OUKA
The University of Osaka
Institutional Knowledge Archive

Title	ROBUST APPROACHES TO LINEAR PROGRAMMING PROBLEMS WITH UNCERTAIN COEFFICIENTS
Author(s)	高, 振中
Citation	大阪大学, 2023, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/92213
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Robust Approaches to Linear Programming Problems with Uncertain Coefficients

Zhenzhong Gao

March 2023

Robust Approaches to Linear Programming Problems with Uncertain Coefficients

A dissertation submitted to
THE GRADUATE SCHOOL OF ENGINEERING SCIENCE
OSAKA UNIVERSITY
in partial fulfilment of the requirements for the degree of
DOCTOR OF PHILOSOPHY IN ENGINEERING

By

Zhenzhong Gao
March 2023

Abstract

When formulating real-world problems as linear programming (LP) problems, decision makers may not be able to know the accurate and precise values of some parameters. We call such parameters uncertain coefficients. Using speculated values for these uncertain coefficients may lead to a great loss when implementing the obtained solution back to the real-world problem. Therefore, it is important to collect the information about the uncertain coefficients, such as historical data and ranges of conceivable values, and to analyse their effect while obtaining a solution. A desirable solution can fulfil the decision maker's requirements or preserve the optimality for all conceivable values of uncertain coefficients. The approach to obtaining such a solution is considered a robust approach to the programming problems.

A lot of researchers have been focusing on this subject and proposing numerous approaches by representing the uncertain coefficients as intervals, fuzzy numbers and random numbers. These approaches have earned remarkable achievements, despite many aspects being left unsolved. This thesis focuses on two of those unsolved problems - one is the robust optimality analysis, and the other is the trade-off between robustness and satisfaction.

For the first problem, we treat an LP problem with uncertain objective function coefficients (OFCs). In the problem, the conceivable range of OFC vectors is given by a subset or a fuzzy subset. To such a problem, it is known that the necessarily optimal solution is the most reasonable solution. As it remains optimal against the certain fluctuations of OFCs, the necessarily optimal solution can be regarded as a robust optimal solution. In Chapter 3, we demonstrate that the necessary optimality degree of a non-degenerate basic feasible (NBF) solution can be obtained easily by representing the uncertain OFC vector to a fuzzy vector and using the tolerance approach. The necessary optimality degree evaluates to what extent the solution remains optimal against the fluctuations of OFCs. Several types of fuzzy subsets showing the conceivable range of OFC vectors are considered. For each type of fuzzy subset, an efficient calculation method of necessary optimality degree is proposed. Numerical examples are given to illustrate the proposed methods.

In real life, we come across cases where some OFCs' ranges depend on the remaining OFCs' values. For example, the prices of products rise together in tandem with raw materials, the gross profit of exported products increases while that of imported products decreases because they depend on the currency exchange rates, and so on. Such dependencies cannot be expressed well by the types of fuzzy subsets treated in Chapter 3. A convex polytope can express such dependencies among marginal ranges of uncertain OFCs to a certain degree. Hence, in Chapter 4, we extend the robust optimality analysis when the conceivable range of OFC vectors is specified by a convex polytope. In this case, the tolerance approach cannot be applied. To test the necessary optimality degree of an NBF solution, we need to check the inclusion of the convex polytope in the convex cone assuring the optimality of the NBF solution. For solving this test problem, we propose an algorithm based on the outer approximation approach. By numerical simulations, we demonstrate the efficiency of the proposed algorithm compared to a conventional vertex-listing method for checking the inclusion.

For the second problem, we consider LP problems where the coefficients of constraints are uncertain in Chapter 5. Such a problem is formulated as LP problems with set-inclusive constraints to ensure that the left-hand side coefficients' values are included in the given satisfactory ranges for all conceivable coefficient values. When the conceivable ranges of coefficients are given by fuzzy subsets, the set-inclusive constraints are treated by a necessity measure as it shows the degree of inclusion between fuzzy subsets. However, the necessity measure cannot express well the decision maker's requirement about the trade-off between the robustness level and satisfaction level of the constraints. We extend the necessity measure to incorporate the trade-off between the robustness level and satisfaction level of the constraints. We apply the extended necessity measure to LP problems with fuzzy coefficients. After the problem is formulated and reduced to the conventional programming problem, we propose an algorithm to solve it. Numerical examples are given to illustrate the proposed approach.

Contents

Abstract **i**

1 Introduction **1**

1.1 Structure of this thesis 6

1.2 Notations 7

2 Preliminaries **9**

2.1 Linear Programming under Uncertainty 9

2.1.1 LP problems and optimal solutions 9

2.1.2 LP problems with uncertain OFCs 11

2.1.3 The tolerance approach 15

2.2 Fuzzy Subsets and Fuzzy Numbers 17

2.2.1 Fuzzy subsets and h-level sets 17

2.2.2 Fuzzy numbers 18

2.2.3 Extension principle 21

2.3 Possibility and Necessity Measures 23

3 Robust Optimality Analysis of the LP Problem with fuzzy OFCs **25**

3.1 Fuzzy Vectors 25

3.2 Fuzzy LP Problems 27

3.2.1 Case of non-interactive symmetric fuzzy OFC vector 28

3.2.2 Case of non-interactive fuzzy OFC vector 31

3.2.3 Case of oblique fuzzy OFC vector 38

3.3 Concluding Remarks 40

4 Robust Optimality Analysis of the LP Problem with OFCs included in a Convex Polytope 43

4.1 LP Problems with OFCs included in a Convex Polytope 43

4.1.1 Construction of the initial convex polytope 45

4.1.2 Cutting and updating strategy 46

4.2 Algorithm for Robust Optimality Analysis 51

4.3 Computational Analysis and Simulations 55

4.4 Concluding Remarks 57

5 A Necessity Measure Approach to Balancing Robustness and Satisfaction in Fuzzy LP Problems 59

5.1 The Extended Necessity Measure 59

5.2 An Application to Fuzzy Linear Programming 65

5.2.1 Problem formulation and reduction 65

5.2.2 A solution algorithm 69

5.2.3 Numerical examples 72

5.3 Concluding Remarks 75

6 Conclusion 77

Bibliography 79

Acknowledgement 83

Publication List 85

List of Figures

2.1 Necessarily and Possibly Optimal Solutions 14
2.2 Illustration of Example 2.2 17
2.3 Illustration of an L-R fuzzy number $\tilde{A} = (a^{\mathrm{L}}, a^{\mathrm{R}}, \alpha, \beta)_{LR}$ 19
2.4 Pseudo-inverse functions L^* and L_* of reference function L 21
2.5 Possibility and Necessity Measures: $\Pi_{\tilde{A}}(\tilde{B})$ and $N_{\tilde{A}}(\tilde{B})$ 24

3.1 Non-interactive fuzzy vectors and oblique fuzzy vectors in $\mathbb{R}^2$ 27
3.2 Illustration of Example 3.1 31
3.3 Membership function of Φ 34
3.4 The necessarily optimal solution in Example 3.4 41

4.1 A constraint $\mathbf{L}$ to a convex polytope Φ 47
4.2 Degenerate vertex with no redundant constraint 48
4.3 Illustration of Example 4.1 54
4.4 The simulation result of the outer approximation approach 57

5.1 The result for fuzzy coefficients of (5.31) 74
5.2 The result for fuzzy coefficients of (5.38) 74
5.3 The result for fuzzy coefficients of (5.39) 75

List of Tables

4.1 Some Simulation Results 57

Chapter 1

Introduction

LINEAR programming (LP) problems can represent many real-world problems, such as production planning, investment and budget allocation, and help the decision maker make the best decision. It is assumed that all coefficients and constraints of the problem are known precisely. However, this assumption is not always satisfied. For example, coefficients can be uncertain due to measurement restrictions, noise, modelling errors and imprecise knowledge [39], and constraints can be vague due to human factors such as the just-noticeable difference (JND) [38]. In such situations, the coefficients can be treated as random variables or fuzzy numbers, and the constraints can be considered soft constraints, like goals rather than restrictions. Since the uncertain coefficients can cause significant loss, the decision maker wants to treat them properly and is interested in a solution that preserves its optimality against the fluctuation of the uncertain coefficients. Such a solution is considered a robust optimal solution [14, 28].

The study of uncertainty has been carried on for centuries, dating back to Archimedes' era when measuring the perimeter of a circle. Archimedes [8] used an interval to represent the uncertain result of the perimeter, which is still a favoured representation by current researchers and engineers. In the last century, researchers proposed many approaches to LP problems with uncertain coefficients in more detailed and analytical ways. For example, a classical approach called sensitivity analysis [3] is useful for evaluating the influence of small and local fluctuations in the coefficients of LP problems. By using shadow prices, it can analyse the maximum perturbation on a single coefficient that maintains the optimality of the current optimal solution. The bigger the maximum perturbation is, the more robust the optimal solution is.

The drawback of the sensitivity analysis is that it cannot treat perturbations on multiple coefficients. Bradley et al. [3] pointed out a particular convex cone defined at a non-degenerate basic feasible (NBF) solution for the uncertain objective function coeffi-

cient (OFC) vector in an LP problem. If all possible realisations of the OFC vector are included in the convex cone, the NBF solution is undoubtedly the optimal solution. This rule is called the 100% rule, and we call the convex cone the optimality assurance cone (OAC) in this thesis. This 100% rule is not yet very useful as it requires an enormous computational effort to check the inclusion of all possible realisations of the OFC vectors in the solution's OAC.

The tolerance approach, proposed by Wendell [40], aims at overcoming the high-computational problem in 100% rules. Similar to the sensitivity analysis, it finds a maximal hyperrectangle included in the OAC. If all possible realisations of the OFC vector are in the obtained hyperrectangle, the NFB solution is stably optimal, and the inclusion of realisations in the rectangle is confirmed very easily. Wondolowski [41] and Filippi [5] modified this approach by enlarging the maximal hyperrectangle included in the OAC. The studies on the tolerance approach have continued to improve its restrictions. Hladí [14, 13] considered constraints with uncertain parameters and showed that the problem becomes a co-NP hard one so that there is no guarantee that the analysis can be completed in a reasonable time. Hladí [11] extended the tolerance approach to the case where the right-hand side values are uncertain, like the uncertain OFCs. He calculated a maximal hyperrectangle within which the optimal basis does not change based on the support set [35] and the optimal partition invariancy [7]. In this approach, a candidate solution is essential. Hladí [12] suggested an optimal solution with respect to the centre values of the conceivable ranges of OFCs.

Independently from the tolerance approach, a different approach has been proposed in a different setting. In the setting of the tolerance approach, the range of possible realisations of OFC is not given in advance. However, when some experts can estimate the range of possible realisations of coefficients as intervals or fuzzy numbers, LP problems are formulated as LP problems with interval or fuzzy coefficients. While an interval coefficient shows the conceivable range of the coefficient value, a fuzzy coefficient vector defines the possibility degree of a coefficient vector by its membership function. To such problems, two major approaches are proposed. One is the satisficing approach, and the other is the optimisation approach.

The satisficing approach was studied much more than the optimisation approach [21]. In the satisficing approach, we assume that the decision maker specifies the target values or goals of the objective functions. As intervals can be regarded as a particular case of fuzzy numbers, we only describe LP problems with fuzzy coefficients. Then the problems are reduced to goal attainment problems. Many methods in the satisficing approach have been proposed, but they can be reasonably understood by using the possibility theory [21, 20]. Using possibility and necessity measures [26] defined in the possibility theory, possible and necessary satisfaction degrees can be defined for constraints with fuzzy coefficients, as

well as possible and necessary attainment degrees for goals given for objective functions with fuzzy coefficients. Then the LP problems with fuzzy coefficients can be treated by utilising those degrees. Basically, three models are considered. The first model is to maximise the satisfaction degree and the attainment degree equally, which is called a symmetric model. The second model is to maximise the attainment degrees under the satisfaction degrees being more than the levels required by the decision maker, which is called a modality maximisation model, where the modality is used as a generic word for possibility and necessity. The third model is to maximise target values under the attainment degrees showing to what extent the objective function values possibly/surely satisfy goals described by the target values are more than the required levels and the satisfaction degrees are more than the required levels, which is called a modality fractile model.

A vector of fuzzy numbers is usually used to represent the fuzzy coefficient vector. In a vector of fuzzy numbers, any of its h-level sets, the subsets of vectors whose all components take membership grades of the corresponding fuzzy numbers not less than $h \in (0, 1]$, becomes a hyperrectangle. Namely, the conceivable range of a coefficient does not depend on the values of other coefficients. In the real world, however, the conceivable range of a coefficient may depend on the values of other coefficients. For example, consider the conceivable prices of two similar products, which depend on the variable popularity. If a product is popular and its price is high, the price of the other product is expected to be high, as a similar product can also be popular. Several fuzzy vector models [17] have been proposed to express such dependencies. For example, the oblique fuzzy vector [23] and the fuzzy convex polytope [16] were proposed for representing conceivable ranges of coefficient vectors. The oblique fuzzy vector expresses the relationship among coefficients by an obliquity matrix. An h-level set of an oblique fuzzy vector is obtained by the oblique transformation of a hyperrectangle showing an h-level set of a vector of fuzzy numbers with the obliquity matrix. It is shown that the linear function value of an oblique fuzzy vector defined by symmetric fuzzy numbers can be calculated easily [23]. Therefore, LP problems with oblique fuzzy vectors can be solved by a simplex method with Bender's decomposition technique. On the other hand, a fuzzy convex polytope is helpful in expressing the conceivable range of coefficient vectors when conceivable ranges of multiple fractional linear function values of the uncertain coefficients are known as closed intervals. It is shown that LP problems with fuzzy convex polytopes can be solved by a bisection method and a relaxation procedure converged simultaneously [16].

On the other hand, in the optimisation approach, the concept of optimality is extended to the interval and fuzzy coefficient cases to obtain an optimal solution in an extended senseor to evaluate a candidate solution from the view of optimality. Inuiguchi and Sakawa [24] defined the notion of possible and necessary optimalities for LP problems with

interval and fuzzy OFCs. In the interval coefficient case, a necessarily optimal solution is the most reasonable solution because it is optimal for all conceivable coefficient values of OFCs. On the other hand, a possibly optimal solution is a minimal reasonable solution because it is only optimal for at least one possible coefficient value. A necessarily optimal solution is preferred, but it does not always exist in the interval coefficient case. The fuzzy coefficient case is advantageous as it enables us to define the necessary optimality degree. The necessary optimality degree is defined by the minimum level of h-level sets of fuzzy OFCs included in the OAC. Let h^* be the minimum level, then the necessary optimality degree is defined by $1-h^*$. Inuiguchi and Sakawa [24] studied the possibility test problem and necessity optimality test problem on a feasible solution to the LP problem with fuzzy OFCs. They showed that the possibility optimality test problem is reduced to an LP problem while the necessity optimality test problem becomes a non-convex programming problem. Therefore, the necessity optimality test problem has been considered a complex problem. The notion of possible and necessary optimality is extended to the case where multiple objective functions are given. Inuiguchi and Sakawa [26] extended the concept of efficiency to LP problems with fuzzy multiple objective functions. They are possible and necessary efficiencies, extensions of possible and necessary optimalities, respectively.

To overcome the non-existence of the necessarily optimal solution in the interval coefficient case, researchers considered the minimax regret criterion [25, 33, 30] and the best-worst achievement rate criterion [27, 10]. If there exist necessarily optimal solutions, the minimax regret solution and the best-worst achievement rate solution coincide with them. Otherwise, they are possibly optimal solutions minimising the deviations from the necessary optimality [25, 27]. In the fuzzy coefficient case, those solution concepts become necessarily soft-optimal solutions, where a set of soft-optimal solutions are a fuzzy subset of feasible solutions whose objective function value is not very different from the optimal value. The decision maker defines this fuzzy subset [28]. Although those solution concepts have good properties, the reduced problems for calculating those solutions become non-convex. Therefore, it takes a lot of computation time to obtain those solutions.

In this thesis, we consider two unsolved problems - one is the robust optimality analysis of an obtained solution, and the other is obtaining a solution that balances the trade-off between the robustness and satisfaction in LP problems with uncertain coefficients.

For the first unsolved problem, we investigate solution methods for necessary optimality test problems. For LP problems with uncertain OFCs, we propose two methods for solving necessary optimality test problems at non-degenerate basic feasible (NBF) solutions. Although the proposed method is useful only for an NBF solution, the necessary optimality test problem of a degenerate basic solution can be solved by applying the proposed methods for all combinations of basic variables associated with the degenerate basic solution. As described earlier, the necessary optimality test problem is a non-convex pro-

gramming problem. Therefore, it is not an easy problem. One of the two proposed methods utilises a tolerance approach, while the other uses an outer approximation approach.

The first method applies a tolerance approach to the necessary optimality test problems. We consider the test problems for two types of fuzzy OFC vectors, i.e., a vector of fuzzy numbers and an oblique fuzzy vector. When the conceivable range of OFC vectors is given by a vector of symmetric fuzzy numbers, it is shown that the problem can be solved very easily by the tolerance approach. When the conceivable range of OFC vectors is given by a vector of general fuzzy numbers, it is shown that the problem can be solved by a bisection method with the tolerance approach. Moreover, when the conceivable range of OFC vectors is given by an oblique fuzzy vector, it is shown that the problem can be solved in the same way as those in the case of a vector of fuzzy numbers with the transformation of the OAC by the obliquity matrix associated with the oblique fuzzy vector. By the application of the tolerance approach, it is revealed that the necessary optimality test problem of an NBF solution for several types of conceivable range of OFC vectors can be solved without much difficulty.

Unfortunately, the tolerance approach to the necessary optimal test problem described is not applicable to any types of conceivable range of OFC vectors. As one of the useful representations of the conceivable range of uncertain coefficient vectors is a convex polytope as it expresses the knowledge about ranges of linear fractional function values of uncertain coefficients, which include knowledge about the sum of uncertain coefficients, the difference of two uncertain coefficients and the ratio between two uncertain coefficients. Then, the second method is developed for solving the necessary optimality test problems of an NBF solution when the conceivable range of the OFC vector is given by a convex polytope. Applying the outer approximation approach, we propose a solution algorithm for the problem. By a numerical experiment, we demonstrate that the proposed algorithm is more computationally efficient than the conventional vertex-listing method.

For the second unsolved problem, we treat LP problems with fuzzy coefficients. The left-hand side function coefficients of the constraints of the LP problem are fuzzy numbers. As the right-hand side of each constraint, we assume that the decision maker gives a fuzzy subset expressing the allowable range of the left-hand side function value. Namely, the problem becomes a set-inclusive programming problem. As the necessity measure shows a degree of inclusion between fuzzy subsets, the problem is formulated as a necessity measure maximisation problem. However, the necessity measure does not well reflect the decision maker's preference about the trade-off between the robustness and the satisfaction. The robustness implies to what extent the fluctuations of coefficients are considered, while the satisfaction implies to what extent the obtained range of left-hand side function value is allowable. To improve the representability of the decision maker's preference, we extend the necessity measure by introducing two parameters corresponding to robustness and

satisfaction. Then the extended necessity measure maximisation problem is formulated and a solution algorithm is given.

1.1 Structure of this thesis

This thesis is organised as follows. In Chapter 2, we review the underlying theory and methods. We begin with descriptions of LP problems and the properties of optimal solutions. The possibly and necessarily optimal solutions are introduced as solution concepts for LP problems with uncertain OFCs. The propositions about the necessary optimality for an NBF solution are given by using the optimality assurance cone. The tolerance approach is briefly reviewed. Then the fuzzy subsets and fuzzy numbers are explained. Moreover, possibility and necessity measures are described with their properties with respect to h-level sets.

In Chapter 3, we treat an LP problem with a fuzzy OFC vector. The notion of interaction among fuzzy numbers is introduced, and non-interactive and oblique fuzzy vectors are defined. Then, the solution methods for the necessary optimal test problems of an NBF solution under two types of fuzzy OFC vectors are investigated using a tolerance approach. The test problem is solved by the bisection method and a tolerance approach when the fuzzy OFC vector is a non-interactive fuzzy vector. Particularly, when both increasing and decreasing parts of membership functions of fuzzy numbers are all defined by a non-increasing function, it is shown that the test problem is solved only by a tolerance approach. By the inverse oblique transformation, the same results are obtained when the fuzzy OFC vector is an oblique fuzzy vector.

In Chapter 4, we study the necessary optimality test problem of an NBF solution in LP problems with uncertain OFC coefficients. The conceivable range of the OFC vector is assumed to be a convex polytope. The knowledge about the ranges of individual coefficients, the sum of OFCs, the difference between two OFCs, the ratio between two OFCs, as well as the linear fractional function value of OFCs. In this case, unfortunately, the tolerance approach cannot be applied well. Then we need to solve the non-convex test problem. Based on the outer approximation, we propose a solution algorithm. By a numerical experiment, we demonstrate that the proposed algorithm is more computationally efficient than a conventional vertex-listing approach.

In Chapter 5, we treat constraints with fuzzy numbers. Namely, the coefficients of the left-hand side functions of the constraints are not known precisely but are vaguely expressed by fuzzy numbers. For each left-hand side function of the constraints, we assume that the decision maker specifies the acceptable values by a fuzzy set. Therefore, the problem is to obtain a solution such that all the left-hand side function fuzzy values

are included in the corresponding fuzzy subsets showing the acceptable values. As the necessity measure shows the degree of inclusion, we can formulate this problem as a necessity measure maximisation problem. However, unfortunately, the usual necessity measure does not always represent the decision maker's preference given by the trade-off between the robustness level and the satisfaction level. Then, an extended necessity measure that can represent well the trade-off requested by the decision maker is defined and applied to the necessity measure maximisation problem. An approximate solution algorithm is proposed. A numerical example is given to exemplify the solution procedure.

Finally, in Chapter 6, we summarise the results of the studies described in this thesis and indicate future directions.

1.2 Notations

In this thesis, we use the following notations:

$\mathbb{R}^n$	An n-dimensional Euclidean space
$\boldsymbol{I}$	An identity matrix with convenient dimension
$\lvert \cdot \rvert$	An entry-wise absolute operator to vectors and matrices
$\boldsymbol{A}^{-\mathrm{T}}$	the inverse of a transposed matrix $\boldsymbol{A}$ (non-singular)
$\mathrm{diag}(\boldsymbol{v})$	the diagonal matrix with the diagonal entries being vector $\boldsymbol{v}$
$\mathrm{Card}(\mathbf{P})$	the cardinality of a finite set $\mathbf{P}$
$\mathrm{cl}(\mathbf{P})$	the closure of a set $\mathbf{P}$

Chapter 2

Preliminaries

IN this chapter, we introduce the relevant and necessary foundations for developing the theories and methods described in the following chapters. First, we briefly review the theory of linear programming (LP). The feasibility and optimality conditions of a basic solution are shown. LP problems with uncertain objective function coefficients (OFCs) are described using the theory of LP. Possible and necessary optimalities are introduced and analysed by using the optimality assurance cone (OAC) defined by the optimality condition of a basic feasible solution. The tolerance approach to LP problems is reviewed with a numerical example. Next, we introduce the fuzzy theory focusing on fuzzy subsets, fuzzy numbers, h-level sets and the extension principle. Moreover, we review the necessity measure based on fuzzy subsets. We demonstrate that h-level sets rather than membership functions are useful in the analysis and manipulation of fuzzy numbers as well as possibility and necessity measures.

2.1 Linear Programming under Uncertainty

2.1.1 LP problems and optimal solutions

In this thesis, the following standard form of LP problem is considered:

$$\text{minimize } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \text{ subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}, \tag{2.1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ denotes a decision variable vector. $\boldsymbol{c} \in \mathbb{R}^n$ denotes the objective function coefficient (OFC) vector, while $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ denotes the left-hand side coefficient matrix of the constraints and $\boldsymbol{b} \in \mathbb{R}^m$ denotes the right-hand side coefficient vector of the constraints.

As an optimal solution to an LP problem exists in basic feasible solutions [3], we consider an optimal basic solution $\boldsymbol{x}^*$. Let $\mathbb{I}_B(\boldsymbol{x}^*) \subseteq \{1, 2, \ldots, n\}$ be the basis of the

basic solution $\boldsymbol{x}^*$. Namely, $\text{Card}(\mathbb{I}_B(\boldsymbol{x}^*)) = m$. Then $\boldsymbol{x}^*$ can be divided into two parts by $\mathbb{I}_B(\boldsymbol{x}^*)$ - the basic variable sub-vector $\boldsymbol{x}_B^*$ and the non-basic sub-vector $\boldsymbol{x}_N^*$. Matrices $\boldsymbol{A}_B \in \mathbb{R}^{m\times m}$ and $\boldsymbol{A}_N \in \mathbb{R}^{m\times(n-m)}$ refer to the basic and non-basic sub-matrices of $\boldsymbol{A}$ corresponding to the index sets of $\boldsymbol{x}_B^*$ and $\boldsymbol{x}_N^*$, respectively. Similar to the division of $\boldsymbol{A}_B \in \mathbb{R}^{m\times m}$ and $\boldsymbol{A}_N \in \mathbb{R}^{m\times(n-m)}$, coefficient vectors $\boldsymbol{c}_B \in \mathbb{R}^m$ and $\boldsymbol{c}_N \in \mathbb{R}^{n-m}$ refer to the basic and non-basic sub-vectors of $\boldsymbol{c}$, respectively. Owing to $\boldsymbol{A}_B$ being non-singular, the value $\boldsymbol{x}^*$ is determined as $\boldsymbol{x}_B^* = \boldsymbol{A}_B^{-1}\boldsymbol{b}$ and $\boldsymbol{x}_N^* = \boldsymbol{0}$. When $\boldsymbol{x}_B^* = \boldsymbol{A}_B^{-1}\boldsymbol{b} \geq \boldsymbol{0}$, we say $\boldsymbol{x}^*$ is *feasible*. Moreover, if $\boldsymbol{x}_B^* = \boldsymbol{A}_B^{-1}\boldsymbol{b} > \boldsymbol{0}$, $\boldsymbol{x}^*$ is said to be a *non-degenerate feasible* solution. Otherwise, it is said to be a *degenerate feasible* solution.

To confirm the *optimality* of a basic feasible (BF) solution, the following proposition is given [3]:

Proposition 2.1. *A basic solution $\boldsymbol{x}^*$ is optimal if and only if the following conditions are valid:*

$$\boldsymbol{c}_N - \boldsymbol{A}_N^{\mathrm{T}}\boldsymbol{A}_B^{-\mathrm{T}}\boldsymbol{c}_B \geq \boldsymbol{0} \text{ and } \boldsymbol{A}_B^{-1}\boldsymbol{b} \geq \boldsymbol{0}, \tag{2.2}$$

where the optimal solution is obtained by $\boldsymbol{x}_B^* = \boldsymbol{A}_B^{-1}\boldsymbol{b}$ and $\boldsymbol{x}_N^* = \boldsymbol{0}$ with the optimised value $\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{b}$.

Proposition 2.1 guarantees the invariance of the basic optimal solution no matter how $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ vary as long as they satisfy (2.2). However, the value of $\boldsymbol{x}^*$ can be changed by the varied $\boldsymbol{A}$ and $\boldsymbol{b}$ due to $\boldsymbol{x}_B^* = \boldsymbol{A}_B^{-1}\boldsymbol{b}$. Moreover, let $\boldsymbol{A}$ and $\boldsymbol{b}$ be constant. If $\boldsymbol{c}$ is varied but satisfies (2.2), the optimal solution is invariant while the optimal value varies because it equals to $\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{A}_B^{-1}\boldsymbol{b}$.

For a *degenerate basic feasible* (DBF) solution, the optimal solution can be invariant even when $\boldsymbol{c}$ is out of the range of (2.2) as far as $\boldsymbol{A}$ and $\boldsymbol{b}$ are constant. This is because there are multiple index sets of the basic solution associated with the degenerate basic solution $\boldsymbol{x}^*$. Namely, corresponding to another index set, a different range of (2.2) exists and even if $\boldsymbol{c}$ fluctuates in it, the optimal solution does not change. Hence, for each index set of basic variables associated with the optimal DBF solution, a different range of $\boldsymbol{c}$ satisfying (2.2) is obtained when $\boldsymbol{A}$ and $\boldsymbol{b}$ are constant. A detailed illustration can be found in Chapter 4.

To avoid the complexity in the analysis caused by the multiplicity of optimal bases, we concentrate on the analysis of a *non-degenerate basic feasible* (NBF) solution $\boldsymbol{x}^*$ as it has a unique range expressed by (2.2). However, we can extend the proposed analysis to the case of a DBF solution by using some proper enumeration technique of all index sets of basic variables with which the associated basic solution is optimal. For example, we may utilise the support set and the optimal partition invariancy [11]. Gao and Inuiguchi [6] also proposed an analysis that tries to unify these ranges.

2.1.2 LP problems with uncertain OFCs

When considering the LP problem with a constant feasible region and an uncertain *objective function coefficient* (OFC) vector $\boldsymbol{\gamma}$, Problem (2.1) becomes:

$$\text{minimize } \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}, \text{ subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}, \tag{2.3}$$

where $\boldsymbol{\gamma} \in \mathbb{R}^n$ denotes the uncertain coefficients. $\boldsymbol{x}$, $\boldsymbol{A}$ and $\boldsymbol{b}$ are defined the same as Problem (2.1).

It is noted that many approaches [22, 37] to the LP problem with uncertain coefficients $\boldsymbol{A}$ and $\boldsymbol{b}$ have been proposed. When the conceivable ranges of $\boldsymbol{A}$ and $\boldsymbol{b}$ are given by intervals or fuzzy subsets, the constraints of LP problems with uncertain coefficients $\boldsymbol{A}$ and $\boldsymbol{b}$ are reduced to linear inequalities (see Inuiguchi et al. [21, 22]). Therefore, when $\boldsymbol{A}$ and $\boldsymbol{b}$ are also uncertain in Problem (2.3), we first apply the results in the previous approaches [21, 22], so that the constraints with uncertain coefficients $\boldsymbol{A}$ and $\boldsymbol{b}$ can be reduced to linear inequalities and the given problem can be reduced to an LP problem only with uncertain OFCs.

When the variation range of $\boldsymbol{\gamma}$ is specified by a crisp set $\Phi \subseteq \mathbb{R}^n$, we apply the notion of *possible and necessary optimalities* [24] to a feasible solution:

Definition 2.1. Let $\Phi \subseteq \mathbb{R}^n$ denote a bounded set composed of all conceivable OFC vectors $\boldsymbol{c}$. Let $\boldsymbol{x}^*$ be a feasible solution, then $\boldsymbol{x}^*$ is

- *possibly optimal* for Φ if and only if it is optimal for at least one $\boldsymbol{c} \in \Phi$, or
- *necessarily optimal* for Φ if and only if it is optimal for every $\boldsymbol{c} \in \Phi$.

We note that infeasible solutions are neither possibly optimal nor necessarily optimal.

When the LP problem has a constant feasible region, the following definition shows the range of uncertain coefficient vector $\boldsymbol{\gamma}$ in which a given feasible solution is optimal.

Definition 2.2. Let $\boldsymbol{x}^*$ be a feasible solution. Then the *optimality assurance cone* (OAC) of $\boldsymbol{x}^*$, denoted as $\mathscr{N}(\boldsymbol{x}^*)$, is defined by

$$\mathscr{N}(\boldsymbol{x}^*) := \Big\{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^* = \min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}\Big\}. \tag{2.4}$$

$\mathscr{N}(\boldsymbol{x}^*)$ is the set of OFC vectors to which the feasible solution $\boldsymbol{x}^*$ is optimal under constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$.

It is apparent that $\mathscr{N}(\boldsymbol{x}^*)$ is a set of OFC vectors maintaining the optimality of $\boldsymbol{x}^*$ and it is a convex cone. Namely, if $\mathscr{N}(\boldsymbol{x}^*)$ is non-empty, a feasible solution $\boldsymbol{x}^*$ is optimal as long as the OFC vector lies in $\mathscr{N}(\boldsymbol{x}^*)$. This is why we call $\mathscr{N}(\boldsymbol{x}^*)$ the OAC of $\boldsymbol{x}^*$, which derives the following equivalences [24]:

$$\boldsymbol{x}^* \text{ is possibly optimal } \Leftrightarrow \ \Phi \cap \mathscr{N}(\boldsymbol{x}^*) \neq \emptyset, \tag{2.5}$$

$$\boldsymbol{x}^* \text{ is necessarily optimal } \Leftrightarrow \ \Phi \subseteq \mathscr{N}(\boldsymbol{x}^*). \tag{2.6}$$

Proposition 2.2 gives the explicit expression of $\mathscr{N}(\boldsymbol{x}^*)$ when $\boldsymbol{x}^*$ is an NBF solution.

