle^{-1}.$$

Therefore we obtain

$$\langle t+r\rangle|v(t,x)|_{2K-8} \le C\varepsilon. \tag{4.37}$$

Conclusion.

Because $K+1 \le 2K-8$ and $K \le 2K-4$ for $K \ge 9$, from (4.22), (4.31), (4.33) and (4.37), we find that there is a positive constant C_0 and $\varepsilon_0 = \varepsilon_0(M)$ such that we have

$$\mathcal{E}(T) \le C_0\varepsilon$$

for $\varepsilon \in (0,\varepsilon_0]$. If M is sufficiently large to satisfy $M \ge 2C_0$, then we have $\mathcal{E}(T) \le M\varepsilon/2$ for $\varepsilon \in (0,\varepsilon_0]$, as desired. This completes the proof. □

4.2.2 Proof of Theorem 2.3

In this subsection, we put $\Gamma = {}^S\Gamma$, and we simply write $|\cdot|_s$, $\|\cdot\|_s$, and $\mathscr{T}_I[\cdot]$ for $|\cdot|_{\Gamma,s}$, $\|\cdot\|_{\Gamma,s}$, and $\mathscr{T}_{c_I}[\cdot]$, respectively. We also write

$$[\phi(t,x)]_{I,s} := [\phi(t,x)]_{\Gamma,c_I,s} = |\phi(t,x)|_{\Gamma,s} + \langle c_I t - r\rangle|\partial\phi(t,x)|_{\Gamma,s-1}$$

for a smooth function ϕ and an integer $s \ge 1$. We put

$$\Lambda_{T,R} = \Lambda_{T,R}^{c_1,c_P}, \mathcal{L}_{T,R}^{I} = \mathcal{L}_{T,R}^{c_I} = \mathcal{L}_{T,R}^{c_I;c_1,c_P},$$

and

$$t_0^I(\sigma) = t_0^{c_I}(\sigma; c_1, c_P, R).$$

We also set

$$\mathcal{W}_-(t,r) = \mathcal{W}_-(t,r;c_1,\dots,c_P) = \min\{\langle r\rangle, \langle r-c_1 t\rangle, \dots, \langle r-c_P t\rangle\}.$$

Then we have

$$\langle r\rangle^{-1}\langle c_I t - r\rangle^{-1} \le C\langle t+r\rangle^{-1}\mathcal{W}_-(t,r)^{-1}, \quad (t,x) \in [0,\infty)\times\mathbb{R}^2, 1 \le I \le P.$$

Notations and the goal

The proof is quite similar to that for Theorem 2.1, and we concentrate on different points. Let $u = (u^I)_{1 \le I \le P} \in C^\infty([0,T) \times \mathbb{R}^2; \mathbb{R}^N)$ be a solution to (1.4) with (1.6), where $N = N^1 + \cdots + N^P$.

Let M be sufficiently large, and ε be sufficiently small compared to M. Our goal is, as before, to show that $\mathcal{E}(T) \le M\varepsilon$ implies $\mathcal{E}(T) \le M\varepsilon/2$, where

$$\mathcal{E}(T) = \sup_{(t,x)\in[0,T)\times\mathbb{R}^2} \sum_{I=1}^{P} \langle r\rangle^{\frac{1}{2}} \langle r - c_I t\rangle^{1-\kappa} \left\{ |\partial u^I(t,x)| + \langle t+r\rangle^{-\rho} |\partial u^I(t,x)|_K \right\} \tag{4.38}$$

with large $K \ge 3$, $0 < \kappa < 1/4$ and a sufficiently small positive number ρ compared to κ.

The energy estimates.

Let $0 \le s \le 2K+1$. Because of the Leibniz rule, we have

$$\begin{aligned} |F(\partial u)|_s &\le C(|\partial u|^2 |\partial u|_s + |\partial u|_K^2 |\partial u|_{s-1}) \\ &\le CM^2\varepsilon^2 \langle t\rangle^{-1} |\partial u|_s + CM^2\varepsilon^2 \langle t\rangle^{2\rho-1} |\partial u|_{s-1}, \end{aligned}$$

where we put $|\phi|_{-1} = 0$ for a smooth function ϕ. Proceeding as before, we get

$$\|\partial u\|_{2K+1} \le C\varepsilon \langle t\rangle^{\delta} \tag{4.39}$$

with $\delta = 4(K+1)\rho$, provided that ε is small enough to satisfy $CM^2\varepsilon^2 \le 2\rho$.

Rough decay estimates.

It follows from Lemma 3.1 and (4.39) that

$$|\partial u(t,x)|_{2K-1} \le C\varepsilon \langle t\rangle^{\delta} \langle r\rangle^{-\frac{1}{2}}.$$

From this we get

$$\begin{aligned} \langle r\rangle^{\frac{1}{2}} |F(\partial u)|_{2K-1} &\le C\langle r\rangle^{\frac{1}{2}} |\partial u|_K^2 |\partial u|_{2K-1} \\ &\le CM^2\varepsilon^3 \langle t+r\rangle^{2\rho+\delta-1} \mathcal{W}_-(t,x)^{2\kappa-2}, \end{aligned} \tag{4.40}$$

which leads to

$$\mathcal{B}_{\rho-(4\rho+\delta)+\rho,1-2\kappa,2K-1}[F^I(\partial u)](t) \le CM^2\varepsilon^3 \le C\varepsilon.$$

Then, we have

$$\langle t+r\rangle^{\frac{1}{2}-\delta'} \langle r - c_I t\rangle^{\rho} |u^I(t,x)|_{2K-1} \le C\varepsilon, \tag{4.41}$$

$$\langle t+r\rangle^{-\delta'} \langle r\rangle^{\frac{1}{2}} \langle r - c_I t\rangle^{1+\rho} |\partial u^I(t,x)|_{2K-2} \le C\varepsilon \tag{4.42}$$

for $1 \le I \le P$, similarly to (4.18) and (4.19), where

$$\delta' = 4\rho + \delta = 4(K+2)\rho.$$

It follows from (4.41) and (4.42) that

$$[u^I(t,x)]_{I,2K-1} \le C\varepsilon \langle t+r\rangle^{\delta'} \langle r\rangle^{-\frac{1}{2}} \langle c_I t - r\rangle^{-\rho}. \tag{4.43}$$

Let $\Lambda^{\mathrm{c}}_{T,R} = \big([0,\infty)\times\mathbb{R}^2\big) \setminus \Lambda_{T,R}$. Then we have $t < 2$, $c_1 t > 2r$ or $r > c_P t + R$ in $\Lambda^{\mathrm{c}}_{T,R}$. If $t < 2$ or $2r \le c_1 t (\le c_I T)$, we have $\langle t+r\rangle \le C\langle c_I t - r\rangle$ for $1 \le I \le P$. The same is true for $1 \le I < P$ when $r > c_P t + R$, because c_P is strictly greater than c_I. On the other hand, if $I = P$ and $r > c_P t + R$, then $\partial u^I(t,x) = 0$ by (4.8). Therefore (4.42) leads to

$$\begin{aligned} |\partial u^I(t,x)|_K &\le |\partial u^I(t,x)|_{2K-3} \\ &\le C\varepsilon \langle r\rangle^{-\frac{1}{2}} \langle t+r\rangle^{\rho+\delta'-1} \le C\varepsilon\langle r\rangle^{-\frac{1}{2}} \langle r - c_I t\rangle^{\kappa-1} \end{aligned}$$

for $(t,x) \in \Lambda^{\mathrm{c}}_{T,R}$, as $K \ge 2$ and $\rho + \delta' = (4K+9)\rho \le \kappa$ for sufficiently small ρ. In other words, we have

$$\langle r\rangle^{\frac{1}{2}} \langle r - c_I t\rangle^{1-\kappa} |\partial u^I(t,x)|_K \le C\varepsilon, \quad (t,x) \in \Lambda^{\mathrm{c}}_{T,R}. \tag{4.44}$$

Better decay estimates.

Let $(t,x) \in \Lambda_{T,R}$ in this step. Recall that t, r and $\langle t+r\rangle$ are equivalent to each other in $\Lambda_{T,R}$. We put

$$\sigma_I := \sigma_{c_I} = r - c_I t.$$

Then we have $(t, \sigma_I, \omega) \in \mathcal{L}^I_{T,R}$. For each $I \in \{1,\dots,P\}$, we apply the arguments in Section 3.2 with

$$\phi = u^I, c = c_I, \Psi = {}^*F^I, \Theta = F^I(\partial u) - {}^*F^I(\partial u^I)$$

in (3.16) (cf. (MSW) in Section 3.2). In this case, we have

$$\Phi(t,x) = \mathcal{D}_{c_I}\big(r^{\frac{1}{2}} u^I(t,x)\big),\ \Phi^{(\alpha)}(t,x) = \mathcal{D}_{c_I}\big(r^{\frac{1}{2}} \Gamma^\alpha u^I(t,x)\big)$$

for $|\alpha| \le K$. We use the $*$-notation with $* = *_{c_I}$. Then by (4.43), we have

$$t^{-\frac{1}{2}} [u^I(t,x)]_{I,K+1} \le C\varepsilon t^{\delta'-1} \langle\sigma_I\rangle^{-\rho} \le C\varepsilon \langle \sigma_I\rangle^{\kappa-1} \tag{4.45}$$

and

$$|\Phi^*(t,\sigma_I,\omega)|,\ |\Phi^{(\alpha)*}(t,\sigma_I,\omega)| \le C\varepsilon t^{\delta'} \langle\sigma_I\rangle^{-1-\rho}. \tag{4.46}$$

Especially we have

$$|\Phi^*(t_0^I(\sigma_I),\sigma_I,\omega)|, |\Phi^{(\alpha)*}(t_0^I(\sigma_I),\sigma_I,\omega)| \le C\varepsilon\langle\sigma_I\rangle^{\delta'-1-\rho} \le C\varepsilon\langle\sigma_I\rangle^{\kappa-1}, \tag{4.47}$$

provided that ρ is small enough to satisfy $\delta' - \rho = (4K+7)\rho \le \kappa$.

