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ON THE DIFFERENTIAL $d_3^{p,0}$ OF U-COBORDISM SPECTRAL SEQUENCE

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For a finite CW-pair (X, A), there is the extraordinary cohomology group of unitary cobordism group denoted by

$$U^k(X, A)=\varinjlim_m \{S^{2m-k}(X/A), MU(m)\},$$

[3]. Consider the spectral sequence $\{E_r^{p,q}\}$ associated to the cohomology group $U^*(X, A)$ with $E_2^{p,q}=H^p(X, A; U^q)$, where $U^q=U^q(\text{a point})$. If q is odd then $U^q=0$. Hence, the differential

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

is zero homomorphism and $E_3^{p,q}\approx H^q(X, A; U^p)$. In this paper we compute the differential $d_3^{p,0}$ and study the admissible multiplication of mod 2 cohomology theory of unitary cobordism.

1. Preliminaries

The spectral sequence $\{E_r^{p,q}\}$ of $U^*(X, A)$ is obtained as follows; Define

$$Z_r^{p,q}=Im\{U^{p+q}(X^{p+r-1}, X^{p-1}) \to U^{p+q}(X^p, X^{p-1})\},$$
$$B_r^{p,q}=Im\{U^{p+q-1}(X^{p-1}, X^{p-r}) \to U^{p+q}(X^p, X^{p-1})\},$$

where X^p is the p-skeleton of (X, A), then $E_r^{p,q}=Z_r^{p,q}/B_r^{p,q}$, and the differential

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

is the following composition homomorphism

$$Z_r^{p,q}/B_r^{p,q}\to Z_r^{p,q}/Z_{r+1}^{p,q}\underset{(*)}{\approx} B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}\to Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

Consider the commutative diagram,

$$\begin{array}{ccccc} & & & & U^{p+q}(X^{p+r}, X^{p-1}) \\ & & & & \downarrow j_1^* \\ & & U^{p+q}(X^{p+r-1}, X^{p-1}) & \xrightarrow{j^*} & U^{p+q}(X^p, X^{p-1}) \\ & & \downarrow \delta & & \downarrow \delta_1 \\ U^{p+q}(X^{p+r-1}, X^p) & \xrightarrow{\delta_2} & U^{p+q+1}(X^{p+r}, X^{p+r-1}) & \xrightarrow{j_2^*} & U^{p+q+1}(X^{p+r}, X^p) . \end{array}$$

Then, the isomorphism (*) is given by

$$Imj^*/Imj_1^* \approx Im\delta_1^* \circ j^* \approx Imj_2^* \circ \delta \approx Im\delta/Im\delta_2 .$$

Therefore,

$$d_r^{p,q}(\{j^*(x)\}) = \{\delta(x)\} ,$$

where $x \in U^{p+q}(X^{p+r-1}, X^{p-1})$.

We define the homomorphism

$$\mu : \tilde{U}^{p+r}(S^p) \to \tilde{H}^p(S^p) \otimes U^r$$

by

$$\mu(x) = e_p^* \otimes S_*^{-p}(x) \qquad \cdots\cdots\cdots (1, 1),$$

where e_p^* is the generator of $\tilde{H}^p(S^p)$ and S_*^{-p} is the inverse of the suspension isomorphism $S_*^p : \tilde{U}^r(S^0) \approx \tilde{U}^{p+r}(S^p)$. The homomorphism μ is the isomorphism, and it follows immediately that

Lemma 1.1. $U^{p+r}(X^p, X^{p-1}) \approx H^p(X^p, X^{p-1}) \otimes U^r$.

We denote this isomorphism by

$$\mu : U^{p+r}(X^p, X^{p-1}) \to H^p(X^p, X^{p-1}) \otimes U^r .$$

2. On the elements of $U^{p-2}(X^p, X^{p-1})$

Consider the element $x \in \tilde{U}^{p-2}(S^p)$, which is the class of a map $f: S^{2m-p+2}S^p \to MU(m)$, where $MU(m)$ is the Thom space of the m-dimensional complex universal bundle ξ_m. Denote by $c_1(\xi_m)$ the 1-st Chern class of ξ_m. Applying the homomorphism μ of Lemma 1.1 to the element x, we can represent the element $\mu(x)$ as follows;

Lemma 2.1. $\mu(x) = -\frac{1}{2}\{S_*^{-(2m-p+2)} f^* \phi_\xi(c_1(\xi_m))\} \otimes [CP(1)]$,

where ϕ_ξ is the Thom isomorphism

$$\phi_\xi : H^*(BU(m)) \to \tilde{H}^{*+2m}(MU(m)) ,$$

S_*^{-k} is the inverse of the k-fold suspension isomorphism S_*^k, and $CP(1)$ is the 1-dimensional complex projective space.