Proposition 2.2. *Let $\boldsymbol{x}^*$ be an NBF solution to Problem (2.3). Then $\mathscr{N}(\boldsymbol{x}^*)$ is equivalent to $\mathscr{M}(\boldsymbol{x}^*)$ defined as:*

$$\mathscr{M}(\boldsymbol{x}^*) := \left\{ \boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{c}_N - \boldsymbol{A}_N^{\mathrm{T}} \boldsymbol{A}_B^{-\mathrm{T}} \boldsymbol{c}_B \geq \boldsymbol{0} \right\}. \tag{2.7}$$

For convenience, we use a simple expression with matrix $\boldsymbol{M}(\boldsymbol{x}^*) = (m_{ij}(\boldsymbol{x}^*))$:

$$\mathscr{M}(\boldsymbol{x}^*) = \left\{ \boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{c} \geq \boldsymbol{0} \right\}, \tag{2.8}$$

where $\boldsymbol{M}(\boldsymbol{x}^*) \in \mathbb{R}^{(n-m)\times n}$ and $m_{ij}(\boldsymbol{x}^*)$ is the (i,j)-component. We note that we have $\boldsymbol{M}(\boldsymbol{x}^*) = \left(-\boldsymbol{A}_N^{\mathrm{T}} \boldsymbol{A}_B^{-\mathrm{T}},\ \boldsymbol{I} \right)$ when $\boldsymbol{c}$ is permutated as $\boldsymbol{c}^{\mathrm{T}} = \left(\boldsymbol{c}_B^{\mathrm{T}},\ \boldsymbol{c}_N^{\mathrm{T}} \right)$.

Proof. It is straightforwardly obtained from Proposition 2.1 and the fact that a unique index set of basic variables is associated with the NBF solution $\boldsymbol{x}^*$. □

Proposition 2.2 indicates that $\mathscr{N}(\boldsymbol{x}^*) = \mathscr{M}(\boldsymbol{x}^*)$ if $\boldsymbol{x}^*$ is an NBF solution. Otherwise, they are not equivalent. Gao and Inuiguchi [6] indicated that when $\boldsymbol{x}^*$ is degenerate, we have $\bigcup_i \mathscr{M}_i(\boldsymbol{x}^*) = \mathscr{N}(\boldsymbol{x}^*)$, where $\mathscr{M}_i(\boldsymbol{x}^*)$ is an OAC of $\boldsymbol{x}^*$ obtained from the i-th basis $\mathbb{I}_{B_i}(\boldsymbol{x}^*)$ among many bases corresponding to $\boldsymbol{x}^*$.

Since $\mathscr{N}(\boldsymbol{x}^*) = \mathscr{M}(\boldsymbol{x}^*)$ for an NBF solution $\boldsymbol{x}^*$, (2.5) and (2.6) can be re-written as:

$$\boldsymbol{x}^* \text{ is possibly optimal } \Leftrightarrow \ \Phi \cap \mathscr{M}(\boldsymbol{x}^*) \neq \emptyset, \tag{2.9}$$

$$\boldsymbol{x}^* \text{ is necessarily optimal } \Leftrightarrow \ \Phi \subseteq \mathscr{M}(\boldsymbol{x}^*). \tag{2.10}$$

Owing to $\Phi \neq \emptyset$, $\Phi \subseteq \mathscr{M}(\boldsymbol{x}^*)$ implies $\Phi \cap \mathscr{M}(\boldsymbol{x}^*) \neq \emptyset$, suggesting that a necessarily optimal solution is also possibly optimal. To illustrate the OAC as well as the possible and necessary optimalities, we use the following example originally given by Inuiguchi and Sakawa [24]:

Example 2.1. Consider the following LP problem:

$$\begin{aligned} \text{minimize} \quad & c_1 x_1 + c_2 x_2, \\ \text{subject to} \quad & 3x_1 + 4x_2 + x_3 = 42, \\ & 3x_1 + x_2 + x_4 = 24, \\ & x_2 + x_5 = 9, \\ & x_i \geq 0, \ i = 1, 2, \ldots, 5, \end{aligned}$$

where c_1 and c_2 are not known exactly but the conceivable range of $(c_1, c_2)^{\mathrm{T}}$ is known as $\check{\Phi} \subseteq \mathbb{R}^2$. Namely, we know $(c_1, c_2)^{\mathrm{T}} \in \check{\Phi}$. As $c_3 = c_4 = c_5 = 0$ in this problem, $\Phi \subseteq \mathbb{R}^5$ is obtained as $\Phi = \left\{(c_1, c_2, 0, 0, 0)^{\mathrm{T}} : (c_1, c_2)^{\mathrm{T}} \in \check{\Phi}\right\}$.

We consider an NBF solution $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$ with the index set of basic variables $\{1, 2, 5\}$. Then, the OAC of $\boldsymbol{x}^*$ is obtained as

$$\mathscr{M}(\boldsymbol{x}^*) = \left\{ (c_1, c_2, c_3, c_4, c_5)^{\mathrm{T}} \;\middle|\; \begin{array}{c} \frac{1}{9}c_1 - \frac{1}{3}c_2 + c_3 + \frac{1}{3}c_5 \geq 0 \\ -\frac{4}{9}c_1 + \frac{1}{3}c_2 + c_4 - \frac{1}{3}c_5 \geq 0 \end{array} \right\}. \tag{2.11}$$

Because we have $c_3 = 0$, $c_4 = 0$ and $c_5 = 0$, we regard x_3, x_4 and x_5 as slack variables. Then, the problem can be depicted on an x_1-x_2 coordinate system. Accordingly, the constraints are rewritten as

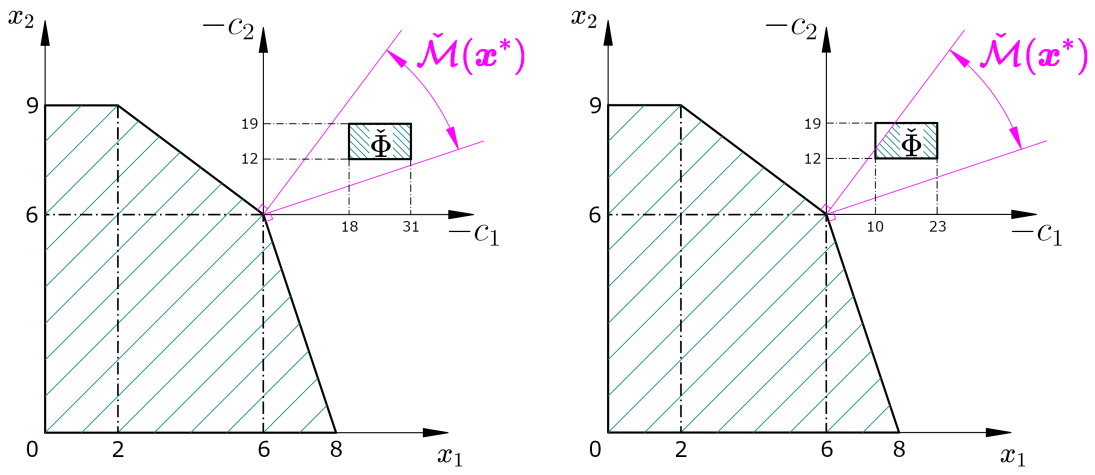
$$3x_1 + 4x_2 \leq 42, \ 3x_1 + x_2 \leq 24, \ x_2 \leq 9, \ x_1 \geq 0, \ x_2 \geq 0.$$

Moreover, due to $c_3 = c_4 = c_5 = 0$, the projection of $\mathscr{M}(\boldsymbol{x}^*)$ on the c_1-c_2 coordinate system is obtained as

$$\check{\mathcal{M}}(\boldsymbol{x}^*) = \left\{ (c_1, c_2)^{\mathrm{T}} \;\middle|\; \begin{array}{c} \frac{1}{9}c_1 - \frac{1}{3}c_2 \geq 0 \\ -\frac{4}{9}c_1 + \frac{1}{3}c_2 \geq 0 \end{array} \right\}. \tag{2.12}$$

To illustrate the possible and necessary optimalities, we give two cases (a) and (b) shown in Figure 2.1.

When $\check{\Phi}$ is given as shown in Figure 2.1a, we know that $\boldsymbol{x}^*$ is necessarily optimal because $\check{\Phi} \subseteq \check{\mathscr{M}}(\boldsymbol{x}^*)$ implies $\Phi \subseteq \mathscr{M}(\boldsymbol{x}^*)$. On the other hand, when $\check{\Phi}$ is given as shown in Figure 2.1b, we know that $\boldsymbol{x}^*$ is only possibly optimal because $\check{\Phi} \cap \check{\mathcal{M}}(\boldsymbol{x}^*) \neq \emptyset$ implies $\Phi \cap \mathcal{M}(\boldsymbol{x}^*) \neq \emptyset$ but there exists $(c_1, c_2)^{\mathrm{T}} \in \check{\Phi}$ such that $(c_1, c_2)^{\mathrm{T}} \notin \check{\mathcal{M}}(\boldsymbol{x}^*)$.



(a) Necessary Optimality

(b) Possible Optimality

Figure 2.1: Necessarily and Possibly Optimal Solutions

Since the possible optimality satisfies the minimal requirement for a feasible solution while the necessary optimality satisfies the ideal requirement [24], we consider obtaining a necessarily optimal solution as far as it exists. We have shown the necessary and sufficient condition for an NBF solution being necessarily optimal as (2.10), which can be extended to the following proposition if Φ is a convex polytope:

Proposition 2.3. *Consider Problem (2.3) with $\boldsymbol{\gamma} \in \Phi$, where Φ is a convex polytope composed of all conceivable OFC vectors. Let $\mathbf{V}(\Phi)$ denote the vertex set of Φ. Then an NBF solution $\boldsymbol{x}^*$ is necessarily optimal if and only if $\mathbf{V}(\Phi) \subseteq \mathscr{M}(\boldsymbol{x}^*)$.*

Proof. Since Φ is a convex polytope, it can be represented by

$$\Phi = \mathrm{Conv}(\mathbf{V}(\Phi)) = \left\{ \boldsymbol{c} \ : \ \boldsymbol{c} = \sum_{k=1}^{l} \lambda_k \boldsymbol{c}^k, \ \ \sum_{k=1}^{l} \lambda_k = 1, \ \lambda_k \geq 0, \ k = 1, 2, \ldots, l \right\},$$

where $\mathrm{Conv}(\Phi)$ is the convex hull of Φ. Then, $\Phi \subseteq \mathcal{M}(\boldsymbol{x}^*)$ implies $\mathbf{V}(\Phi) \subseteq \Phi \subseteq \mathcal{M}(\boldsymbol{x}^*)$. On the other hand, $\mathbf{V}(\Phi) \subseteq \mathcal{M}(\boldsymbol{x}^*)$ implies $\Phi = \mathrm{Conv}(\Phi) \subseteq \mathcal{M}(\boldsymbol{x}^*)$ because $\mathcal{M}(\boldsymbol{x}^*)$ is convex. □

Together with (2.8) in Proposition 2.2, Proposition 2.3 indicates that the necessary optimality of an NBF solution $\boldsymbol{x}^*$ can be confirmed by checking $\boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{v}^i \geq \mathbf{0}$ for all components $\boldsymbol{v}^i \in \mathbf{V}(\Phi)$ when Φ is a convex polytope.

However, this procedure can require a lot of computational effort. For example, if the range of each OFC is given by a closed interval, which eventually forms a hyperrectangle in $\mathbb{R}^n$, one needs to check the inequalities for all 2^n vertices of Φ. When n becomes large enough, the computation cannot be done in a reasonable time.

2.1.3 The tolerance approach

To overcome the computational difficulty, Wendell [40] proposed the *tolerance analysis*. Let the range of each OFC be given by a closed interval $[c_i^{\mathrm{L}}, c_i^{\mathrm{R}}]$, $i \in \{1, 2, \ldots, n\}$, and let a solution $\boldsymbol{x}^*$ be obtained with the OFC vector $\boldsymbol{c}^{\mathrm{C}} = (\boldsymbol{c}^{\mathrm{L}} + \boldsymbol{c}^{\mathrm{R}})/2$, where $\boldsymbol{c}^{\mathrm{L}} = (c_1^{\mathrm{L}}, \ldots, c_n^{\mathrm{L}})^{\mathrm{T}}$ and $\boldsymbol{c}^{\mathrm{R}} = (c_1^{\mathrm{R}}, \ldots, c_n^{\mathrm{R}})^{\mathrm{T}}$. If $\boldsymbol{x}^*$ is an NBF solution, then the tolerance approach can examine the condition $\mathbf{V}(\Phi) \subseteq \mathscr{M}(\boldsymbol{x}^*)$ in Proposition 2.3 straightforwardly.

Before the tolerance approach, we describe a lemma shown by Filippi [5]:

Lemma 2.1. *Let $\mathcal{P}^0 = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}^0\boldsymbol{x} \leq \boldsymbol{b}^0\}$ be a polyhedron with $l \times n$ matrix $\boldsymbol{A}^0 = (a_{kj}^0) \neq \boldsymbol{O}$ and $\boldsymbol{b}^0 \geq \boldsymbol{0}$, where $\boldsymbol{O}$ is a zero matrix. For $k = 1, 2, \ldots, l$, we define*

$$\sigma_k = \begin{cases} \dfrac{b_k^0}{\sum_{j=1}^n |a_{kj}^0|}, & \text{if } \sum_{j=1}^n |a_{kj}^0| > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \sigma(\mathcal{P}^0) = \min_{\substack{k=1,2,\ldots,l \\ \sum_{j=1}^n |a_{kj}^0| > 0}} \sigma_k. \tag{2.13}$$

Then, a set $\mathscr{B} \subseteq \mathbb{R}^n$ defined by

$$\mathscr{B} = \left\{ \boldsymbol{s} = (s_1, s_2, \ldots, s_n)^{\mathrm{T}} : s_i \in [-\sigma(\mathcal{P}^0), \sigma(\mathcal{P}^0)] \right\} \tag{2.14}$$

is the largest hyper-box contained in $\mathcal{P}^0$.

Applying Wendell's idea [40], we consider an OFC vector $\boldsymbol{\gamma} = \boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{c}^{\mathrm{S}})\boldsymbol{\tau}$ for Problem (2.3), where $\boldsymbol{c}^{\mathrm{S}} = (\boldsymbol{c}^{\mathrm{R}} - \boldsymbol{c}^{\mathrm{L}})/2$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)^{\mathrm{T}}$ denote the parameter vectors. Let $\boldsymbol{x}^*$ denote the optimal NBF solution with respect to the OFC vector $\boldsymbol{c}^{\mathrm{C}}$, then we obtain $\boldsymbol{c}_N^{\mathrm{C}} - A_N^{\mathrm{T}} A_B^{-\mathrm{T}} \boldsymbol{c}_B^{\mathrm{C}} = \boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{c}^{\mathrm{C}} \geq \boldsymbol{0}$ by Proposition 2.1.

Therefore, by Definition 2.1, we obtain the set of $\boldsymbol{\tau}$ that preserves the optimality of $\boldsymbol{x}^*$ where the OFC vector is perturbed by $\boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{c}^{\mathrm{S}})\boldsymbol{\tau}$ as:

$$\begin{aligned} \mathscr{M}_\tau(\boldsymbol{x}^*) &= \{\boldsymbol{\tau} \in \mathbb{R}^n : \boldsymbol{M}(\boldsymbol{x}^*)(\boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{c}^{\mathrm{S}})\boldsymbol{\tau}) \geq \boldsymbol{0}\} \\ &= \{\boldsymbol{\tau} \in \mathbb{R}^n : -\boldsymbol{M}(\boldsymbol{x}^*)\mathrm{diag}(\boldsymbol{c}^{\mathrm{S}})\boldsymbol{\tau} \leq \boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{c}^{\mathrm{C}}\}, \end{aligned} \tag{2.15}$$

and by applying Lemma 2.1, we obtain the following theorem:

Theorem 2.1. *For $k = 1, 2, \ldots, n-m$ and let $\boldsymbol{M}(\boldsymbol{x}^*) = (m_{ij})(\boldsymbol{x}^*)$, we define*

$$\tau_k = \begin{cases} \dfrac{\sum_{j=1}^{n} m_{kj}(\boldsymbol{x}^*) c_j^{\mathrm{C}}}{\sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)||c_j^{\mathrm{S}}|}, & \text{if } \sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)||c_j^{\mathrm{S}}| > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.16}$$

$$\tau^{\min} = \min_{\substack{k=1,2,\ldots,n-m \\ \sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)||c_j^{\mathrm{S}}| > 0}} \tau_k. \tag{2.17}$$

Then, the set $\mathscr{B}_{\tau^{\min}} := \left\{(s_1, s_2, \ldots, s_n)^{\mathrm{T}} : s_i \in [-\tau^{\min}, \tau^{\min}],\ i = 1, 2, ..., n\right\} \subseteq \mathbb{R}^n$ becomes the largest hyperrectangle contained in $\mathcal{M}_\tau(\boldsymbol{x}^*)$.

Proof. It can be obtained straightforwardly by applying Lemma 2.1 to $\mathcal{M}_\tau(\boldsymbol{x}^*)$. □

Theorem 2.1 is proved by Wendell [40]. From Theorem 2.1, we obtain the following theorem which shows the relation to the necessary optimality.

Theorem 2.2. *Let $\tau^{\min}$ denote the result in Theorem 2.1. If $\tau^{\min} \geq 1$, then $\boldsymbol{x}^*$ is a necessarily optimal solution to the LP problem with uncertain OFC vector $\boldsymbol{\gamma}$, whose range is given by the hyperrectangle defined with $[c_i^{\mathrm{L}}, c_i^{\mathrm{R}}]$, $i = 1, 2, \ldots, n$.*

Proof. When $\tau^{\min} = 1$, we have $\mathscr{B}_{\tau^{\min}} \subseteq \mathscr{M}_\tau(\boldsymbol{x}^*)$, which is equivalent to

$$\Phi = [\boldsymbol{c}^{\mathrm{L}}, \boldsymbol{c}^{\mathrm{R}}] = \{\boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{c}^{\mathrm{S}})\boldsymbol{\tau} : \boldsymbol{\tau} \in [-1, 1]^n\} \subseteq \mathcal{M}(\boldsymbol{x}^*)$$

by the substitution of $\boldsymbol{c} = \boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{c}^{\mathrm{S}})\boldsymbol{\tau}$ resulting from (2.15). Hence, we obtain $\Phi \subseteq \mathcal{M}(\boldsymbol{x}^*)$, which implies $\boldsymbol{x}^*$ is necessarily optimal by (2.10). □

To illustrate the utilisation of Theorem 2.2, we use the following example similar to Example 2.1:

Example 2.2. Reconsider the LP problem in Example 2.1, where c_1 and c_2 are uncertain coefficients satisfying $c_1 \in [-31, -15]$ and $c_2 \in [-19, -9]$. Check whether there exists a necessarily optimal solution.

In this case, we obtain $\boldsymbol{c}^{\mathrm{C}} = (-23, -14)^{\mathrm{T}}$ and $\boldsymbol{c}^{\mathrm{S}} = (8, 5)^{\mathrm{T}}$ with an NBF solution $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$. Since the solution is the same as Example 2.1, we use the result of (2.11) directly. By applying Theorem 2.2, we obtain $\tau_1 = 19/23$ and $\tau_2 = 50/47$, resulting $\tau^{\min} = 19/23 < 1$. Therefore, $\boldsymbol{x}^*$ is not necessarily optimal. If we ignore the slack variable x_3, x_4 and x_5 and draw the problem in Figure 2.2, the result can be confirmed since the vertex $(-31, -9)^{\mathrm{T}}$ is out of $\mathscr{M}(\boldsymbol{x}^*)$.

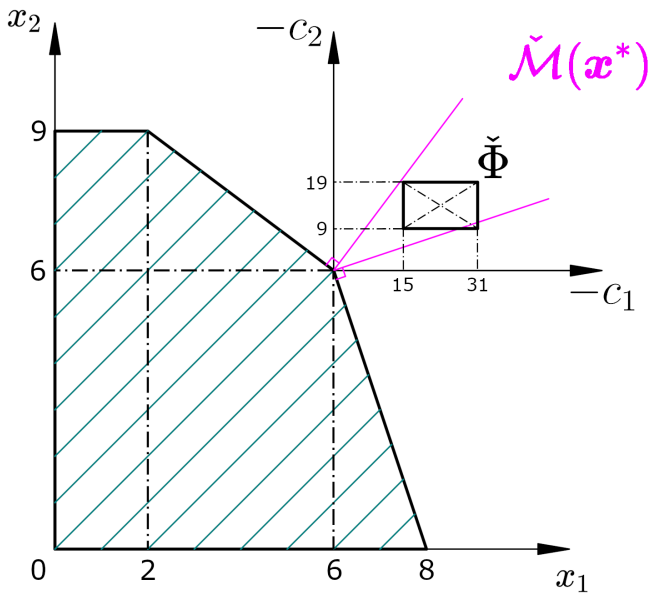


Figure 2.2: Illustration of Example 2.2

2.2 Fuzzy Subsets and Fuzzy Numbers

2.2.1 Fuzzy subsets and *h*-level sets

We give the definition of a fuzzy subset (see Nguyen et al. [34]):

Definition 2.3. A fuzzy subset $\tilde{A}$ of the universal set X is characterised by a function $\mu_{\tilde{A}} : \tilde{A} \to [0, 1]$, so-called the membership function of $\tilde{A}$.

Since the (weak) h-level set and strong h-level set of a fuzzy subset are useful to the analysis of fuzzy systems, we give their definitions:

Definition 2.4. Given a fuzzy subset $\tilde{A}$ of the universal set X, its (weak) h-level set, denoted as $[\tilde{A}]_h$, and strong h-level set, denoted as $(\tilde{A})_h$, are defined as:

$$[\tilde{A}]_h := \{r \in X : \mu_{\tilde{A}}(r) \geq h\}, \text{ for } h \in (0, 1], \tag{2.18}$$

$$(\tilde{A})_h := \{r \in X : \mu_{\tilde{A}}(r) > h\}, \text{ for } h \in [0, 1). \tag{2.19}$$

For the sake of convenience, we define 0-level set, $[\tilde{A}]_0 := X$, and strong 1-level set, $(\tilde{A})_1 := \emptyset$.

To describe the connection between the membership function and the h-level sets of a fuzzy subset $\tilde{A}$, we introduce the *resolution principle* [4].

Proposition 2.4. *(Resolution Principle) Given h-level sets $[\tilde{A}]_h$, $h \in (0,1]$ of a fuzzy subset $\tilde{A}$, we can obtain its membership function by*

$$\mu_{\tilde{A}}(x) = \sup_{h\in[0,1]} \min\left(h, \chi_{[\tilde{A}]_h}(x)\right), \ \forall x \in X, \tag{2.20}$$

where $\chi_{[\tilde{A}]_h}$ is the characteristic function of $[\tilde{A}]_h$, i.e. $\chi_{[\tilde{A}]_h}(x) = 1$ for $x \in [\tilde{A}]_h$ and $\chi_{[\tilde{A}]_h}(x) = 0$ for $x \notin [\tilde{A}]_h$. Similarly, given the strong h-level set $(\tilde{A})_h$, $h \in [0,1)$ of a fuzzy subset $\tilde{A}$, we can obtain its membership function by

$$\mu_{\tilde{A}}(x) = \sup_{h\in[0,1]} \min\left(h, \chi_{(\tilde{A})_h}(x)\right), \ \forall x \in X, \tag{2.21}$$

where $\chi_{(\tilde{A})_h}$ is the characteristic function of $(\tilde{A})_h$, i.e. $\chi_{(\tilde{A})_h}(x) = 1$ for $x \in (\tilde{A})_h$ and $\chi_{(\tilde{A})_h}(x) = 0$ for $x \notin (\tilde{A})_h$.

2.2.2 Fuzzy numbers

A fuzzy subset of $\mathbb{R}$, called a *fuzzy quantity*, has many types of expressions. In this thesis, we focus on the fuzzy numbers, especially *L-R fuzzy numbers* [4], which are characterised by parameters and functions specifying the shapes of the membership function.

A fuzzy number $\tilde{A}$ [4] can be defined by its h-level set satisfying (i) $[\tilde{A}]_1 \neq \emptyset$, (ii) $[\tilde{A}]_h$ is closed for all $h \in (0,1]$, (iii) $[\tilde{A}]_h$ is convex for all $h \in (0,1]$ and (iv) $[\tilde{A}]_h$ is bounded for all $h \in (0,1]$. These conditions are equivalent to the following conditions by using the membership function $\mu_{\tilde{A}}$, respectively: (i) $\exists\, r \in \mathbb{R},\ \mu_{\tilde{A}}(r) = 1$, (ii) $\mu_{\tilde{A}}$ is upper semi-continuous [31], (iii) $\mu_{\tilde{A}}$ is quasi-concave [2] and (iv) $\lim_{r\to\infty} \mu_{\tilde{A}}(r) = \lim_{r\to-\infty} \mu_{\tilde{A}}(r) = 0$.

Since an L-R fuzzy number is much easier in their calculations, its definition is given as follows [4].

Definition 2.5. An *L-R fuzzy number* $\tilde{A} \subseteq \mathbb{R}$ is a fuzzy subset of the real line $\mathbb{R}$ characterised by the following membership function:

$$\mu_{\tilde{A}}(r) = \begin{cases} L\left(\dfrac{a^{\mathrm{L}} - r}{\alpha}\right), & \text{if } r < a^{\mathrm{L}},\ \alpha > 0, \\ 1, & \text{if } a^{\mathrm{L}} \leq r \leq a^{\mathrm{R}}, \\ R\left(\dfrac{r - a^{\mathrm{R}}}{\beta}\right), & \text{if } r > a^{\mathrm{R}},\ \beta > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.22}$$

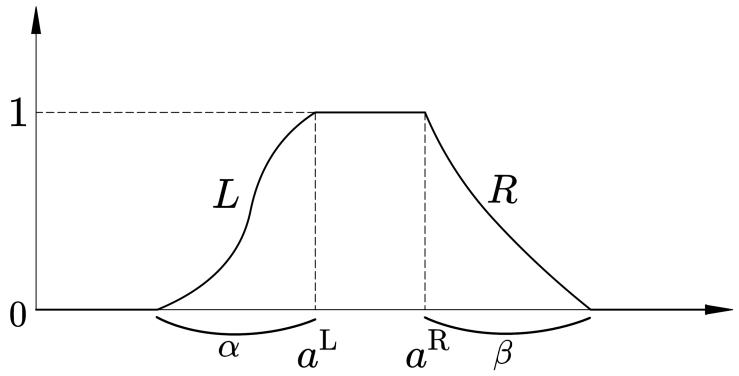


Figure 2.3: Illustration of an L-R fuzzy number $\tilde{A} = (a^{\mathrm{L}}, a^{\mathrm{R}}, \alpha, \beta)_{LR}$

where a^{L} and a^{R}, satisfying $a^{\mathrm{L}} \leq a^{\mathrm{R}}$, are the lower and upper bounds of $[\tilde{A}]_1$, respectively. $\alpha \geq 0$ and $\beta \geq 0$ are the left and right spreads of $\tilde{A}$, respectively. $L : [0, \infty) \to [0, 1]$ and $R : [0, \infty) \to [0, 1]$ are the reference functions satisfying

1. $L(0) = R(0) = 1$, and $\forall r > 0$, $L(r)$, $R(r) < 1$.
2. $\lim_{r\to\infty} L(r) = \lim_{r\to\infty} R(r) = 0$.
3. L and R are upper semi-continuous and non-increasing.

For simplification, we denote an L-R fuzzy number $\tilde{A}$ as $(a^{\mathrm{L}}, a^{\mathrm{R}}, \alpha, \beta)_{LR}$ by the parameters and reference functions.

An L-R fuzzy number is illustrated in Figure 2.3. When its reference functions L and R are the same, i.e., $L = R$, we call it an L-L *fuzzy number* and denote it as $(a^{\mathrm{L}}, a^{\mathrm{U}}, \alpha, \beta)_{LL}$. Moreover, if both $L = R$ and $\alpha = \beta$, we call it a *symmetric fuzzy number* and denote it as $(a^{\mathrm{L}}, a^{\mathrm{U}}, \alpha, \alpha)_{LL}$.

We note that any fuzzy number $\tilde{A}$ can be represented by an L-R fuzzy number $(a^{\mathrm{L}}, a^{\mathrm{R}}, \alpha, \beta)_{LR}$ with

$$\begin{aligned} a^{\mathrm{L}} &= \inf[\tilde{A}]_1, \quad \alpha = \begin{cases} \inf[\tilde{A}]_1 - \inf(\tilde{A})_0, & \text{if } \inf(\tilde{A})_0 > -\infty, \\ \inf[\tilde{A}]_1 - \inf[\tilde{A}]_\epsilon, & \text{otherwise}, \end{cases} \\ a^{\mathrm{R}} &= \sup[\tilde{A}]_1, \quad \beta = \begin{cases} \sup(\tilde{A})_0 - \sup[\tilde{A}]_1, & \text{if } \sup[\tilde{A}]_0 < \infty, \\ \sup[\tilde{A}]_\epsilon - \sup[\tilde{A}]_1, & \text{otherwise}, \end{cases} \end{aligned} \tag{2.23}$$

and

$$L(r) = \mu_{\tilde{A}}(a^{\mathrm{L}} - \alpha r), \ r \geq 0; \quad R(r) = \mu_{\tilde{A}}(\alpha r - a^{\mathrm{R}}), \ r \geq 0, \tag{2.24}$$

where $\epsilon > 0$ denotes a very small positive constant.

Let $L^* : [0,1] \to [0,+\infty]$ and $R^* : [0,1] \to [0,+\infty]$ denote pseudo-inverse functions of reference functions L and R, respectively. By (2.22) in Definition 2.5, we define them as:

$$\begin{aligned} L^*(h) &= \sup\{r \in [0,+\infty) : L(r) \geq h\}, \\ R^*(h) &= \sup\{r \in [0,+\infty) : R(r) \geq h\}. \end{aligned} \tag{2.25}$$

By using these pseudo-inverse functions, we have the following proposition:

Proposition 2.5. *A fuzzy subset $\tilde{A}$ can be denoted as $\tilde{A} = (a^{\mathrm{L}}, b^{\mathrm{R}}, \alpha, \beta)_{LR}$ if and only if the following condition is valid:*

$$\forall h \in (0,1],\ [\tilde{A}]_h = \left[a^{\mathrm{L}} - L^*(h)\alpha,\ a^{\mathrm{R}} + R^*(h)\beta\right]. \tag{2.26}$$

Proof. Let $\tilde{A}$ be an L-R fuzzy number specified by $(a^{\mathrm{L}}, b^{\mathrm{R}}, \alpha, \beta)_{LR}$. By the definitions of L^* and R^* in (2.25), we obtain that $\forall\, h \in (0,1],\ \mu_{\tilde{A}}(r) \geq h$ if and only if r satisfies at least one of the following conditions:

$$\begin{aligned} &\text{(a)}\ \ L\left(\frac{a^{\mathrm{L}} - r}{\alpha}\right) \geq h,\ r < a^{\mathrm{L}} \text{ and } \alpha > 0, \\ &\text{(b)}\ \ R\left(\frac{r - a^{\mathrm{R}}}{\beta}\right) \geq h,\ r > a^{\mathrm{R}} \text{ and } \beta > 0, \\ &\text{(c)}\ \ a^{\mathrm{L}} \leq r \leq a^{\mathrm{R}}, \end{aligned}$$

where condition (a) is equivalent to $a^{\mathrm{L}} - L^*(h)\alpha \leq r < a^{\mathrm{L}}$ and $\alpha > 0$, and condition (b) is equivalent to $a^{\mathrm{R}} < r \leq a^{\mathrm{R}} + R^*(h)\beta$ and $\beta > 0$.

Since we have $\alpha \geq 0$ and $\beta \geq 0$ by the definition of L-R fuzzy numbers, Condition (a), (b) and (c) give the result $r \in [a^{\mathrm{L}} - L^*(h)\alpha,\ a^{\mathrm{R}} + R^*(h)\beta]$. Therefore, we obtain $\forall\, h \in (0,1],\ [\tilde{A}]_h = [a^{\mathrm{L}} - L^*(h)\alpha,\ a^{\mathrm{R}} + R^*(h)\beta]$.