It follows from (4.41) and (4.43) that

$$\mathscr{T}_I[t^{-\frac{3}{2}}|u^I|_{K+2}](t,\sigma_I,\omega) = \mathcal{O}(\varepsilon t^{\delta'-2}\langle\sigma_I\rangle^{-\rho}), \tag{4.48}$$

$$\mathscr{T}_I[t^{-\frac{1}{2}}[u^I]^3_{I,K+1}](t,\sigma_I,\omega) = \mathcal{O}(\varepsilon^3 t^{3\delta'-2}\langle\sigma_I\rangle^{-3\rho}). \tag{4.49}$$

We set

$$\mathcal{W}_-^I(t,x) = \min_{0\le J\le P; J\ne I}\langle r - c_J t\rangle \tag{4.50}$$

with $c_0 = 0$. Note that we have

$$\mathscr{T}_I[\mathcal{W}_-^I](t,\sigma_I,\omega) = \min_{0\le J\le P; J\ne I}\langle (c_J - c_I)t - \sigma_I\rangle.$$

By the definition of ${}^*F^I(\partial u^I)$, we see that $\Theta = F^I(\partial u) - {}^*F^I(\partial u^I)$ is a linear combination of $(\partial_a u_j^J)(\partial_b u_k^{J'})(\partial_c u_l^{J''})$ with $(J,J',J'') \ne (I,I,I)$. Since (4.42) yields

$$t^{\frac{1}{2}} \sum_{(J,J',J'')\ne(I,I,I)} |\partial u^J|_K |\partial u^{J'}|_K |\partial u^{J''}|_K \le C\varepsilon^3 t^{3\delta'-1}\mathcal{W}_-^I(t,x)^{-1-\rho},$$

we obtain

$$t^{\frac{1}{2}}|\Theta(t,x)|_K \le C\varepsilon^3 t^{3\delta'-1}\mathcal{W}_-^I(t,x)^{-1-\rho}. \tag{4.51}$$

Let $\mathcal{R}$ be given by (3.25). Using Lemma 4.3 for the estimate of the integral of $\mathscr{T}_I[t^{\frac{1}{2}}\Theta](t,\sigma_I,\omega)$, we obtain

$$\begin{aligned}\int_t^\infty \tau^\theta \mathscr{T}_I[\mathcal{R}](\tau,\sigma_I,\omega)\,d\tau &\le C\varepsilon t^{\theta+\delta'-1}\langle\sigma_I\rangle^{-\rho} + C\varepsilon^3 t^{\theta+3\delta'-1}\langle\sigma_I\rangle^{-3\rho} \\ &\quad + C\varepsilon^3 t^{\theta+3\delta'-1} \\ &\le C\varepsilon t^{\theta+\frac{\kappa}{4}-\frac{1}{2}}\langle\sigma_I\rangle^{\frac{3\kappa}{4}-\frac{1}{2}}\end{aligned} \tag{4.52}$$

for $0\le\theta\ll 1$ and $|\alpha|\le K$, provided that ρ is small enough to satisfy

$$4\delta' = 16(K+2)\rho < \frac{\kappa}{4}.$$

Especially, we have

$$\int_{t_0^I(\sigma_I)}^\infty \mathscr{T}_I[\mathcal{R}](\tau,\sigma_I,\omega)\,d\tau \le C\varepsilon\langle\sigma_I\rangle^{\kappa-1}. \tag{4.53}$$

It follows from Lemma 3.7, (4.45), (4.47) and (4.53) that

$$|\partial u^I(t,x)| \le C\varepsilon \langle r\rangle^{-\frac{1}{2}}\langle r - c_I t\rangle^{\kappa-1}, \quad (t,x) \in \Lambda_{T,R}. \tag{4.54}$$

Let $\mathcal{G}(\omega, Y)$ be given by (3.32). Then (4.46) leads to

$$\big\|\mathcal{G}\big(\omega, \Phi^*(t,\sigma_I,\omega)\big)\big\| \le 2c_I^2 C_0\varepsilon^2$$

for some $C_0 > 0$. Let $1 \le s \le K$ and $\mathcal{R}^{(\alpha)}$ be given by (3.34). Similarly to (4.53), we obtain

$$\begin{aligned} &\sum_{|\alpha|\le s}\int_{t_0^I(\sigma)}^t \left(\frac{t}{\tau}\right)^{C_0\varepsilon^2} |\mathscr{T}_I[\mathcal{R}^{(\alpha)}](\tau,\sigma_I,\omega)|d\tau \\ &\quad \le Ct^{C_0\varepsilon^2}\int_{t_0^I(\sigma)}^t \tau^{-C_0\varepsilon^2-1}|\tau^{\frac{1}{2}}\mathscr{T}_I[|\partial u^I|_{s-1}](\tau,\sigma_I,\omega)|^3 d\tau \\ &\qquad + C\varepsilon t^{C_0\varepsilon^3}\langle\sigma_I\rangle^{\kappa-1}. \end{aligned} \tag{4.55}$$

It follows from Lemma 3.8, (4.45), (4.47), and (4.55) that

$$\begin{aligned} t^{1/2}|\partial u^I(t,x)|_s \le Ct^{C_0\varepsilon^2}&\int_{t_0^I(\sigma)}^t \tau^{-C_0\varepsilon^2-1}|\tau^{\frac{1}{2}}\mathscr{T}_I[|\partial u^I|_{s-1}](\tau,\sigma_I,\omega)|^3 d\tau \\ &+ C\varepsilon t^{C_0\varepsilon^3}\langle\sigma_I\rangle^{\kappa-1}. \end{aligned}$$

By induction, we can show that

$$t^{1/2}|\partial u^I(t,x)|_s \le C\varepsilon t^{3^{s-1}C_0\varepsilon^2}\langle\sigma_I\rangle^{\kappa-1}.$$

Especially we have

$$|\partial u^I(t,x)|_K \le C\varepsilon\langle t+r\rangle^{\rho}\langle r\rangle^{-\frac{1}{2}}\langle r - c_I t\rangle^{\kappa-1}, \quad (t,x)\in\Lambda_{T,R}, \tag{4.56}$$

provided that ε is small enough to satisfy $3^{K-1}C_0\varepsilon^2 \le \rho$.

Conclusion.

From (4.44), (4.54) and (4.56), there are $C_* > 0$, which is independent of (M,ε,T), and $\varepsilon_1 = \varepsilon_1(M) > 0$ such that we have

$$\mathcal{E}(T) \le C_*\varepsilon$$

for $0 < \varepsilon \le \varepsilon_1(M)$. If $M \ge 2C_* =: M_0$, we obtain $\mathcal{E}(T) \le M\varepsilon/2$. This completes the proof. □

4.3 Proof of the asymptotic behavior

In this section, we will prove the two space dimensional part of Theorems 2.5 and 2.6. For $f, g \in C_0^\infty$ and sufficiently small ε, (1.3) (resp. (1.4)) with (1.6) admits a global classical solution $u = (v, w)$ (resp. $u = (u^I)_{1\le I\le P}$) by Theorem 2.1 (resp. Theorem 2.3). By its proof, we find that $\mathcal{E}(\infty) \le M\varepsilon$ with some $M > 0$, where $\mathcal{E}$ is given by (4.9) (resp. (4.38)), and we find that all the estimates in the proof of Theorem 2.1 (resp. Theorem 2.3) are valid with $T = \infty$.

4.3.1 Proof of Theorem 2.5

We use the same notation as in the proof of Theorem 2.1. Firstly we consider the Klein-Gordon component. By $\mathcal{E}(\infty) \le M\varepsilon$ and (4.15), we get

$$\left\| F_j^{(\mathrm{w})}(\partial w)(t) \right\|_1 \le CM^2\varepsilon^3 t^{2\rho+\delta-1} \to 0, \quad t \to \infty.$$

Hence, with the help of (4.35), Lemma 3.16 implies the desired result for the Klein-Gordon component v.

Secondly we consider the wave component. By (4.20), we have

$$r^{\frac{1}{2}}[w(t,x)]_1 \le C\varepsilon\langle t+r\rangle^{\delta'}\langle t-r\rangle^{-\rho}, \quad (t,x) \in [0,\infty)\times\mathbb{R}^2. \tag{4.57}$$

By (4.26), we have

$$\int_t^\infty \tau^\theta \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)d\tau \le C\varepsilon t^{\theta+\frac{\kappa}{4}-\frac{1}{2}}\langle\sigma\rangle^{\frac{3\kappa}{4}-\frac{1}{2}}, \quad (t,\sigma,\omega) \in \mathcal{L}_{\infty,R} = \mathcal{L}_{\infty,R}^{1;1,1} \tag{4.58}$$

for $0 \le \theta \ll 1$. If we choose sufficiently small ρ, compared to κ, we have

$$\delta' - 1 \le \frac{\kappa}{4} - \frac{1}{2}, \quad \delta' - 1 - \rho \le \left(\frac{\kappa}{4} - \frac{1}{2}\right) + \left(\frac{3\kappa}{4} - \frac{1}{2}\right) = \kappa - 1. \tag{4.59}$$

By (4.57), (4.58) and (4.59), as well as (4.7) and the conditions (B1) or (B2) for $(d,p) = (2,3)$ in the assumption of Theorem 2.5, we see that all the assumptions in Lemma 3.12 are fulfilled and the lemma implies the desired results for the wave component w, by choosing $\theta = \kappa/4$. This completes the proof.