Proof. From (1.1) we have

$$\mu(x) = e_p^* \otimes S_*^{-p}(x), \ S_*^{-p}(X) \in U_2 .$$

Put $[V^2] = S_*^{-p}(x)$. Then, since the generator of U_2 is the cobordism class of 1-dimensional complex projective space $CP(1)$, we can represent $[V^2]$ as

$$[V^2] = a[CP(1)], \ a \in Z .$$

Consider the Chern number $\langle c_1(\tau), [V^2]\rangle$, where τ is the tangent bundle of V^2, $c_1(\tau)$ is the lst Chern class of τ and $[V^2]$ is the fundamental class of V^2. Since

$$\langle c_1(\tau(CP(1))), [CP(1)]\rangle = 2 ,$$

We have the following

$$[V^2] = \frac{1}{2}\langle c_1(\tau), [V^2]\rangle [CP(1)] .$$

Therefore,

$$\mu\{f\} = e_p^* \otimes \frac{1}{2}\langle c_1(\tau), [V^2]\rangle [CP(1)] \qquad \cdots\cdots\cdots (2.1).$$

We can see that $V^2 = f_0^{-1}(BU(m))$, where f_0 is transverse regular on $BU(m)$ and an $\mathcal{E}$-approximation to $f \mid S^{2m+2} - f^{-1}(P)$, where $\mathcal{E}$ is a positive continuous function on $S^{2m+2} - f^{-1}(P)$ and P is the base point of $MU(m)$. Let η be the normal bundle of V^2 in $S^{2m+2} - f^{-1}(P)$. We have the bundle map

$$\bar{f}_0 : \eta \rightarrow \xi_m ,$$

which induces the map $f_0 : V^2 \rightarrow BU(m)$. Let $J : S^{2m+2} \rightarrow D(\eta)/S(\eta)$ be the map given by collapsing $S^{2m+2} - IntD(\eta)$, where $D(\eta)$ and $S(\eta)$ denote the associated disk bundle and sphere bundle of η respectively. Then, $\tilde{f}_0 \circ J$ is homotopic to f [4], where $\tilde{f}_0 : T(\eta) = D(\eta)/S(\eta) \rightarrow MU(m)$ is the map induced by $\bar{f}_0$. Let e_*^p and $[T]$ be the fundamental classes of $\tilde{H}_p(S^p)$ and $H_{2m+2}(D(\eta), S(\eta))$ respectively. Let $U(\xi_m)$ and $U(\eta)$ be the Thom classes of ξ_m and η respectively. Denote by

$$\phi_\eta : H^*(V^2) \rightarrow H^{*+2m}(T(\eta))$$

the Thom isomorphism and by

$$\pi : D(\eta) \rightarrow V^2$$

the projection.

$$\langle c_1(\tau), [V^2]\rangle$$

$$
\begin{aligned}
&= -\langle c_1(\eta), [V^2]\rangle \\
&= -\langle c_1(\eta), \pi_*([T]\cap U(\eta))\rangle \\
&= -\langle \phi_\eta(c_1(\eta)), [T]\rangle \\
&= -\langle \tilde{f}_0^*\phi_\xi(c_1(\xi_m)), [T]\rangle \\
&= -\langle J^*\tilde{f}_0^*\phi_\xi(c_1(\xi_m)), e_*^{2m+2}\rangle \\
&= -\langle f^*\phi_\xi(c_1(\xi_m)), e_*^{2m+2}\rangle .
\end{aligned}
$$

Therefore, by (2.1)

$$
\begin{aligned}
\mu\{f\} &= -\frac{1}{2}\langle f^*\phi_\xi(c_1(\xi_m)), e_*^{2m+2}\rangle e_p^*\otimes[CP(1)] \\
&= -\frac{1}{2}S_*^{-(2m+2-p)}f^*\phi_\xi(c_1(\xi_m))\otimes[CP(1)] \text{ . q.e.d.}
\end{aligned}
$$