On the other hand, if we know $\forall\, h \in (0,1],\ [\tilde{A}]_h = [a^{\mathrm{L}} - L^*(h)\alpha,\ a^{\mathrm{R}} + R^*(h)\beta]$, we obtain $\mu_{\tilde{A}}(x) = \sup\{h : x \in [\tilde{A}]_h\}$, which becomes the membership function of $\tilde{A} = (a^{\mathrm{L}}, b^{\mathrm{R}}, \alpha, \beta)_{LR}$ by the resolution principle in Proposition 2.4. □

We can also define pseudo-inverse functions of reference functions L and R in a different way with $L_* : (0,1] \to [0,+\infty)$ and $R_* : (0,1] \to [0,+\infty)$ such that

$$\begin{aligned} L_*(h) &= \inf\{r \in [0,+\infty) : L(r) \leq h\}, \\ R_*(h) &= \inf\{r \in [0,+\infty) : R(r) \leq h\}. \end{aligned} \tag{2.27}$$

By using the differently defined pseudo-inverse functions, the closure of the strong h-level set of the L-R fuzzy number $\tilde{A} = (a^{\mathrm{L}}, b^{\mathrm{R}}, \alpha, \beta)_{LR}$ can be obtained by

$$\mathrm{cl}[\tilde{A}]_h = \left[a^{\mathrm{L}} - L_*(h)\alpha, a^{\mathrm{R}} + R_*(h)\beta\right], \ \forall\, h \in [0,1). \tag{2.28}$$

The difference between $L^*(h)$ and $L_*(h)$ is depicted in Figure 2.4. As shown in Figure 2.4, the difference appears at the level of the flat part of reference function L.

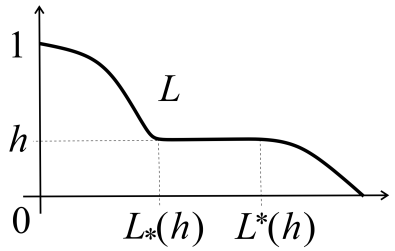


Figure 2.4: Pseudo-inverse functions L^* and L_* of reference function L

We note that $L_*(h)$ defined by (2.27) takes a different value from $L^*(h)$ defined by (2.25) at $h \in [0,1)$ when there exists an interval $[r_1, r_2] \subseteq [0,\infty)$ such that $r_1 < r_2$ and $L(r_1) = L_(r_2) = h$ (see Figure 2.4). We always have $\forall\, h \in [0,1),\ L_*(h) \le L^*(h)$. When $L : [0,\infty) \to [0,1]$ is strictly decreasing, we have $\forall\, h \in [0,1),\ L_*(h) = L^*(h) = L^{-1}(h)$. The difference between $R_*(h)$ and $R^*(h)$ is the same as that between $L_*(h)$ and $L^*(h)$.

2.2.3 Extension principle

For the calculation of function values of fuzzy quantities, the following *extension principle* [4] is defined.

Definition 2.6. Given a function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, the function value $\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ of fuzzy quantities $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ is a fuzzy subset of $\mathbb{R}^m$ defined by the following membership function:

$$\mu_{\varphi(\tilde{A}_1,\ldots,\tilde{A}_n)}(\boldsymbol{y}) = \begin{cases} \sup\limits_{\boldsymbol{x}\in\varphi^{-1}(\boldsymbol{y})} \min(\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)), & \text{if } \varphi^{-1}(\boldsymbol{y}) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{2.29}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $\varphi^{-1}(\boldsymbol{y})$ is the inverse image of φ defined by $\varphi^{-1}(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}) = \boldsymbol{y}\}$.

From the extension principle given in Definition 2.6, we have the following properties:

$$\left(\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)\right)_h = \varphi\left((\tilde{A}_1)_h, (\tilde{A}_2)_h, \ldots, (\tilde{A}_n)_h\right), \ \forall h \in [0,1), \tag{2.30}$$

$$\left[\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)\right]_h \supseteq \varphi\left([\tilde{A}_1]_h, [\tilde{A}_2]_h, \ldots, [\tilde{A}_n]_h\right), \quad \forall h \in (0,1]. \tag{2.31}$$

The left-hand side of (2.30) is the strong h-level set of $\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ while the right-hand side is the function value of the strong h-level set of $\tilde{A}_i$, $i = 1, 2, \ldots, n$. Namely, the strong h-level set of $\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ can be calculated by the function values of strong h-level sets $(\tilde{A}_i)_h$, $i = 1, 2, \ldots, n$. The calculation in the right-hand side is easier than that in the left-hand side because the subset in the right-hand side is obtained by the calculation of crisp subsets. Unfortunately, in the case of (weak) h-level set, we obtain only the inclusion relation.

The following proposition shows that the calculation of (weak) h-level sets of $\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ is calculated by the function values of the h-level sets $[\tilde{A}_i]_h$, $i = 1, 2, \ldots, n$ as long as φ is continuous and $\tilde{A}_i$,$i = 1, 2, \ldots, n$ are L-R fuzzy numbers (see Dubois and Prade [4]).

Proposition 2.6. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then for L-R fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$,..., $\tilde{A}_n$ defined by Definition 2.5, we have*

$$\left[\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)\right]_h = \varphi\left([\tilde{A}_1]_h, [\tilde{A}_2]_h, \ldots, [\tilde{A}_n]_h\right), \ \forall h \in (0,1]. \tag{2.32}$$

By Propositions 2.5 and 2.6, the h-level set of $\varphi(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ can be calculated straightforwardly by the function value of closed intervals $[a_i^{\mathrm{L}} - L^*(h)\alpha_i^{\mathrm{L}}, \ a_i^{\mathrm{R}} + R^*(h)\alpha_i^{\mathrm{R}}]$ when f is continuous and $\tilde{A}_i = (a_i^{\mathrm{L}}, a_i^{\mathrm{R}}, \alpha_i^{\mathrm{L}}, \alpha_i^{\mathrm{R}})_{LR}$, $i = 1, 2, \ldots, n$.

Proposition 2.6 enables us to consider the calculation of $\tilde{\boldsymbol{A}}\boldsymbol{x}$, where $\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})$ is an $m \times n$ matrix composed of L-R fuzzy number components $\tilde{a}_{ij} = (a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{R}}, \alpha_{ij}^{\mathrm{L}}, \alpha_{ij}^{\mathrm{R}})_{LR}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $\boldsymbol{x} \in \mathbb{R}^n$. Since the weighted sum is continuous, we apply Proposition 2.6 when $\boldsymbol{x}^*$ and obtain

$$\begin{aligned}
&\left[\sum_{j=1}^{n} \tilde{a}_{ij} x_j\right]_h = \sum_{j=1}^{n} [\tilde{a}_{ij}]_h \, x_j \\
&= \left[\sum_{j=1}^{n} (a_{ij}^{\mathrm{L}} - L^*(h)\alpha_{ij}^{\mathrm{L}}) x_j, \ \sum_{j=1}^{n} (a_{ij}^{\mathrm{R}} + R^*(h)\alpha_{ij}^{\mathrm{R}}) x_j\right], \ \forall h \in [0,1], \\
&= \left[\sum_{j=1}^{n} a_{ij}^{\mathrm{L}} x_j - L^*(h) \sum_{j=1}^{n} \alpha_{ij}^{\mathrm{L}} x_j, \ \sum_{j=1}^{n} a_{ij}^{\mathrm{R}} x_j + R^*(h) \sum_{j=1}^{n} \alpha_{ij}^{\mathrm{R}} x_j\right], \ \forall h \in [0,1],
\end{aligned} \tag{2.33}$$

where $h = 0$ due to $L^*(0) = R^*(0) = \infty$.

From Proposition 2.5, when $\boldsymbol{x} \geq \mathbf{0}$, we obtain

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j = \left(\sum_{j=1}^{n} a_{ij}^{\mathrm{L}} x_j, \sum_{j=1}^{n} a_{ij}^{\mathrm{R}} x_j, \sum_{j=1}^{n} \alpha_{ij}^{\mathrm{L}} x_j, \sum_{j=1}^{n} \alpha_{ij}^{\mathrm{R}} x_j \right)_{LR}, \ i = 1, 2, \ldots, m. \tag{2.34}$$

(2.34) indicates that $\tilde{\boldsymbol{A}}\boldsymbol{x}$ with $\boldsymbol{x} \geq \mathbf{0}$ is an L-R fuzzy number.

Similarly, when $\boldsymbol{x} \geq \mathbf{0}$, we obtain

$$\begin{aligned}
&\mathrm{cl} \left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \right)_h = \sum_{j=1}^{n} \mathrm{cl} \left(\tilde{a}_{ij} \right)_h x_j \\
&= \left[\sum_{j=1}^{n} (a_{ij}^{\mathrm{L}} - L_*(h) \alpha_{ij}^{\mathrm{L}}) x_j, \ \sum_{j=1}^{n} (a_{ij}^{\mathrm{R}} + R_*(h) \alpha_{ij}^{\mathrm{R}}) x_j \right], \ \forall h \in [0, 1), \\
&= \left[\sum_{j=1}^{n} a_{ij}^{\mathrm{L}} x_j - L_*(h) \sum_{j=1}^{n} \alpha_{ij}^{\mathrm{L}} x_j, \ \sum_{j=1}^{n} a_{ij}^{\mathrm{R}} x_j + R_*(h) \sum_{j=1}^{n} \alpha_{ij}^{\mathrm{R}} x_j \right], \ \forall h \in [0, 1),
\end{aligned} \tag{2.35}$$

where $L_* : [0, 1) \to [0, \infty) \cup \{\infty\}$ and $R_* : [0, 1) \to [0, \infty) \cup \{\infty\}$ are pseudo-inverse functions defined by (2.27).

2.3 Possibility and Necessity Measures

Consider two crisp subsets A and B. A possibility (resp. necessity) measure evaluates to what extent a statement $r \in B$ is possible (resp. necessary) when we know $r \in A$. To put it differently, a possibility (resp. necessity) measure shows the possible (necessary) degree that the condition $r \in B$ is satisfied when we know $r \in A$. Accordingly, the *possibility and necessity measures* are defined as follows.

Definition 2.7. Given two crisp subsets A and B, the possibility measure $\Pi_A(B)$ and necessity measure $N_A(B)$ are defined by

$$\Pi_A(B) = \begin{cases} 1, \text{ if } A \cap B \neq \emptyset, \\ 0, \text{ otherwise}, \end{cases} \tag{2.36}$$

$$N_A(B) = \begin{cases} 1, \text{ if } A \subseteq B, \\ 0, \text{ otherwise}. \end{cases} \tag{2.37}$$

The possibility and necessity measures are generalised to the situation where the given information and statement are expressed by fuzzy subsets $\tilde{A}$ and $\tilde{B}$ (see Inuiguchi et al. [26]).

Definition 2.8. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of a universal set X with membership functions $\mu_{\tilde{A}} : X \to [0,1]$ and $\mu_{\tilde{B}} : X \to [0,1]$, respectively. Then the *possibility and necessity measures* of $\tilde{B}$ under $\tilde{A}$ are defined respectively as:

$$\Pi_{\tilde{A}}(\tilde{B}) = \sup_{r\in\Omega} \min\left(\mu_{\tilde{A}}(r),\ \mu_{\tilde{B}}(r)\right), \tag{2.38}$$

$$N_{\tilde{A}}(\tilde{B}) = \inf_{r\in\Omega} \max\left(1-\mu_{\tilde{A}}(r),\ \mu_{\tilde{B}}(r)\right). \tag{2.39}$$

$\Pi_{\tilde{A}}(\tilde{B})$ shows the possibility degree that a variable r is $\tilde{B}$ when we know r is $\tilde{A}$. On the other hand, $N_{\tilde{A}}(\tilde{B})$ shows the necessity degree that a variable r is $\tilde{B}$ when we know r is $\tilde{A}$.

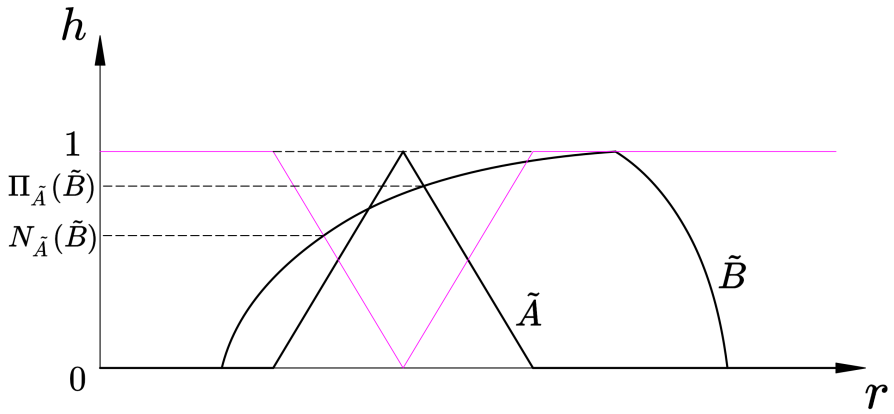


Figure 2.5: Possibility and Necessity Measures: $\Pi_{\tilde{A}}(\tilde{B})$ and $N_{\tilde{A}}(\tilde{B})$

Possibility and necessity measures of fuzzy subset $\tilde{B}$ under fuzzy subset $\tilde{A}$ are depicted in Figure 2.5. They are closely related to the non-emptiness of the intersection $\tilde{A} \cap \tilde{B}$ and the inclusion $\tilde{A} \subseteq \tilde{B}$. These relations are remarkable by the following proposition [19].

Proposition 2.7. *We have the following equivalences:*

$$\Pi_{\tilde{A}}(\tilde{B}) > h \ \Leftrightarrow\ (\tilde{A})_h \cap (\tilde{B})_h \neq \emptyset,\ \forall h \in [0,1), \tag{2.40}$$

$$N_{\tilde{A}}(\tilde{B}) \geq h \ \Leftrightarrow\ (\tilde{A})_{1-h} \subseteq [\tilde{B}]_h,\ \forall h \in (0,1]. \tag{2.41}$$

Since we use the necessity measure in the following chapters, we give the following proposition directly derived from Proposition 2.7 [19]:

Proposition 2.8. *The necessity measure of (2.41) can be represented as:*

$$N_{\tilde{A}}(\tilde{B}) = \sup\left\{h \in [0,1] : (\tilde{A})_{1-h} \subseteq [\tilde{B}]_h\right\}. \tag{2.42}$$

Chapter 3

Robust Optimality Analysis of the LP Problem with fuzzy OFCs

IN this chapter, we study LP problems with fuzzy OFCs. We utilise the necessity measure to evaluate the necessary optimality degree of an NBF solution under fuzzy OFCs. We define the non-interactive and oblique fuzzy vectors and investigate the necessary optimality test problem of an NBF solution. Namely, we calculate the necessary optimality degree of an NBF solution. We show the following results. When fuzzy OFCs are given by a non-interactive fuzzy vector, the test problem is solved by the bisection method with the tolerance approach. Moreover, when the non-interactive fuzzy vector is defined by the same reference function, the test problem is solved simply by the tolerance approach. These results are applicable to the case where the fuzzy OFCs are given by an oblique fuzzy vector with the OAC transformed by the inverse of the obliquity matrix. Numerical examples are given to illustrate the obtained results. Through these results, it is shown that the tolerance approach solves necessity optimality test problems of NBF solutions rather simply under non-interactive and oblique fuzzy vectors.

3.1 Fuzzy Vectors

In literature, a *fuzzy vector* composed of fuzzy numbers is frequently used to represent a conceivable range of the uncertain variable vector. If the conceivable range of each component of the uncertain variable vector is defined independently from the value of other components, it is called a *non-interactive fuzzy vector*. Mathematically, a non-interactive fuzzy vector $\tilde{\boldsymbol{A}} \subseteq \mathbb{R}^n$ is defined by fuzzy numbers $\tilde{A}_i$, $i = 1, \ldots, n$ by the

following membership function:

$$\mu_{\tilde{\boldsymbol{A}}}(\boldsymbol{r}) = \min_{i=1,\ldots,n} \mu_{\tilde{A}_i}(r_i), \tag{3.1}$$

where $\boldsymbol{r} = (r_1, \ldots, r_n)^{\mathrm{T}} \subseteq \mathbb{R}^n$ and $\mu_{\tilde{A}_i}$ denotes the membership function of $\tilde{A}_i$, $i = 1, \ldots, n$.

In general, the conceivable range of an uncertain variable may depend on the values of other uncertain variables. In such a case, we say that uncertain variables are interactive. In the case of non-interactive fuzzy vectors, the conceivable range of each uncertain variable is represented by a fuzzy number independent of other uncertain variable values. Therefore, a non-interactive fuzzy vector cannot appropriately represent the conceivable range of an uncertain variable vector whose components are interactive.

It is known that a fuzzy vector whose uncertain variables are arbitrarily interactive cannot be treated easily. Then, we consider a special fuzzy vector called an *oblique fuzzy vector* [23]. An oblique fuzzy vector represents some interaction among the uncertain variables without great loss of tractability in calculations:

Definition 3.1. An *oblique fuzzy vector* $\tilde{\boldsymbol{A}}$ is defined as a fuzzy subset of $\mathbb{R}^n$ having the following membership function:

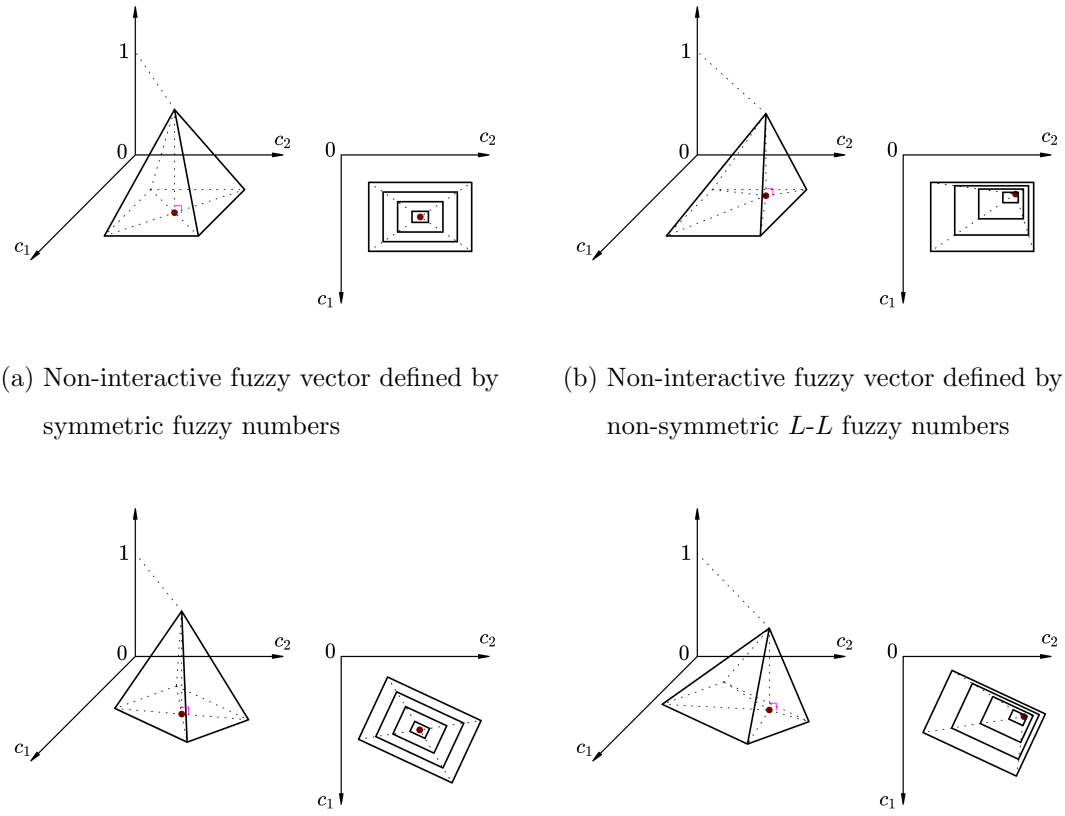
$$\mu_{\tilde{\boldsymbol{A}}}(\boldsymbol{r}) = \min_{i=1,\ldots,n} \mu_{\tilde{B}_i}(\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{r}), \tag{3.2}$$

where $\boldsymbol{d}_i^{\mathrm{T}}$ is the i-th row of a given non-singular matrix $\boldsymbol{D}$, called an *obliquity matrix*, and $\tilde{B}_i$, $i = 1, \ldots, n$ are fuzzy numbers. The uncertain variables whose conceivable ranges are given by $\tilde{B}_i$, $i = 1, \ldots, n$ are assumed to be non-interactive.

Let $\tilde{\boldsymbol{B}} \subseteq \mathbb{R}^n$ be a non-interactive fuzzy vector composed of $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n$. Then (3.2) in Definition 3.1 can be rewritten as:

$$\mu_{\tilde{\boldsymbol{A}}}(\boldsymbol{r}) = \mu_{\tilde{\boldsymbol{B}}}(\boldsymbol{D}\boldsymbol{r}). \tag{3.3}$$

Two non-interactive fuzzy vectors and two oblique fuzzy vectors in $\mathbb{R}^2$ are illustrated in Figure 3.1. In Figure 3.1, the membership function of each fuzzy vector is depicted in a 3-dimensional coordinate system with its level curves depicted in a 2-dimensional coordinate system. Subfigure 3.1a shows a non-interactive fuzzy vector defined by symmetric fuzzy numbers, while subfigure 3.1b shows a non-interactive fuzzy vector defined by non-symmetric L-L fuzzy numbers. On the other hand, subfigure 3.1c shows an oblique fuzzy vector defined by symmetric fuzzy numbers $\tilde{B}_1$ and $\tilde{B}_2$, while subfigure 3.1d shows an oblique fuzzy vector defined by non-symmetric L-L fuzzy numbers $\tilde{B}_1$ and $\tilde{B}_2$.



(a) Non-interactive fuzzy vector defined by symmetric fuzzy numbers

(b) Non-interactive fuzzy vector defined by non-symmetric L-L fuzzy numbers

(c) Oblique fuzzy vector defined by symmetric fuzzy numbers

(d) Oblique fuzzy vector defined by non-symmetric L-L fuzzy numbers

Figure 3.1: Non-interactive fuzzy vectors and oblique fuzzy vectors in $\mathbb{R}^2$

3.2 Fuzzy LP Problems

We consider LP problems (2.3) with fuzzy OFCs, where the conceivable range of γ is specified by a fuzzy subset $\Phi \subseteq \mathbb{R}^n$. To this problem, we investigate the calculations of necessary optimality test problems of an NBF solution.

By Proposition 2.7, the *possible and necessary optimalities* under fuzzy subset Φ can be defined as follows:

Definition 3.2. The *possibly and necessarily optimal solution sets* under fuzzy OFC sub-

set Φ are defined by fuzzy subsets, where their membership functions are defined by

$$\mu_{\Pi S}(\boldsymbol{x}) = \begin{cases} \Pi_\Phi(\mathscr{N}(\boldsymbol{x})), & \text{if } \boldsymbol{x} \text{ is feasible,} \\ 0, & \text{otherwise,} \end{cases} \tag{3.4}$$

$$\mu_{NS}(\boldsymbol{x}) = \begin{cases} N_\Phi(\mathscr{N}(\boldsymbol{x})), & \text{if } \boldsymbol{x} \text{ is feasible,} \\ 0, & \text{otherwise,} \end{cases} \tag{3.5}$$

where Π_Φ and N_Φ are possibility and necessity measures under Φ, and $\mathscr{N}(\boldsymbol{x})$ is a set of OFC vectors to which $\boldsymbol{x}$ is optimal, i.e., the OAC defined by (2.4).

The correspondences of the possible and necessary optimalities under crisp subset Φ (Definition 2.1) and those under fuzzy subset Φ (Definition 3.2) can be understood from the properties shown in Proposition 2.7. Namely, the possible optimality is associated with the non-empty intersection while the necessary optimality is associated with the satisfaction of the inclusion relation.

Since we are interested in a solution robust against the fluctuations of OFCs, we concentrate on the necessary optimality. Applying Proposition 2.7 to the necessarily optimal solution set, we obtain the following equivalences for any $h \in [0, 1]$,

$$\begin{aligned} \boldsymbol{x} \in [NS]_h &\Leftrightarrow \mu_{NS}(\boldsymbol{x}) \geq h \\ &\Leftrightarrow N_\Phi(\mathscr{N}(\boldsymbol{x})) \geq h \\ &\Leftrightarrow (\Phi)_{1-h} \subseteq [\mathscr{N}(\boldsymbol{x})]_h. \end{aligned} \tag{3.6}$$

Together with the fact that $\mathscr{N}(\boldsymbol{x})$ is crisp and closed by its definition, (3.6) implies

$$\boldsymbol{x} \in [NS]_h \;\Leftrightarrow\; \mathrm{cl}(\Phi)_{1-h} \subseteq \mathscr{N}(\boldsymbol{x}),\ \forall\, h \in [0, 1]. \tag{3.7}$$

The result in (3.7) indicates that the necessary optimality degree of a feasible solution $\boldsymbol{x}$ is obtained as the maximal value of $h \in [0, 1]$ such that $\mathrm{cl}(\Phi)_{1-h} \subseteq \mathscr{N}(\boldsymbol{x})$:

$$\mu_{NS}(\boldsymbol{x}) = \max\left\{h \in [0, 1] : \mathrm{cl}(\Phi)_{1-h} \subseteq \mathscr{N}(\boldsymbol{x})\right\}. \tag{3.8}$$

For an NBF solution $\boldsymbol{x}^*$, we have $\mathscr{N}(\boldsymbol{x}^*) = \mathscr{M}(\boldsymbol{x}^*)$. Then, from (3.8), we have

$$\mu_{NS}(\boldsymbol{x}^*) = \max\left\{h \in [0, 1] : \mathrm{cl}(\Phi)_{1-h} \subseteq \mathscr{M}(\boldsymbol{x}^*)\right\}. \tag{3.9}$$

3.2.1 Case of non-interactive symmetric fuzzy OFC vector

First, we describe the result when Φ is a non-interactive fuzzy vector defined by symmetric fuzzy numbers. Namely, we consider an LP problem with fuzzy OFC vector Φ, where Φ

is composed of symmetric fuzzy numbers $\tilde{C}_i = (c_i^{\mathrm{C}}, c_i^{\mathrm{C}}, \alpha_i, \alpha_i)_{LL},\ i = 1, 2, \ldots, n$. The membership function of Φ is expressed as

$$\begin{aligned}\mu_\Phi(\boldsymbol{c}) &= \min_{i=1,2,\ldots,n} \mu_{\tilde{C}_i}(c_i) \\ &= \begin{cases} \min\limits_{i:\alpha_i>0} L\left(\dfrac{|c_i - c_i^{\mathrm{C}}|}{\alpha_i}\right), & \text{if } \forall j \in \{i : \alpha_i = 0\}, c_j = c_j^{\mathrm{C}}, \\ 0, & \text{otherwise,} \end{cases}\end{aligned} \tag{3.10}$$

where μ_Φ and $\mu_{\tilde{C}_i}$ are the membership functions of Φ and $\tilde{C}_i$, respectively.

In this case, the necessary optimality degree $\mu_{NS}(\boldsymbol{x}^*)$ of an NBF solution $\boldsymbol{x}^*$ can be obtained easily as shown in the following theorem.

Theorem 3.1. *Let $\boldsymbol{x}^*$ be an NBF solution optimal to the LP problem with OFC vector $\boldsymbol{c}^{\mathrm{C}} = (c_1^{\mathrm{C}}, c_2^{\mathrm{C}}, ..., c_n^{\mathrm{C}})^{\mathrm{T}}$. When Φ is a non-interactive fuzzy vector composed of symmetric fuzzy numbers $\tilde{C}_i = (c_i^{\mathrm{C}}, c_i^{\mathrm{C}}, \alpha_i, \alpha_i)_{LL}$, $i = 1, 2, \ldots, n$, and defined by (3.10), we obtain the necessary optimality degree $\mu_{NS}(\boldsymbol{x}^*)$ straightforwardly by*

$$\mu_{NS}(\boldsymbol{x}^*) = 1 - L\left(\min_{\substack{k=1,2,\ldots,n-m \\ \sum_{j=1}^n |m_{kj}(\boldsymbol{x}^*)||\alpha_j|>0}} \frac{\sum_{j=1}^n m_{kj}(\boldsymbol{x}^*) c_j^{\mathrm{C}}}{\sum_{j=1}^n |m_{kj}(\boldsymbol{x}^*)||\alpha_j|}\right), \tag{3.11}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$. Furthermore, if $\mu_{NS}(\boldsymbol{x}^*) = 1$, it is regarded as a perfect necessarily optimal solution.