4.3.2 Proof of Theorem 2.6

We use the same notation as in the proof of Theorem 2.3. We fix $I = 1,\ldots,P$. From (4.43), we get

$$r^{\frac{1}{2}}[u^I(t,x)]_{I,1} \le C\varepsilon\langle t+r\rangle^{\delta'}\langle c_I t-r\rangle^{-\rho}, \quad (t,x) \in [0,\infty)\times\mathbb{R}^2. \tag{4.60}$$

We can write (4.52) as

$$\int_t^\infty \tau^\theta \mathscr{T}_I[\mathcal{R}](\tau, \sigma_I, \omega)\, d\tau \le C\varepsilon t^{\theta+\frac{\kappa}{4}-\frac{1}{2}} \langle \sigma_I \rangle^{\frac{3\kappa}{4}-\frac{1}{2}}, \quad 0 \le \theta \ll 1 \tag{4.61}$$

for $(t, \sigma, \omega) \in \mathcal{L}^I_{\infty,R} = \mathcal{L}^{c_I;c_1,c_P}_{\infty,R}$. Observe that (4.60) and (4.61) correspond to (4.57) and (4.58), respectively. We also have (4.8), which corresponds to (4.7). Therefore, proceeding as above, we obtain the desired results by Lemma 3.12. This completes the proof.

Chapter 5

Three space dimensional case

5.1 Key decay estimates for the wave equations

In this chapter, we will prove our theorems for the three space dimensional case. As in Section 4.1, we firstly summarize known decay results for solutions to linear wave equations

$$\begin{aligned} &\Box_c w(t,x) = \Psi(t,x), && (t,x) \in (0,\infty)\times\mathbb{R}^3,\\ &w(0,x) = w_0(x),\ (\partial_t w)(0,x) = w_1(x), && x \in \mathbb{R}^3 \end{aligned} \tag{5.1}$$

with $c > 0$. We employ the weighted L^∞-L^∞ estimates in Asakura [5] and Yokoyama [62] (see also [32] and Kubota-Yokoyama [50] for the expression below).

To state the weighted L^∞-L^∞ estimates, we define

$$\mathcal{X}_\kappa(t,x) = \begin{cases} \log\left(1 + \dfrac{\langle t+r\rangle}{\langle t-r\rangle}\right), & \text{if } \kappa = 1,\\ 1, & \text{if } \kappa > 1. \end{cases}$$

Let $c^1, \dots, c^n$ be given positive constants, and let $\mathcal{W}_-(t,r)$ be given by (4.3).

The first one is a decay estimate of the solutions to the homogeneous wave equation.

Lemma 5.1. *Let $c > 0$, and w be a smooth solution to* (5.1) *with $\Psi = 0$. For $\theta > 0$ and $\kappa \ge 1$, there is a positive constant C such that*

$$\langle t+r\rangle^{1-\theta}\langle ct-r\rangle^{\kappa-1}|w(t,x)| \le C\mathcal{X}_\kappa(ct,x)\mathcal{A}^\dagger_{\kappa-\theta}[w_0,w_1]$$

for $(t,x) \in [0,\infty)\times\mathbb{R}^3$, where we set

$$\mathcal{A}^\dagger_\rho[w_0,w_1] := \sup_{x\in\mathbb{R}^3} \langle r\rangle^\rho\big(\langle r\rangle|w_0(x)|_{\partial_x,1} + r|w_1(x)|\big). \tag{5.2}$$

The second pair is for the inhomogeneous wave equation with zero data. To describe decay estimates, we introduce some notation. Let Γ be a set of vector fields. For $\kappa \in \mathbb{R}, 0 < \mu < 1$, a non-negative integer s and a smooth function $\Psi = \Psi(t,x)$, we set

$$\mathcal{B}^{\Gamma,\dagger}_{\kappa,\,\mu,\,s}[\Psi](t) := \sup_{(\tau,x)\in[0,t)\times\mathbb{R}^3} r\langle \tau + r\rangle^{\kappa+\mu}\mathcal{W}_-(\tau,x)^{1-\mu}|\Psi(\tau,x)|_{\Gamma,s} \tag{5.3}$$

for $t \geq 0$. We write $\mathcal{B}^{\dagger}_{\kappa,\mu,0}[\Psi](t)$ for $\mathcal{B}^{\Gamma,\dagger}_{\kappa,\mu,0}[\Psi](t)$ as the choice of Γ makes no difference.

Lemma 5.2. *Let $c > 0, 0 < T \leq \infty$, and w be a smooth solution to* (5.1) *with $w(0) = (\partial_t w)(0) = 0$. For $\xi \geq 0, \zeta \geq 1$, and $0 < \eta < 1$, there is a positive constant C, being independent of T, such that*

$$\langle t + r\rangle^{1-\xi}\langle ct - r\rangle^{\zeta-1}|w(t,x)| \leq \mathcal{X}_\zeta(ct,x)\mathcal{B}^{\dagger}_{\zeta-\xi,\,\eta,\,0}[\Psi](t), \tag{5.4}$$

$$\langle t + r\rangle^{-\xi}\langle r\rangle\langle ct - r\rangle^{\zeta}|\partial w(t,x)| \leq C\mathcal{B}^{(\Omega,\partial),\dagger}_{\zeta-\xi,\,\eta,\,1}[\Psi](t) \tag{5.5}$$

for $(t,x) \in (0,T)\times\mathbb{R}^3$, where $(\Omega,\partial) = (\Omega_{23},\Omega_{31},\Omega_{12},\partial_0,\partial_1,\partial_2,\partial_3)$.

In the rest of this section, we introduce different versions of decay estimates for derivatives of solutions to wave equations. To begin with, let us explain our motivation, which is closely related to the difficulty in our problems (1.3) and (1.4) in three space dimensions. Let Γ be the full set of our vector fields, including S and L_k, in d space dimensions. Then it is known that we have

$$\langle t - r\rangle|\partial\phi(t,x)| \leq C|\phi(t,x)|_{\Gamma,1} \tag{5.6}$$

for any smooth function ϕ on $\mathbb{R}^d$ (see [32] for instance). Therefore, if we have the estimate

$$|w(t,x)|_{\Gamma,s+1} \leq C\varepsilon\langle t + r\rangle^{\rho-\frac{d-1}{2}}$$

with some $0 < \rho \ll 1$, we also obtain

$$|\partial w(t,x)|_{\Gamma,s} \leq C\varepsilon\langle t + r\rangle^{\rho-\frac{d-1}{2}}\langle t - r\rangle^{-1}.$$

Especially, in a region close to the t-axis, say $r < t/2$, we have the estimate $|\partial w(t,x)|_{\Gamma,s} \leq C\varepsilon\langle t + r\rangle^{\rho-1-\frac{d-1}{2}}$. This fact is effectively used for the study of (1.7) in [37, 38], in combination with decay estimates coming from the profile system in a region away from the t-axis.

On the other hand, in our restricted sets ${}^L\Gamma$ or ${}^S\Gamma$, (5.6) is unavailable, and if we use (5.5) when $d = 3$ and (4.5) when $d = 2$ to obtain decay estimates for ∂w, we can only expect an estimate like

$$|\partial w(t,x)|_{\Gamma,s} \leq C\varepsilon\langle t + r\rangle^{\rho}\langle r\rangle^{-\frac{d-1}{2}}\langle ct - r\rangle^{-1}$$

for u with the propagation speed c, where $\Gamma = {}^L\Gamma$ or $\Gamma = {}^S\Gamma$. This estimate only gives a weaker result $|\partial w(t,x)|_{\Gamma,s} \le C\varepsilon\langle t+r\rangle^{\rho-1}\langle r\rangle^{-\frac{d-1}{2}}$ near the t-axis. As we have done in the previous chapter, this causes no serious problem when $d = 2$, because $\rho - 1$ is relatively small compared to $-(d-1)/2 = -1/2$, and we can show $|\partial w(t,x)|_{\Gamma,s} \le C\varepsilon\langle t\rangle^{-\frac{d-1}{2}}$ (or even a faster decay estimate) near the t-axis (see (4.42) and (4.44)). However , when $d = 3$, we only obtain a bound $C\varepsilon\langle t\rangle^{\rho-1}$ on the t-axis, that is insufficient as $\rho - 1 > -1 = -(d-1)/2$.

To turn around this difficulty, we assume the additional conditions (A1) or (A2) in Theorems 2.2 and 2.4, and we will use the following decay estimates for x-derivatives when $\Gamma = {}^L\Gamma$ and t-derivatives when $\Gamma = {}^S\Gamma$. Recall the definition (5.3).

Proposition 5.1. *Let $c > 0$, $0 < T \le \infty$ and w be a smooth solution to* (5.1) *with $w(0) = (\partial_t w)(0) = 0$. Let $\xi \ge 0, \zeta \ge 1$, and $0 < \eta < 1$.*

(1) *If $c = 1$, then there is a positive constant C, being independent of T, such that we have*

$$\langle t+r\rangle^{1-\xi}\langle t-r\rangle^{\zeta}|\partial_x w(t,x)| \le C\mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t), \quad (t,x) \in (0,T) \times \mathbb{R}^3.$$

(2) *For $c > 0$, there is a positive constant C, being independent of T, such that we have*

$$\langle t+r\rangle^{1-\xi}\langle ct-r\rangle^{\zeta}|\partial_t w(t,x)| \le C\mathcal{B}^{{}^S\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t), \quad (t,x) \in (0,T) \times \mathbb{R}^3.$$

Proof. If $r > t/2$ or $t < 1$, then we have $\langle r\rangle^{-1} \le C\langle t+r\rangle^{-1}$, and the two estimates above are immediate consequences of (5.5). Hence we suppose that $r \le t/2$ and $t \ge 1$ from now on. Note that we have $\langle t+r\rangle^{-1} \le Ct^{-1}$ then.