Theorem 2.2. *If $x\in U^{p-2}(X^p, X^{p-1})$ and x is represented by a map f: $S^{2m-p+2}(X^p/X^{p-1})\to MU(m)$, then*

$$
\mu(x) = -\frac{1}{2}S_*^{-(2m+2-p)}f^*\phi_\xi c_1(\xi_m)\otimes[CP(1)] .
$$

Proof. For the element $x\in U^{p-2}(X^p, X^{p-1})\approx \tilde{U}^{p-2}(\bigvee_j S_j^p)$,

$$
x = \sum_j p_j^* i_j^*(x),
$$

where i_j: $S_j^p\subset \bigvee_k S_k^p$ is inclusion and p_j: $\bigvee_k S_k^p\to S_j^p$ is projection.

$$
\begin{aligned}
\mu(x) &= \sum_j (p_j^*\otimes 1)\mu(i_j^*(x)) \\
&= \sum_j (p_j^*\otimes 1)\mu\{f\circ i_j\} \\
&= -\sum_j p_j^*\Big(\frac{1}{2}S_*^{-(2m+2-p)}(f\circ i_j)^*\phi_\xi(c_1(\xi_m))\Big)\otimes [CP(1)]
\end{aligned}
$$

by Lemma 2.1,

$$
\begin{aligned}
&= -\frac{1}{2}\sum_j p_j^* i_j^* S_*^{-(2m+2-p)}f^*\phi_\xi(c_1(\xi_m))\otimes[CP(1)] \\
&= -\frac{1}{2}S_*^{-(2m+2-p)}f^*\phi_\xi(c_1(\xi_m))\otimes[CP(1)] \text{ . q.e.d.}
\end{aligned}
$$

3. The differential $d_3^{p,0}$

In §1, we have seen that $d_3^{p,0}\{j^*(x)\}=\{\delta(x)\}$, where

$$
j^* : U^p(X^{p+2}, X^{p-1}) \to U^p(X^p, X^{p-1}) ,
$$

and

$$\delta : U^p(X^{p+2}, X^{p-1}) \to U^{p+1}(X^{p+3}, X^{p+2})$$

are the maps induced by injection j: $(X^p, X^{p-1}) \to (X^{p+2}, X^{p-1})$ and the coboundary homomorphism of the exact sequence of the triple $(X^{p+3}, X^{p+2}, X^{p-1})$ respectively. By Lemma 1.1, μ: $U^{p+r}(X^p, X^{p-1}) \approx H^p(X^p, X^{p-1}) \otimes U^r$, and we can see easily that

$$(\delta \otimes id) \circ \mu = \mu \circ d_1^{p,r},$$

where $\delta : H^p(X^p, X^{p-1}) \to H^{p+1}(X^{p+1}, X^p)$ is the coboundary homomorphism and $d_1^{p,r} : E_1^{p,r} \to E_1^{p+1,r}$ is the differential.

Considering $H^p(X^p, X^{p-1}) = C^p(X, A)$ as the cochain group, we have

$$E_2^{p,r} = H^p(X, A; U^r).$$

Since $E_3^{p,r} \approx E_2^{p,r}$, we identify the homomorphism $d_3^{p,0} : E_3^{p,0} \to E_3^{p+3,-2}$ with the homomorphism which applies $[\mu(j^*(x))] \in H^p(X, A)$ to $[\mu\delta(x)] \in H^{p+3}(X, A; U^{-2})$. Let $x \in U^p(X^{p+2}, X^{p-1})$ be represented by a map

$$f: S^{2m-p}(X^{p+2}/X^{p-1}) \to MU(m).$$

Then, $\delta(x)$ is represented by the following composition

$$g: S^{2m-p-1}(X^{p+3}/X^{p+2}) \xrightarrow{r} S^{2m-p}(X^{p+2}/X^{p-1}) \xrightarrow{f} MU(m),$$

where r is the composition map of homotopy equivalence $X^{p+3}/X^{p+2} \simeq (X^{p+3}/X^{p-1}) \cup C(X^{p+2}/X^{p-1})$ and the natural map induced by the projection $(X^{p+3}/X^{p-1}) \cup C(X^{p+2}/X^{p-1}) \to S(X^{p+2}/X^{p-1})$. The map r gives the boundary homomorphism