Proof. Since $\tilde{C}_i = (c_i^{\mathrm{C}}, c_i^{\mathrm{C}}, \alpha_i, \alpha_i)_{LL}$, we have $[\tilde{C}_i]_h = [c_i^{\mathrm{C}} - L^*(h)\alpha_i, c_i^{\mathrm{C}} + L^*(h)\alpha_i]$ for any $h \in (0, 1]$. Hence, we obtain

$$\begin{aligned}[\Phi]_h &= [\tilde{C}_1]_h \times [\tilde{C}_2]_h \times \cdots \times [\tilde{C}_n]_h \\ &= \{\boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{\alpha})\boldsymbol{\tau} : \tau_i \in [-L^*(h), L^*(h)],\ i = 1, 2, \ldots, n\}.\end{aligned} \tag{3.12}$$

By Theorem 2.1, we obtain $\tau^{\min}$ of (2.17) with substitutions of $\boldsymbol{\alpha}$ and $\boldsymbol{x}^*$ for $\boldsymbol{c}^{\mathrm{S}}$ and $\hat{\boldsymbol{x}}$. Then, we obtain the following result:

$$\hat{\mathscr{B}}_{\tau^{\min}} = \{\boldsymbol{c}^{\mathrm{C}} + \mathrm{diag}(\boldsymbol{\alpha})\boldsymbol{\tau} : \tau_i \in [-\tau^{\min}, \tau^{\min}],\ i = 1, 2, \ldots, n\} \subseteq \mathscr{M}(\boldsymbol{x}^*), \tag{3.13}$$

where $\tau = \tau^{\min}$ maximises τ subject to $\hat{\mathscr{B}}_\tau \subseteq \mathscr{M}(\boldsymbol{x}^*)$. Let $\bar{h} = L(\tau^{\min})$, then we obtain that $L^*(\bar{h}) \geq \tau^{\min}$ and $L^*(\bar{h} + \epsilon) \leq \tau^{\min}$ for any $\epsilon \in (0, 1 - \bar{h}]$. Since we have $(\Phi)_h = \bigcup_{\epsilon\in(0,1-h]} [\Phi]_{h+\epsilon}$ for any $h \in [0, 1)$, from (3.12), (3.13) and $L^*(\bar{h} + \epsilon) \leq \tau^{\min}$ for any $\epsilon \in (0, 1 - \bar{h}]$, we obtain

$$(\Phi)_{\bar{h}} = \bigcup_{\epsilon\in(0,1-\bar{h}]} [\Phi]_{\bar{h}+\epsilon} \subseteq \hat{\mathscr{B}}_{\tau^{\min}}. \tag{3.14}$$

Owing to the closedness of $\hat{\mathscr{B}}_{\tau^{\min}}$, we obtain $\mathrm{cl}(\Phi)_{\bar{h}} \subseteq \hat{\mathscr{B}}_{\tau^{\min}}$. On the other hand, from $L^*(\bar{h}) \geq \tau^{\min}$ and $\bar{h} = L(\tau^{\min})$, we have

$$\mathrm{cl}(\Phi)_h \supset [\Phi]_{\bar{h}} \supseteq \hat{\mathscr{B}}_{\tau^{\min}}, \ \forall h < \bar{h}. \tag{3.15}$$

Therefore, we obtain $\mathrm{cl}(\Phi)_{\bar{h}} \subseteq \hat{\mathscr{B}}_{\tau^{\min}}$ and $\mathrm{cl}(\Phi)_h \supset hat\mathscr{B}_{\tau^{\min}}$, $\forall h < \bar{h}$. Together with the fact that $\tau = \tau^{\min}$ maximises τ subject to $\hat{\mathscr{B}}_\tau \subseteq \mathscr{M}(\boldsymbol{x}^*)$, we obtain $\mathrm{cl}(\Phi)_{\bar{h}} \subseteq \mathscr{M}(\boldsymbol{x}^*)$ and $\mathrm{cl}(\Phi)_h \not\subseteq \mathscr{M}(\boldsymbol{x}^*)$, $\forall h < \bar{h}$. By (3.9), we obtain $\mu_{NS}(\boldsymbol{x}^*) = 1 - \bar{h} = 1 - L(\tau^{\min})$. From the definition of $\tau^{\min}$ with substitutions of $\boldsymbol{\alpha}$ and $\boldsymbol{x}^*$ for $\boldsymbol{c}^{\mathrm{S}}$ and $\hat{\boldsymbol{x}}$, we obtain (3.11). □

As shown in Theorem 3.1, the necessary optimality degree of an NBF solution of Problem (2.3) can be straightforwardly obtained by (3.11) when the conceivable ranges of OFCs are described by a non-interactive fuzzy vector defined by symmetric fuzzy numbers. We use the following example to show the simple calculation of $\mu_{NS}(\boldsymbol{x}^*)$ shown in Theorem 3.1:

Example 3.1. Reconsider the LP problem in Example 2.1, where the conceivable ranges of c_1 and c_2 are symmetric fuzzy numbers $\tilde{C}_1$ and $\tilde{C}_2$ defined by $\tilde{C}_1 = (-23, -23, 8, 8)_{LL}$ and $\tilde{C}_2 = (-14, -14, 5, 5)_{LL}$, respectively. The reference function $L : [0, +\infty) \to [0, 1]$ is defined by

$$L(r) = \begin{cases} 1 - r, & \text{if } r \in [0, 1], \\ 0, & \text{otherwise.} \end{cases} \tag{3.16}$$

The OFCs of decision variables x_3, x_4 and x_5 are zeros, resulting in $\tilde{C}_i = (0, 0, 0, 0)_{LL}$, $i = 3, 4, 5$. We assume that the conceivable range of OFC vectors is given by a non-interactive fuzzy vector Φ composed of symmetric fuzzy numbers $\tilde{C}_i$, $i = 1, 2, ..., 5$. Accordingly, we obtain $\boldsymbol{c}^{\mathrm{C}} = (-23, -14, 0, 0, 0)^{\mathrm{T}}$. The optimal solution to the LP problem with OFC vector $\boldsymbol{c}^{\mathrm{C}}$ is obtained as $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$. Since $\boldsymbol{x}^*$ is an NBF solution, its OAC is obtained as:

$$M(\boldsymbol{x}^*) = \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ -\frac{4}{9} & \frac{1}{3} & 0 & 1 & -\frac{1}{3} \end{pmatrix}$$

Next, we calculate the necessary optimality degree $\mu_{NS}(\boldsymbol{x}^*)$. By using $\boldsymbol{c}^{\mathrm{C}} = (-23, -14, 0, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{\alpha} = (8, 5, 0, 0, 0)^{\mathrm{T}}$, we obtain

$$\tau_1 = \frac{-\frac{1}{9} \cdot 23 + \frac{1}{3} \cdot 14}{\frac{1}{9} \cdot 8 + \frac{1}{3} \cdot 5} = \frac{19}{23},$$
$$\tau_2 = \frac{\frac{4}{9} \cdot 23 - \frac{1}{3} \cdot 14}{\frac{4}{9} \cdot 8 + \frac{1}{3} \cdot 5} = \frac{50}{47}.$$

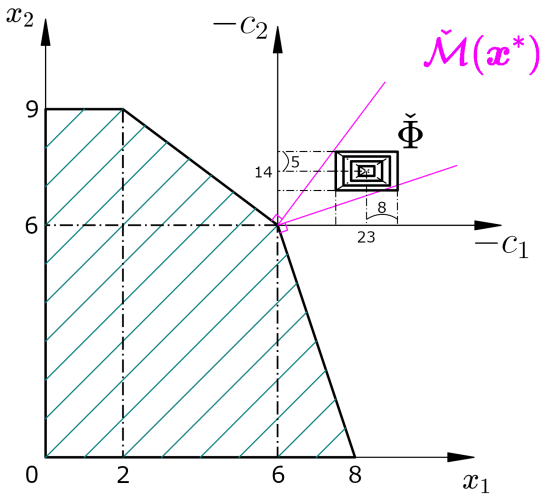


Figure 3.2: Illustration of Example 3.1

Thus, we obtain $\tau_{\min} = 19/23$. Since $\tau_{\min} < 1$, $\mu_{NS}(\boldsymbol{x}^*) = 1 - L(\tau_{\min}) = 19/23$.

The situation of $\boldsymbol{x}^*$ is depicted in Figure 3.2. In Figure 3.2, the feasible region is projected on the x_1-x_2 coordinate system while the assurance cone $\mathcal{M}(\boldsymbol{x}^*)$ and Φ are projected on the c_1-c_2 coordinate system. We confirm that the left-lower vertex of the projected $\mathrm{cl}(\Phi)_0$ is located outside of the projected OAC of $\boldsymbol{x}^*$.

3.2.2 Case of non-interactive fuzzy OFC vector

We describe the results when Φ is defined by non-interactive fuzzy numbers defined with different reference functions. Since any fuzzy number can be represented as an L-R fuzzy number, we investigate the calculation method of $\mu_{NS}(\boldsymbol{x}^*)$ when Φ is defined by non-interactive L-R fuzzy numbers. Namely, Φ is composed of L-R fuzzy numbers $\tilde{C}_i = (c_i^{\mathrm{L}}, c_i^{\mathrm{R}}, \alpha_i, \beta_i)_{L_i R_i}$, $i = 1, 2, \ldots, n$ and its membership function is defined by

$$\mu_\Phi(\boldsymbol{c}) = \min_{i=1,\ldots,n} \mu_{\tilde{C}_i}(c_i), \tag{3.17}$$

where $\boldsymbol{c} = (c_1, c_2, ..., c_n)^{\mathrm{T}}$.

Let $L_{i*} : (0, 1] \to [0, +\infty)$ and $R_{i*} : (0, 1] \to [0, +\infty)$ be the pseudo-inverse functions of L_i and R_i defined by (2.27) with substitutions of L_i and R_i for L and R. Then we obtain that $\mathrm{cl}(\tilde{C}_i)_h = [c_i^{\mathrm{L}} - L_{i*}(h)\alpha_i, c_i^{\mathrm{R}} + R_{i*}(h)\beta_i]$ for $h \in (0, 1]$, where we note that

$\mathrm{cl}(\tilde{C}_i)_1 = \emptyset$. Let $q_i^{\mathrm{C}} : (0,1] \to \mathbb{R}$ and $q_i^{\mathrm{S}} : (0,1] \to [0,+\infty)$ be functions defined by

$$q_i^{\mathrm{C}}(h) = \frac{(c_i^{\mathrm{R}} + R_{i*}(h)\beta_i) + (c_i^{\mathrm{L}} - L_{i*}(h)\alpha_i)}{2}, \tag{3.18}$$

$$q_i^{\mathrm{S}}(h) = \frac{(c_i^{\mathrm{R}} + R_{i*}(h)\beta_i) - (c_i^{\mathrm{L}} - L_{i*}(h)\alpha_i)}{2}, \tag{3.19}$$

for $h \in (0,1]$. Namely, $q_i^{\mathrm{C}}(h)$ and $q_i^{\mathrm{S}}(h)$ are the centre and spread of $\mathrm{cl}(\tilde{C}_i)_h$. We obtain the following theorem:

Theorem 3.2. *Let $\boldsymbol{x}^*$ be an NBF solution to Problem (2.3) with OFC vector $(\boldsymbol{c}^{\mathrm{L}} + \boldsymbol{c}^{\mathrm{R}})/2$, where $\boldsymbol{c}^{\mathrm{L}} = (c_1^{\mathrm{L}}, \ldots, c_n^{\mathrm{L}})^{\mathrm{T}}$ and $\boldsymbol{c}^{\mathrm{R}} = (c_1^{\mathrm{R}}, \ldots, c_n^{\mathrm{R}})^{\mathrm{T}}$. Calculate $\tau^{\min}$ by (2.17) with substitutions of $q_i^{\mathrm{C}}(h)$, $q_i^{\mathrm{S}}(h)$ and $\boldsymbol{x}^*$ for c_i^{C}, c_i^{S} and $\hat{\boldsymbol{x}}$ of (2.16). Then we obtain*

(a) $\tau^{\min} \geq 1 \;\Rightarrow\; \mu_{NS}(\boldsymbol{x}^*) \geq 1 - h$, *and*

(b) $\tau^{\min} < 1 \;\Rightarrow\; \mu_{NS}(\boldsymbol{x}^*) < 1 - h$.

Proof. Since $\mathrm{cl}(\Phi)_h = \mathrm{cl}(\tilde{C}_1)_h \times \mathrm{cl}(\tilde{C}_2)_h \times \cdots \times \mathrm{cl}(\tilde{C}_n)_h$, $\mathrm{cl}(\Phi)_h$ is a hyperrectangle defined by $\mathrm{cl}(\tilde{C}_i)_h = [c_i^{\mathrm{L}} - L_{i*}(h)\alpha_i, c_i^{\mathrm{R}} + R_{i*}(h)\beta_i] = [c_i^{\mathrm{C}} - c_i^{\mathrm{S}}, c_i^{\mathrm{C}} + c_i^{\mathrm{S}}]$, $i = 1, 2, ..., n$.

First we prove (a). When $\tau^{\min} \geq 1$, as shown in the proof of Theorem 2.2, we have $\mathrm{cl}(\Phi)_h \subseteq \mathcal{M}(\boldsymbol{x}^*)$. Hence by (3.9), we obtain $\mu_{NS}(\boldsymbol{x}^*) \geq 1 - h$.

Then we prove (b). The fact $\tau^{\min} < 1$ implies that the hyperrectangle defined by $[c_i^{\mathrm{C}} - c_i^{\mathrm{S}}, c_i^{\mathrm{C}} + c_i^{\mathrm{S}}]$, $i = 1, 2, ..., n$ is not included in $\mathcal{M}(\boldsymbol{x}^*)$. Namely, $\mathrm{cl}(\Phi)_h \not\subseteq \mathcal{M}(\boldsymbol{x}^*)$. Hence by (3.9), we obtain $\mu_{NS}(\boldsymbol{x}^*) < 1 - h$. □

By Theorem 3.2, we propose an algorithm of calculating $\mu_{NS}(\boldsymbol{x}^*)$, where $\boldsymbol{x}^*$ is an NBF solution. The algorithm is based on the *bisection method*, and described as follows, where we set c_i^{C} and c_i^{S}, $i = 1, 2, \ldots, n$ as those described in Theorem 3.2:

Algorithm 3.1. (*An algorithm for the robust optimality analysis in LP problems with non-interactive fuzzy OFC vectors*)

- **Initialisation**:

  1. Set $\delta > 0$ as the threshold for terminating the algorithm. Set $\varepsilon > 0$ as a very small positive number.
  2. Obtain an optimal solution $\boldsymbol{x}^*$ to LP problem with OFCs $c_i = q_i^{\mathrm{C}}(1)$, $i = 1, 2, \ldots, n$. This algorithm is applied only when $\boldsymbol{x}^*$ is an NBF solution.

3. If $\boldsymbol{x}^*$ is an NBF solution, obtain the matrix $M(\boldsymbol{x}^*)$ of (2.8).

- **Iteration**:

Step 1. Set $h = 1$. Calculate $\tau^{\min}$ by (2.16) with $c_i^{\mathrm{C}} = q_i^{\mathrm{C}}(h)$ and $c_i^{\mathrm{S}} = q_i^{\mathrm{S}}(h)$, $i = 1, 2, \ldots, n$.

1. If $\sum_{k=1}^{n-m} \sum_{j=1}^{n} |M_{kj}(\boldsymbol{x}^*)||q_i^{\mathrm{S}}(h)| > 0$ and $\tau^{\min} < 1$, there exists no necessarily optimal solution. Namely, we obtain $\mu_{NS}(\boldsymbol{x}^*) = 0$ and terminate the algorithm.
2. If $\sum_{k=1}^{n-m} \sum_{j=1}^{n} |M_{kj}(\boldsymbol{x}^*)||q_i^{\mathrm{S}}(h)| = 0$, calculate $\tau^{\min}$ by (2.16) with $c_i^{\mathrm{C}} = q_i^{\mathrm{C}}(h)$ and $c_i^{\mathrm{S}} = 1$, $i = 1, 2, \ldots, n$.
3. Calculate h^{M} by the following equation:

$$h^{\mathrm{M}} = \max_{i=1,\ldots,n} \max \left(\mu_{\tilde{C}_i}(q_i^{\mathrm{C}}(h) - \tau^{\min} q_i^{\mathrm{S}}(h)), \mu_{\tilde{C}_i}(q_i^{\mathrm{C}}(h) + \tau^{\min} q_i^{\mathrm{S}}(h)) \right). \tag{3.20}$$

4. If $h^{\mathrm{M}} = 0$, we obtain $\mu_{NS}(\boldsymbol{x}^*) = 1$ and terminate the algorithm. Otherwise, set $h^{\mathrm{H}} = h^{\mathrm{M}}$ and proceed to Step 2.

Step 2. Let $h = \varepsilon$. Calculate $\tau^{\min}$ by (2.16) with $c_i^{\mathrm{C}} = q_i^{\mathrm{C}}(h)$ and $c_i^{\mathrm{S}} = q_i^{\mathrm{S}}(h)$, $i = 1, 2, \ldots, n$. If $\tau^{\min} \geq 1$, we obtain $\mu_{NS}(\boldsymbol{x}^*) \approx 1$ and terminate the algorithm. Otherwise, update h^{M} by (3.20) with $h = \varepsilon$. Set $h^{\mathrm{L}} = \varepsilon$ and update $h^{\mathrm{H}} = \min(h^{\mathrm{H}}, h^{\mathrm{M}})$.

Step 3. If $h^{\mathrm{H}} - h^{\mathrm{L}} < \varepsilon$, we know that $\mu_{NS}(\boldsymbol{x}^*) \approx 1 - h^{\mathrm{H}}$ and we terminate the algorithm.

Step 4. Let $h = (h^{\mathrm{H}} + h^{\mathrm{L}})/2$. Calculate $\tau^{\min}$ by (2.16) with $c_i^{\mathrm{C}} = q_i^{\mathrm{C}}(h)$ and $c_i^{\mathrm{S}} = q_i^{\mathrm{S}}(h)$, $i = 1, 2, \ldots, n$.

1. If $\tau^{\min} < 1$, we update $h^{\mathrm{L}} = h$.
2. Update h^{M} of (3.20) with h and then update $h^{\mathrm{H}} = \min(h^{\mathrm{H}}, h^{\mathrm{M}})$. Then return to Step 3.

In Algorithm 3.1, when h^{L} and h^{H} take some values, $1 - h^{\mathrm{H}} \leq \mu_{NS}(\boldsymbol{x}^*) < 1 - h^{\mathrm{L}}$ can always be satisfied. We use the property of $\tau^{\min}$ shown in Theorem 2.1, i.e., $\mathscr{B}_{\tau^{\min}} \subseteq \mathscr{M}_{\tau}(\boldsymbol{x}^*)$. To put this differently, we have

$$\{\boldsymbol{q}^{\mathrm{C}}(h) + \mathrm{diag}(\boldsymbol{q}^{\mathrm{S}}(h))\boldsymbol{\tau} : \boldsymbol{\tau} \in [-\tau_{\min}, \tau_{\min}]^n\} \subseteq \mathcal{M}(\boldsymbol{x}^*), \tag{3.21}$$

whenever $\tau_{\min}$ is calculated in Algorithm 3.1. (3.20) calculates the maximum h satisfying

$$\mathrm{cl}(\Phi)_h \subseteq \{\boldsymbol{q}^{\mathrm{C}}(h) + \mathrm{diag}(\boldsymbol{q}^{\mathrm{S}}(h))\boldsymbol{\tau} : \boldsymbol{\tau} \in [-\tau_{\min}, \tau_{\min}]^n\}. \tag{3.22}$$

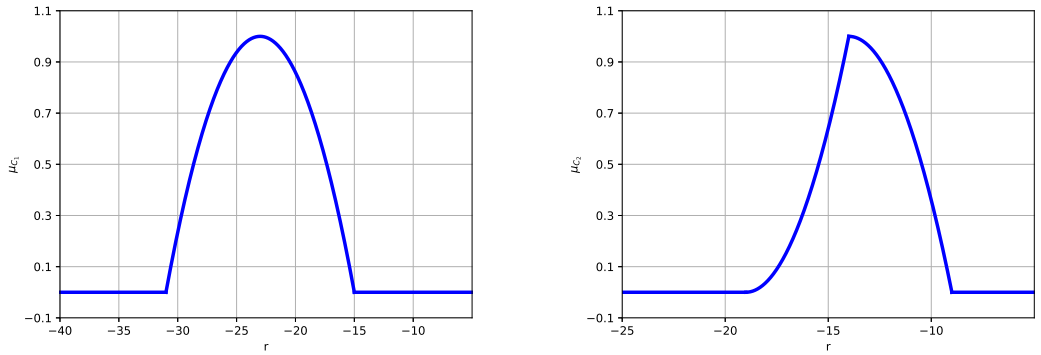
We utilise (3.20) to accelerate the bisection procedure, which makes h^{H} converge faster. When $\sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)||q_i^{\mathrm{S}}(h) = 0$ for all $k \in \{1, 2, ..., n-m\}$, $\tau^{\min}$ is not defined. To treat this case properly, 2 of Step 1 is added. The value of $\tau^{\min}$ at Steps 2 and 4 can be negative, which implies the case where $\sum_{j=1}^{n} m_{kj}(\boldsymbol{x}^*)q_j^{\mathrm{C}}(h) < 0$, i.e., the vector composed of centres of h-level set $(\tilde{C}_j)_h$ is not in $\mathcal{M}(\boldsymbol{x}^*)$.

To describe Algorithm 3.1, we use Example 2.1 again as follows:

Example 3.2. Reconsider the LP problem described in Example 2.1. In this example, we define $\tilde{C}_1$ and $\tilde{C}_2$ as L-R fuzzy numbers with the following membership functions:

$$\mu_{\tilde{C}_1}(r) = \begin{cases} 1 - \frac{(r+23)^2}{64}, & r \in (-31, -15), \\ 0, & \text{otherwise}, \end{cases}, \quad \mu_{\tilde{C}_2}(r) = \begin{cases} \frac{(r+19)^2}{25}, & r \in (-19, -14), \\ 1 - \frac{(r+14)^2}{25}, & r \in [-14, -9), \\ 0, & \text{otherwise}. \end{cases}$$

The membership functions are depicted in Figure 3.3.



(a) Membership function of $\tilde{C}_1$ (b) Membership function of $\tilde{C}_2$

Figure 3.3: Membership function of Φ

By using reference functions

$$L(r) = \max(0, 1 - r^2), \quad \text{and} \quad R(r) = (\max(0, 1 - r))^2, \tag{3.23}$$

we have $\tilde{C}_1 = (-23, -23, 8, 8)_{LL}$ and $\tilde{C}_2 = (-14, -14, 5, 5)_{LR}$. Since $\tilde{C}_i = (0, 0, 0, 0)_{LL}$, $i = 3, 4, 5$. Then the conceivable range of the OFC vectors is given as a non-interactive fuzzy vector Φ composed of L-R fuzzy numbers $\tilde{C}_i$, $i = 1, 2, ..., 5$. We obtain $\boldsymbol{c}^{\mathrm{C}} =$

$(-23, -14, 0, 0, 0)^{\mathrm{T}}$. Then, $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$ is the unique optimal NBF solution to LP problem with OFCs $c_i = q_i^{\mathrm{C}}(1) = c_i^{\mathrm{C}}$, $i = 1, 2, \ldots, 5$.

Since $\tilde{C}_2$ is not a symmetric fuzzy number, we cannot use the results of Theorem 3.1, and thus, Algorithm 3.1 is applied to calculating $\mu_{NS}(\boldsymbol{x}^*)$. The calculation process is roughly shown as follows (we round values to three decimal places):

Initialisation: Set $\delta = 1 \times 10^{-3}$ and $\varepsilon = 1 \times 10^{-4}$. We obtain $\boldsymbol{q}^{\mathrm{C}}(1) = \boldsymbol{c}^{\mathrm{C}} = (-23, -14, 0, 0, 0)^{\mathrm{T}}$ and the NBF solution $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$. $M(\boldsymbol{x}^*)$ is shown in Example 3.1.

Step 1: Set $h = 1$, which gives $\boldsymbol{q}^{\mathrm{C}}(h) = (-23, -14, 0, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{q}^{\mathrm{S}}(h) = (0, 0, 0, 0, 0)^{\mathrm{T}}$. As we have $\sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)||q_i^{\mathrm{S}}(h)| = 0$, we recalculate $\tau^{\min}$ with $c_i^{\mathrm{S}} = 1$, $i = 1, 2, ..., n$ in (2.16). We obtain $\tau^{\min} = 4.75$, and from (3.20), we obtain $h^{\mathrm{M}} = 0.647$. Then we set $h^{\mathrm{H}} = h^{\mathrm{M}} = 0.647$.

Step 2: Set $h = \varepsilon$, which gives $\boldsymbol{q}^{\mathrm{C}}(h) = (-23, -13.992, 0, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{q}^{\mathrm{S}}(h) = (8, 4.992, 0, 0, 0)^{\mathrm{T}}$. We obtain $\tau^{\min} = 0.826 < 1$. Namely, $\boldsymbol{x}^*$ is not a completely necessarily optimal, i.e., $\mu_{NS}(\boldsymbol{x}^*) < 1$. From (3.20), $h^{\mathrm{M}} = 0.318$. Then, we set $h^{\mathrm{L}} = \varepsilon$ and $h^{\mathrm{H}} = \min(0.647, 0.318) = 0.318$.

Step 3 $h^{\mathrm{H}} - h^{\mathrm{L}} = 0.318 \geq \delta$. Continue the procedure.

Step 4 Set $h = (h^{\mathrm{H}} + h^{\mathrm{L}})/2 = 0.159$. $\boldsymbol{q}^{\mathrm{C}}(h) = (-23, -13.211, 0, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{q}^{\mathrm{S}}(h) = (7.337, 3.796, 0, 0, 0)^{\mathrm{T}}$. From (2.16), we obtain $\tau^{\min} = 0.888 < 1$. We update $h^{\mathrm{L}} = 0.159$. From (3.20), $h^{\mathrm{M}} = 0.336$. $h^{\mathrm{H}} = \min(0.318, 0.336) = 0.318$. Return to Step 3.

Step 3 $h^{\mathrm{H}} - h^{\mathrm{L}} = 0.159 \geq \delta$. Continue the procedure.

Step 4 Set $h = (h^{\mathrm{H}} + h^{\mathrm{L}})/2 = 0.238$. $\boldsymbol{q}^{\mathrm{C}}(h) = (-23, -13.098, 0, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{q}^{\mathrm{S}}(h) = (6.982, 3.461, 0, 0, 0)^{\mathrm{T}}$. From (2.16), we obtain $\tau^{\min} = 0.938 < 1$. We update $h^{\mathrm{L}} = 0.238$. From (3.20), $h^{\mathrm{M}} = 0.330$. $h^{\mathrm{H}} = \min(0.318, 0.330) = 0.318$. Return to Step 3.

Step 3 $h^{\mathrm{H}} - h^{\mathrm{L}} = 0.079 \geq \delta$. Continue the procedure.

Continue Steps 3 and 4 for $h = 0.278$, 0.298, 0.308, 0.312, 0.315, 0.317, ..., 0.318. After 11th iteration. we obtain the approximate $\mu_{NS}(\boldsymbol{x}^*) \approx 1 - 0.318 = 0.682$. Hence, we obtain $\mu_{NS}(\boldsymbol{x}^*) \approx 0.682$.

The next theorem shows that $\mu_{NS}(\boldsymbol{x}^*)$ is calculated easily also when Φ is composed of non-interactive L-L fuzzy numbers $\tilde{C}_i = (c_i^{\mathrm{L}}, c_i^{\mathrm{R}}, \alpha_i, \beta_i)_{LL}$, $i = 1, 2, ..., n$ even they are not symmetric, i.e., $\alpha_i \neq \beta_i$, $i = 1, 2, ..., n$.

Theorem 3.3. *Assume that $L_i : (0,1] \to [0, +\infty)$ and $R_i : (0,1] \to [0, +\infty)$, $i = 1, 2, \ldots, n$ share the same reference function $L : (0,1] \to [0, +\infty)$. Let $\boldsymbol{x}^*$ be an NBF solution optimal to Problem (2.3) with OFC vector $(\boldsymbol{c}^{\mathrm{L}} + \boldsymbol{c}^{\mathrm{R}})/2$, where $\boldsymbol{c}^{\mathrm{L}} = (c_1^{\mathrm{L}}, \ldots, c_n^{\mathrm{L}})^{\mathrm{T}}$ and $\boldsymbol{c}^{\mathrm{R}} = (c_1^{\mathrm{R}}, \ldots, c_n^{\mathrm{R}})^{\mathrm{T}}$. If $\tau^{\min}$ with substitutions of $q_i^{\mathrm{C}}(1)$, $q_i^{\mathrm{S}}(1)$ and $\boldsymbol{x}^*$ for c_i^{C}, c_i^{S} and $\hat{\boldsymbol{x}}$ of (2.16) is not less than 1 or $q_i^{\mathrm{S}}(1) = 0$, $i \in N$, we obtain the necessary optimality degree as*

$$\mu_{NS}(\boldsymbol{x}^*) = 1 - L\left(\min_{\substack{k=1,\ldots,n-m \\ w_k>0}} \frac{r_k}{w_k}\right), \tag{3.24}$$

where w_k and r_k, for $k = 1, 2, \ldots, n-m$, are defined by

$$w_k = \sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)>0}}^{n} m_{kj}(\boldsymbol{x}^*)\alpha_j - \sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)<0}}^{n} m_{kj}(\boldsymbol{x}^*)\beta_j, \tag{3.25}$$

$$r_k = \sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)>0}}^{n} m_{kj}(\boldsymbol{x}^*)c_j^{\mathrm{L}} + \sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)<0}}^{n} m_{kj}(\boldsymbol{x}^*)c_j^{\mathrm{R}}. \tag{3.26}$$

Proof. From Theorem 3.2(a), we consider the minimal h such that $\tau^{\min} \geq 1$, where $\tau^{\min}$ is obtained from (2.17) with substitutions of $q_i^{\mathrm{C}}(h)$, $q_i^{\mathrm{S}}(h)$ and $\boldsymbol{x}^*$ for c_i^{C}, c_i^{S} and $\hat{\boldsymbol{x}}$. Under these substitutions, for $k = 1, 2, \ldots, n-m$, the denominator of τ_k of (2.16) is always non-negative, and thus, the numerator of τ_k of (2.16) should be non-negative for having $\tau^{\min} \geq 1$. Therefore, from (2.16) and (2.17) with substitutions of $q_j^{\mathrm{C}}(h)$ defined by (3.18), $q_j^{\mathrm{S}}(h)$ defined by (3.19) and $\boldsymbol{x}^*$ for c_j^{C}, c_j^{S} and $\hat{\boldsymbol{x}}$, we have the following equivalences:

$$\begin{aligned}
& \tau^{\min} \geq 1 \\
\Leftrightarrow\ & \sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)||c_j^{\mathrm{R}} - c_j^{\mathrm{L}} + L_*(h)(\alpha_j + \beta_j)| \\
& \leq \sum_{j=1}^{n} m_{kj}(\boldsymbol{x}^*)(c_j^{\mathrm{L}} + c_j^{\mathrm{R}} + L_*(h)(\beta_j - \alpha_j)),\ k = 1, 2, \ldots, n-m
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)|(c_j^{\mathrm{R}} - c_j^{\mathrm{L}}) - \sum_{j=1}^{n} m_{kj}(\boldsymbol{x}^*)(c_j^{\mathrm{L}} + c_j^{\mathrm{R}}) \\
&\quad \leq L_*(h) \left(\sum_{j=1}^{n} m_{kj}(\boldsymbol{x}^*)(\beta_j - \alpha_j) - \sum_{j=1}^{n} |m_{kj}(\boldsymbol{x}^*)|(\alpha_j + \beta_j) \right), \ k = 1, 2, \ldots, n-m \\
&\Leftrightarrow -2 \left(\sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)>0}}^{n} m_{kj}(\boldsymbol{x}^*) c_j^{\mathrm{L}} + \sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)<0}}^{n} m_{kj}(\boldsymbol{x}^*) c_j^{\mathrm{R}} \right) \\
&\quad \leq -2L_*(h) \left(\sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)>0}}^{n} m_{kj}(\boldsymbol{x}^*) \alpha_j - \sum_{\substack{j=1 \\ m_{kj}(\boldsymbol{x}^*)<0}}^{n} m_{kj}(\boldsymbol{x}^*) \beta_j \right), \ k = 1, 2, \ldots, n-m \\
&\Leftrightarrow r_k \geq L_*(h) w_k, \ k = 1, 2, \ldots, n-m. \qquad (3.27)
\end{aligned}
$$

Owing to $\alpha_j, \beta_j \geq 0$, $j = 1, 2, \ldots, n$, we have $w_k \geq 0$, $k = 1, 2, \ldots, n-m$. Moreover, $r_k \geq 0$ is guaranteed by the assumption $\tau^{\min} \geq 1$ with substitutions of $q_i^{\mathrm{C}}(1)$, $q_i^{\mathrm{S}}(1)$ and $\boldsymbol{x}^*$ for c_i^{C}, c_i^{S} and $\hat{\boldsymbol{x}}$ in (2.16) for $i = 1, 2, \ldots, n$. Those ensure the last equivalence. From $L_*(h) = \inf\{r \in [0, +\infty) : L(r) \leq h\}$, $L_*(h) \leq v$ is equivalent to $h \geq L(v)$. Hence, from (3.27), the minimal h satisfying $\tau^{\min} \geq 1$ is obtained as

$$
\check{h} = L \left(\min_{\substack{k=1,\ldots,n-m, \\ w_k>0}} \frac{r_k}{w_k} \right).
$$

Hence, we obtain $\mu_{NS}(\boldsymbol{x}^*) = 1 - \check{h}$ and the theorem is proven. □

Example 3.3. Reconsider Example 3.2 where $\tilde{C}_1$ and $\tilde{C}_2$ are non-interactive L-L fuzzy numbers with $\tilde{C}_1 = (-23, -23, 4, 8)_{LL}$, $\tilde{C}_2 = (-14, -14, 2, 5)_{LL}$ and $\tilde{C}_i = (0, 0, 0, 0)_{LL}$, $i = 3, 4, 5$. As for the reference function L, we assume a non-linear one with

$$
L(r) = \begin{cases} (1-r)^2, & \text{if } r \in [0, 1], \\ 0 \quad , & \text{otherwise.} \end{cases}
$$

Since $\boldsymbol{c}^{\mathrm{C}} = (-23, -14, 0, 0, 0)^{\mathrm{T}}$, the NBF solution is still $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$.

Then, we calculate $\mu_{NS}(\boldsymbol{x}^*)$. The matrix $M(\boldsymbol{x}^*)$ defining the associated OAC $\mathscr{M}(\boldsymbol{x}^*)$ is the same as Example 3.1. As all $\tilde{C}_i$, $i = 1, 2, ..., 5$ are L-L fuzzy numbers, we apply

Theorem 3.3, We obtain

$$\begin{aligned} w_1 &= \frac{1}{9}\cdot 4 - (-\frac{1}{3})\cdot 5 + 1\cdot 0 + \frac{1}{3}\cdot 0 = \frac{19}{9}, \\ w_2 &= -(-\frac{4}{9})\cdot 8 + \frac{1}{3}\cdot 2 + 1\cdot 0 - (-\frac{1}{3})\cdot 0 = \frac{38}{9}, \\ r_1 &= \frac{1}{9}\cdot(-23) - \frac{1}{3}\cdot(-14) + 1\cdot 0 + \frac{1}{3}\cdot 0 = \frac{19}{9}, \\ r_2 &= -\frac{4}{9}\cdot(-23) + \frac{1}{3}\cdot(-14) + 1\cdot 0 - \frac{1}{3}\cdot 0 = \frac{50}{9}. \end{aligned}$$

Then we obtain $r_1/w_1 = 19/19 = 1$ and $r_2/w_2 = 25/19 > 1$. $\check{h} = L(\min(1, 25/19)) = L(1) = 0$. Therefore, we obtain $\mu_{NS}(\boldsymbol{x}^*) = 1 - \check{h} = 1$.