First we prove (1). Let $1 \le k \le 3$. Then we have

$$t\partial_k w = L_k w - x_k\partial_t w. \tag{5.7}$$

We get $\Box(L_k w) = L_k\Psi$, and

$$\begin{aligned}(L_k w)(0,x) &= (t\partial_k w(t,x) + x_k\partial_t w(t,x))|_{t=0} = 0,\\ (\partial_t L_k w)(0,x) &= \big(\partial_k w(t,x) + t\partial_k\partial_t w(t,x) + x_k\partial_t^2 w(t,x)\big)|_{t=0} = x_k\Psi(0,x).\end{aligned}$$

From the definitions (5.2) and (5.3), we obtain

$$\begin{aligned}&\mathcal{A}^{\dagger}_{\zeta-\xi}[0, x_k\Psi(0)] \le \sup_{x\in\mathbb{R}^3} r\langle r\rangle^{\zeta-\xi+\eta}\langle r\rangle^{1-\eta}|\Psi(t,x)| \le \mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t),\\ &\mathcal{B}^{(\Omega,\partial),\dagger}_{\zeta-\xi,\eta,0}[L_k\Psi](t) \le C\mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t).\end{aligned}$$

Since $\mathcal{X}_\zeta(t,x) \le C$ for $r < t/2$ and $\zeta \ge 1$, Lemma 5.1 and (5.4) yield

$$\begin{aligned}|L_k w(t,x)| &\le C\langle t+r\rangle^{\xi-1}\langle t-r\rangle^{1-\zeta}\mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t) \\ &\le C\langle t+r\rangle^{\xi}\langle t-r\rangle^{-\zeta}\mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t).\end{aligned} \tag{5.8}$$

It apparently follows from (5.5) that

$$|x_k\partial_t w(t,x)| \le C\langle t+r\rangle^{\xi}\langle t-r\rangle^{-\zeta}\mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t). \tag{5.9}$$

By (5.7), (5.8) and (5.9), we obtain

$$\langle t+r\rangle|\partial_x w(t,x)| \le Ct|\partial_x w(t,x)| \le C\langle t+r\rangle^{\xi}\langle t-r\rangle^{-\zeta}\mathcal{B}^{{}^L\Gamma,\dagger}_{\zeta-\xi,\eta,1}[\Psi](t),$$

which shows (1) for $r \le t/2$ and $t \ge 1$.

To prove (2), we use

$$t\partial_t w = Sw - x\cdot\nabla_x w, \tag{5.10}$$

instead of (5.7). We have $\Box_c(Sw) = (S+2)\Psi$, and

$$\begin{aligned}(Sw)(0,x) &= (t\partial_t w(t,x) + x\cdot\nabla_x w(t,x))|_{t=0} = 0, \\ (\partial_t Sw)(0,x) &= (\partial_t w(t,x) + t\partial_t^2 w(t,x) + x\cdot\nabla_x(\partial_t w)(t,x))|_{t=0} = 0.\end{aligned}$$

The rest of the proof is similar to the above. This completes the proof. □

5.2 Proof of SDGE

As in Section 4.2, we always suppose that $f(x) = g(x) = 0$ for all $|x| \ge R$ in (1.6) (cf. (4.6)). Then, we have (4.7) and (4.8).

The proof goes a similar way to the two space dimensional case, but even if we set aside the additional assumptions (A1) and (A2), we still need one more step in the decay estimates for solutions to wave equations in three space dimensions.

5.2.1 Proof of Theorem 2.2

We use the same notation as in the proof of Theorem 2.1. For the sake of readability, we describe the notation again: We put $\Gamma = {}^L\Gamma$, and we simply write $|\cdot|_s$, $\|\cdot\|_s$, $[\cdot]_s$, and $\mathscr{T}[\cdot]$ for $|\cdot|_{\Gamma,s}$, $\|\cdot\|_{\Gamma,s}$, $[\cdot]_{\Gamma,1,s}$, and $\mathscr{T}_1[\cdot]$, respectively. We also put

$$\Lambda_{T,R} = \Lambda^{1,1}_{T,R}, \quad \mathcal{L}_{T,R} = \mathcal{L}^1_{T,R} = \mathcal{L}^{1;1.1}_{T,R}$$

and

$$t_0(\sigma) = t_0^1(\sigma;1,1,R) = \max\{\langle r\rangle, \langle t-r\rangle\}, \quad \sigma \le R.$$

We also set $\mathcal{W}_-(t,r) = \mathcal{W}_-(t,r;1) = \min\{\langle r\rangle, \langle t-r\rangle\}$.

Notations and the goal.

For a smooth solution $u = (v, w)$ to the Cauchy problem (1.3) with (1.6) on $[0, T) \times \mathbb{R}^3$, we define

$$\begin{aligned}\mathcal{E}(T) = &\sup_{(t,x)\in[0,T)\times\mathbb{R}^3} \big(\langle t + r\rangle^{3/2} |v(t, x)|_{K+1} + \langle t + r\rangle\langle t - r\rangle^{1-\rho} |\partial_x w(t, x)|\big) \\ &+ \sup_{(t,x)\in[0,T)\times\mathbb{R}^3} \langle t + r\rangle^{-\rho}\langle r\rangle\langle t - r\rangle |\partial w(t, x)|_K, \end{aligned} \tag{5.11}$$

where $K \geq 12$ is a positive integer, and ρ is a sufficiently small positive number.

We suppose that $\mathcal{E}(T) \leq M\varepsilon$ with some large M. We also suppose that ε is relatively small compared to M. Then we have

$$\big|\big(v(t, x), \partial v(t, x), \partial_x w(t, x)\big)\big| \leq CM\varepsilon\langle t + r\rangle^{-1}\langle t - r\rangle^{-\frac{1}{2}}, \tag{5.12}$$

$$\big|\big(v(t, x), \partial u(t, x)\big)\big|_K \leq CM\varepsilon\langle t + r\rangle^{\rho-1}\mathcal{W}_-(t, r)^{-\rho-\frac{1}{2}}. \tag{5.13}$$

As before, our goal is to obtain $\mathcal{E}(T) \leq M\varepsilon/2$.

The energy estimates.

Let $0 \leq s \leq 2K + 1$. We put $|\phi(t, x)|_{-1} = \|\phi(t)\|_{-1} = 0$ for any smooth function ϕ. By the Leibniz formula and the condition (A1), we obtain from (5.12) and (5.13) that

$$\begin{aligned}|F(v, \partial v, \partial_x w)|_s &\leq C\big(|(v, \partial v, \partial_x w)|\, |(v, \partial u)|_s + |(v, \partial u)|_{[s/2]}\, |(v, \partial u)|_{s-1}\big) \\ &\leq CM\varepsilon(1 + t)^{-1}\, |(v, \partial u)|_s + CM\varepsilon(1 + t)^{\rho-1}\, |(v, \partial u)|_{s-1}, \end{aligned} \tag{5.14}$$

since we have $[s/2] \leq K$ for $0 \leq s \leq 2K+1$. The standard energy inequalities for the Klein-Gordon and wave equations yield

$$\begin{aligned}\|(v, \partial u)(t)\|_s \leq C\varepsilon + CM\varepsilon &\int_0^t (1 + \tau)^{-1}(\|(v, \partial u)(\tau)\|_s)\, d\tau \\ &+ CM\varepsilon \int_0^t (1 + \tau)^{\rho-1}(\|(v, \partial u)(\tau)\|_{s-1})\, d\tau.\end{aligned}$$

Using the Gronwall lemma, we obtain

$$\|(v, \partial u)(t)\|_s \leq C\varepsilon(1 + t)^{CM\varepsilon+s\rho}$$

by mathematical induction in s. Then we have

$$\langle t\rangle^{\rho}\|(v, \partial u)(t)\|_{2K} + \|(v, \partial u)(t)\|_{2K+1} \leq C\varepsilon\langle t\rangle^{\delta}, \tag{5.15}$$

where

$$\delta = 2(K + 1)\rho,$$

provided that ε is small enough to satisfy $CM\varepsilon \leq \rho$. Note that we can assume δ to be sufficiently small by choosing small ρ.

Remark 11. (5.14) is the only point where we explicit use the assumption (A1). If we assume that F has also higher-order nonlinear terms $F^{\mathrm{c}}(v, \partial u)$, depending also on $\partial_t u$ and satisfying $F(v, \partial u) = \mathcal{O}\big(|v|^3 + |\partial u|^3\big)$ near $(v, \partial u) = (0, 0)$, then

$$\begin{aligned} |F^{\mathrm{c}}(v, \partial u)|_s &\le C|(v, \partial u)|_{[s/2]}^2 |(v, \partial u)|_s \\ &\le CM^2\varepsilon^2(1+t)^{2\rho-2}|(v, \partial u)|_s \\ &\le CM\varepsilon(1+t)^{-1}|(v, \partial u)|_s, \end{aligned}$$

and (5.14) remains true. Consequently, all the estimates in this step are valid. Contribution by F^{c} in the sequel can be easily treated.

Rough decay estimates.