$$-\delta : \tilde{H}^*(X^{p+2}/X^{p-1}) \to \tilde{H}^{*+1}(X^{p+3}/X^{p+2})$$

and the following diagram is commutative,

$$\begin{array}{ccc} \tilde{H}^*(X^{p+2}/X^{p+1}) & \xrightarrow{\hat{j}^*} & \tilde{H}^*(X^{p+2}/X^{p-1}) \\ & \searrow \delta \quad \delta \swarrow & \\ & \tilde{H}^{*+1}(X^{p+3}/X^{p+2}). & \end{array}$$

where $\hat{j}$ is the injection $\hat{j}$: $(X^{p+2}, X^{p-1}) \to (X^{p+2}, X^{p+1})$.

Considering the cohomology exact sequence of the triple $(X^{p+2}, X^{p+1}, X^{p-1})$, we have the following

Lemma 3.1. $\hat{j}^* : \tilde{H}^{p+2}(X^{p+2}/X^{p+1}) \to \tilde{H}^{p+2}(X^{p+2}/X^{p-1})$

is an epimorphism.

Lemma 3.2. *There exists an element* $y \in \tilde{H}^{p+2}(X^{p+2}/X^{p+1}) = C^{p+2}(X, A)$ *such that* $\rho_2 y$ *is a cocycle,*

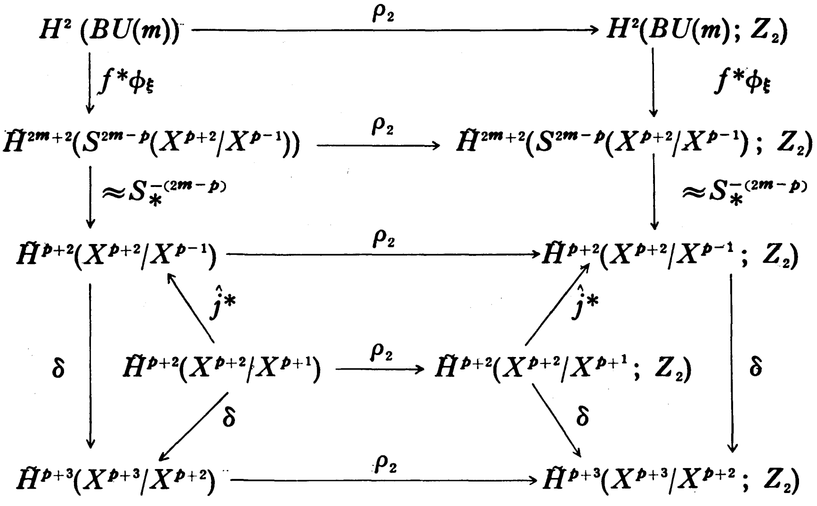
$$S_*^{-(2m-p)} f^* \phi_\xi \rho_2 c_1(\xi_m) = \hat{j}^* \rho_2(y) \qquad \cdots\cdots\cdots (3, 1),$$

and

$$\beta[\rho_2(y)] = [-\frac{1}{2} S_*^{-(2m-1-p)} g^* \phi_\xi c_1(\xi_m)] \qquad \cdots\cdots\cdots (3.2),$$

where ρ_2 *is the reduction modulo 2,* β *is the Bockstein homomorphism and* [ ] *denotes the cohomology class of* $H^*(X, A)$.

Proof. We consider the following commutative diagram.



By Lemma 3.1, there exists the element $y \in \tilde{H}^{p+2}(X^{p+2}/X^{p+1})$ such that

$$\hat{j}^*(y) = S_*^{-(2m-p)} f^* \phi_\xi c_1(\xi_m),$$

and (3.1) follows. By the definition of the map g and (3.1),

$$\delta \hat{j}^*(y) = -S_*^{-(2m-p-1)} g^* \phi_\xi c_1(\xi_m).$$

Then, we note that Theorem 2.2 implies that there exists the element $\frac{1}{2} S_*^{-(2m-p-1)} g^* \phi_\xi c_1(\xi_m)$ in the cochain group $C^{p+3}(X, A) = \tilde{H}^{p+3}(X^{p+3}/X^{p+2})$. Therefore, $\rho_2 \delta \hat{j}^*(y) = 0$, that is, $\rho_2(y)$ is cocycle. Then, we have

$$\beta[\rho_2(y)] = \left[\frac{1}{2} \delta(y)\right]$$

$$= \left[\frac{1}{2}\,\delta j^*(y)\right]$$

$$= -\left[\frac{1}{2} S_*^{-(2m-p-1)} g^* \phi_\xi c_1(\xi_m)\right]. \quad \text{q.e.d.}$$