3.2.3 Case of oblique fuzzy OFC vector

In this subsection, we consider the case where Φ is an oblique fuzzy vector [23] defined by the following membership function:

$$\mu_\Phi(\boldsymbol{c}) = \min_{i=1,\ldots,n} \mu_{\Psi_i}(\boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{c}) = \mu_\Psi(\boldsymbol{D}\boldsymbol{c}), \tag{3.28}$$

where $\boldsymbol{D}$ is the obliquity matrix and $\boldsymbol{d}_i^{\mathrm{T}}$ is the i-th row of $\boldsymbol{D}$, i.e.,

$$\boldsymbol{D} = (\boldsymbol{d}_1,\ \boldsymbol{d}_2,\ \ldots,\ \boldsymbol{d}_n)^{\mathrm{T}}, \tag{3.29}$$

and $\mu_{\Psi_i} : \mathbb{R} \to [0,1]$ is the membership function of the fuzzy number $\Psi_i,\ i = 1, 2, \ldots, n$. Since the obliquity matrix is non-singular, there always exists $\boldsymbol{D}^{-1}$. Together with the assumption that Ψ is a non-interactive fuzzy vector composed of fuzzy numbers $\Psi_1,\ \Psi_2,\ \ldots,\ \Psi_n$, we obtain the following theorem:

Theorem 3.4. *Let Φ be an oblique fuzzy vector defined by (3.28) and $\boldsymbol{x}^*$ be an NBF solution. Then we obtain*

$$\mathrm{cl}(\Phi)_{1-h} \subseteq \mathcal{M}(\boldsymbol{x}^*) \ \Leftrightarrow\ \mathrm{cl}(\Psi)_{1-h} \subseteq \mathcal{M}_D(\boldsymbol{x}^*), \tag{3.30}$$

where $\mathcal{M}(\boldsymbol{x}^*)$ is the cone defined by (2.8) while $\mathcal{M}_D(\boldsymbol{x}^*)$ is the cone defined by

$$\mathcal{M}_D(\boldsymbol{x}^*) = \{\boldsymbol{r} \in \mathbb{R}^n : \boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{D}^{-1}\boldsymbol{r} \geq \boldsymbol{0}\}. \tag{3.31}$$

Proof. By (2.8), we have

$$\mathrm{cl}(\Phi)_{1-h} \subseteq \mathcal{M}(\boldsymbol{x}^*) \ \Leftrightarrow\ \boldsymbol{c} \in \mathrm{cl}(\Phi)_{1-h} \text{ implies } \boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{c} \geq \boldsymbol{0} \tag{3.32}$$

Since by (3.28), $\boldsymbol{c} \in \mathrm{cl}(\Phi)_{1-h}$ is equivalent to $\boldsymbol{Dc} \in \mathrm{cl}(\Psi)_{1-h}$. To put this differently, $\boldsymbol{r} \in \mathrm{cl}(\Psi)_{1-h}$ is equivalent to $\boldsymbol{D}^{-1}\boldsymbol{r} \in \mathrm{cl}(\Phi)_{1-h}$, which implies

$$\begin{aligned} \mathrm{cl}(\Phi)_{1-h} \subseteq \mathcal{M}(\boldsymbol{x}^*) &\Leftrightarrow \ \boldsymbol{r} \in \mathrm{cl}(\Psi)_{1-h} \ \text{ implies } \ \boldsymbol{M}(\boldsymbol{x}^*)\boldsymbol{D}^{-1}\boldsymbol{r} \geq \boldsymbol{0} \\ &\Leftrightarrow \ \mathrm{cl}(\Psi)_{1-h} \subseteq \mathcal{M}_D(\boldsymbol{x}^*). \end{aligned}$$

□

Applying Theorem 3.4 to (3.9), we obtain

$$\mu_{NS}(\boldsymbol{x}^*) = \max\{h : \mathrm{cl}(\Psi)_{1-h} \subseteq \mathcal{M}_D(\boldsymbol{x}^*)\}. \tag{3.33}$$

Namely, Theorem 3.4 ensures the necessary optimality degree $\mu_{NS}(\boldsymbol{x}^*)$ under an oblique fuzzy vector Φ can be calculated in the same way as that under a non-interactive fuzzy vector Ψ by replacing $\mathcal{M}(\boldsymbol{x}^*)$ with $\mathcal{M}_D(\boldsymbol{x}^*)$. More concretely, $\mu_{NS}(\boldsymbol{x}^*)$ is calculated by Algorithm 3.1 by replacing Φ and $\mathcal{M}(\boldsymbol{x}^*)$ with Ψ and $\mathcal{M}_D(\boldsymbol{x}^*)$, respectively.

When Ψ is a vector with symmetric fuzzy numbers $\Psi_i = (g_i^{\mathrm{C}}, g_i^{\mathrm{C}}, \alpha_i, \alpha_i)_{LL}$, we can apply (3.10) with substitutions of Ψ and $\mathcal{M}_D(\boldsymbol{x}^*)$ for Φ and $\mathcal{M}(\boldsymbol{x}^*)$. Similarly, when Ψ is defined by fuzzy numbers $\Psi_i = (g_i^{\mathrm{L}}, g_i^{\mathrm{R}}, \alpha_i, \beta_i)_{LL}$, $i = 1, 2, ..., n$, we can apply (3.24) with substitutions of Ψ and $\mathcal{M}_D(\boldsymbol{x}^*)$ for Φ and $\mathcal{M}(\boldsymbol{x}^*)$.

Example 3.4. Reconsider the problem in Example (2.1) again with an oblique fuzzy vector Φ. The oblique fuzzy vector Φ is defined by an obliquity matrix $\boldsymbol{D}$,

$$\boldsymbol{D} = \begin{pmatrix} 10 & 1 & 0 & 0 & 0 \\ -8 & 15 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.34}$$

and non-interactive L-L fuzzy numbers $\Psi_1 = (-399, -399, 18, 140)_{LL}$, $\tilde{G}_2 = (-139, -139, 112, 204)_{LL}$ and $\Psi_i = (0, 0, 0, 0)_{LL}$, $i = 2, 3, 4$ (see (3.2)). Let $g_i^{\mathrm{C}} = (g_i^{\mathrm{L}} + g_i^{\mathrm{R}})/2$, $i = 1, 2, ..., 5$ we have $\boldsymbol{g}^{\mathrm{C}} = (g_1^{\mathrm{C}}, g_2^{\mathrm{C}}, ..., g_5^{\mathrm{C}})^{\mathrm{T}} = (-399, -139, 0, 0, 0)^{\mathrm{T}}$. Then we have $\boldsymbol{c}^{\mathrm{C}} = \boldsymbol{D}^{-1}\boldsymbol{g}^{\mathrm{C}} = (-37, -29, 0, 0, 0)^{\mathrm{T}}$. We consider an NBF solution $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$ solves the LP problem with objective function vector $\boldsymbol{c}^{\mathrm{C}}$. We calculate $\mu_{NS}(\boldsymbol{x}^*)$ using

(3.30) or equivalently, (3.33). The cone $\mathcal{M}_D(\boldsymbol{x}^*)$ is characterised by the following matrix:

$$
\begin{aligned}
M(\boldsymbol{x}^*)D^{-1} &= \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ -\frac{4}{9} & \frac{1}{3} & 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 10 & 1 & 0 & 0 & 0 \\ -8 & 15 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ -\frac{4}{9} & \frac{1}{3} & 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{15}{158} & \frac{-1}{158} & 0 & 0 & 0 \\ \frac{4}{79} & \frac{5}{79} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{158} & -\frac{31}{1422} & 1 & 0 & \frac{1}{3} \\ -\frac{2}{79} & \frac{17}{711} & 0 & 1 & -\frac{1}{3} \end{pmatrix}.
\end{aligned}
$$

As Ψ is composed by L-L fuzzy numbers Ψ_i, $i = 1, 2, ..., 5$, we may apply Theorem 3.3 to obtaining $\mu_{NS}(\boldsymbol{x}^*)$ by (3.33). We obtain w_1, w_2, r_1 and r_2 as

$$
\begin{aligned}
w_1 &= -(-\frac{1}{158}) \cdot 140 - (-\frac{31}{1422}) \cdot 204 + 1 \cdot 0 + \frac{1}{3} \cdot 0 = \frac{16}{3}, \\
w_2 &= -(-\frac{2}{79}) \cdot 140 + \frac{17}{711} \cdot 112 + 1 \cdot 0 - (-\frac{1}{3}) \cdot 0 = \frac{56}{9}, \\
r_1 &= -\frac{1}{158} \cdot (-399) - \frac{31}{1422} \cdot (-139) + 1 \cdot 0 + \frac{1}{3} \cdot 0 = \frac{50}{9}, \\
r_2 &= -\frac{2}{79} \cdot (-399) + \frac{17}{711} \cdot (-139) + 1 \cdot 0 - \frac{1}{3} \cdot 0 = \frac{61}{9}.
\end{aligned}
$$

Then, we compute $r_1/w_1 = 25/24 > 1$ and $r_2/w_2 = 61/56 > 1$, which give the result as $\check{h} = L(\min(25/24, 61/56)) = 0$. Therefore, we obtain $\mu_{NS}(\boldsymbol{x}^*) = 1 - \check{h} = 1$.

The situation of the necessarily optimal solution $\boldsymbol{x}^* = (6, 6, 0, 0, 3)^{\mathrm{T}}$ is depicted in Figure 3.4 on the x_1-x_2 coordinate system. In this figure, we observe $\mathrm{cl}(\check{\Phi})_0 \subseteq \check{\mathcal{M}}(\boldsymbol{x}^*)$, where $\check{\Phi}$ and $\check{\mathcal{M}}(\boldsymbol{x}^*)$ are the projected sets of Φ and $\mathcal{M}(\boldsymbol{x}^*)$ on the c_1-c_2 coordinate system. This confirms that $\mu_{NS}(\boldsymbol{x}^*) = 1$.

3.3 Concluding Remarks

In this chapter, we propose the notion of an NBF solution's necessary optimality degree for an LP problem with fuzzy OFCs as well as several methods to obtain the results in

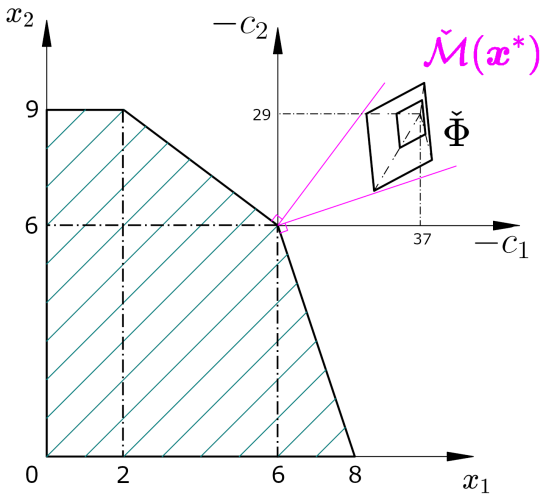


Figure 3.4: The necessarily optimal solution in Example 3.4

different situations with several kinds of fuzzy OFC vectors. We showed that this problem can be obtained by the tolerance analysis.

The main results obtained in this chapter are as follows:

- When the fuzzy OFC vector is a non-interactive fuzzy vector composed of symmetric fuzzy numbers, the necessary optimality degree of an NBF solution is obtained by simple calculations.
- When the fuzzy OFC vector is a non-interactive fuzzy vector composed of L-L fuzzy numbers, the necessary optimality degree of an NBF solution is also obtained by simple calculations.
- When the fuzzy OFC vector is a general non-interactive fuzzy vector, the necessary optimality degree of an NBF solution is calculated by an iterative procedure based on a bisection method.
- When the fuzzy OFC vector is an oblique fuzzy vector composed of an obliquity matrix and non-interactive fuzzy numbers, the necessary optimality degree of an NBF solution is calculated in the same way as the case of a non-interactive fuzzy vector by replacing the OAC with the linearly transformed OAC by the inverse of the obliquity matrix.

From those results, we found that the necessary optimality degree of an NBF solution is obtained easily when the fuzzy OFC vector is a non-interactive fuzzy vector or an oblique fuzzy vector.

We can extend the proposed approach to the case of degenerate basic feasible solutions. The proposed approach would work well also in that case if all basic representations of the solution are obtained. The consideration of an efficient calculation, in this case, would be a future topic. Moreover, since the oblique fuzzy vector is one of the simplest interactive fuzzy vectors, we may try to extend the proposed approach to the cases of a more general interactive fuzzy vector.

Chapter 4

Robust Optimality Analysis of the LP Problem with OFCs included in a Convex Polytope

IN this chapter, we study the LP problem with the uncertain OFC vector whose conceivable range is given by a convex polytope. First, we define a convex polytope representing a conceivable range of the uncertain OFC vector and review the outer approximation approach. By modifying the outer approximation approach, we build an algorithm with an initialisation process and a cutting strategy. To illustrate the procedure, we give a numerical example. We demonstrate the computational efficiency of the proposed algorithm compared with a conventional vertex-listing algorithm by a series of numerical experiments.

4.1 LP Problems with OFCs included in a Convex Polytope

Unlike the robust optimality analysis developed in the previous chapter, the solution method for the necessary optimality test of an NBF solution is not a simple task when conceivable range of the uncertain OFC vector is a convex polytope. Namely, the problem is set to be Problem (2.3), where the conceivable range of $\boldsymbol{\gamma}$ is specified by a convex polytope $\Phi \subseteq \mathbb{R}^n$ defined as

$$\Phi = \Big\{ \boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g} \Big\}, \tag{4.1}$$

where $\boldsymbol{H} \in \mathbb{R}^{p \times n}$ and $\boldsymbol{g} \in \mathbb{R}^p$ are constant matrix and vector, respectively. Since Φ is bounded, p must be substantially greater than n.

Proposition 2.3 indicates that if all vertices of the convex polytope Φ are included by the OAC of an NBF solution, then the solution is necessarily optimal. However, vertex listing requires a lot of computational effort.

Intuitively, the process of listing all vertices of the convex polytope defined in (4.1) is equivalent to the combination of n constraints selected from p constraints of Φ. The complexity order of listing vertices process would be extremely large as it can be $O(p\mathrm{C}n)$ potentially. Although it is known that the maximum vertex number of the convex polytope defined in (4.1) is $O(p^{n/2})$ [9], much less than $O(p\mathrm{C}n)$, it still requires enormous computational effort.

By Definition 2.1, we obtain the following lemma. It shows that if a feasible solution is necessarily optimal for a convex polytope, it is also necessarily optimal for a subset of the convex polytope.

Lemma 4.1. *If a feasible solution $\boldsymbol{x}^* \in \mathbb{R}^n$ is necessarily optimal for a convex polytope Λ that includes Φ, i.e., $\Phi \subseteq \Lambda$, then it is also necessarily optimal for Φ.*

The advantage of Lemma 4.1 is that it is not necessary to find all vertices. Instead, we should build a superset of the given convex polytope which is a simple convex polytope having small number of vertices, and then we cut the superset iteratively by adding a linear inequality constraint such that the superset approaches to the given convex polytope.

To develop a solution algorithm for the necessary optimality test problem based on Lemma 4.1, the satisfaction of the following requirements is essential:

(i) Let Φ be the given convex polytope, and Λ^k be the convex polytope including Φ at k-th iteration. Since the inclusion of Λ^k in the OAC is tested by using all vertices of Λ^k, the initial convex polytope including Φ should be characterised by a small number of vertices.

(ii) In each iteration of the alogorithm, we need to construct Λ^{k+1} such that $\Phi \subseteq \Lambda^{k+1} \subsetneqq \Lambda^k$. Moreover, it is useful if the vertex set of Λ^{k+1} is obtained simply from that of Λ^k.

Considering these requirements, we found that the *outer approximation approach* is useful. For developing the algorithm, we concentrated on the following two aspects:

(i) the construction of the initial convex polytope Λ^0,

(ii) the cutting approach to the construction of Λ^{k+1} from Λ^k.

Before discussing the aspects (i) and (ii), we describe the determination of the solution to test its necessary optimality. We solve the following $(n+1)$ LP problems:

$$\text{maximize } \boldsymbol{e}^{\mathrm{T}}\boldsymbol{c}, \text{ subject to } \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g}, \tag{4.2}$$

$$\text{minimize } c_i, \text{ subject to } \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g},\ i = 1, 2, \ldots, n, \tag{4.3}$$

where $\boldsymbol{e} := (1, 1, \ldots, 1)^{\mathrm{T}}$. Let $\boldsymbol{c}^0$ and $\boldsymbol{c}^i$, $i = 1, 2, ..., n$ be optimal solutions to Problems (4.2) and (4.3), respectively. Then we define

$$\bar{\boldsymbol{c}} = \frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{c}^i. \tag{4.4}$$

We have $\bar{\boldsymbol{c}} \in \Phi$ because $\boldsymbol{c}^i$ satisfies $H\boldsymbol{c}^i \leq \boldsymbol{g}$, i.e., $\boldsymbol{c}^i \in \Phi$ and Φ is convex. Then, we consider an optimal solution to the LP problem,

$$\text{maximize } \bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}, \text{ subject to } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \tag{4.5}$$

for the solution whose necessary optimality is tested if it is an NBF solution.

4.1.1 Construction of the initial convex polytope

Horst et al. [15] showed several methods for constructing the initial convex polytope. However, since Φ and Λ^k for any k are convex, we prefer the one with the least computation in obtaining all vertices of Λ^k. At the initialisation process, we need to check whether Φ is bounded and obtain all vertices of the initial cone Λ^0. To accomplish these initialisation tasks, we use the following proposition to build the initial convex polytope Λ^0.

Proposition 4.1. *Let α_0 and α_i, $i = 1, 2, \ldots, n$ be the optimal values of LP problems (4.2) and (4.3), respectively. Since Φ is bounded, α_0 and α_i, $i = 1, 2, \ldots, n$ are all finite. Then, the initial simplex $\Lambda^0 \supseteq \Phi = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g}\}$ is built by*

$$\Lambda^0 = \Big\{\boldsymbol{c} \in \mathbb{R}^n :\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{c} \leq \alpha_0,\ c_i \geq \alpha_i,\ i = 1, 2, ..., n\Big\}, \tag{4.6}$$

where the $(n+1)$ vertices are obtained as $(\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$ and

$$\Big(\alpha_1, \alpha_2, \ldots, \alpha_0 - \sum_{\substack{j=1 \\ j\neq k}}^{n} \alpha_j, \ldots, \alpha_n\Big)^{\mathrm{T}},\ k = 1, 2, \ldots, n.$$

Although there exist other methods to build the initial convex polytope, such as a hyperrectangle, we confirm that the simplex is advantageous in the following three points:

- Although all initialisation methods can check the boundedness of Φ, the simplex Λ^0 has the fewest polyhedral boundaries. The number of LP problems to be solved is $(n+1)$, which is also the number of polyhedral boundaries of Λ^0.

- In the proposed cutting approach described later, the number of cuttings, i.e., the number of added linear constraints after Λ^0 is built, is less than the number of constraints of Φ. Accordingly, the shape of the initial convex polytope does not affect the solution algorithm, and, thus, the simplest one is sufficient.

- The vertex set of Λ^0 can be simultaneously obtained when solving $(n+1)$ LP problems (4.2) and (4.3). Moreover, since the simplex has the least number of vertices, it requires the least computational effort.

After the initial bounded convex polytope Λ^0 with the vertex set $\mathbf{V}(\Lambda^0)$ is obtained, the iteration process starts.

4.1.2 Cutting and updating strategy

Assuming that we already obtained the convex polytope Λ^k with its vertex set $\mathbf{V}(\Lambda^k)$ at k-th iteration, we want to update the convex polytope and its vertex set by a cut, namely, we want to generate Λ^{k+1} and $\mathbf{V}(\Lambda^{k+1})$ from Λ^k and $\mathbf{V}(\Lambda^k)$. Since Φ is a convex polytope, we utilise one of its constraints for cutting Λ^k at each iteration. The maximum number of cutting hyperplanes is less than or equal to p, i.e., the number of the constraints of Φ, so that $\Lambda^p = \Phi$ holds. This fact guarantees the termination of the algorithm. However, to accomplish this task, we still need to solve several problems:

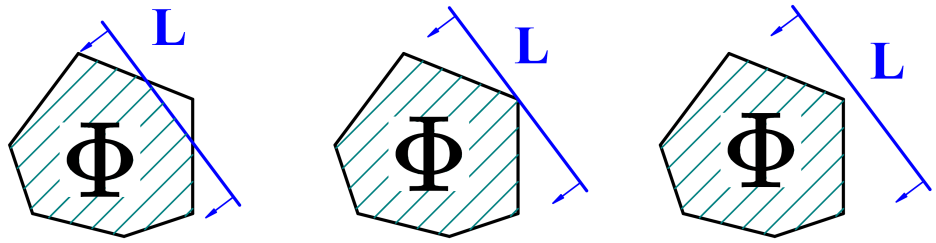
1. When cutting Λ^k, some constraints may become inessential because the added constraint makes the feasible region smaller. The inessential constraints, called *redundant constraints*, increase the calculation load in obtaining new vertices. Hence, it is required to remove them efficiently.

2. After cutting Λ^k and removing redundant constraints, we need to add the newly generated vertices and remove the vertices that do not satisfy the added constraint efficiently.

Identifying Redundant Constraints

Removing redundant constraints is not a simple task. To develop an algorithm that can efficiently identify them, we review some essential knowledge.

Definition 4.1. A constraint $\mathbf{L}$ included in the constraints defining the convex polytope Λ^k is *redundant* if Λ^k remains the same after the removal of $\mathbf{L}$. Moreover, if all vertices of Λ^k belong to the *interior* of $\mathbf{L}$, then $\mathbf{L}$ is *strictly redundant*.



(a) **L** is not redundant (b) **L** is redundant but not strictly redundant (c) **L** is strictly redundant

Figure 4.1: A constraint **L** to a convex polytope Φ

The redundant constraint and strictly redundant constraint are illustrated in Figure 4.1. Before the detailed descriptions about the identification of redundant constraints, we give the definition of the *degeneracy* for the clarification of the difference between redundancy and degeneracy.

Definition 4.2. A vertex $\boldsymbol{v} \in \Phi$ is *degenerate* if the number of hyperplanes composing the boundaries of Φ passing $\boldsymbol{v}$ is more than n.

A vertex of a polytope in $\mathbb{R}^n$ is determined by n independent hyperplanes. Therefore, if there are more than n hyperplanes passing a vertex, it is possible to remove some of the hyperplanes while keeping the vertex as a vertex of the resulting polyhedral set. It is noted that *the existence of a degenerate vertex does not always imply the existence of redundant constraints*. The main reason is that a constraint corresponding to a hyperplane that can be removed for a degenerate vertex may not be removable for the polyhedral set. A typical example is shown in Figure 4.2, where vertex $(3, 3, 2)^{\mathrm{T}}$ on hyperplane (i), (ii), (iii) and (iv) is degenerate but we cannot remove any of their corresponding constraints.

Since the degenerate vertices are much less than non-degenerate ones and the degeneracy can cause a great trouble, we assume that there exists no degenerate vertex in the convex polytope. As it is mentioned that the redundant constraint should be removed, we need an efficient algorithm to identify it.

Horst and Tuy [15] showed the following useful proposition to identify a redundant constraint.

Proposition 4.2. *Let* $l_j(\boldsymbol{c}) = \boldsymbol{h}'_j.\boldsymbol{c} - g'_j,\ j \in \mathbb{I}_K \subseteq \mathbb{Z}$ *denote linear functions. A constraint* $\mathbf{L}_j = \{\boldsymbol{c} \in \mathbb{R}^n : l_j(\boldsymbol{c}) \leq 0\},\ j \in \mathbb{I}_K$, *used to build a convex polytope* $\mathbf{P} = \{\boldsymbol{c} \in \mathbb{R}^n : l_i(\boldsymbol{c}) \leq$

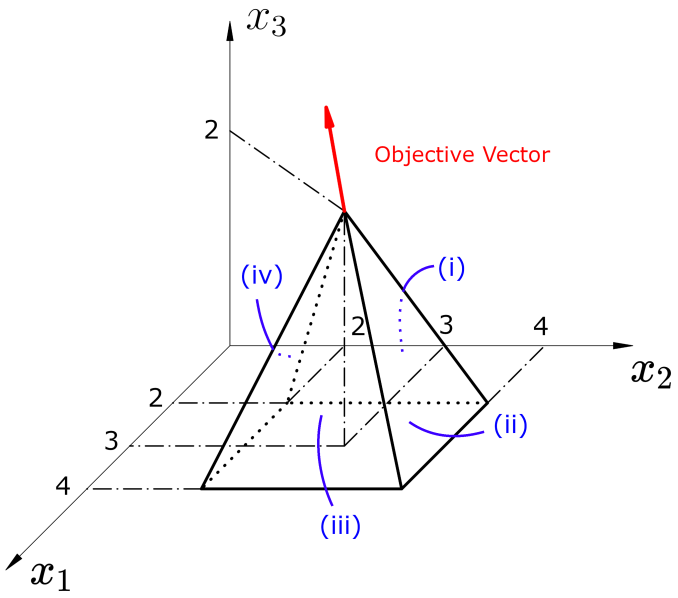


Figure 4.2: Degenerate vertex with no redundant constraint

$0, i \in \mathbb{I}_K\}$, *is redundant if there exists an* $i_0 \in \mathbb{I}_K \backslash \{j\}$ *such that*

$$\begin{aligned} &\Big\{\boldsymbol{c} \in \mathbb{R}^n : l_j(\boldsymbol{c}) = 0, \big(\forall i \in \mathbb{I}_K \backslash \{i_0\}, l_i(\boldsymbol{c}) \leq 0\big)\Big\} \\ &\quad \subseteq \Big\{\boldsymbol{c} \in \mathbb{R}^n : l_{i_0}(\boldsymbol{c}) \geq 0\Big\}. \end{aligned} \tag{4.7}$$

Therefore, we say that the constraint $\mathbf{L}_j$ *is redundant for* $\mathbf{P}$ *relative to* $\mathbf{L}_{i_0}$.

Proposition 4.2 gives an method to identify a redundant constraint. However, since it utilises the inclusion relation, the realisation is difficult. Moreover, since we use the vertex set of a convex polytope rather than the constraints, we prefer obtaining the updated vertex set to the updated constraints. Therefore, it is necessary to develop an algorithm for identifying redundant constraints by the vertex set of the polytope when a cut is applied. Hence, we use the following theorem [15].

Theorem 4.1. *Given a polytope* $\mathbf{P} = \{\boldsymbol{c} \in \mathbb{R}^n : l_i(\boldsymbol{c}) \leq 0, i \in \mathbb{I}_K\}$ *and a valid cut* $\mathbf{L}_k = \{\boldsymbol{c} \in \mathbb{R}^n : l_k(\boldsymbol{c}) \leq 0\}$, *let* $\mathbf{V}_k^-(\mathbf{P}) := \{\boldsymbol{c} \in \mathbf{V}(\mathbf{P}) : l_k(\boldsymbol{c}) < 0\} \neq \emptyset$ *denote the vertex set of* $\mathbf{P}$ *in the interior of* $\mathbf{L}_k$. *Then a constraint* $\mathbf{L}_j = \{\boldsymbol{c} \in \mathbb{R}^n : l_j(\boldsymbol{c}) \leq 0\}$, $j \in \boldsymbol{K}$ *of* $\mathbf{P}$ *is redundant for* $\mathbf{P} \cap \mathbf{L}_k$ *relative to* $\mathbf{L}_k$ *if and only if the following condition is valid:*

$$\forall \boldsymbol{c} \in \mathbf{V}_k^-(\mathbf{P}),\ l_j(\boldsymbol{c}) < 0. \tag{4.8}$$

Furthermore, when the convex polytope $\mathbf{P}$ has an interior point, i.e., $\operatorname{Int} \Phi \neq \emptyset$, we obtain a more useful result described as follows.

Theorem 4.2. *Let* $\mathbf{P} = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}'\boldsymbol{c} \leq \boldsymbol{g}'\}$, $\boldsymbol{H}' \in \mathbb{R}^{p\times n}$ *and* $\boldsymbol{g}' \in \mathbb{R}^p$, *denote a convex polytope, and let* $\mathbf{V}(\mathbf{P})$ *be its vertex set. Assume* $\mathbf{P}$ *has an interior point, i.e.,* $\mathrm{Int}\Phi \neq \emptyset$. *Then for the i-th constraint in* $\mathbf{P}$ *with* $\mathbf{L}_i = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{h}'_{i\cdot}\boldsymbol{c} \leq g'_i\}$, *let* $\mathbf{V}_i^{=}(\mathbf{P}) := \{\boldsymbol{v} \in \mathbf{V}(\mathbf{P}) : \boldsymbol{h}'_{i\cdot}\boldsymbol{v} = g'_i\}$, *where* $\boldsymbol{h}'_{i\cdot}$ *is the i-th row of* $\boldsymbol{H}$ *and* g'_i *is the i-th component of* $\boldsymbol{g}'$. *If* $\mathrm{Card}(\mathbf{V}_i^{=}(\mathbf{P})) < n$, *it is redundant for* $\mathbf{P}$. *Moreover, if* $\mathrm{Card}(V_i^{=}(\mathbf{P})) = 0$, *i.e.,* $\forall \boldsymbol{v} \in \mathbf{V},\ \boldsymbol{h}'_{i\cdot}\boldsymbol{v} < g'_i$, *it is strictly redundant.*

Proof. Since $\mathbf{P} \subseteq \mathbb{R}^n$ is a bounded, closed and convex polytope, the hyperplane including each $(n-1)$-face is determined by at least n vertices of $\mathbf{P}$. When $\mathbf{P}$ is defined $\boldsymbol{h}'_{i\cdot}\boldsymbol{c} \leq g'_i,\ i \in \mathbb{I}_K$, the hyperplane including a $(n-1)$-face is expressed as $\boldsymbol{h}'_{i^*\cdot}\boldsymbol{c} \leq g'_{i^*}$ with a certain $i^* \in \mathbb{I}_K$. If a hyperplane $\boldsymbol{h}'_{j\cdot}\boldsymbol{c} \leq g'_j, j \in \mathbb{I}_K$ does not include n vertices of $\mathbf{P}$, it is not the boundary of $\mathbf{P}$. As we have $\mathbf{P} \subseteq \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{h}'_{j\cdot}\boldsymbol{c} \leq g'_j\}$, we know constraint $\boldsymbol{h}'_{j\cdot}\boldsymbol{c} \leq g'_j$ is redundant. □

By Theorem 4.2, we obtain the following lemma.

Lemma 4.2. *Assume* $\mathrm{Int}\,\Phi \neq \emptyset$. *Then* Λ^0 *defined by Proposition 4.1 has no redundant constraint or degenerate vertex.*

Proof. Since the convex polytope constructed by Proposition 4.1 is apparently a simplex, the result is trivial due to $\mathrm{Int}\,\Phi \neq \emptyset$. □

From Lemma 4.2, we do not need to identify redundant constraints in the initialisation process.

Updating Vertex Set

After developing the algorithm for identifying redundant constraints by the vertex set, we discuss the process for updating the vertex set when applying a valid cut on a polytope. We assume that there exists no degenerate vertex in the convex polytope to be updated. Let $\mathbf{L} = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{c} \leq \beta\}$ be the valid cut applied to the convex polytope Λ^k, where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$.

Similar to the procedure in removing redundant constraints, we divide $\mathbf{V}(\Lambda^k)$ into $\mathbf{L}$ to $\mathbf{V}^-(\Lambda^k)$ and $\mathbf{V}^+(\Lambda^k)$, where $\mathbf{V}^-(\Lambda^k) := \{\boldsymbol{v} \in \mathbf{V}(\Lambda^k) : \boldsymbol{v} \in \mathbf{L}\}$ and $\mathbf{V}^+(\Lambda^k) := \{\boldsymbol{v} \in \mathbf{V}(\Lambda^k) : \boldsymbol{v} \notin \mathbf{L}\}$. In the proposed method, the vertices on $\mathbf{L}^{=} = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{c} = \beta\}$ are generated from either $\mathbf{V}^-(\Lambda^k)$ or $\mathbf{V}^+(\Lambda^k)$. As the results are the same, selecting the

vertex set with smaller size from $\mathbf{V}^{-}(\Lambda^k)$ and $\mathbf{V}^{+}(\Lambda^k)$ would be effective in reducing the computational cost. Let $\widehat{\mathbf{V}}$ denote the selected vertex set.