Using Lemma 3.1 , we obtain from (5.15) that

$$|\partial w(t,x)|_{2K-1} \le C\varepsilon \langle r\rangle^{-1}\langle t\rangle^{\delta}. \tag{5.16}$$

By (5.14) and (5.15), we get

$$\|F(v, \partial u)\|_{2K+1} \le CM\varepsilon^2\langle t\rangle^{\delta-1} \le C\varepsilon\langle t\rangle^{\delta-1}.$$

Hence, it follows from Lemma 3.13, we obtain

$$|v(t,x)|_{2K-3} \le C\varepsilon\langle t+r\rangle^{\delta-\frac{3}{2}}. \tag{5.17}$$

Let $s \le 2K-4$. By (5.16) and (5.17), we get

$$\begin{aligned} \langle r\rangle|F(v,\partial u)|_s &\le C\langle r\rangle(|v|_{K+1}|v|_{2K-3} + |v|_{K+1}|\partial w|_s + |\partial w|_K|v|_{2K-3}) \\ &\quad + C\langle r\rangle|\partial w|_K|\partial w|_{2K-4} \qquad\qquad (5.18) \\ &\le CM\varepsilon^2\langle t+r\rangle^{\delta+\rho-1}\mathcal{W}_-(t,r)^{-1} + CM\varepsilon\langle t+r\rangle^{-\frac{3}{2}}\langle r\rangle|\partial w|_s. \end{aligned}$$

If we use (5.16) again to estimate the last term, we obtain

$$\langle r\rangle|F(v,\partial u)|_{2K-4} \le CM\varepsilon^2\langle t+r\rangle^{\delta+\rho-1}\mathcal{W}_-(t,r)^{-\rho-\frac{1}{2}},$$

which yields

$$\mathcal{B}^{\Gamma,\dagger}_{1-(\frac{1}{2}+\delta),\frac{1}{2}-\rho,2K-4}[F](t) \le CM\varepsilon^2 \le C\varepsilon.$$

It follows from (5.5) with $\xi = (1/2)+\delta$, $\zeta = 1$ and $\eta = (1/2)-\rho$ that

$$|\partial w(t,x)|_{2K-5} \le C\varepsilon\langle t+r\rangle^{\delta+\frac{1}{2}}\langle r\rangle^{-1}\langle t-r\rangle^{-1}. \tag{5.19}$$

By (5.18) and (5.19) we get

$$\langle r\rangle|F(v,\partial u)|_{2K-5} \le CM\varepsilon^2\langle t+r\rangle^{\delta+\rho-1}\mathcal{W}_-(t,r)^{-1}.$$

Since we have

$$\mathcal{B}^{\Gamma,\dagger}_{1+\rho-(\delta+3\rho),\rho,2K-5}[F](t) \le CM\varepsilon^2 \le C\varepsilon,$$

(5.4) and (5.5) with $\xi = \delta + 3\rho$, $\zeta = 1 + \rho$ and $\eta = \rho$, lead to

$$|w(t,x)|_{2K-5} \le C\varepsilon\langle t+r\rangle^{\delta+3\rho-1}\langle t-r\rangle^{-\rho} \le C\varepsilon\langle t+r\rangle^{2\delta-1}, \tag{5.20}$$

$$\begin{aligned} |\partial w(t,x)|_{2K-6} &\le C\varepsilon\langle t+r\rangle^{\delta+3\rho}\langle r\rangle^{-1}\langle t-r\rangle^{-1-\rho} \\ &\le C\varepsilon\langle t+r\rangle^{2\delta}\langle r\rangle^{-1}\langle t-r\rangle^{-1}. \end{aligned} \tag{5.21}$$

Better estimates for the Klein-Gordon components.

We will make use of Lemma 3.15 to obtain the better decay estimates for the Klein-Gordon components. We put

$$\widetilde{v}_j = v_j - m_j^{-2}F_j^{(\mathrm{w})}(\partial w).$$

Since, as before, each term in $F_j(v,\partial v,\partial w) - F_j^{(\mathrm{w})}(\partial w)$ has at least one factor of the form $\partial^\alpha v_k$ with $|\alpha| \le 1$, it follows from (5.13) and (5.17) that

$$\begin{aligned} |F_j(v,\partial u) - F_j^{(\mathrm{w})}(\partial w)|_{2K-6} &\le C|v|_{2K-5}|(v,\partial u)|_{2K-6} \\ &\le C\varepsilon\langle t\rangle^{\delta-\frac{3}{2}}|(v,\partial u)|_{2K-6} \end{aligned}$$

Hence, by (5.15), We obtain

$$\begin{aligned} \|F_j(v,\partial u) - F_j^{(\mathrm{w})}(\partial w)\|_{2K-6} &\le C\||v|_{2K-5}\|_{L^\infty}\|(v,\partial u)\|_{2K-6} \\ &\le C\varepsilon^2(1+t)^{2\delta-\frac{3}{2}} \end{aligned}$$

From (5.13) and (5.15), we also have

$$\||(v,\partial u)|_{K-2}^2|(v,\partial u)|_{2K-5}\|_{L^2} \le CM^2\varepsilon^3(1+t)^{2\rho+\delta-2}.$$

It follows from (5.20) and (5.21) that

$$\begin{aligned} &\|\langle t+|\cdot|\rangle^{-1}[w]_{2K-5}|\partial w|_{2K-6}\|_{L^2} \\ &\quad \le C\varepsilon^2\left(\int_0^{t+R}|\langle t+r\rangle^{4\delta-1}\langle r\rangle^{-2}\langle t-r\rangle^{-1}|^2 r^2\,dr\right)^{\frac{1}{2}} \\ &\quad \le C\varepsilon^2(1+t)^{4\delta-2}. \end{aligned}$$

Gathering the above estimates, we see from Lemma 3.15 that

$$\|(\Box + m_j^2)\widetilde{v}_j\|_{2K-6} \le C\varepsilon^2(1+t)^{2\delta-\frac{3}{2}}, \tag{5.22}$$

which, together with Lemma 3.13, implies

$$|\widetilde{v}(t,x)|_{2K-10} \le C\varepsilon\langle t+r\rangle^{-\frac{3}{2}}.$$

Since (5.21) implies

$$\begin{aligned} |v(t,x)-\widetilde{v}(t,x)|_{2K-10} &\le |F_{\mathrm{K}}^{(\mathrm{w})}(\partial w)|_{2K-10} \le C|\partial w|_{2K-10}^2 \\ &\le C\varepsilon^2 \langle t+r\rangle^{4\delta-2} \\ &\le C\varepsilon\langle t+r\rangle^{-\frac{3}{2}}, \end{aligned}$$

we obtain

$$|v(t,x)|_{2K-10} \le C\varepsilon\langle t+r\rangle^{-\frac{3}{2}}. \tag{5.23}$$

Better decay estimates for the wave components.

Let $(t,x)\in\Lambda_{T,R}$, and $\sigma=r-t$ for a while. Then we have $(t,\sigma,\omega)\in\mathcal{L}_{T,R}$. Recall that t,r and $\langle t+r\rangle$ are equivalent to each other in $\Lambda_{T,R}$. We apply the arguments in Section 3.2 with

$$\phi=w,\ c=1,\ \Psi=F_{\mathrm{W}}^{(\mathrm{w})},\ \Theta=F_{\mathrm{W}}(v,\partial u)-F_{\mathrm{W}}^{(\mathrm{w})}(\partial w)$$

in (3.16) (cf. (WKG) in Section 3.2). Then Φ and $\Phi^{(\alpha)}$ in Section 3.2 are

$$\Phi(t,x)=\mathcal{D}\big(rw(t,x)\big),\quad \Phi^{(\alpha)}(t,x)=\mathcal{D}\big(r\Gamma^\alpha w(t,x)\big).$$

with $\mathcal{D}=(\partial_r-\partial_t)/2$. By (5.20) and (5.21), we get

$$[w(t,x)]_{2K-5} \le C\varepsilon\langle t+r\rangle^{2\delta-1}. \tag{5.24}$$

Let $s\le 2K-11$. By the definition of $F_{\mathrm{W}}^{(\mathrm{w})}(\partial w)$, each term in $\Theta = F_{\mathrm{W}}(v,\partial u)-F_{\mathrm{W}}^{(\mathrm{w})}(\partial w)$ has at least one factor like $\partial^\alpha v_k$ with $|\alpha|\le 1$. Therefore, we obtain from (5.23) that

$$\begin{aligned} \mathscr{T}[r|\Theta|_s] &\le C\mathscr{T}[r|v|_{2K-10}(|v|_{2K-10}+|\partial w|_s)] \\ &\le C\left(\varepsilon^2 t^{-2}+\varepsilon t^{-\frac{1}{2}}\mathscr{T}[|\partial w|_s]\right). \end{aligned} \tag{5.25}$$

By (5.20) and (5.24), we obtain

$$\mathscr{T}\left[\langle t-r\rangle^{-1}[w(t,x)]_{2K-10}^2\right]=\mathcal{O}\left(\varepsilon^2 t^{4\delta-2}\langle\sigma\rangle^{-1}\right), \tag{5.26}$$

$$\mathscr{T}\left[t^{-1}|w(t,x)|_{2K-9}\right]=\mathcal{O}\left(\varepsilon t^{2\delta-2}\right). \tag{5.27}$$

Let $\mathcal{R}$ be given by (3.25). Recall that $t_0(\sigma)$ is equivalent to $\langle\sigma\rangle$ (cf. (3.9)). Gathering the above estimates, (5.25), (5.26) and (5.27), with the help of (3.9), we get from (5.21) that

$$\int_{t_0(\sigma)}^\infty \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\,d\tau \le C\varepsilon\langle\sigma\rangle^{2\delta-1}. \tag{5.28}$$