It is well known that $\rho_2\, c_1(\xi_m) = W_2\,(\xi_m)$, where W_2 is 2-dimensional Stiefel-Whitney class, and $W_2\,(\xi_m) = \phi_\xi^{-1} Sq^2 \phi_\xi(1)$, [5].
Therefore, it follows that

Corollary 3.3. $\hat{j}^* \rho_2(y) = S_*^{-(2m-p)} f^* Sq^2 \rho_2 \phi_\xi(1)$

Theorem 3.4. $d_3^{p,0}[\mu\{j^*(x)\}] = \beta Sq^2[\rho_2 \mu\{j^*(x)\}] \otimes [CP(1)]$.

Proof. By Theorem 2.2 and (3.2)

$$[\mu\{\delta(x)\}] = [-\frac{1}{2} S_*^{-2m+p+1} g^* \phi_\xi c_1(\xi_m)] \otimes [CP(1)]$$

$$= \beta[\rho_2(y)] \otimes [CP(1)] \qquad \cdots\cdots\cdots (3.3).$$

Consider the following commutative diagram ;

$$\begin{array}{ccccc}
 & & & & \tilde{H}^{p+2}(X^{p+2}/X^{p+1}\,;\,Z_2) \\
 & & & & \downarrow \hat{j}^* \\
\tilde{H}^p(X^p/X^{p-1}\,;\,Z_2) & \xleftarrow{j^*} & \tilde{H}^p(X^{p+2}/X^{p-1}\,;\,Z_2) & \xrightarrow{Sq^2} & \tilde{H}^{p+2}(X^{p+2}/X^{p-1}\,;\,Z_2) \\
\downarrow i_1^* & & \downarrow i_2^* & & \downarrow i_2^* \\
\tilde{H}^p(X^p/A\,;\,Z_2) & \xleftarrow{k^*} & \tilde{H}^p(X^{p+2}/A\,;\,Z_2) & \xrightarrow{Sq^2} & \tilde{H}^{p+2}(X^{p+2}/A\,;\,Z_2) \\
 & \nwarrow p_1^* & \uparrow p_2^* & & \uparrow p_2^* \\
 & & \tilde{H}^p(X/A\,;\,Z_2) & \xrightarrow{Sq^2} & \tilde{H}^{p+2}(X/A\,;\,Z_2)
\end{array}$$

$$p_2^*[\rho_2(y)] = i_2^* \hat{j}^*(\rho_2(y))$$
$$= i_2^* S_*^{-(2m-p)} f^* Sq^2 \rho_2 \phi_\xi(1)$$

by Corollary 3.3,

$$= Sq^2 i_2^* S_*^{-(2m-p)} f^* \rho_2 \phi_\xi(1).$$

Put $\hat{\mu}(x) = S_*^{-(2m-p)} f^* \phi_\xi(1)$. Let $[j^* \rho_2 \hat{\mu}(x)]$ be the cohomology class of $\tilde{H}^{p+1}(X/A; Z_2)$ represented by $j^* \rho_2 \hat{\mu}(x)$, that is,

$$i_1^* j^* \rho_2 \hat{\mu}(x) = p_1^*[j^* \rho_2 \hat{\mu}(x)]\,.$$

Using the same way as Lemma 2.1, we have $j^* \hat{\mu}(x) = \mu\{j^*(x)\}$.

Then,

$$k^*p_2^*[\rho_2\mu j^*(x)] = k^*i_2^*\rho_2\hat{\mu}(x)\,.$$

Since k^* is injective, $p_2^*[\rho_2\mu j^*(x)] = i_2^*\rho_2\hat{\mu}(x)$. On the other hand

$$\begin{aligned} p_2^*Sq^2[\rho_2\mu j^*(x)] &= Sq^2p_2^*[\rho_2\mu j^*(x)] \\ &= Sq^2i_2^*\rho_2\hat{\mu}(x) \\ &= p_2^*[\rho_2(y)]\,. \end{aligned}$$

Since p_2^* is injective, $Sq^2[\rho_2\mu j^*(x)]=[\rho_2(y)]$. Hence by (3.3) theorem follows.