For a vertex $\boldsymbol{v} \in \widehat{\mathbf{V}}$, we select the constraints of Λ^k active at $\boldsymbol{v}$, i.e.,

$$\sum_{j=1}^{n} g_{ij} c_j \leq g_i, \ i \in \mathbb{I}_{\boldsymbol{v}}, \tag{4.9}$$

where $\mathbb{I}_{\boldsymbol{v}}$ is the index set of the active constraints at $\boldsymbol{v}$. As all vertices of Λ^k are non-degenerate, $\mathrm{Card}(\mathbb{I}_{\boldsymbol{v}}) = n$ holds.

To generate vertices on $\mathbf{L}^{=}$, we add the constraint $\mathbf{L}$, i.e.,

$$\left\{ \boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{c} \leq \beta \right\}. \tag{4.10}$$

Then we consider a linear inequality system composed of (4.9) and (4.10). Adding slack variables $y_1, \ y_2, \ldots, \ y_n \geq 0$, we consider the following simplex tableau corresponding to the linear inequality system composed of (4.9) and (4.10). $c_1, \ c_2, \ldots, \ c_n$ are free variables while $y_1, \ y_2, \ldots, y_n$ are non-negative variables:

c_1	c_2	$\ldots$	c_n	y_1	y_2	$\ldots$	y_n	y_{n+1}	RHS
$h_{i_1 1}$	$h_{i_1 2}$	$\ldots$	$h_{i_1 n}$	1	0	$\ldots$	0	0	g_{i_1}
$h_{i_2 1}$	$h_{i_2 2}$	$\ldots$	$h_{i_2 n}$	0	1	$\ldots$	0	0	g_{i_2}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$h_{i_n 1}$	$h_{i_n 2}$	$\ldots$	$h_{i_n n}$	0	0	$\ldots$	1	0	g_{i_n}
α_1	α_2	$\ldots$	α_n	0	0	$\ldots$	0	1	β

(4.11)

At $\boldsymbol{v}$, we know that $c_1, \ c_2, \ldots, \ c_n, \ y_{n+1}$ are the basic variables of $\boldsymbol{v}$. From this fact, the simplex tableau corresponding to $\boldsymbol{v}$ becomes the following one:

c_1	c_2	$\ldots$	c_n	y_1	y_2	$\ldots$	y_n	y_{n+1}	RHS
1	0	$\ldots$	0	$\check{h}_{i_1 1}$	$\check{h}_{i_1 2}$	$\ldots$	$\check{h}_{i_1 n}$	0	$\check{g}_{i_1}$
0	1	$\ldots$	0	$\check{h}_{i_2 1}$	$\check{h}_{i_2 2}$	$\ldots$	$\check{h}_{i_2 n}$	0	$\check{g}_{i_2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$\check{h}_{i_n 1}$	$\check{h}_{i_n 2}$	$\ldots$	$\check{h}_{i_n n}$	0	$\check{g}_{i_n}$
0	0	$\ldots$	0	$\check{h}_{i_{n+1} 1}$	$\check{h}_{i_{n+1} 2}$	$\ldots$	$\check{h}_{i_{n+1} n}$	1	$\hat{\beta}$

(4.12)

From this tableau, we generate vertices on $\mathbf{L}^{=}$ by excluding y_{n+1} from the basis and including a slack variable $y_i, \ i \in \mathbb{I}_{\boldsymbol{v}}$ into the basis. Owing to $\mathrm{Card}(\mathbb{I}_{\boldsymbol{v}}) = n$, we can obtain n vertices on $\mathbf{L}^{=}$. Moreover, as the linear inequality system (4.9) and (4.10) has $(n+1)$ constraints, each vertex on $\mathbf{L}^{=}$ is generated only by one pivot operation.

Applying the procedure described above for every vertex in $\widehat{\mathbf{V}}$, we obtain vertices on $\mathbf{L}^{=}$. However, we note that not all of these generated vertices are the vertices of Λ^{k+1},

because only a part of constraints of Λ^k, i.e., only (4.9) for $\boldsymbol{v}$ are considered. Therefore, the confirmation of the satisfaction of all constraints of Λ^k is necessary. Only the generated vertices satisfying all constraints of Λ^k are added to the vertex set of Λ^{k+1}.

4.2 Algorithm for Robust Optimality Analysis

Now we describe the algorithm for analysing the robust optimality of an NBF solution as follows.

Algorithm 4.1. (*An algorithm for robust optimality analysis of the LP problem with the OFC vector included in a convex polytope*)

- **Initialisation**:

  1. Check whether $\Phi = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g}\}$ is bounded by Proposition 4.1. If it is not, terminate the procedure. Otherwise, build the initial polytope $\Lambda^0 := \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{e}^{\mathrm{T}}\boldsymbol{c} \leq \alpha_0, c_i \geq \alpha_i, i = 1, 2, ..., n\} = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}^0\boldsymbol{c} \leq \boldsymbol{g}^0\}$ and its vertex set $\mathbf{V}(\Lambda^0)$. Then set $k \leftarrow 0$.
  2. Choose the point $\bar{\boldsymbol{c}}$ by (4.4) as the OFC vector and obtain an NBF solution $\boldsymbol{x}^*$, which gives the OAC as $\mathcal{M}(\boldsymbol{x}^*) := \{\boldsymbol{c} \in \mathbb{R}^n : \ \boldsymbol{c}_N - \boldsymbol{A}_N^{\mathrm{T}}\boldsymbol{A}_B^{-\mathrm{T}}\boldsymbol{c}_B \leq \boldsymbol{0}\} = \{\boldsymbol{c} \in \mathbb{R}^n : \ \boldsymbol{M}\boldsymbol{c} \leq \boldsymbol{0}\}$.

- **Iteration**:

  Step 1. If $\mathbf{V}(\Lambda^k) \subseteq \mathcal{M}(\boldsymbol{x}^*)$, terminate with $\boldsymbol{x}^*$ being necessarily optimal.

  Step 2. Pick a vertex $\boldsymbol{v} \in \mathbf{V}(\Lambda^k)$ such that $\boldsymbol{v} \notin \mathcal{M}(\boldsymbol{x}^*)$. If $\boldsymbol{v} \in \Phi$, terminate with no necessarily optimal solution exists.

  Step 3. (*Adding a New Constraint*) Choose a constraint in Φ as $\mathbf{L}_i = \{\boldsymbol{c} \in \mathbb{R}^n : h_{i\cdot}^k\boldsymbol{c} \leq \boldsymbol{g}_i^k\}$ such that $\boldsymbol{v} \notin \mathbf{L}_i$. (we can update Φ by removing $\mathbf{L}_i$ from it.) Add $\mathbf{L}_i$ to Λ^k to obtain Λ^{k+1}:

$$\Lambda^{k+1} = \Lambda^k \cap \mathbf{L}_i = \left\{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}^{k+1}\boldsymbol{c} \leq \boldsymbol{g}^{k+1}\right\}. \tag{4.13}$$

  Step 4. (*Divide the Vertex Set*) Divide $\mathbf{V}(\Lambda^k)$ by $\mathbf{L}_i$ into $\mathbf{V}^+(\Lambda^k)$ and $\mathbf{V}^-(\Lambda^k)$:

$$\mathbf{V}^+(\Lambda^k) = \left\{\boldsymbol{c} \in \mathbf{V}(\Lambda^k) : \boldsymbol{h}_{i\cdot}^{k+1}\boldsymbol{c} > g_i^{k+1}\right\}, \tag{4.14}$$

$$\mathbf{V}^-(\Lambda^k) = \left\{\boldsymbol{c} \in \mathbf{V}(\Lambda^k) : \boldsymbol{h}_{i\cdot}^{k+1}\boldsymbol{c} \leq g_i^{k+1}\right\}. \tag{4.15}$$

Step 5. (*Update the Vertex Set*) Choose a vertex set with smaller size from $\mathbf{V}^-(\Lambda^k)$ and $\mathbf{V}^+(\Lambda^k)$. Let $\widehat{\mathbf{V}}_k$ be the selected vertex set. For every $\boldsymbol{v}^j \in \widehat{\mathbf{V}}_k$, generate new vertices on the boundary of $\mathbf{L}_i$ and remove the vertices not satisfying the constraints of Λ^k. Let $\mathbf{V}(\boldsymbol{v}^j)$ be the set of the obtained vertices by the vertex $\boldsymbol{v}^j$. Then obtain the new vertex set $\mathbf{V}(\Lambda^{k+1})$ with:

$$\mathbf{V}(\Lambda^{k+1}) = \mathbf{V}^-(\Lambda^k) \bigcup \Big(\cup_j \mathbf{V}(\boldsymbol{v}^j) \Big) \tag{4.16}$$

Step 6. (*Remove the Redundant Constraints*) Apply Theorem 4.1 if $\operatorname{Int} \Phi \neq \emptyset$ and Theorem 4.2 otherwise to Λ^{k+1}. Namely, for each constraint $\mathbf{L}_j$ in Λ^{k+1}, let $\mathbf{V}_j^{=}(\Lambda^{k+1}) = \{\boldsymbol{v} \in \mathbf{V}(\Lambda^{k+1}) : \boldsymbol{h}_{j\cdot}^{k+1}\boldsymbol{c} = g_j\}$. If $\operatorname{Card}(\mathbf{V}_j^{=}(\Lambda^{k+1})) < n$, remove $\mathbf{L}_j$ from Λ^{k+1}.

Step 7. Set $k \leftarrow k+1$ and repeat the iteration.

Algorithm 4.1 is illustrated in the following example.

Example 4.1. We consider a production problem. We assume that products A and B are manufactured at a factory in Japan, where A is exported to Australia and B is exported to the USA. Formulating the manufacturing condition with decision variables x_1 and x_2 expressing the production amounts of products A and B, respectively, we find that the constraints become the same as Example 3.1. The price of A is set in AUD, and the price of B is set in USD, while we consider the gross profit in JPY. Since A and B are exported to different foreign countries, their profits are also affected by the exchange rate. From past experience, the profit of A, i.e., c_1, is never beyond 5 thousand JPY and the profit of B, i.e., c_2, is never beyond 3.5 thousand JPY. Moreover, in the past five years, it has been observed that c_1 and c_2 satisfied a linear constraint with $3c_1 + 4c_2 \geq 23$, where 23 denotes 23 thousand JPY. We would like to know the amounts of products A and B that maximise the gross profit.

At first, we formulate the problem as the following LP problem with uncertain OFC vector:

$$\begin{aligned} \text{maximize} \quad & c_1 x_1 + c_2 x_2, \\ \text{subject to} \quad & 3x_1 + 4x_2 \leq 42, \\ & 3x_1 + x_2 \leq 24, \\ & x_2 \leq 9, \ x_1, x_2 \geq 0, \end{aligned} \tag{4.17}$$

with c_1, c_2 satisfying

$$\left\{ \left(c_1, c_2\right)^{\mathrm{T}} : 3c_1 + 4c_2 \geq 23, c_1 \leq 5, c_2 \leq 3.5 \right\} \tag{4.18}$$

The problem is depicted in Figure 4.3. We solve the problem by Algorithm 4.1. For the sake of simplicity, the process using slack variables is omitted in this example:

Initialisation: The convex polytope $\Phi = \{\boldsymbol{c} \in \mathbb{R}^2 : \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g}\}$ takes the following values:

$$\boldsymbol{H} = \begin{bmatrix} -3 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{g} = \begin{bmatrix} -23 \\ 5 \\ 3.5 \end{bmatrix}.$$

We verify the boundedness. The constraints of Φ are denoted by (1), (2) and (3) in Figure 4.3, respectively. Matrix $\boldsymbol{H}^0$ and vector $\boldsymbol{g}^0$ of the initial polytope $\Lambda^0 = \{\boldsymbol{c} \in \mathbb{R}^2 : \boldsymbol{H}^0\boldsymbol{c} \leq \boldsymbol{g}^0\}$ are obtained as

$$\boldsymbol{H}^0 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \boldsymbol{g}^0 = \begin{bmatrix} 8.5 \\ -3 \\ -2 \end{bmatrix}.$$

The constraints of Λ^0 are denoted by (4), (5) and (6) in Figure 4.3, respectively. The vertex set is obtained as $\mathbf{V}(\Lambda^0) = \left\{(3,2)^{\mathrm{T}}, (6.5,2)^{\mathrm{T}}, (3,5.5)^{\mathrm{T}}\right\}$. Consider an NBF solution $\boldsymbol{x}^* = (6,6)^{\mathrm{T}}$. The associated OAC is obtained as

$$\mathscr{M}(\boldsymbol{x}^*) = \left\{ \left(c_1, c_2\right)^{\mathrm{T}} \;\middle|\; \begin{array}{c} \frac{1}{9}c_1 - \frac{1}{3}c_2 \leq 0 \\ -\frac{4}{9}c_1 + \frac{1}{3}c_2 \leq 0 \end{array} \right\}.$$

Iteration $k = 0$:

Step 1. Since $(6.5,2)^{\mathrm{T}} \notin \mathscr{M}(\boldsymbol{x}^*)$, $\mathbf{V}(\Lambda^0) \nsubseteq \mathscr{M}(\boldsymbol{x}^*)$, proceed to the next step.

Step 2. Since $(6.5,2)^{\mathrm{T}} \notin \Phi$, proceed to the next step.

Step 3. Since $(6.5,2)^{\mathrm{T}}$ does not satisfy Constraint (2), add it to Λ^0 and update to Λ^1 with Constraints (4), (5), (6) and (2).

Step 4. Divide $\mathbf{V}(\Lambda^0)$ into

$$\mathbf{V}^+(\Lambda^0) = \{(6.5,2)^{\mathrm{T}}\} \text{ and } \mathbf{V}^-(\Lambda^0) = \{(3,2)^{\mathrm{T}}, (3,5.5)^{\mathrm{T}}\}.$$

Step 5. By pivoting, we obtain new vertices $(5,2)^{\mathrm{T}}$ and $(5,3.5)^{\mathrm{T}}$. Both satisfy the constraints of Λ^0. Hence, we obtain $\mathbf{V}(\Lambda^1) = \{(3,2)^{\mathrm{T}}, (3,5.5)^{\mathrm{T}}, (5,2)^{\mathrm{T}}, (5,3.5)^{\mathrm{T}}\}$.

Step 6. Since every constraint in Λ^1 contains 2 vertices, there exists no redundant constraint. proceed to the next iteration.

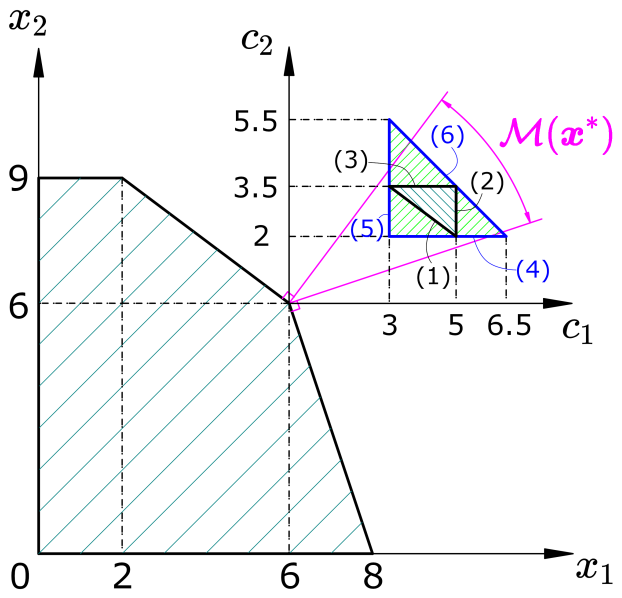


Figure 4.3: Illustration of Example 4.1

Iteration $k = 1$:

Step 1. Since $(3, 5.5)^{\mathrm{T}} \notin \mathscr{M}(\boldsymbol{x}^*)$, $\mathbf{V}(\Lambda^1) \nsubseteq \mathscr{M}(\boldsymbol{x}^*)$, proceed to the next step.

Step 2. Since $(3, 5.5)^{\mathrm{T}} \notin \Phi$, proceed to the next step.

Step 3. Since $(3, 5.5)^{\mathrm{T}}$ does not satisfy Constraint (3), add it to Λ^1 and update to Λ^2 with Constraints (4), (5), (6), (2) and (3).

Step 4. Divide $\mathbf{V}(\Lambda^1)$ into

$$\mathbf{V}^+(\Lambda^1) = \{(3, 5.5)^{\mathrm{T}}\} \text{ and } \mathbf{V}^-(\Lambda^1) = \{(3, 2)^{\mathrm{T}}, (5, 2)^{\mathrm{T}}, (5, 3.5)^{\mathrm{T}}\}.$$

Step 5. By pivoting, we obtain new vertices $(3, 3.5)^{\mathrm{T}}$ and $(5, 3.5)^{\mathrm{T}}$. Both satisfy the constraints of Λ^1. Hence, we obtain $\mathbf{V}(\Lambda^2) = \{(3, 2)^{\mathrm{T}}, (5, 2)^{\mathrm{T}}, (5, 3.5)^{\mathrm{T}}, (3, 3.5)^{\mathrm{T}}\}$;

Step 6. Since Constraint (6) contains only 1 vertex $(5, 3.5)^{\mathrm{T}}$, remove it and update Λ^2 with Constraints (4), (5), (2) and (3). Proceed to the next iteration.

Iteration $k = 2$:

Step 1. Since $\mathbf{V}(\Lambda^2) \subseteq \mathscr{M}(\boldsymbol{x}^*)$, terminate the procedure and state that $\boldsymbol{x}^* = (6, 6)^{\mathrm{T}}$ to be necessarily optimal.

As the result, we verify that the solution $(6, 6)^{\mathrm{T}}$ is necessarily optimal. Namely, the plan corresponds to $(6, 6)^{\mathrm{T}}$ attain the maximum gross profit no matter the profits of A and B take values in the considered range.

4.3 Computational Analysis and Simulations

In this section, we compare the computational efficiency of the proposed approach with the conventional vertex-listing method. We refer to the LP Problem (2.3) with convex polytope Φ as the $(m \times n)$-scale LP problem, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{\gamma} \in \Phi$. $\Phi \subseteq \mathbb{R}^n$ is the convex polytope defined by (4.1), i.e., $\Phi = \{\boldsymbol{c} \in \mathbb{R}^n : \boldsymbol{H}\boldsymbol{c} \leq \boldsymbol{g}\}$, where $\boldsymbol{H} \in \mathbb{R}^{p \times n}$ and $\boldsymbol{g} \in \mathbb{R}^p$.

The conventional vertex-listing method we compare is composed of the following procedure:

(i) List all vertices of Φ. Let $\mathbf{V}(\Phi)$ be the set of vertices.

(ii) Obtain an NBF solution $\boldsymbol{x}^*$ with $\mathscr{M}(\boldsymbol{x}^*)$ by the median point of $\mathbf{V}(\Phi)$.

(iii) Check whether all vertices in $\mathbf{V}(\Phi)$ are in $\mathscr{M}(\boldsymbol{x}^*)$.

We expect a lot of computational effort at (i) especially when the problem size is large. As we mentioned earlier, the maximum vertex number of a convex polytope defined in (4.1) is $O(p^{n/2})$, where p is the number of the constraints of Φ, we use it as the worst complexity order for listing all vertices. Moreover, in each iteration of listing a new vertex, the matrix inversion calculation is involved, of which the worst complexity order is $O(n^3)$. It is noted that the complexity order of the matrix inversion calculation becomes lower and lower in recent years. However, we use $O(n^3)$ for the complexity of matrix inversion computation. Therefore, the complexity order of vertex-listing method is

$$O(p^{n/2}) \cdot O(n^3) = O(p^{n/2} \cdot n^3). \tag{4.19}$$

Let us see the complexity of the proposed approach. In the initialisation, it is necessary to solve $(m \times n)$-scale LP problems for $(n+1)$ times to find the initial polytope with the vertex set. Then an $(m \times n)$-scale LP problem is solved once to find an NBF solution with its OAC. In the iteration process, we need multiple assumptions to estimate the complexity. For example, although it is known that the number of iterations is up to the number of the constraints of Φ, i.e., p, the algorithm usually terminates at a much earlier step. Moreover, unlike the vertex-listing method, the number of vertices and removable redundant constraints at each iteration are unknown. Therefore, we can only analyse its worst case approximately as follows:

(i) The number of iterations equals to its maximum, i.e., p.

(ii) Only one vertex is removed at each iteration. However, when dividing the vertex set, we assume all vertices are used, including the removed one.

(iii) No redundant constraint is removed during all iterations.

(iv) All vertices generated by a cut, i.e., n vertices, are assumed to be added.

Since the computation time for checking whether a vertex belongs to the convex polytope is too short compared with the update process of vertex sets, we ignore it.

By Tableaus (4.11) and (4.12), it is not hard to understand that the process of pivoting an $(l \times k)$ tableau has the same computational complexity with solving an $(l \times k)$-scale LP problem. Moreover, by algebraic analysis, it is known that the complexity order of solving an $(l \times k)$-scale LP problem equals to the one of a non-singular $(l \times l)$ matrix inversion when $l \leq k$. Therefore, the complexity order of the proposed algorithm is

$$O\left(\sum_{j=1}^{p}\left[(n+2)+(j-1)\cdot(n-1)\right]\right)\cdot O(n^3) = O(p^2 \cdot n^4). \tag{4.20}$$

Since in a large-scale problem p is much greater than n and n is much greater than 4, the results of (4.19) and (4.20) show that, even in the worst case, our approach can still treat the robust optimality analysis more efficient than the vertex-listing method.

Furthermore, to validate our approach by comparisons generally, we execute a series of computer simulations, which are carried out by `Python3.8.9` with `numpy1.22.1` and `scipy1.7.3` in `MacOS` with `Intel(R) Core(TM) i7-4770 HQ CPU@2.20GHz` and `16GB 1600MHz DDR3 RAM`.

Since the case where a necessarily optimal solution exists usually requires more computational effort than a non-existence case, we show the results only in the former case. The result is shown in Table 4.1, where m and n are the numbers of the constraints and variables of the LP problem defined in (2.3), p is the number of the constraints of the convex polytope in (4.1). The result shows that the proposed approach performs better even when the scale of the LP problem is not large. Moreover, when the scale becomes too large to be solved by the vertex-listing method, the proposed approach solves the problems in a reasonable time.

The computation load of the proposed approach with respect to the size is shown in Figure 4.4. We find that its expected computational amount is almost linear to the scale of the LP problem. However, the simulation also indicates that the proposed approach is not always better than the conventional one. When there exists degeneracy in the convex polytope, our approach can only terminate without giving any result. For example, in Figure 4.4 around the scale of 1500, the computational time becomes very small because the program encounters a degenerate case and has to give up the analysis.

Table 4.1: Some Simulation Results

m	n	p	$T_1/(s)$	$T_2/(s)$	Nec.	m	n	p	$T_1/(s)$	$T_2/(s)$	Nec.
3	5	10	0.0698	0.0539	True	4	7	18	2.3431	0.1839	True
3	5	12	0.1118	0.0526	True	4	7	18	2.1856	0.1408	True
3	5	12	0.1157	0.0638	True	4	7	20	5.4029	0.1502	True
3	5	12	0.1825	0.0992	True	4	7	20	5.3161	0.1621	True
3	5	12	0.1828	0.0738	True	4	7	20	5.2396	0.1856	True
3	6	15	0.5978	0.1632	True	4	8	21	11.5032	0.2000	True
3	6	15	0.491	0.1519	True	4	8	24	40.4563	0.2802	True
3	6	15	0.4212	0.0968	True	4	8	25	82.0311	0.2356	True
3	6	15	0.4165	0.0919	True	4	8	25	71.5362	0.2114	True
3	6	16	0.6579	0.136	True	4	8	25	59.9396	0.2183	True

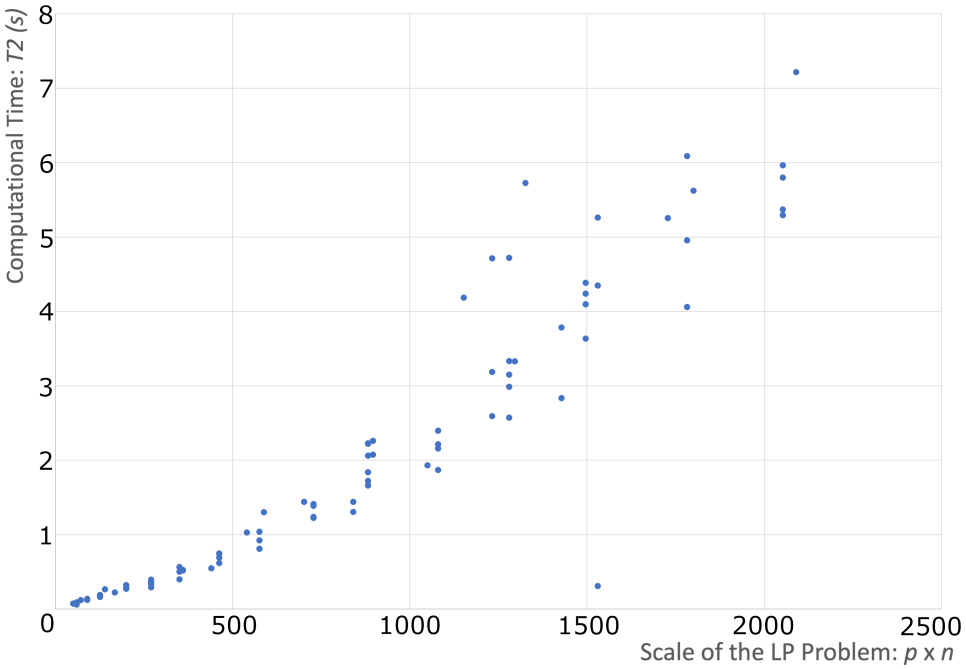


Figure 4.4: The simulation result of the outer approximation approach

4.4 Concluding Remarks

In this chapter, we proposed a robust optimality analysis in the LP problem with an uncertain OFC vector whose conceivable range is given by a convex polytope.

For an NBF solution, we proposed an algorithm based on the outer approximation approach for testing its necessary optimality. Analysing the problem, we showed theo-

rems useful for the construction of the algorithm, i.e., identifying redundant constraints, removing the redundant constraints, and updating the vertex set. Then we proposed the algorithm and analysed its computational complexity. Moreover, the advantage of the proposed algorithm over the conventional vertex-listing method is demonstrated theoretically by the computational complexity as well as by numerical simulations.

In the proposed approach, we assumed the non-degeneracy of the basic feasible solution. It is possible to extend the proposed approach to the case of a degenerate basic feasible solution. Moreover, we assumed that the degeneracy never occurs at any vertex of the convex polytope showing the conceivable range of uncertain OFCs. To overcome these limitations would be our future work.

Chapter 5

A Necessity Measure Approach to Balancing Robustness and Satisfaction in Fuzzy LP Problems

IN this chapter, we treat a problem of finding a solution satisfying set-inclusive constraints with fuzzy coefficients. As the necessity measure shows an inclusion degree between fuzzy subsets, the necessity measure is utilised for treating the set-inclusive constraints of the problem. However, the conventional necessity measure does not express well the trade-off between robustness and satisfaction. Then, an extended necessity measure is proposed. After investigating its properties, we apply the extended necessity measure to the problem with set-inclusive constraints with fuzzy coefficients. The reduced mathematical programming problem is shown, and an approximate solution algorithm is proposed. A numerical example shows the differences of solutions by a fluctuation of the spreads of fuzzy coefficients.

5.1 The Extended Necessity Measure

Proposition 2.8 shows an interpretation of the necessity measure using h-level sets. It gives an intuitive understanding for the decision maker. From Proposition 2.8, we obtain $N_{\tilde{A}}(\tilde{B}) \geq h$ if and only if $(\tilde{A})_{1-h} \subseteq [\tilde{B}]_h$ for fuzzy subsets $\tilde{A}$ and $\tilde{B}$.

Consider a set-inclusive constraint $\tilde{A}(\boldsymbol{x}) \subseteq \tilde{B}$, where the fuzzy quantity $\tilde{A}(\boldsymbol{x})$ shows the conceivable range of a certain function value determined by a variable $\boldsymbol{x}$, and $\tilde{B}$ is a fuzzy subset showing a satisfiable range of the left-hand side function values. $\tilde{A}(\boldsymbol{x})$ and $\tilde{B}$

are both fuzzy quantities, but their meanings of the uncertainty are different.

$\tilde{A}(\boldsymbol{x})$ shows the conceivable range of the uncertain function value, while $\tilde{B}$ shows a vague range of satisfiable values. We explain this difference by using h-level sets. As $h \in [0,1]$ increasing, we consider more values as conceivable values of the left-hand side values by $(\tilde{A})_{1-h}$. On the other hand, as $h \in [0,1]$ increasing, $[\tilde{B}]_h$, the range of satisfiable values becomes smaller, i.e., we accept less values as satisfiable. Namely, in the constraint $(\tilde{A}(\boldsymbol{x}))_{1-h} \subseteq [\tilde{B}]_h$, the level h of $(\tilde{A}(\boldsymbol{x}))_{1-h}$ evaluates to what extent the possible fluctuation of the left-hand side value we take into account, i.e., the robustness degree, while the level h of $[\tilde{B}]_h$ evaluates to what extent the decision maker is intolerant, i.e., the satisfaction degree.

Introducing the interpretations of level h described above, we investigate the necessity measure satisfying $N_{\tilde{A}}(\tilde{B}) \geq h$ if and only if $(\tilde{A})_{1-h} \subseteq [\tilde{B}]_h$. Because the conventional necessity measure $N_{\tilde{A}}(\tilde{B})$ is obtained by

$$N_{\tilde{A}}(\tilde{B}) = \sup\left\{h \in [0,1] : (\tilde{A})_{1-h} \subseteq [\tilde{B}]_h\right\},$$

which evaluates a trade-off between the robustness degree and the satisfaction degree. However, the trade-off expressed the necessity measure is univocal with only robustness degree $(1-h)$ to satisfaction degree h. This fact would be too restrictive for representing the decision maker's preference between robustness and satisfaction. We illustrate the restriction of the conventional necessity measure.

Consider two fuzzy subsets $\tilde{A}$ and $\tilde{B}$ satisfying $(\tilde{A})_{0.24} \subseteq [\tilde{B}]_{0.76}$ and $(\tilde{A})_{0.2} \nsubseteq [\tilde{B}]_{0.8}$, where 0.76 is the maximum h. However, we find that $(\tilde{A})_{0.2} \subseteq [\tilde{B}]_{0.75}$ holds, which may be much better than $(\tilde{A})_{0.24} \subseteq [\tilde{B}]_{0.76}$ for the decision maker. Hence, the solution corresponding to $(\tilde{A})_{0.2} \subseteq [\tilde{B}]_{0.75}$ should be preferred for the decision maker. Nevertheless, by the conventional necessity measure, we obtain a solution corresponding to $(\tilde{A})_{0.24} \subseteq [\tilde{B}]_{0.76}$.

Then, we propose an extended necessity measure. To express the trade-off between the robustness degree and the satisfaction degree freely, we use two parameters. Accordingly, we consider an extended necessity measure by using v and w representing the robustness degree and the satisfaction degree, respectively. Before the definition of the proposed extended necessity measure, we give the following remark and lemma.

Remark 5.1. The necessity measure of $\tilde{B}$ on $\tilde{A}$, i.e. $N_{\tilde{A}}(\tilde{B})$, should at least satisfy the following two conditions:

(i) $N_{\tilde{A}}(\tilde{B}) = 1 \;\Leftrightarrow\; (\tilde{A})_0 \subseteq [\tilde{B}]_1$,

(ii) $N_{\tilde{A}}(\tilde{B}) > 0 \;\Leftrightarrow\; [\tilde{A}]_1 \subseteq (\tilde{B})_0$.

Lemma 5.1. *The set-inclusion relation* $(\tilde{A})_v \subseteq [\tilde{B}]_w$ *implies* $(\tilde{A})_{v'} \subseteq [\tilde{B}]_{w'}$ *for all* $v, v', w, w' \in [0,1]$ *such that* $v' \geq v$ *and* $w' \leq w$.

Proof. Trivial from the definitions of h- and strong h-level sets. □

Now, we give the definition of the proposed extended necessity measure. Inspired from Proposition 2.8, it is defined as follows.