Recall the $*$-notation (3.23). By (3.21) and (3.29), we also obtain

$$\begin{aligned} |\Phi^*(t,\sigma,\omega)|, |\Phi^{(\alpha)*}(t,\sigma,\omega)| &\le C\mathscr{T}\big[t\langle t-r\rangle^{-1}[w]_{2K-10}\big] \\ &\le C\varepsilon t^{2\delta}\langle\sigma\rangle^{-1}. \end{aligned} \tag{5.29}$$

Especially, we have

$$\begin{aligned} \left|\Phi^*\big(t_0(\sigma),\sigma,\omega\big)\right|, \left|\Phi^{(\alpha)*}\big(t_0(\sigma),\sigma,\omega\big)\right| &\le C\varepsilon t_0(\sigma)^{2\delta}\langle\sigma\rangle^{-1} \\ &\le C\varepsilon\langle\sigma\rangle^{2\delta-1}. \end{aligned} \tag{5.30}$$

It follows from Lemma 3.7, (5.24), (5.28) and (5.30) that

$$r|\partial w(t,x)| \le C\varepsilon\langle t-r\rangle^{2\delta-1}. \tag{5.31}$$

From (5.29), we get $|\Phi^*(t,\sigma,\omega)| \le C\varepsilon$, which implies

$$\|\mathcal{G}\big(\omega,\Phi^*(t,\sigma,\omega)\big)\| \le 2C_0\varepsilon$$

with some positive constant C_0, where $\mathcal{G}(\omega,Y)$ is given by (3.32). Let $\mathcal{R}^{(\alpha)}$ be given by (3.34). Similarly to (5.28) we obtain

$$\begin{aligned} \sum_{|\alpha|\le s}\int_{t_0(\sigma)}^t \left(\frac{t}{\tau}\right)^{C_0\varepsilon} &|\mathscr{T}[\mathcal{R}^{(\alpha)}](\tau,\sigma,\omega)|d\tau \\ \le Ct^{C_0\varepsilon}&\int_{t_0(\sigma)}^t \tau^{-C_0\varepsilon-1}\big(\tau\mathscr{T}[|\partial w|_{s-1}](t,\sigma,\omega)\big)^2\,d\tau \\ &+ C\varepsilon t^{C_0\varepsilon}\langle\sigma\rangle^{2\delta-1}. \end{aligned} \tag{5.32}$$

It follows from Lemma 3.8, (5.24), (5.30), and (5.32) that

$$\begin{aligned} t|\partial w(t,x)|_s \le Ct^{C_0\varepsilon}&\int_{t_0(\sigma)}^t \tau^{-C_0\varepsilon-1}\big(\tau\mathscr{T}[|\partial w|_{s-1}](t,\sigma,\omega)\big)^2\,d\tau \\ &+ C\varepsilon t^{C_0\varepsilon}\langle\sigma\rangle^{2\delta-1}. \end{aligned}$$

By induction, we can show that

$$t|\partial w(t,x)|_s \le C\varepsilon t^{2^{s-1}C_0\varepsilon}\langle t-r\rangle^{2\delta-1}.$$

for $1 \le s \le 2K-11$. Especially, if we choose sufficiently small ε, we have

$$t|\partial w(t,x)|_{2K-11} \le C\varepsilon t^{\nu}\langle t-r\rangle^{2\delta-1} \tag{5.33}$$

with $\nu = \rho/4$.

From here, we consider $(t,x) \in [0,T)\times\mathbb{R}^3$. If $r \ge t+R$, then we have $w(t,x) = 0$. On the other hand, if $t < 2$ or $t > 2r$, then we have $\langle t+r\rangle \le C\langle t-r\rangle$. Therefore, by (5.21), we get

$$|\partial w(t,x)|_{2K-11} \le C\varepsilon\langle r\rangle^{-1}\langle t-r\rangle^{2\delta-1}, \quad (t,x)\in\Lambda_{T,R}^{\mathrm{c}}.$$

Combining this estimate in $\Lambda_{T,R}^{\mathrm{c}}$ with the estimate (5.33) in $\Lambda_{T,R}$, we get

$$|\partial w(t,x)|_{2K-11} \le C\langle t+r\rangle^{\nu}\langle r\rangle^{-1}\langle t-r\rangle^{2\delta-1}, \quad (t,x) \in [0,T)\times\mathbb{R}^3. \tag{5.34}$$

We use (5.23) and (5.34) to obtain

$$\begin{aligned}\langle r\rangle|F(v,\partial u)|_{2K-11} &\le C\langle r\rangle(|v|_{2K-10}^2 + |\partial w|_{2K-11}^2) \\ &\le C\varepsilon^2\big(\langle t+r\rangle^{-2} + \langle t+r\rangle^{2\nu}\langle r\rangle^{-1}\langle t-r\rangle^{4\delta-2}\big) \\ &\le C\varepsilon^2\langle t+r\rangle^{2\nu-1}\mathcal{W}_-(t,r)^{-1-2\nu}.\end{aligned} \tag{5.35}$$

Since we have $\mathcal{B}^{\Gamma,\dagger}_{(1+\nu)-4\nu,\nu,2K-11}[F(v,\partial u)] \le C\varepsilon^2$, Lemmas 5.1 and 5.2 with $\xi = 4\nu$, $\zeta = 1+\nu$ and $\eta = \nu$, lead to

$$|w(t,x)|_{2K-11} \le C\varepsilon\langle t+r\rangle^{\rho-1}\langle t-r\rangle^{-\nu}, \tag{5.36}$$

$$|\partial w(t,x)|_{2K-12} \le C\varepsilon\langle t+r\rangle^{\rho}\langle r\rangle^{-1}\langle t-r\rangle^{-1-\nu}, \tag{5.37}$$

as $\nu = \rho/4$. It also follows from Proposition 5.1 that

$$|\partial_x w(t,x)| \le C\varepsilon\langle t+r\rangle^{\rho-1}\langle t-r\rangle^{-1-\nu}, \tag{5.38}$$

which implies

$$|\partial_x w(t,x)| \le C\varepsilon\langle t+r\rangle^{-1}\langle t-r\rangle^{\rho-1-\nu}, \quad (t,x) \in \Lambda_{T,R}^{\mathrm{c}}. \tag{5.39}$$

Now we return to the argument using the profile system again. Let $(t,x) \in \Lambda_{T,R}$ in the rest of this step. (5.36) and (5.37) yield

$$[w(t,x)]_1 \le C\varepsilon\langle t+r\rangle^{\rho-1}, \tag{5.40}$$

and

$$|\Phi^*\big(t_0(\sigma),\sigma,\omega\big)| \le C\varepsilon\langle\sigma\rangle^{\rho-1}. \tag{5.41}$$

Using also (5.25), we obtain

$$|\mathcal{R}(t,x)| \le \langle t-r\rangle^{-1}[w]_1^2 + t^{-1}[w]_2 + t|\Theta| \le C\varepsilon t^{\rho-\frac{3}{2}}\langle t-r\rangle^{-\frac{1}{2}},$$

which leads to

$$\int_t^\infty \tau^\theta\mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\,d\tau \le C\varepsilon^{\rho+\theta-\frac{1}{2}}\langle\sigma\rangle^{-\frac{1}{2}}, \quad t \ge t_0(\sigma) \tag{5.42}$$

for $0 \le \theta \ll 1$. Especially, we have

$$\int_{t_0(\sigma)}^\infty \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\,d\tau \le C\varepsilon\langle\sigma\rangle^{\rho-1}. \tag{5.43}$$

It follows from (5.40), (5.41), (5.43) and Lemma 3.7 that

$$t|\partial w(t,x)| \le C\varepsilon\langle t-r\rangle^{\rho-1}, \quad (t,x) \in \Lambda_{T,R},$$

which combined with (5.39), implies

$$|\partial_x w(t,x)| \le C\varepsilon\langle t+r\rangle^{-1}\langle t-r\rangle^{\rho-1}, \quad (t,x) \in [0,T)\times\mathbb{R}^3. \tag{5.44}$$

Conclusion.

Because $K+1 \le 2K-10$ and $K \le 2K-12$ for $K \ge 12$, from (5.23), (5.37) and (5.44), we get

$$\mathcal{E}(T) \le C_0\varepsilon$$

with a positive constant C_0, provided the ε is sufficiently small. If M is sufficiently large to satisfy $M \ge 2C_0$, we obtain the desired result $\mathcal{E}(T) \le M\varepsilon/2$. This completes the proof. □

5.2.2 Proof of Theorem 2.4

We use the same notation as in the proof of Theorem 2.3. For the sake of readability, we describe the notation again: We put $\Gamma = {}^S\Gamma$, and we simply write $|\cdot|_s$, $\|\cdot\|_s$ and $\mathscr{T}_I[\cdot]$ for $|\cdot|_{\Gamma,s}$, $\|\cdot\|_{\Gamma,s}$ and $\mathscr{T}_{c_I}[\cdot]$, respectively. We also write

$$[\phi(t,x)]_{I,s} := [\phi(t,x)]_{\Gamma,c_I,s} = |\phi(t,x)|_{\Gamma,s} + \langle c_I t - r\rangle|\partial\phi(t,x)|_{\Gamma,s-1}$$

for a smooth function ϕ and an integer $s \ge 1$. We put

$$\Lambda_{T,R} = \Lambda_{T,R}^{c_1,c_P}, \quad \mathcal{L}_{T,R}^I = \mathcal{L}_{T,R}^{c_I} = \mathcal{L}_{T,R}^{c_I;c_1,c_P},$$

and

$$t_0^I(\sigma) = t_0^{c_I}(\sigma; c_1, c_P, R).$$

We also set

$$\mathcal{W}_-(t,r) = \mathcal{W}_-(t,r;c_1,\dots,c_P) = \min\bigl\{\langle r\rangle, \langle r - c_1 t\rangle, \dots, \langle r - c_P t\rangle\bigr\}.$$

Then we have

$$\langle r\rangle^{-1}\langle c_I t - r\rangle^{-1} \le C\langle r + r\rangle^{-1}\mathcal{W}_-(t,r)^{-1}, \quad (t,x) \in [0,\infty)\times\mathbb{R}^3,\ 1 \le I \le P.$$

Notations and the goal.