4. Application

Araki-Toda [1] showed the existence theorem of the admissible multiplications in the mod q-cohomology theories, that is; In case $q\not\equiv 2 \pmod 4$ admissible multiplications exist always; In case $q\equiv 2 \pmod 4$, if we assume that $\eta^{**}=0$ in $\tilde{h}$ and μ is commutative then admissible ones exist, where μ is the multiplication in $\tilde{h}$, and $\eta: S^3\to S^2$ is the Hopf map. In mod 2 U^*-cohomology theory, it is known that $\tilde{U}^k(S^m)=U^{k-m}$ and the canonical multiplication induces the isomorphism $\tilde{U}^{n+i}(X\wedge S^n)\approx\tilde{U}^i(X)\otimes\tilde{U}^n(S^n)$. Hence, it follows immediately that $\eta^{**}=0$. Therefore, there exist the admissible multiplications in mod 2 U^*-cohomology theory. Moreover, Araki-Toda [2] showed the existence theorem of the commutative admissible multiplications in the mod q-cohomology theories.

Let $\bar{\eta}$ be a generator of $\{S^2M_2, S^2\}$, $M_2=S^1\underset{2}{\cup}e^2$, which is represented by a map f: $S^4M_2\to S^4$ such that

$$f\circ S^4i = S^2\eta \qquad \cdots\cdots\cdots (4.1),$$

where i: $S^1\subset M_2$ and η is the Hopf map.

Theorem 4.1. (Araki-Toda). *Let $\tilde{h}$ be equipped with a commutative and associative multiplication and $\eta^{**}=0$ in $\tilde{h}$. The necessary and sufficient condition for the existence of commutative admissible multiplication in $\tilde{h}(\,;Z_2)$ is that $\bar{\eta}^*(1)=0$.*

Applying Theorem 4.1 to the mod 2 U^*-cohomology theory, we have the following,

Corollary 4.2. *The mod 2 U^*-cohomology theory has no commutative admissible multiplication.*

Proof. Let L be the mapping cone of f, that is,

$$L = S^4\underset{f}{\cup}C(S^4M_2)\,.$$

By (4.1), there exists the following commutative diagram,

$$\begin{array}{ccccc} \tilde{U}^2(S^2) & \xrightarrow{\bar{\eta}^*} & \tilde{U}^2(S^2M_2) & & \\ \wr & & \wr & & \\ & \overset{f^*}{\nearrow} & \tilde{U}^4(S^4M_2) & & \\ & & \wr & & \\ \tilde{U}^4(S^4) & \xrightarrow[\delta]{} & \tilde{U}^5(S^5M_2) & \longrightarrow & \tilde{U}^5(L) \end{array} \qquad \cdots\cdots\cdots (4.2)$$

the lower sequence is exact, considering the cofibration $S^4 \to L \to S^5M_2$. It is well known that

$$H^i(L\,;\,Z) \approx \begin{cases} Z \text{ for } i=0,4 \\ Z_2 \text{ for } i=7 \\ 0 \text{ others,} \end{cases}$$

and $S_q^3 | H^4(L; Z_2)$ is non trivial. By Theorem 3.4, $d_3^{4,0}$ is non trivial. Let $\{J^{p,5-p}\}$ be the filtration of $\tilde{U}^5(L)$ with $J^{p,5-p}/J^{p+1,4-p} \approx E_\infty^{p,5-p}$. Then,

$$\tilde{U}^5(L) \approx J^{0,5} \text{ and } J^{i,5-i}/J^{i+1,4-i} \approx 0 \text{ for } 0 \leqq i \leqq 6\,.$$

Since $d_r^{7,-2}=0$ and if $r>3$ then $d_r^{7-r,-3+r}=0$,

$$J^{7,-2}/J^{8,-3} \approx \cdots \approx E_4^{7,-2},\ J^{8,-3} = 0\,.$$

Since $d_3^{4,0}$ is non trivial, $E_4^{7,-2}=0$. Therefore, $\tilde{U}^5(L) \approx 0$, and by (4.2) $\bar{\eta}^*$ is onto. Note that $\tilde{U}^2(S^2) \approx Z$ and $\tilde{U}^2(S^2M_2) \approx Z_2$, we have $\bar{\eta}^*(1) \neq 0$. q.e.d.

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