Definition 5.1. The extended necessity measure maximisation for two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as

$$N^E_{\tilde{A}}(\tilde{B}) = \sup\Big\{f(v,w) : (\tilde{A})_v \subseteq [\tilde{B}]_w,\ v, w \in [0,1]\Big\}, \tag{5.1}$$

where $f : [0,1] \times [0,1] \to [0,1]$ is called the trade-off function, which is upper semi-continuous [31], and is *strictly decreasing* on the first argument and *strictly increasing* on the second argument. Moreover, $f(1,w) = 0,\ \forall w \in [0,1]$ and $f(v,0) = 0,\ \forall v \in [0,1]$ because $(\tilde{A})_1 = \emptyset$ and $[\tilde{B}]_1 = X$ (universal set) vanish the condition $(\tilde{A})_v \subseteq [\tilde{B}]_w$ and this fact ensures $N^E_{\tilde{A}}(\tilde{B}) \geq \max(f(1,1), f(0,0))$. If $f(1,1) > 0$ or $f(0,0) > 0$ holds, the necessity degree is positive and it destroys the meaning of the necessity measure in every case.

We note that the extended necessity measure N^E is reduced to the conventional necessity measure N when $f(v,w) = \min(1-v, w)$. Moreover, when considering a function $f : [0,1] \times [0,1] \to [0,1]$ satisfying the conditions of Definition 5.1, function $I : [0,1] \times [0,1] \to [0,1]$ defined by $I(w,v) = 1 - f(v,w)$ can be regarded as an implication function [29, 18] because we have $I(0,v) = 1,\ \forall v \in [0,1]$ and $I(w,1) = 1,\ \forall w \in [0,1]$ and I is decreasing on the first argument and increasing on the second argument. In other words, f can be defined from an implication function I by $f(v,w) = 1 - I(w,v)$.

To show a property similar to Proposition 2.7 for $N^E_{\tilde{A}}(\tilde{B})$, we introduce upper and lower semi-continuities of set-valued maps. For the introduction, we use the following definitions [1].

Definition 5.2. Let F be a set-valued map from X to Y. The *domain* of F is the subset of elements $x \in X$ such that $F(x)$ is not empty:

$$\mathrm{Dom}(F) := \Big\{x \in X : F(x) \neq \emptyset\Big\}. \tag{5.2}$$

Definition 5.3. Let $d(x, y)$ denote the *distance* between $x \in X$ and $y \in X$, then the *ball of radius* η *around* $x \in X$ $B_X(x, \eta)$ is defined by

$$B_X(x, \eta) := \{x' \in X : d(x', x) \leq \eta\}, \tag{5.3}$$

and the distance from $x \in X$ to $K \subseteq X$ is defined by

$$d(x, K) = \inf_{y \in K} d(x, y). \tag{5.4}$$

Definition 5.4. The *ball of radius* η *around* $K \subseteq X$ is defined by

$$B_X(K, \eta) := \{x \in X : d(x, K) \leq \eta\}, \tag{5.5}$$

where the balls $B_X(K, \eta)$ are *neighbourhood*s of K.

The ball of radius η around $y \in Y$ $B_Y(y, \eta)$, and that around $O \subseteq Y$ $B_Y(O, \eta)$ are defined in the same ways.

Now, we define upper and lower continuities of a set-valued map F in the following two definitions.

Definition 5.5. A set-valued map $F : X \to Y$ is called *upper semi-continuous* at $x \in \mathrm{Dom}(F)$ if and only if for any neighbourhood $\mathscr{U}$ of $F(x)$

$$\exists\, \eta > 0 \text{ such that } \forall\, x' \in B_X(x, \eta),\ F(x') \subseteq \mathscr{U}. \tag{5.6}$$

Moreover, F is *upper semi-continuous* if and only if it is upper semi-continuous at any point of $\mathrm{Dom}(F)$.

Definition 5.6. A set-valued map $F : X \to Y$ is called *lower semi-continuous* at $x \in \mathrm{Dom}(F)$ if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in \mathrm{Dom}(F)$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y. Moreover, it is said to be *lower semi-continuous* if it is lower semi-continuous at every point $x \in \mathrm{Dom}(F)$.

We note that the upper and lower semi-continuities of a set-valued map are different from those of a real-valued function used previously.

To obtain a property similar to Proposition 2.7 for $N^E_{\tilde{A}}(\tilde{B})$, we consider the following set-valued maps $S : [0, 1) \to X$ and $W : (0, 1] \to X$:

$$S_{\tilde{A}}(h) = (\tilde{A})_h, \qquad W_{\tilde{B}}(h) = [\tilde{B}]_h. \tag{5.7}$$

Namely, $S_{\tilde{A}}(h)$ shows the strong h-level set of $\tilde{A}$ while $W_{\tilde{B}}(h)$ shows the h-level set of $\tilde{B}$.

Lemma 5.2. *For any $y \in S_{\tilde{A}}(h)$ and $h \in [0,1)$, for a sufficiently small $\varepsilon > 0$, there exist $w \in B_Y(y,\varepsilon)$ and $h' > h$ such that $w \in S_{\tilde{A}}(h')$.*

Proof. Consider $y \in S_{\tilde{A}}(h)$ and $h \in [0,1)$, arbitrarily. Suppose that there are no $w \in B_Y(y,\varepsilon)$ and $h' > h$ such that $w \in F(h')$ for a sufficiently small $\varepsilon > 0$. This implies that for all $w \in B_Y(y,\varepsilon)$, $w \notin S_{\tilde{A}}(h') = (\tilde{A})_{h'}$ for any $h' > h$. On the other hand, from $y \in S_{\tilde{A}}(h) = (\tilde{A})_h$, we obtain $\mu_{\tilde{A}}(y) > h$. Let $\bar{h} = (\mu_{\tilde{A}}(y) + h)/2$. Then we have $\bar{h} > h$ and $\mu_{\tilde{A}}(y) > \bar{h}$. Namely, we have $y \in B_Y(y,\varepsilon)$ such that $y \in S_{\tilde{A}}(\bar{h})$ and $\bar{h} > h$. This is contradictory to the previous result under the supposition. Then we have the lemma. □

Using Lemma 5.2, we can know that $S_{\tilde{A}}$ is lower semi-continuous by the following proposition.

Proposition 5.1. *$S_{\tilde{A}} : [0,1) \to X$ is lower semi-continuous.*

Proof. Let $h \in [0,1)$, arbitrarily. For any $y \in S_{\tilde{A}}(h)$, consider a sequence $h_n \in \mathrm{Dom}(S_{\tilde{A}})$ converges to h. As $S_{\tilde{A}}(h)$ is a strong h-level set, we have $S_{\tilde{A}}(h) \subseteq S_{\tilde{A}}(h_n)$ for $h_n \leq h$ and $S_{\tilde{A}}(h) \supseteq S_{\tilde{A}}(h_n)$ for $h_n > h$. We define a sequence y_n by $y_n = \mathrm{argmin}_{z \in \mathrm{cl}(S_{\tilde{A}}(h_n))} d(z,y)$, for $h_n > h$, where $\mathrm{cl}(D)$ denotes the closure of a subset D. Then we construct a sequence x_n by

$$x_n = \begin{cases} y, & \text{if } h_n \leq h, \\ \text{a point in } S_{\tilde{A}}(h_n) \cap B_Y\left(y_n, \frac{\varepsilon}{n}\right), & \text{if } h_n > h, \end{cases} \tag{5.8}$$

where $\varepsilon > 0$ is a sufficiently small number. $S_{\tilde{A}}(h_n) \cap B_Y\left(y_n, \frac{\varepsilon}{n}\right) \neq \emptyset$ is ensured by the definition of y_n. By the definition of the sequence x_n, we have $x_n \in S_{\tilde{A}}(h_n)$. As y_n is the closest to y in $\mathrm{cl}(S_{\tilde{A}}(h_n))$, even when $h_n > h$, we know that x_n converges to y from Lemma 5.2. Therefore, x_n converges to y. This implies that $S_{\tilde{A}} : [0,1) \to X$ is lower semi-continuous. □

Next we show $W_{\tilde{B}}$ is lower semi-continuous.

Proposition 5.2. *$W_{\tilde{B}} : [0,1) \to X$ is upper semi-continuous.*

Proof. Consider $h \in (0,1]$. For an arbitrary neighbourhood $\mathcal{U}$ of $W_{\tilde{B}}(h)$, we define $\Delta_{\mathcal{U}}$ by

$$\Delta_{\mathcal{U}} = \sup\{h : [\tilde{B}]_h \subseteq \mathcal{U}\} = \sup\{h : (\tilde{B})_h \subseteq \mathcal{U}\} = 1 - N_{\tilde{B}}(\mathcal{U}). \tag{5.9}$$

Then, we obtain

$$\forall h' \in B_{(0,1]}(h, \Delta_{\mathcal{U}}),\ W_{\tilde{B}}(h) \subseteq \mathcal{U}. \tag{5.10}$$

Therefore, $W_{\tilde{B}} : [0,1) \to X$ is upper semi-continuous. □

As we have proven that $S_{\tilde{C}}$ is lower semi-continuous and $W_{\tilde{C}}$ is upper semi-continuous, we can obtain the necessary and sufficient conditions for $N^E_{\tilde{A}}(\tilde{B}) \geq h$. At first, we obtain the following lemma:

Lemma 5.3. *The set* $Q = \{(v, w) : (\tilde{A})_v \subseteq [\tilde{B}]_w,\ v, w \in [0,1]\}$ *is a closed set.*

Proof. Consider a sequence $\{(v_i, w_i)\} \subseteq Q$ such that (v_i, w_i) converges to $(\bar{v}, \bar{w}) \in Q$. From $(\bar{v}, \bar{w}) \in Q$, we have $(\tilde{A})_{\bar{v}} \subseteq [\tilde{B}]_{\bar{w}}$, i.e. $S_{\tilde{A}}(\bar{v}) \subseteq W_{\tilde{B}}(\bar{w})$. When $S_{\tilde{A}}(\bar{v}) = \emptyset$, $S_{\tilde{A}}(\bar{v}) \subseteq W_{\tilde{B}}(\bar{w})$, i.e. $(\tilde{A})_{\bar{v}} \subseteq [\tilde{B}]_{\bar{w}}$ is satisfied.

Then we consider the case where $S_{\tilde{A}}(\bar{v}) \neq \emptyset$. From Proposition 5.1, $S_{\tilde{A}}$ is lower semi-continuous. Then for any $y \in S_{\tilde{A}}(\bar{v})$ and for the sequence of elements v_i converging to $\bar{v}$, there exists a sequence of elements $y_i \in S_{\tilde{A}}(v_i)$ converging to y. On the other hand, from $S_{\tilde{A}}(\bar{v}) \subseteq W_{\tilde{B}}(\bar{w})$ and $S_{\tilde{A}}(\bar{v}) \neq \emptyset$, we have $W_{\tilde{B}}(\bar{w}) \neq \emptyset$. From Proposition 5.2, $W_{\tilde{B}}$ is upper semi-continuous. Then, for any neighbourhood $\mathcal{U}$ of $W_{\tilde{B}}(\bar{w})$, there exists η such that for any $w' \in B_{(0,1]}(\bar{w}, \eta)$, $W_{\tilde{B}}(w') \subseteq \mathcal{U}$. For a sufficiently large i, we obtain $W_{\tilde{B}}(w_i) \subseteq \mathcal{U}$ because w_i converges to $\bar{w}$. Therefore, for any neighbourhood $\mathcal{U}$ of $W_{\tilde{B}}(\bar{w})$, for a sufficiently large i, we have $y_i \in S_{\tilde{A}}(v_i) \subseteq W_{\tilde{B}}(w_i) \subseteq \mathcal{U}$. This implies

$$\bar{y} = \lim_{i \to +\infty} y_i \in \bigcap_{\text{all neighbourhoods } \mathcal{U} \text{ of } \mathrm{W}_{\tilde{\mathrm{B}}}(\bar{\mathrm{w}})} \mathcal{U} = \mathrm{cl}(W_{\tilde{B}}(\bar{w})). \tag{5.11}$$

Because $y \in S_{\tilde{A}}(\bar{v})$ is arbitrary, we obtain $S_{\tilde{A}}(\bar{v}) \subseteq \mathrm{cl}(W_{\tilde{B}}(\bar{w}))$.

When $W_{\tilde{B}}(\bar{w})$, i.e. $[\tilde{B}]_{\bar{w}}$ is not closed, from the definition of the h-level set, there exists a point z in $\mathrm{cl}(W_{\tilde{B}}(\bar{w})) = \mathrm{cl}([\tilde{B}]_{\bar{w}})$, where the membership function $\mu_{\tilde{B}}$ is discontinuous and the membership function value $\mu_{\tilde{B}}(z)$ is smaller than $\bar{w}$. Suppose $z \in S_{\tilde{A}}(\bar{v}) = (\tilde{A})_{\bar{v}}$. Then, from the definition of the strong h-level set, we have $\mu_{\tilde{A}}(z) > \bar{v}$. Because (v_i, w_i) converges to $(\bar{v}, \bar{w})$, for a sufficiently large i, we have $v_i < \mu_{\tilde{A}}(z)$ and $w_i > \mu_{\tilde{B}}(z)$. Namely, $z \in (\tilde{A})_{v_i}$ and $z \notin [\tilde{B}]_{w_i}$, which implies $S_{\tilde{A}}(v_i) = (\tilde{A})_{v_i} \not\subseteq [\tilde{B}]_{w_i} = W_{\tilde{B}}(w_i)$. This contradicts to $S_{\tilde{A}}(v_i) \subseteq W_{\tilde{B}}(w_i)$. Hence, $z \notin S_{\tilde{A}}(\bar{v}) = (\tilde{A})_{\bar{v}}$. Therefore, even if $[\tilde{B}]_{\bar{w}}$ is not closed, we obtain $S_{\tilde{A}}(\bar{v}) \subseteq W_{\tilde{B}}(\bar{w})$, i.e. $(\tilde{A})_{\bar{v}} \subseteq [\tilde{B}]_{\bar{w}}$. As we proved that the accumulating point of Q belongs to Q. Q is a closed set. □

Applying the Weierstrass theorem [36], we obtain the following theorem.

Theorem 5.1. *The extended necessity measure $N^E_{\tilde{A}}(\tilde{B})$ defined in Definition 5.1 can also be expressed as*

$$N^E_{\tilde{A}}(\tilde{B}) = \max\Big\{f(v,w) : (\tilde{A})_v \subseteq [\tilde{B}]_w,\ v,w \in [0,1]\Big\}. \tag{5.12}$$

Namely, the supremum is replaced by the maximum. Or equivalently, we obtain

$$N^E_{\tilde{A}}(\tilde{B}) \geq h \ \Leftrightarrow\ \exists\, v,w \in [0,1] \text{ with } (\tilde{A})_v \subseteq [\tilde{B}]_w \text{ such that } f(v,w) \geq h. \tag{5.13}$$

Proof. Using $Q = \{(v,w) : (\tilde{A})_v \subseteq [\tilde{B}]_w,\ v,w \in [0,1]\}$, we have

$$N^E_{\tilde{A}}(\tilde{B}) = \sup\{f(v,w) : (v,w) \in Q\}. \tag{5.14}$$

From Lemma 5.3, Q is closed set while f is upper semi-continuous. From the Weierstrass theorem, $N^E_{\tilde{A}}(\tilde{B})$ is attained. Namely, there exists $(\bar{v},\bar{w}) \in Q$ such that $f(\bar{v},\bar{w}) = N^E_{\tilde{A}}(\tilde{B})$. Then the supremum is replaced with the maximum, and the equivalence shown in (5.13) is obtained. □

5.2 An Application to Fuzzy Linear Programming

5.2.1 Problem formulation and reduction

The conventional LP problem is usually composed of an objective function and constraints. However, when the coefficients in constraints contain uncertainty, the conventional definitions of feasibility and optimality on a solution are not always applicable. Instead, some modifications, extensions and/or interpretations of the problem are required.

As we concentrate on the treatment of uncertainty in the feasible set of the LP problem in this chapter, we consider the solution satisfying given constraints with no objective function. The feasible solution set of the conventional LP problem is expressed as follows:

$$\mathscr{S}(\boldsymbol{A},\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}, \tag{5.15}$$

where $\boldsymbol{x}$ is the decision-variable vector, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ are the coefficient matrix and vector, respectively.

When the coefficients are not known precisely but imprecisely by the set of its conceivable values, we can express the feasible solution set $\mathscr{S}_I(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})$ of the problem formally

by utilising the imprecise information about the coefficients:

$$\mathscr{S}_I(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = \big\{\boldsymbol{x} \in \mathbb{R}^n : \tilde{\boldsymbol{A}}\boldsymbol{x} \asymp \tilde{\boldsymbol{b}},\ \boldsymbol{x} \geq \boldsymbol{0}\big\}, \tag{5.16}$$

where $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ are coefficient matrix and vector specified imprecisely or vaguely by the set of their conceivable values, respectively. We note that the $\asymp$ represents some *relation* (or *connection*) between the left-hand side and right-hand side in constraints. Typically, it can represent the relation of *equal to*, *included in* or *belongs to* in different situations. For instance, if $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ are given precisely, then the relation $\asymp$ represents *equal to*. On the other hand, if coefficients $\tilde{\boldsymbol{A}}$ are imprecise and acceptable values $\tilde{\boldsymbol{b}}$ are vague and both specified by crisp sets, then the relation $\asymp$ means *subset of*. If $\tilde{\boldsymbol{A}}$ is precise but $\tilde{\boldsymbol{b}}$ is specified vaguely by a crisp set, the relation $\asymp$ means *belong to*.

In this chapter, we assume conceivable values for coefficients $\tilde{\boldsymbol{A}}$ and also for acceptable values $\tilde{\boldsymbol{b}}$ are specified by fuzzy numbers. In this situation, each solution is crisp, i.e. a point in $\mathbb{R}^n$, attached with a membership value showing the *satisfactory* (or *reliable*) degree due to the imprecision and vagueness of $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$. The feasible set $\mathscr{S}_F(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})$ of this problem with fuzzy numbers is formally written as

$$\mathscr{S}_F(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = \big\{\boldsymbol{x} \in \mathbb{R}^n : \tilde{\boldsymbol{A}}\boldsymbol{x} \subsetneqq \tilde{\boldsymbol{b}},\ \boldsymbol{x} \geq \boldsymbol{0}\big\}, \tag{5.17}$$

where components of $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ are fuzzy numbers and $\subsetneqq$ formally represents that the left-hand side fuzzy numbers are required to be included in the right-hand side fuzzy numbers. From the fuzzy set of solutions, we select a solution having the maximum membership value as the best solution.

To deal with the problem formally given by Problem (5.17), we use a necessity measure because a necessity measure can be regarded as a degree of inclusion between two fuzzy sets. Then Problem (5.17) can be formulated as

$$\begin{aligned} &\text{maximize } N_{\tilde{\boldsymbol{A}}\boldsymbol{x}}(\tilde{\boldsymbol{b}}), \\ &\text{subject to } \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{5.18}$$

which is known as the necessity measure maximisation problem [22, 16]. As we proposed the extended necessity measure N^E for introducing the trade-off between the robustness level and satisfaction level, we use it instead of the necessity measure N of Problem (5.18). Accordingly, our problem becomes

$$\begin{aligned} &\text{maximize } N^E_{\tilde{\boldsymbol{A}}\boldsymbol{x}}(\tilde{\boldsymbol{b}}), \\ &\text{subject to } \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{5.19}$$

Hence, we investigate the reduced problem of Problem (5.19) and its solution algorithm. By applying Theorem 5.1 and the fact that $[\tilde{\boldsymbol{b}}]_w$ is closed for every $w \in [0, 1]$,

Problem (5.19) is reduced to the following mathematical programming problem:

$$\begin{aligned} &\text{maximize } f(v,w), \\ &\text{subject to } \mathrm{cl}(\tilde{\boldsymbol{A}}\boldsymbol{x})_v \subseteq [\tilde{\boldsymbol{b}}]_w, \\ &\qquad v \in [0,1],\ w \in [0,1],\ \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{5.20}$$

where f is the trade-off function defined in Definition 5.1. Since $[\tilde{\boldsymbol{b}}]_w$ is a closed set for all $w \in [0,1]$, the closure of $(\tilde{\boldsymbol{A}}\boldsymbol{x})_v$ ensures that the feasible region of Problem (5.20) is closed.

As f is increasing in the first argument and decreasing in the second argument and we have $(\tilde{\boldsymbol{A}}\boldsymbol{x})_v = (\tilde{\boldsymbol{A}})_v\boldsymbol{x}$, we obtain

(i) the smaller v is, or equivalently, the wider $(\tilde{\boldsymbol{A}})_v$ is, and

(ii) the larger w is, or equivalently, the narrower $[\tilde{\boldsymbol{b}}]_w$ is,

the larger $f(v,w)$ is. If $(\tilde{\boldsymbol{A}}\boldsymbol{x})_v \subseteq [\tilde{\boldsymbol{b}}]_w$, or equivalently, $\mathrm{cl}(\tilde{\boldsymbol{A}}\boldsymbol{x})_v \subseteq [\tilde{\boldsymbol{b}}]_w$ is satisfied, we obtain $N_{\tilde{\boldsymbol{A}}\boldsymbol{x}}(\tilde{\boldsymbol{b}}) \geq f(v,w)$. The set $\tilde{\boldsymbol{A}}$ shows the conceivable range of uncertain coefficients, then it is more robust that the larger part of $\tilde{\boldsymbol{A}}$ is included in $\tilde{\boldsymbol{b}}$. Namely, v represents the robustness degree. On the other hand, $\tilde{\boldsymbol{b}}$ shows the set of acceptable values and the membership grade of $\tilde{\boldsymbol{b}}$ shows the degree of desirability. It is more satisfactory for the decision maker that the set of conceivable left-hand side values $\tilde{\boldsymbol{A}}\boldsymbol{x}$, is included in the set of more desirable values. Therefore, w represents the satisfaction degree. Enlarging the robustness degree v, the satisfaction degree w should be decreased for fulfilling $(\tilde{\boldsymbol{A}}\boldsymbol{x})_v \subseteq [\tilde{\boldsymbol{b}}]_w$, and vice versa. To put it differently, we face the trade-off the robustness degree v against the satisfaction degree w. Problem (5.20) finds the best trade-off between the robustness degree v and the satisfaction degree w in the sense of the given trade-off function f. If the decision maker has no specific requirement, s/he can build it as $f(v,w) = \kappa(1-v) + \eta w$ for $(v,w) \in [0,1) \times (0,1]$ with $\kappa > 0$ and $\eta > 0$ represent the weights satisfying $\kappa + \eta = 1$ on robustness and satisfaction, respectively.

In reality, we may have usual constraints such that coefficients of the left-hand side linear functions and the right-hand side values are precisely specified. Such constraints are called *hard constraints*. To make the proposed approach more flexible in applications, we consider two kinds of constraints: one is hard constraints expressed by $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}$ and the other is fuzzy inclusion constraints $\tilde{\boldsymbol{A}}\boldsymbol{x} \subsetneq \tilde{\boldsymbol{b}}$ treated by the extended necessity measure N^E, where $\boldsymbol{C} \in \mathbb{R}^{l\times n}$ and $\boldsymbol{d} \in \mathbb{R}^l$, and $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ are $m \times n$ matrix and m vector with fuzzy number components while $\boldsymbol{x} \in \mathbb{R}^n$ denotes the decision vector. In this setting, Problem

(5.20) is generalised to

$$\begin{aligned} &\text{maximize } f(v,w), \\ &\text{subject to } \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}, \\ &\qquad \mathrm{cl}(\tilde{\boldsymbol{A}}\boldsymbol{x})_v \subseteq [\tilde{\boldsymbol{b}}]_w, \\ &\qquad v \in [0,1],\ w \in [0,1],\ \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{5.21}$$

or equivalently, expressing intervals $\mathrm{cl}(\tilde{\boldsymbol{A}}\boldsymbol{x})_v = \mathrm{cl}(\tilde{\boldsymbol{A}})_v\boldsymbol{x}$ from (2.30), and $[\tilde{\boldsymbol{b}}]_w$ by their lower and upper bounds,

$$\begin{aligned} &\text{maximize } f(v,w), \\ &\text{subject to } \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}, \\ &\qquad \inf(\mathrm{cl}(\tilde{\boldsymbol{A}})_v)\boldsymbol{x} \geq \inf([\tilde{\boldsymbol{b}}]_w), \\ &\qquad \sup(\mathrm{cl}(\tilde{\boldsymbol{A}})_v)\boldsymbol{x} \leq \sup([\tilde{\boldsymbol{b}}]_w), \\ &\qquad v,w \in [0,1],\ \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{5.22}$$

where we define $\inf\emptyset = -\infty$ and $\sup\emptyset = \infty$ for convenience.

Let $\boldsymbol{C} = (c_{kj})$, $k = 1,2,\ldots,l$, $j = 1,2,\ldots,n$, $\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})$ with $\tilde{a}_{ij} = (a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{R}}, \alpha_{ij}^{\mathrm{L}}, \alpha_{ij}^{\mathrm{R}})_{L_{ij}R_{ij}}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ and $\tilde{\boldsymbol{b}} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)^{\mathrm{T}}$ with $\tilde{b}_i = (b_i^{\mathrm{L}}, b_i^{\mathrm{R}}, \beta_i^{\mathrm{L}}, \beta_i^{\mathrm{R}})_{L_iR_i}$. Then Problem (5.16) is reduced to the following programming problem:

$$\begin{aligned} &\text{maximize } f(v,w), \\ &\text{subject to } \sum_{j=1}^{n} c_{kj}x_j = d_k,\ k = 1,2,\ldots,l, \\ &\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{L}} - L_{ij*}(v)\alpha_{ij}^{\mathrm{L}}\right) x_j \geq b_i^{\mathrm{L}} - L_i{}^*(w)\beta_i^{\mathrm{L}},\ i = 1,2,\ldots,m, \\ &\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{R}} + R_{ij*}(v)\alpha_{ij}^{\mathrm{R}}\right) x_j \leq b_i^{\mathrm{R}} + R_i{}^*(w)\beta_i^{\mathrm{R}},\ i = 1,2,\ldots,m, \\ &\qquad 0 \leq v \leq 1,\ 0 \leq w \leq 1,\ x_j \geq 0,\ j = 1,2,\ldots,n, \end{aligned} \tag{5.23}$$

where we define pseudo-inverse functions L_{ij*}, $R_{ij*} : [0,1] \to [0,\infty) \cup \{-\infty, \infty\}$ and $L_i{}^*$, $R_i{}^* : [0,1] \to [0,\infty) \cup \{\infty\}$ by

$$L_{ij*}(v) = \sup\left\{r \in [0,+\infty) : L_{ij}(r) > v\right\},\ \text{for } v \in [0,1),\ L_{ij*}(1) = -\infty \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{5.24}$$

$$R_{ij*}(v) = \sup\left\{r \in [0,+\infty) : R_{ij}(r) > v\right\},\ \text{for } v \in [0,1),\ R_{ij*}(1) = -\infty \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{5.25}$$

$$L_i{}^*(w) = \sup\left\{r \in [0,+\infty) : L_i(r) \geq w\right\},\ i = 1,2,\ldots,n, \tag{5.26}$$

$$R_i{}^*(w) = \sup\left\{r \in [0,+\infty) : R_i(r) \geq w\right\},\ i = 1,2,\ldots,n. \tag{5.27}$$

We note that the domains of L_{ij_*} and R_{ij_*} include $\{1\}$ by defining $L_{ij_*}(1) = R_{ij_*}(1) = -\infty$. Accordingly, $\{-\infty\}$ is included in the range of L_{ij_*} and R_{ij_*}.

As a special case, we consider Problem (5.23) where $\tilde{a}_{ij}$, $\tilde{b}_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are all triangular fuzzy numbers in the numerical example. Namely, we consider a case where $L_{ij}(r) = R_{ij}(r) = L_i(r) = R_i(r) = \max(1 - r, 0)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. In this case, we have $L_{ij_*}(h) = R_{ij_*}(h) = L_i{}^*(h) = R_i{}^*(h) = 1 - h$ for $h \in (0, 1]$, $L_{ij_*}(0) = R_{ij_*}(0) = 1$ and $L_i{}^*(0) = R_i{}^*(0) = \infty$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. In this special case, Problem (5.23) is reduced to

$$
\begin{aligned}
&\text{maximize } f(v, w),\\
&\text{subject to } \sum_{j=1}^{n} c_{kj} x_j = d_k,\ k = 1, 2, \ldots, l,\\
&\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{L}} - (1 - v)\alpha_{ij}^{\mathrm{L}}\right) x_j \geq b_i^{\mathrm{L}} - (1 - w)\beta_i^{\mathrm{L}},\ i = 1, 2, \ldots, m,\\
&\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{R}} + (1 - v)\alpha_{ij}^{\mathrm{R}}\right) x_j \leq b_i^{\mathrm{R}} + (1 - w)\beta_i^{\mathrm{R}},\ i = 1, 2, \ldots, m,\\
&\qquad 0 \leq v < 1,\ 0 < w \leq 1,\ x_j \geq 0,\ j = 1, 2, \ldots, n. \qquad (5.28)
\end{aligned}
$$

We note that $v = 1$ and $w = 0$ are excluded because the second and third constraints in (5.28) should always be satisfied. When $\sum_{j=1}^{n} c_{kj} x_j = d_k$, $k = 1, 2, \ldots, l$ are satisfied with some $\boldsymbol{x} \geq \boldsymbol{0}$, the optimal value of Problem (5.28) should be no less than $f(0, 0) = f(1, 1) = 0$. In what follows, we assume that there exists $\boldsymbol{x} \geq \boldsymbol{0}$ satisfying $\sum_{j=1}^{n} c_{kj} x_j = d_k$, $k = 1, 2, \ldots, l$, and accordingly, the optimal value of Problem (5.28) is not less than 0.

5.2.2 A solution algorithm

As we have mentioned, Problem (5.23) is a nonlinear programming problem due to the multiplication of $L_{ij_*}(v)x_j$ and $R_{ij_*}(v)x_j$. Even in the special case, i.e. in the problem with triangular fuzzy numbers, Problem (5.28) is still nonlinear due to the multiplication of v and x_j. Moreover, Problem (5.23) cannot be guaranteed to be a convex programming problem. Therefore, solving Problem (5.23) is not a simple task, which is the same for solving Problem (5.28).

However, Problem (5.28) gives us a hint that the constraints of Problem (5.28) become linear in respect of the decision variables w and x_j, $j = 1, 2, \ldots, n$ when $v \in [0, 1]$ is fixed as $v = \bar{v}_s \in [0, 1]$. Moreover, $f(\cdot, w)$ is increasing in respect of w. Therefore, the objective function $f(v, w)$ can be replaced simply with w for obtaining the optimal solution when $v = \bar{v}_s$. Namely, when v is fixed as $v = \bar{v}_s$, Problem (5.28) is reduced to the following LP

problem:

$$
\begin{aligned}
&\text{maximize } w,\\
&\text{subject to } \sum_{j=1}^{n} c_{kj}x_j = d_k,\ k=1,2,\ldots,l,\\
&\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{L}} - (1-\bar{v}_s)\alpha_{ij}^{\mathrm{L}}\right) x_j - \beta_i^{\mathrm{L}} w \geq b_i^{\mathrm{L}} - \beta_i^{\mathrm{L}},\ i=1,2,\ldots,m,\\
&\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{R}} + (1-\bar{v}_s)\alpha_{ij}^{\mathrm{R}}\right) x_j + \beta_i^{\mathrm{R}} w \leq b_i^{\mathrm{R}} + \beta_i^{\mathrm{R}},\ i=1,2,\ldots,m,\\
&\qquad 0 \leq w \leq 1,\ x_j \geq 0,\ j=1,2,\ldots,n.
\end{aligned}
\tag{5.29}
$$

In Problem (5.29), $w = 0$ is included while it should be excluded as in Problem (5.28). However, this makes the feasible region of Problem (5.29) is closed and LP techniques can be applied. If the optimal value of Problem (5.29) is zero, i.e. $w = 0$ or Problem (5.29) is infeasible, we treat $w = 0$ because the second and third constraints vanish in the original problem (5.21) when $w = 0$ (remember we assume that there exists $\boldsymbol{x} \geq \boldsymbol{0}$ satisfying the first constraints). Let $\bar{w}_s$ and $\bar{x}_{sj}$, $j = 1, 2, \ldots, n$ be an optimal solution. Then, we know that the optimal value of Problem (5.28) is at least $f(\bar{v}_s, \bar{w}_s)$. Preparing multiple fixed points $\bar{v}_s \in [0, 1]$, $s = 1, 2, \ldots, N$ for v and taking the optimal solution $\bar{v}_s$ $\bar{w}_s$ and $\bar{x}_{sj}$, $j = 1, 2, \ldots, n$ having the largest $f(\bar{v}_s, \bar{w}_s)$, we obtain an approximate optimal solution.