The proof is quite similar to that for Theorem 2.2, and we concentrate on different points. Let $u = (u^I)_{1\le I\le P} \in C^\infty\bigl([0,T)\times\mathbb{R}^3;\mathbb{R}^N\bigr)$ be a solution to (1.4) with (1.6), where $N = N^1 + \cdots + N^P$.

Let M be sufficiently large, and ε be sufficiently small compared to M. Our goal is, as before, to show that $\mathcal{E}(T) \le M\varepsilon$ implies $\mathcal{E}(T) \le M\varepsilon/2$, where

$$\begin{aligned}\mathcal{E}(T) = &\sup_{(t,x)\in[0,T)\times\mathbb{R}^3} \sum_{I=1}^P \langle t+r\rangle\langle c_I t - r\rangle^{1-\rho}|\partial_t u^I(t,x)| \\ &+ \sup_{(t,x)\in[0,T)\times\mathbb{R}^3} \sum_{I=1}^P \langle t+r\rangle^{-\rho}\langle r\rangle\langle c_I t - r\rangle|\partial u^I(t,x)|_K\end{aligned} \tag{5.45}$$

with a large integer $K \ge 4$ and a sufficiently small positive number ρ.

The energy estimates and rough decay estimates.

Using the condition (A2), we get

$$\begin{aligned}|F(\partial_t u)|_s &\le C|\partial_t u||\partial u|_s + |\partial u|_K|\partial u|_{s-1}\\ &\le CM\varepsilon\big((1+t)^{-1}|\partial u|_s + (1+t)^{\rho-1}|\partial u|_{s-1}\big)\end{aligned}$$

for $s \le 2K-1$, where we put $|\phi|_{-1} = 0$ for a smooth function ϕ. Proceeding as before, we get

$$\|\partial u(t)\|_{2K+1} \le C\varepsilon\langle t\rangle^{\delta} \tag{5.46}$$

with $\delta = 2(K+1)\rho$, provided that ε is small enough to satisfy $CM\varepsilon \le \rho$.

Using lemma 3.1 and (5.46), we obtain

$$|\partial u(t,x)|_{2K-1} \le C\varepsilon\langle t\rangle^{\delta}\langle r\rangle^{-1}.$$

From this we get

$$\langle r\rangle|F(\partial u)|_{2K-1} \le C\langle r\rangle|\partial u|_K|\partial u|_{2K-1} \le CM\varepsilon^2\langle t+r\rangle^{\delta+\rho-1}\mathcal{W}_-(t,r)^{-1},$$

which leads to

$$|u^I(t,x)|_{2K-1} \le C\varepsilon\langle t+r\rangle^{2\delta-1}, \tag{5.47}$$

$$|\partial u^I(t,x)|_{2K-2} \le C\varepsilon\langle t+r\rangle^{2\delta}\langle r\rangle^{-1}\langle c_I t - r\rangle^{-1} \tag{5.48}$$

for $1 \le I \le P$, similarly to (5.20) and (5.21).

Better decay estimates.

For a while, we suppose $(t,x) \in \Lambda_{T,R}$. Recall that t, r and $\langle t+r\rangle$ are equivalent to each other in $\Lambda_{T,R}$. We put

$$\sigma_I := \sigma_{c_I} = r - c_I t.$$

Then we have $(t,\sigma_I,\omega) \in \mathcal{L}^I_{T,R}$. For each $I \in \{1,\dots,P\}$, we apply the arguments in Section 3.2 with

$$\phi = u^I,\ c = c_I,\ \Psi = {}^*F^I,\ \Theta - F^I(\partial u) - {}^*F^I(\partial u^I)$$

in (3.16) (cf. (MSW) in Section 3.2). In this case, we have

$$\Phi(t,x) = \mathcal{D}_{c_I}\big(ru^I(t,x)\big),\ \Phi^{(\alpha)}(t,x) = \mathcal{D}_{c_I}\big(r\Gamma^\alpha u^I(t,x)\big),\quad |\alpha| \le 2K-3.$$

We use the $*$-notation with $* = *_{c_I}$. By (5.47) and (5.48), we get

$$[u^I(t,x)]_{I,2K-1} \le C\varepsilon\langle t+r\rangle^{2\delta-1}, \tag{5.49}$$

which, together with (5.47), implies

$$\mathscr{T}_I\big[\langle c_I t - r\rangle^{-1}[u^I(t,x)]^2_{I,2K-2} + t^{-1}|u^I(t,x)|_{2K-1}\big] = \mathcal{O}\left(\varepsilon t^{2\delta-\frac{3}{2}}\langle\sigma_I\rangle^{-\frac{1}{2}}\right). \tag{5.50}$$

It follows from (5.49) that

$$|\Phi^*(t,\sigma_I,\omega)|, |\Phi^{(\alpha)*}(t,\sigma_I,\omega)| \le C\varepsilon t^{2\delta}\langle\sigma_I\rangle^{-1}. \tag{5.51}$$

Especially we have

$$|\Phi^*(t_0^I(\sigma_I),\sigma_I,\omega)|, |\Phi^{(\alpha)*}(t_0^I(\sigma_I),\sigma_I,\omega)| \le C\varepsilon\langle\sigma_I\rangle^{2\delta-1}. \tag{5.52}$$

As before, we put

$$\mathcal{W}_-^I(t,x) = \min_{0\le J\le P; J\ne I}\langle r - c_J t\rangle$$

with $c_0 = 0$. By the definition of ${}^*F^I(\partial u^I)$, we see that $\Theta = F^I(\partial u) - {}^*F^I(\partial u^I)$ is a linear combination of $(\partial_a u_j^J)(\partial_b u_k^{J'})$ with $(J,J') \ne (I,I)$. For $s \le 2K-2$, since (5.48) yields

$$\begin{aligned}\mathscr{T}_I\Big[t\sum_{(J,J')\ne(I,I)}|\partial u^J|_s|\partial u^{J'}|_s\Big] &\le C\varepsilon^2\langle t+r\rangle^{4\delta-1}\mathcal{W}_-^I(t,r)^{-1}\mathcal{W}_-(t,r)^{-1}\\ &\le C\varepsilon^2 t^{5\delta-1}\mathcal{W}_-^I(t,r)^{-1-\delta},\end{aligned}$$

we obtain

$$t|\Theta(t,x)|_s \le C\varepsilon^2 t^{5\delta-1}\mathcal{W}_-^I(t,r)^{-1-\delta}. \tag{5.53}$$

Let $\mathcal{R}$ be given by (3.25). Using Lemma 4.3 for the estimate of the integral of $\mathscr{T}_I[t\Theta](t,\sigma_I,\omega)$, we obtain

$$\int_{t_0^I(\sigma_I)}^\infty \mathscr{T}_I[\mathcal{R}](\tau,\sigma_I,\omega)\,d\tau \le C\varepsilon\langle\sigma_I\rangle^{5\delta-1}. \tag{5.54}$$

It follows from Lemma 3.7, (5.49), (5.52) and (5.54) that

$$t|\partial u^I(t,x)| \le C\varepsilon\langle c_I t - r\rangle^{5\delta-1}, \tag{5.55}$$

in place of (5.31).

Let $\mathcal{G}(\omega,Y)$ be given by (3.32). Then (5.51) leads to

$$\|\mathcal{G}\big(\omega,\Phi^*(t,\sigma_I,\omega)\big)\| \le 2c_I^2 C_0\varepsilon$$

for some $C_0 > 0$. Let $1 \le s \le 2K-3$ and $\mathcal{R}^{(\alpha)}$ be given by (3.34). Similarly to (5.54), we obtain

$$\begin{aligned}&\sum_{|\alpha|\le s}\int_{t_0^I(\sigma)}^t\left(\frac{t}{\tau}\right)^{C_0\varepsilon}|\mathscr{T}_I[\mathcal{R}^{(\alpha)}](\tau,\sigma_I,\omega)|\,d\tau\\ &\quad\le Ct^{C_0\varepsilon}\int_{t_0^I(\sigma)}^t\tau^{-C_0\varepsilon-1}\big(\tau\mathscr{T}_I[|\partial u^I|_{s-1}](\tau,\sigma_I,\omega)\big)^2\,d\tau\\ &\qquad+C\varepsilon t^{C_0\varepsilon}\langle\sigma_I\rangle^{5\delta-1}.\end{aligned} \tag{5.56}$$

It follows from Lemma 3.8, (5.49), (5.52) and (5.56) that

$$
\begin{aligned}
t|\partial u^I(t,x)|_s \le{}& Ct^{C_0\varepsilon}\int_{t_0^I(\sigma)}^t \tau^{-C_0\varepsilon-1}\big(\tau\mathscr{T}_I[|\partial u^I|_{s-1}](\tau,\sigma_I,\omega)\big)^2\,d\tau \\
&+ C\varepsilon t^{C_0\varepsilon}\langle\sigma_I\rangle^{5\delta-1}.
\end{aligned}
$$

By induction, we can show that

$$
t|\partial u^I(t,x)|_{2K-3} \le C\varepsilon t^{\nu}\langle c_I t - r\rangle^{5\delta-1} \tag{5.57}
$$

with $\nu=\rho/8$, in place of (5.33), provided the ε is sufficiently small. Here we have chosen smaller ν than before for the later purpose.