Extending this idea to the general case, we may solve Problem (5.23) when $L_i = R_i = L$, $i = 1, 2, \ldots, m$, i.e. $\tilde{b}_i$, $i = 1, 2, \ldots, m$ are L-L fuzzy numbers, in the same way described above for Problem (5.28). Taking care that $L^*(w)$ is decreasing in respect of w while $f(\cdot, w)$ is increasing in respect of w, Problem (5.23) is reduced to the following LP problem when v is fixed as $v = \bar{v}_s$:

$$
\begin{aligned}
&\text{minimize } u,\\
&\text{subject to } \sum_{j=1}^{n} c_{kj}x_j = d_k,\ k=1,2,\ldots,l,\\
&\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{L}} - L_{ij_*}(\bar{v}_s)\alpha_{ij}^{\mathrm{L}}\right) x_j + \beta_i^{\mathrm{L}} u \geq b_i^{\mathrm{L}},\ i=1,2,\ldots,m,\\
&\qquad \sum_{j=1}^{n} \left(a_{ij}^{\mathrm{R}} + R_{ij_*}(\bar{v}_s)\alpha_{ij}^{\mathrm{R}}\right) x_j - \beta_i^{\mathrm{R}} u \leq b_i^{\mathrm{R}},\ i=1,2,\ldots,m,\\
&\qquad u \geq 0,\ x_j \geq 0,\ j=1,2,\ldots,n.
\end{aligned}
\tag{5.30}
$$

Let $\bar{u}_s$ and $\bar{x}_{sj}$, $j = 1, 2, \ldots, n$ be an optimal solution. Then, we know w corresponding to this solution is $\bar{w}_s = L(\bar{u}_s$ and that the optimal value of Problem (5.23) is at least $f(\bar{v}_s, \bar{w}_s) = f(\bar{v}_s, L(\bar{u}_s))$. Preparing multiple fixed points $\bar{v}_s \in [0, 1]$, $s = 1, 2, \ldots, N$ for

v and taking the optimal solution $\bar{v}_s$ $\bar{w}_s = L(\bar{u}_s)$ and $\bar{x}_{sj}$, $j = 1, 2, \ldots, n$ having the largest $f(\bar{v}_s, \bar{w}_s)$, we obtain an approximate optimal solution to Problem (5.23) when $\tilde{b}_i$, $i = 1, 2, \ldots, m$ are all L-L fuzzy numbers.

Algorithm 5.1. (*An algorithm for Problem (5.23)* with $L_i = R_i = L$)

- **Initialisation**:

  1. Set the precision tolerance $\epsilon > 0$ and initialise $\bar{v} \leftarrow 1 - \epsilon$.
  2. Initialise $f^* \leftarrow 0$ as we assume $\exists\, \boldsymbol{x},\ \boldsymbol{Cx} = \boldsymbol{d}$.

- **Iteration**:

  While $\bar{v} \geq 0$, repeat the following steps:

  Step 1. If Problem (5.30) with $\bar{v}_s = \bar{v}$ has no feasible solution, then terminate with $\bar{v} \leftarrow 0$.

  Step 2. Let $\bar{u}$, $\bar{x}_i$, $i = 1, 2, \ldots, n$ be an optimal solution to Problem (5.30) with $\bar{v}_s = \bar{v}$.

  Step 3. If $f(\bar{v}, L(\bar{u})) > f^*$, update $\boldsymbol{x}^* \leftarrow (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^{\mathrm{T}}$, $v^* \leftarrow \bar{v}$, $w^* \leftarrow L(\bar{u})$ and $f^* \leftarrow f(\bar{v}, L(\bar{u}))$.

  Step 4. Update $\bar{v} \leftarrow \bar{v} - \epsilon$.

Based on the discussion above, we obtain a solution algorithm shown in Algorithm 5.1. We determine ϵ as a sufficiently small positive number showing the precision tolerance. In this algorithm, we explore the maximum $f(\bar{v}, \bar{w})$ by solving LP problems decreasing the fixed $v = \bar{v}$ from $1 - \epsilon$ by a step size ϵ while $v = \bar{v}$ is non-negative. The precision of the obtained optimal value and the computational effort depend on the step size ϵ. The smaller ϵ is, the more precise optimal value is obtained, and at the same time, the more computational effort is required. As we have trade-off relation between the precision and the computational effort, it would not be very easy to find a suitable ϵ. Nevertheless, showing the fundamental solution algorithm shown in Algorithm 5.1 would be significant as it calls several modifications based on some efficient exploration strategies. For example, when we assume the optimal value of $\bar{w}$ in respect of a fixed $\bar{v}$ does not fluctuate greatly, a zoom in strategy may work well. In the zoom in strategy, we explore the optimal solution with a rather big step size (a rough exploration) and we focus on the promising region of $\bar{v}$ by decreasing the step size based on the result of rough exploration. We may continue this focusing process until the obtained optimal value becomes stable. Other strategies

such as the multi-start local search algorithms [32] could also be applied for enhancing the exploration efficiency. Because the search area is one-dimensional and bounded, the exploration would be rather efficient.

5.2.3 Numerical examples

We give three numerical examples of Problem (5.28) which are different only in the values of spread parameters. By those examples, we demonstrate the varieties of $f(\bar{v}, \bar{w})$ obtained by solving Problem (5.29) even when simple problems with triangular fuzzy numbers and $f(v, w) = \kappa(1 - v) + \eta w$ for $v < 1$ and $w > 0$, and $f(1, w) = f(v, 0) = 0$ for $v, w \in [0, 1]$.

Example 5.1. Let $m = 2$, $n = 2$ and $l = 1$. Reconsider Problem (5.19), or equivalently, Problem (5.20) with $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{b}}$, $\boldsymbol{C}$ and $\boldsymbol{d}$ defined by

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} (9,9,0.5,0.5)_{LL} & (8,8,0.8,0.8)_{LL} \\ (1,1,0.1,0.1)_{LL} & (-1,-1,0,0)_{LL} \end{pmatrix}, \quad \tilde{\boldsymbol{b}} = \begin{pmatrix} (16,16,2,2)_{LL} \\ (1,1,4,4)_{LL} \end{pmatrix}, \tag{5.31}$$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 4 \end{pmatrix}, \quad \boldsymbol{d} = (7), \tag{5.32}$$

where $L(r) = \max(1 - r, 0)$ and thus, L-L fuzzy numbers are all triangular fuzzy numbers. We note that L-L fuzzy number $(-1,-1,0,0)_{LL}$ is nothing but real number 1 as both left and right spreads are zeros. The trade-off function f is defined by $f(v, w) = 0.15(1 - v) + 0.85w$ for $v < 1$ and $w > 0$ and $f(1, w) = f(v, 0) = 0$ for $v, w \in [0, 1]$.

Problem (5.28) corresponding to the parameters above becomes as follows:

$$\begin{aligned} &\text{maximize } 0.15(1 - v) + 0.85w, \\ &\text{subject to } 3x_1 + 4x_2 = 7, \\ &\qquad (9 - (1 - v)0.5)x_1 + (8 - (1 - v)0.8)x_2 \geq 16 - (1 - w)2, \\ &\qquad (1 - (1 - v)0.1)x_1 - x_2 \geq 1 - (1 - w)4, \\ &\qquad (9 + (1 - v)0.5)x_1 + (8 + (1 - v)0.8)x_2 \leq 16 + (1 - w)2, \\ &\qquad (1 + (1 - v)0.1)x_1 - x_2 \leq 1 + (1 - w)4, \\ &\qquad 0 \leq v < 1,\ 0 < w \leq 1,\ x_1 \geq 0,\ x_2 \geq 0. \end{aligned} \tag{5.33}$$

Introducing surplus variables y_1, y_2 and slack variables y_3, y_4 and multiplying the

third and forth constraints by (-1), Problem (5.33) is rewritten as

$$\begin{aligned}&\text{maximize } 0.15(1-v)+0.85w,\\ &\text{subject to}\end{aligned} \tag{5.34}$$

$$\begin{pmatrix} 3 & 4 & 0 \\ -9 & -8 & 2 \\ -1 & 1 & 4 \\ 9 & 8 & 2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ w \end{pmatrix} + (1-v) \begin{pmatrix} 0 & 0 \\ 0.5 & 0.8 \\ 0.1 & 0 \\ 0.5 & 0.8 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ 3 \\ 18 \\ 5 \end{pmatrix},$$

$$0 \le v < 1,\ 0 < w \le 1,\ x_1,\ x_2,\ y_1,\ y_2,\ y_3,\ y_4 \ge 0. \tag{5.35}$$

When v is fixed at $\bar{v} \in [0,1)$, the problem corresponding to Problem (5.29) is obtained as

$$\begin{aligned}&\text{maximize } w,\\ &\text{subject to}\end{aligned} \tag{5.36}$$

$$\begin{pmatrix} 3 & 4 & 0 \\ -9 & -8 & 2 \\ -1 & 1 & 4 \\ 9 & 8 & 2 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ w \end{pmatrix} + (1-\bar{v}) \begin{pmatrix} 0 & 0 \\ 0.5 & 0.8 \\ 0.1 & 0 \\ 0.5 & 0.8 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ 3 \\ 18 \\ 5 \end{pmatrix},$$

$$0 \le w \le 1,\ x_1,\ x_2,\ y_1,\ y_2,\ y_3,\ y_4 \ge 0. \tag{5.37}$$

By applying Algorithm 5.1 with setting $\epsilon = 0.01$, we obtain the optimal solution as $x_1^* = 0.670$, $x_2^* = 1.248$, $v^* = 0.400$, $w^* = 0.595$ and the optimal value $f(v^*, w^*) = 0.596$. Let $x_1(v)$, $x_2(v)$ and $w(v)$ show an optimal solution x_1, x_2 and w of Problem (5.37) for $\bar{v} = v$. The functions $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ obtained by solving Problem (5.37) for many $\bar{v}_s = v$ are shown in Figure 5.1. $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ look like piecewise linear functions although they are nonlinear in each segment. At the corner points, the basis of the optimal basic solution to Problem (5.37) is changed. Functions $w(v)$ and $f(v, w(v))$ look like concave functions.

To see functions $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ more, we change $\tilde{\boldsymbol{A}}$ to

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} (9,9,0.1,0.1)_{LL} & (8,8,0,0)_{LL} \\ (1,1,0.5,0.5)_{LL} & (-1,-1,0.8,0.8)_{LL} \end{pmatrix}, \tag{5.38}$$

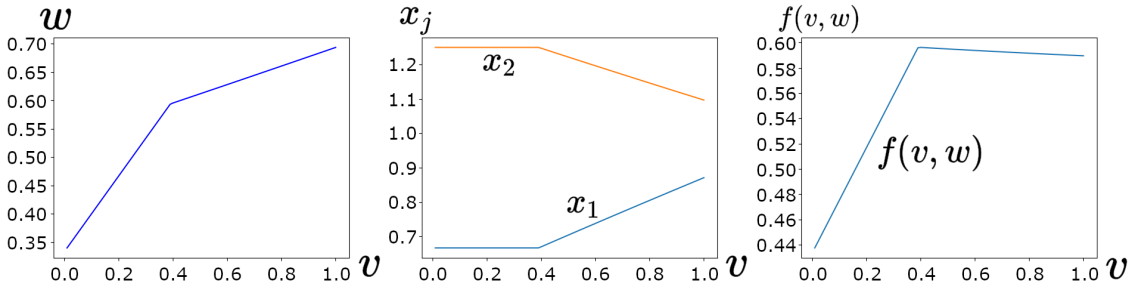


Figure 5.1: The result for fuzzy coefficients of (5.31)

where $(8, 8, 0, 0)_{LL}$ is nothing but a real number 8. By Algorithm 5.1 with setting $\epsilon = 0.01$, we obtain the optimal solution as $x_1^* = 0.87$, $x_2^* = 1.10$, $v^* = 0.99$, $w^* = 0.69$ and the optimal value $f(v^*, w^*) = 0.59$. Functions $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ for $\tilde{\boldsymbol{A}}$ of Equation (5.38) are shown in Fig. 5.2. As shown in Fig. 5.2, $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ are smooth curve segment close to line segments. Functions $w(v)$ and $f(v, w(v))$ are convex functions.

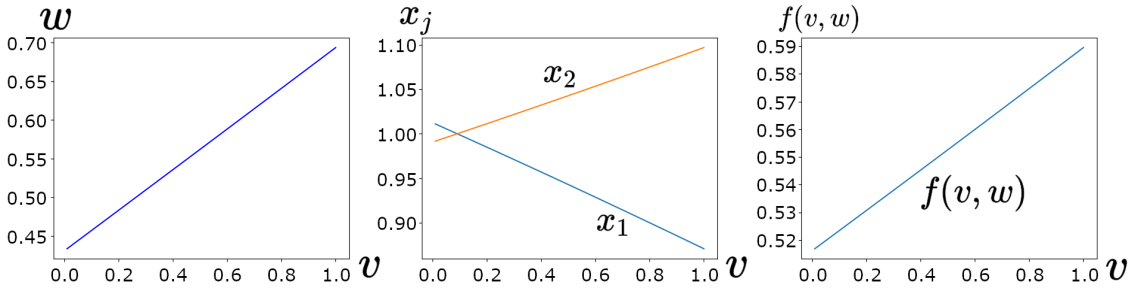


Figure 5.2: The result for fuzzy coefficients of (5.38)

Further, we change $\tilde{\boldsymbol{A}}$ to

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} (9, 9, 0.2, 0.2)_{LL} & (8, 8, 0.8, 0.8)_{LL} \\ (1, 1, 1.2, 1.2)_{LL} & (-1, -1, 0, 0)_{LL} \end{pmatrix}, \tag{5.39}$$

where $(-1, -1, 0, 0)_{LL}$ is nothing but a real number -1. The optimal solution is $x_1^* = 0.87$, $x_2^* = 1.10$, $v^* = 0.99$, $w^* = 0.69$ and the optimal value is $f(v^*, w^*) = 0.59$. Functions $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ for $\tilde{\boldsymbol{A}}$ of Equation (5.39) are shown in Fig. 5.3. From Fig. 5.3, we know that $x_1(v)$, $x_2(v)$, $w(v)$ and $f(v, w(v))$ are again smooth curve segment more bent than the previous case. We observe that $w(v)$ and $f(v, w(v))$ are convex.

As we observed in three cases of $\tilde{\boldsymbol{A}}$, the shapes of $w(v)$ and $f(v, w(v))$ change variously.

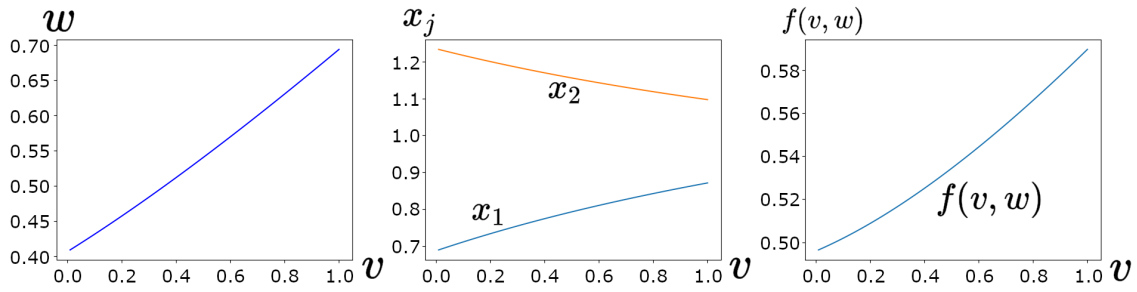


Figure 5.3: The result for fuzzy coefficients of (5.39)

We note that the changes of $\tilde{\boldsymbol{A}}$ are only in the spread parameters of their components. Furthermore, all L-R fuzzy numbers are triangular fuzzy numbers and trade-off function is linear in the domain $[0,1) \times (0,1]$. We have known that $w(v)$ and $f(v, w(v))$ vary widely even in such a simple case. This fact may imply that solving Problem (5.23) is not an easy task in general cases.

5.3 Concluding Remarks

In this chapter, we proposed an extended necessity measure to obtain an acceptable solution with a trade-off between robustness and satisfaction defined by decision makers. Since uncertain coefficients make the LP system not crisp anymore, the constraints with the set-inclusion relation become more suitable to interpret the problem's solution set. Since the h-level set of fuzzy numbers can represent the set-inclusion relation straightforwardly, we used them to represent the uncertain coefficients and considered using the necessity measure approach.

Since the conventional necessity measure cannot express the trade-off between robustness and satisfaction of a solution because of using one parameter, we consider using two parameters to represent a suitable trade-off reflecting the decision maker's preference. One of the two parameters shows the robustness degree against the ambiguity of coefficients of constraints, while the other shows the satisfaction degree of fulfilment of constraints. The extended necessity measure is defined successfully as the maximal trade-off under satisfying the inclusion relation.

Applying the proposed extended necessity measure to a fuzzy LP problem, we reduced the problem to a crisp programming problem. We found that the reduced problem becomes an LP problem by fixing the robustness parameter. Using this good property with the boundedness of the range of the robustness parameter, we gave a fundamental solution

algorithm. To illustrate the potential difficulty of the problem, we gave numerical examples showing the variety of the optimal value function in respect of the robustness parameter.

As the given solution algorithm is based on a simple one-dimensional exploration, we may introduce some efficient exploration strategies such as a zoom-in strategy, a multi-start local search strategy and so on. The solution algorithms enhancing the computation efficiency with those strategies and their comparisons by numerical experiments are the future topics of this study. On the other hand, regarding the extended necessity measure proposed in this chapter, the following future topics remain: (1) a method for the elicitation of a trade-off function f suitable for the decision maker and (2) the investigation of the relations between the extended necessity measure and a necessity measure defined by a general implication function [29, 18].

Chapter 6

Conclusion

THROUGHOUT this thesis, we studied two unsolved problems of the LP problem with uncertain coefficients. We proposed robust approaches to them. The contributions of the studies are summarised as follows.

In Chapter 3, we studied the LP problem with fuzzy OFCs. Using fuzzy numbers to represent the uncertain coefficients, we considered the necessary optimality test problem, which evaluates to what extent a feasible solution stays optimal against the fluctuation of the OFCs. We proposed methods using the tolerance approach to compute the optimal value of the necessary optimality problem of an NBF solution. We showed that the necessary optimality degree is obtained simply by the calculation based on the tolerance approach when representing all uncertain OFCs with fuzzy numbers sharing the same reference function. When the fuzzy numbers have different reference functions, we proposed an algorithm using the bisection method together with the tolerance approach. Moreover, when the fuzzy OFC vector is given by an oblique fuzzy vector, we showed that the necessary optimality test problem is solved in the same way as the case where uncertain coefficients are independently given by fuzzy numbers with a linear transformation using the obliquity matrix.

In Chapter 4, we continued the study of the necessary optimality test problem when the range of the uncertain OFC vector is given by a convex polytope. Since the conventional vertex-listing method requires a lot of computational effort, we proposed an algorithm based on the outer approximation approach. We constructed the algorithm by dividing the whole process into the initialisation process and the iteration process. We considered the computational complexity of the proposed algorithm and compared it to the conventional vertex-listing method. We showed that the proposed approach could be more efficient. Through numerical simulations, we also verify the computational efficiency of the proposed algorithm, especially large-scale problems.

In Chapter 5, we treated set-inclusive constraints with fuzzy coefficients. We formulated the problem of finding a solution satisfying the set-inclusive constraints with fuzzy constraints as a necessity measure maximisation problem. Using the expression of the conventional necessity measure by h-level sets, we showed that the level h of strong $(1-h)$-level set of a fuzzy subset to be included is interpreted as a robustness degree, while the level h of h-level set of a fuzzy subset to include is interpreted as a satisfaction degree in the setting of set-inclusive constraints with fuzzy coefficients. Moreover, we demonstrated the insufficiency of the conventional necessity measure in representing the decision maker's preference about the trade-off between the robustness degree and the satisfaction degree. Then we proposed an extended necessity measure enabling a flexible representation of the trade-off between the robustness degree and the satisfaction degree. The properties of the extended necessity measure were investigated. We considered the extended necessity measure maximisation problem for the set-inclusive constraints with fuzzy coefficients. The extended necessity measure maximisation problem was reduced to a mathematical programming problem. An approximate solution algorithm for the reduced problem was proposed. Numerical examples were given to illustrate the proposed approach using the extended necessity measure.

There are still open problems for future study. In the setting of LP problems with uncertain OFCs, we did not study the necessary optimality test problems of degenerate basic feasible solutions. As the OAC of an NBF solution was obtained easily, we assumed the non-degeneracy of the basic feasible solutions to be analysed. However, we may obtain degenerate basic feasible solutions in real-world problems as we encountered in the numerical simulations described in Chapter 4. When the basic solution to be analysed is degenerate, there are many bases corresponding to the solution, and for each basis, we obtain an OAC. We should apply the proposed method in this thesis to each OAC in the case of degenerate basic feasible solutions. However, this treatment would become complex, and some systematic and efficient approach is required.

Another open problem is a solution method for obtaining an exact solution to the extended necessity measure maximisation problem described in Chapter 5. As the reduced problem of the extended necessity measure maximisation problem becomes a non-convex programming problem, some effective global optimisation techniques should be applied to obtain an exact solution.

Bibliography

[1] Aubin, J.-P. Set-valued analysis. In *Mutational and Morphological Analysis*. Birkhäuser Boston, Boston, MA, 1999, pp. 205–264.

[2] Borwein, J. M., and Lewis, A. S. *Convex Analysis and Nonlinear Optimization*. Springer, New York, 2000.

[3] Bradley, S. P., Hax, A. C., and Magnanti, T. L. *Applied mathematical programming*. Addison-Wesley Publishing Company, Massachusetts, 1977.

[4] Dubois, D., and Prade, H. Fuzzy numbers: An overview. In *The Analysis of Fuzzy Information, Vol. 1: Mathematics and Logic*, J. C. Bezdek, Ed. CRC Press, Boca Raton, FL, 1987, pp. 3–39.

[5] Filippi, C. A fresh view on the tolerance approach to sensitivity analysis in linear programming. *European journal of operational research 167*, 1 (2005), 1–19.

[6] Gao, Z., and Inuiguchi, M. An analysis to treat the degeneracy of a basic feasible solution in interval linear programming. In *International Symposium on Integrated Uncertainty in Knowledge Modelling and Decision Making* (IUKM 2022), K. Honda, T. Entani, S. Ubukata, V.-N. Huynh, and M. Inuiguchi, Eds., Springer, pp. 130–142.

[7] Greenberg, H. J. The use of the optimal partition in a linear programming solution for postoptimal analysis. *Operations Research Letters 15*, 4 (1994), 179–185.

[8] Heath, T. L., et al. *The works of Archimedes*. Courier Corporation, Mineola, New York, 2002.

[9] Henk, M., Richter-Gebert, J., and Ziegler, G. M. Basic properties of convex polytopes. In *Handbook of Discrete and Computational Geometry, 2nd Edition* (Boca Raton, FL, 2004), J. Goodman and J. O'Rourke, Eds., CRC Press, pp. 243–270.

[10] Henriques, C. O., and Antunes, C. An interactive method of tackling uncertainty in interval multiple objective linear programming. *Journal of Mathematical Sciences 161*, 6 (2009), 854–866.

[11] Hladík, M. Multiparametric linear programming: support set and optimal partition invariancy. *European Journal of Operational Research 202*, 1 (2010), 25–31.

[12] Hladík, M. Tolerance analysis in linear systems and linear programming. *Optimization Methods and Software 26*, 3 (2011), 381–396.

[13] Hladík, M. Complexity of necessary efficiency in interval linear programming and multiobjective linear programming. *Optimization Letters 6*, 5 (2012), 893–899.

[14] Hladík, M. Robust optimal solutions in interval linear programming with forall-exists quantifiers. *European Journal of Operational Research 254*, 3 (2016), 705–714.

[15] Horst, R., and Tuy, H. *Global optimization: Deterministic approaches*, 3rd, rev. & enl. ed. Springer-Verlag, Berlin Heidelberg New York, 1996.

[16] Inuiguchi, M. Necessity measure optimization in linear programming problems with fuzzy polytopes. *Fuzzy Sets and Systems 158*, 17 (2007), 1882–1891.

[17] Inuiguchi, M. Approaches to linear programming problems with interactive fuzzy numbers. In *Fuzzy Optimization: Recent Advances and Applications*, W. A. Lodwick and J. Kacprzyk, Eds. Springer-Verlag, Berlin Heidelberg, 2010, pp. 145–161.

[18] Inuiguchi, M., Greco, S., Słowinski, R., and Tanino, T. Possibility and necessity measure specification using modifiers for decision making under fuzziness. *Fuzzy Sets and Systems 137*, 1 (2003), 151–175.

[19] Inuiguchi, M., and Ichihashi, H. Relative modalities and their use in possibilistic linear programming. *Fuzzy sets and Systems 35*, 3 (1990), 303–323.

[20] Inuiguchi, M., Ichihashi, H., and Kume, Y. Relationships between modality constrained programming problems and various fuzzy mathematical programming problems. *Fuzzy Sets and Systems 49*, 3 (1992), 243–259.

[21] Inuiguchi, M., Ichihashi, H., and Kume, Y. Modality constrained programming problems: A unified approach to fuzzy mathematical programming problems in the setting of possibility theory. *Information Sciences 67*, 1-2 (1993), 93–126.

[22] Inuiguchi, M., and Ramík, J. Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy sets and systems 111*, 1 (2000), 3–28.

[23] Inuiguchi, M., Ramík, J., and Tanino, T. Oblique fuzzy vectors and their use in possibilistic linear programming. *Fuzzy Sets and Systems 135*, 1 (2003), 123–150.

[24] Inuiguchi, M., and Sakawa, M. Possible and necessary optimality tests in possibilistic linear programming problems. *Fuzzy Sets and Systems 67*, 1 (1994), 29–46.

[25] Inuiguchi, M., and Sakawa, M. Minimax regret solution to linear programming problems with an interval objective function. *European Journal of Operational Research 86*, 3 (1995), 526–536.

[26] Inuiguchi, M., and Sakawa, M. Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test. *Fuzzy sets and systems 78*, 2 (1996), 231–241.

[27] Inuiguchi, M., and Sakawa, M. An achievement rate approach to linear programming problems with an interval objective function. *Journal of the Operational Research Society 48*, 1 (1997), 25–33.

[28] Inuiguchi, M., and Sakawa, M. Robust optimization under softness in a fuzzy linear programming problem. *International Journal of Approximate Reasoning 18*, 1-2 (1998), 21–34.

[29] Inuiguchi, M., and Tanino, T. Necessity measures and parametric inclusion relations of fuzzy sets. *Fundamenta Informaticae 42*, 3-4 (2000), 279–302.

[30] Inuiguchi, M., and Tanino, T. On computation methods for a minimax regret solution based on outer approximation and cutting hyperplanes. *International Journal of Fuzzy Systems 3*, 4 (2001), 548–557.

[31] Kolmogorov, A. N., and Fomin, S. V. *Introductory real analysis*. Courier Corporation, New York, 1975.

[32] Martí, R. Multi-start methods. In *Handbook of metaheuristics*, F. Glover and G. A. Kochenberger, Eds. Springer Science & Business Media, Boston, MA, 2003, pp. 355–368.

[33] Mausser, H. E., and Laguna, M. Minimising the maximum relative regret for linear programmes with interval objective function coefficients. *Journal of the Operational Research Society 50*, 10 (1999), 1063–1070.

[34] Nguyen, H. T., Walker, C., and Walker, E. A. *A first course in fuzzy logic*, 4th ed. CRC Press, Boca Raton, FL, 2018.

[35] Rockafellar, R. T., and Wets, R. J.-B. *Variational Analysis*. Springer-Verlag, Berlin Heidelberg, 2009.

[36] Rusnock, P., and Kerr-Lawson, A. Bolzano and uniform continuity. *Historia Mathematica 32*, 3 (2005), 303–311.

[37] Stancu-Minasian, I. *Stochastic Programming with Multiple Objective Functions.* Editura Academiei and Reidel, Dordrecht, 1984.

[38] Stern, M. K., and Johnson, J. H. Just noticeable difference. In *The Corsini Encyclopedia of Psychology*, I. B. Weiner and W. E. Craighead, Eds. John Wiley & Sons, Inc, Hoboken, NJ, USA, 2010, pp. 1–2.

[39] Wendell, R. E. Using bounds on the data in linear programming: The tolerance approach to sensitivity analysis. *Mathematical Programming 29*, 3 (1984), 304–322.

[40] Wendell, R. E. The tolerance approach to sensitivity analysis in linear programming. *Management Science 31*, 5 (1985), 564–578.

[41] Wondolowski Jr, F. R. A generalization of wendell's tolerance approach to sensitivity analysis in linear programming. *Decision Sciences 22*, 4 (1991), 792–811.

Acknowledgement

This thesis would not have been completed without the help and instruction from my supervisor, Professor Masahiro Inuiguchi of Osaka University. As a beginner starting my PhD course, he always gave me intuitive illustrations and inspiring suggestions to help me learn the fundamental knowledge and guide my research studies throughout the past three years. Without his kind help, I could not even step into the realm of operational research.

Then, I would like to show my gratitude to all members of the Inuiguchi laboratory. When I came across difficulties, they were always by my side and gave me warm help, which smoothed my everyday life. Among these warm-hearted people, I really thank Associate Professor Naoki Hayashi and Assistant Professor Hirosato Seki for their suggestions on my studies.

Also, I must show my respects to Professor Toshimitsu Ushio and Professor Youji Iiguni for their careful check and precious suggestions for this thesis. They helped me reconsider the writing style of my thesis, which gave me a new view of what I had written and how to write it better.

Finally, I want to thank my parents and all people around me.

Publication List

Journal Paper

1. M. Inuiguchi, Z. Gao, and C. O. Henriques, "Robust optimality analysis of non-degenerate basic feasible solutions in linear programming problems with fuzzy objective coefficients," *Fuzzy Optimization and Decision Making*, pp. 1–29, 2022, DOI: 10.1007/s10700-022-09383-2

2. Z. Gao and M. Inuiguchi, "An extended necessity measure maximisation incorporating the trade-off between robustness and satisfaction in fuzzy LP problems," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, pp. 1-26, 2022, DOI: 10.1142/S0218488523500010

3. Z. Gao and M. Inuiguchi, "Robust optimality analysis for linear programming problems with uncertain objective function coefficients: An outer approximation approach," *Kybernetika*, (Submitted)

Proceedings of International Conferences

1. Z. Gao and M. Inuiguchi, "A minimax criterion approach to treat the inexactness in feasible set of a linear programming problem," in *19th World Congress of the International Fuzzy Systems Association (IFSA), 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), and 11th International Summer School on Aggregation Operators (AGOP).* Atlantis Press, 2021, pp. 258–265, DOI: 10.2991/asum.k.210827.035

2. Z. Gao and M. Inuiguchi, "A necessity measure of fuzzy inclusion relation in linear programming problems," in *International Conference on Modeling Decisions for Artificial Intelligence.* Springer, 2021, pp. 131–140, DOI: 10.1007/978-3-030-85529-1_11

3. Z. Gao and M. Inuiguchi, "Estimating the optimal probability of a candidate basic solution in stochastic linear programming," in *2021 60th Annual Conference of the*

Society of Instrument and Control Engineers of Japan (SICE). IEEE, 2021, pp. 640–643.

4. Z. Gao and M. Inuiguchi, "An analysis to treat the degeneracy of a basic feasible solution in interval linear programming," in *International Symposium on Integrated Uncertainty in Knowledge Modelling and Decision Making.* Springer, 2022, pp. 130–142, DOI: 10.1007/978-3-030-98018-4_11

5. Z. Gao and M. Inuiguchi, "Optimality analysis for stochastic LP problems," in *International Conference on Modeling Decisions for Artificial Intelligence.* Springer, 2022, pp. 3–14, DOI: 10.1007/978-3-031-13448-7_1