Combining (5.48) in $\Lambda_{T,R}^{\mathrm{c}}$, and (5.57) in $\Lambda_{T,R}$, we obtain

$$
|\partial u^I(t,x)|_{2K-3} \le C\varepsilon\langle t+r\rangle^{\nu}\langle r\rangle^{-1}\langle c_I t - r\rangle^{5\delta-1}, \quad (t,x)\in[0,T)\times\mathbb{R}^3, \tag{5.58}
$$

which yields

$$
\begin{aligned}
\langle r\rangle|F(\partial u)|_{2K-3} &\le C\varepsilon^2\langle t+r\rangle^{2\nu-1}\mathcal{W}_-(t,r)^{10\delta-2} \\
&\le C\varepsilon^2\langle t+r\rangle^{\frac{\rho}{2}-1-2\nu}\mathcal{W}_-(t,r)^{\nu-1},
\end{aligned}
$$

as $\nu=\rho/8$. Since $\mathcal{B}^{\Gamma,\dagger}_{(1+\nu)-\frac{\rho}{2},\nu,2K-3}[F(\partial u)] \le C\varepsilon^2$, in correspondence with (5.36), (5.37), and (5.38), we obtain

$$
|u^I(t,x)|_{2K-3} \le C\varepsilon\langle t+r\rangle^{\frac{\rho}{2}-1}\langle c_I t - r\rangle^{-\nu}, \tag{5.59}
$$

$$
|\partial u^I(t,x)|_{2K-4} \le C\varepsilon\langle t+r\rangle^{\frac{\rho}{2}}\langle r\rangle^{-1}\langle c_I t - r\rangle^{-1-\nu}, \tag{5.60}
$$

as well as

$$
|\partial_t u^I(t,x)|_{2K-4} \le C\varepsilon\langle t+r\rangle^{\frac{\rho}{2}-1}\langle c_I t - r\rangle^{-1-\nu} \tag{5.61}
$$

for $(t,x)\in[0,T)\times\mathbb{R}^3$.

Now we return to the argument using the profile system again. Let $(t,x)\in\Lambda_{T,R}$. From (5.59) and (5.60), we obtain

$$
[u^I(t,x)]_{I,1} \le C\varepsilon\langle t+r\rangle^{\frac{\rho}{2}-1}\langle c_I t - r\rangle^{-\nu}, \tag{5.62}
$$

$$
\big|\Phi^*\big(t_0^I(\sigma_I),\sigma_I,\omega\big)\big| \le C\varepsilon\langle\sigma_I\rangle^{\frac{\rho}{2}-1-\nu}, \tag{5.63}
$$

and

$$
\mathscr{T}_I\big[\langle c_I t - r\rangle^{-1}[u^I(t,x)]_{I,2}^2 + t^{-1}|u^I(t,x)|_1\big] \le C\varepsilon t^{\frac{\rho}{2}-\frac{3}{2}}\langle\sigma_I\rangle^{-\frac{1}{2}}. \tag{5.64}
$$

It follows from (5.60) that

$$
\mathscr{T}_I\Big[t\sum_{(J,J')\neq(I,I)}|\partial u^J||\partial u^{J'}|\Big] \le C\varepsilon^2 t^{\rho-1}\mathcal{W}_-^I(t,r)^{-1-\nu},
$$

which yields

$$t|\Theta(t,x)| \le C\varepsilon^2 t^{\rho-1}\mathcal{W}_-^I(t,r)^{-1-\nu}. \tag{5.65}$$

Similarly to (5.54), we obtain

$$\begin{aligned}\int_t^\infty \tau^\theta \mathscr{T}_I[\mathcal{R}](\tau,\sigma,\omega)\,d\tau &\le C\varepsilon t^{\frac{\rho-1}{2}+\theta}\langle\sigma_I\rangle^{-\frac{1}{2}} + C\varepsilon t^{\rho-1\theta} \\ &\le C\varepsilon t^{\frac{\rho-1}{2}+\theta}\langle\sigma_I\rangle^{\frac{\rho-1}{2}}, \quad t \ge t_0^I(\sigma_I)\end{aligned} \tag{5.66}$$

for $0 \le \theta \ll 1$. Especially we have

$$\int_{t_0^I(\sigma_I)}^\infty \mathscr{T}_I[\mathcal{R}](\tau,\sigma,\omega)d\tau \le C\varepsilon\langle\sigma_I\rangle^{\rho-1}. \tag{5.67}$$

From (5.62), (5.63), (5.67) and Lemma 3.7 we obtain

$$t|\partial u^I(t,x)| \le C\varepsilon\langle c_I t - r\rangle^{\rho-1}, \quad (t,x) \in \Lambda_{T,R}. \tag{5.68}$$

Combining (5.61) in $\Lambda_{T,R}^{\mathrm{c}}$ and (5.68) in $\Lambda_{T,R}$, we obtain

$$|\partial_t u(t,x)| \le C\varepsilon\langle t+r\rangle^{-1}\langle c_I t - r\rangle^{\rho-1}, \quad (t,x) \in [0,T) \times \mathbb{R}^3. \tag{5.69}$$

Finally, we obtain the desired result $\mathcal{E}(T) \le M\varepsilon/2$ from (5.60) and (5.69), provided that M is sufficiently large and ε is sufficiently small. This completes the proof. □

5.3 Proof of the asymptotic behavior

In this section, we will prove the three space dimensional part of the Theorems 2.5 and 2.6. For $f, g \in C_0^\infty$ and sufficiently small ε, (1.3) (resp. (1.4)) with (1.6) admits a global classical solution $u = (v,w)$ (resp. $u = (u^I)_{1\le I\le P}$) by Theorem 2.2 (resp. Theorem 2.4). By its proof, we find that $\mathcal{E}(\infty) \le M\varepsilon$ with some $M > 0$, where $\mathcal{E}$ is given by (5.11) (resp. (5.45)), and we find that all the estimates in the proof of Theorem 2.2 (resp. Theorem 2.4) are valid with $T = \infty$.

We will apply Lemma 3.12 to obtain the asymptotic behavior of solutions to wave equations. Note that (3.39) is satisfied for the applications below, because of the KMS conditions for $(d,p) = (3,2)$. Hence we can take $\theta = 0$ in our applications below.

5.3.1 Proof of Theorem 2.5

We use the same notations as in the proof of Theorem 2.2. Firsrly we consider the Klein-Gordon component. By $\mathcal{E}(\infty) \le M\varepsilon$ and (5.15), we get

$$\left\| F_j^{(\mathrm{w})}(\partial w)(t) \right\|_1 \le CM\varepsilon^2 t^{\delta-1}, \quad t \to \infty.$$

Hence, with the help of (5.22), Lemma 3.16 implies the desired result for the Klein-Gordon component v.

Secondly we consider the wave component. Given $0 < \kappa \ll 1$, we take $\rho = \kappa/2$. Then (5.40) and (5.42) imply

$$\begin{aligned} r[w(t,x)]_1 &\leq C\varepsilon\langle t+r\rangle^{\frac{\kappa}{2}}, \qquad (t,x) \in [0,\infty)\times\mathbb{R}^3, \\ \int_t^\infty \tau^\theta \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\,d\tau &\leq C\varepsilon t^{\frac{\kappa-1}{2}+\theta}\langle\sigma\rangle^{-\frac{1}{2}} \\ &\leq C\varepsilon t^{\frac{\kappa-1}{2}+\theta}\langle\sigma\rangle^{\frac{\kappa-1}{2}}, \quad (t,\sigma,\omega) \in \mathcal{L}^{1;1,1}_{\infty,R},\ 0 \leq \theta \ll 1. \end{aligned}$$

Apparently we have

$$\frac{\kappa}{2} - 1 \leq \frac{\kappa-1}{2} - \frac{1}{2}, \quad \frac{\kappa}{2} - 1 \leq \frac{\kappa-1}{2} + \frac{\kappa-1}{2}. \tag{5.70}$$

Therefore, recalling (4.7), we can apply Lemma 3.12 with $\theta = 0$ to show the desired results. This completes the proof. □

5.3.2 Proof of Theorem 2.6

We use the same notation as in the proof of Theorem 2.4. We fix $I = 1,\ldots,P$. Given $0 < \kappa \ll 1$, we choose $\rho = \kappa$ this time. Then it follow from (5.62) and (5.66) that

$$\begin{aligned} r[u^I(t,x)]_{I,1} &\leq C\varepsilon\langle t+r\rangle^{\frac{\kappa}{2}}, \qquad (t,x) \in [0,\infty)\times\mathbb{R}^3, \\ \int_t^\infty \tau^\theta \mathscr{T}[\mathcal{R}](\tau,\sigma,\omega)\,d\tau &\leq C\varepsilon t^{\frac{\kappa-1}{2}+\theta}\langle\sigma\rangle^{\frac{\kappa-1}{2}}, \quad (t,\sigma,\omega) \in \mathcal{L}^{1;c_1,c_P}_{\infty,R},\ 0 \leq \theta \ll 1. \end{aligned}$$

Since we have (4.8), we can apply Lemma 3.12 with $\theta = 0$ to obtain the desired results as above. This completes the proof. □

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