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Osaka J. Math.
20 (1983), 145–175

ISOPARAMETRIC TRIPLE SYSTEMS OF ALGEBRA TYPE

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(Received May 21, 1981)

Introduction. In this paper we continue our study of isoparametric triple systems. These triple systems have been introduced in [3] and are studied in [3], [4] and [5]. They are in one-to-one correspondence with isoparametric hypersurfaces in spheres which have four distinct principal curvatures.

The classes of isoparametric hypersurfaces which have been considered up to now are the homogeneous ones ([10], [11]), the surfaces of FKM-type ([5], [6]), the surfaces satisfying "condition (A) and (B)" ([9], [10]) and the surfaces where the multiplicity of one of the principal curvatures is ≤ 2 ([12], [10]).

However, until now there exists no classification of all isoparametric hypersurfaces in spheres. It therefore may be useful to investigate special types of hypersurfaces, i.e., special types of isoparametric triple systems. In this paper we classify isoparametric triples of algebra type. Such triples correspond uniquely to those isoparametric hypersurfaces which satisfy the "condition (A)" of [9], but not necessarily the additional "condition (B)" of [9].

The classification is summarized in Theorem 5.18. As a corollary we get that every isoparametric triple of algebra type is equivalent to a hypersurface of FKM-type or to one 8-dimensional homogeneous hypersurface.

The paper is organized as follows: In section 1 we introduce the basic notations and mention some fundamental results concerning isoparametric triple systems. Next, we reduce the problem of describing isoparametric triples of algebra type to the problem of classifying certain families of representations of Clifford algebras. The result indicates that one has to consider the cases $m_1>m_2+1$, $m_1=m_2+1$ and $m_1=m_2$ separately (where m_1 and m_2 are the multiplities of the principal curvatures). This is done in the next 3 sections. In each case we explicitly determine the isomorphism classes of the corresponding triple systems. As an application of our results we show in the last section that every isoparametric triple system which is 'generically' of algebra type is already homogeneous.

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*Research supported by Deutsche Forschungsgemeinschaft.

We thank K. McCrimmon for solving the isotopy problem considered in section 4.

1. Some results from the theory of isoparametric triple systems

In this section we state, without proofs, some of the results of the theory of isoparametric triple systems which was developed in [3].

An *isoparametric triple system* is a tuple $(V, \langle\, ,\rangle, \{\cdots\})$ where $(V, \langle\, ,\rangle)$ is a finite dimensional Euclidean space and

$$\{\cdots\} : V\times V\times V \to V: (x, y, z) \to \{xyz\} =: T(x, y)z$$

is a trilinear map such that the following properties hold

(1.1) $\{x_1x_2x_3\} = \{x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\}$ for any permutation σ, i.e., $\{\cdots\}$ is totally symmetric,

(1.2) $T(x, y)$ is selfadjoint relative to $\langle\cdot, \cdot\rangle$,

(1.3) $\langle\{xxx\}, \{xxx\}\rangle - 9\langle x, x\rangle\langle\{xxx\}, x\rangle + 18\langle x, x\rangle^3 = 0$,

(1.4) there exist positive integers m_1, $m_2>0$ satisfying
trace $T(x, y) = 2(3+2m_1+m_2)\langle x, y\rangle$ and $\dim V = 2(m_1+m_2+1)$.

When no confusion is possible we write V instead of $(V, \langle\cdot, \cdot\rangle, \{\cdots\})$. We also often use the abbreviation $T(x)$ for $T(x, x)$.

To each isoparametric triple system $(V, \langle\cdot, \cdot\rangle, \{\cdots\})$ there is associated its *dual* (*triple system*) $(V, \langle\cdot, \cdot\rangle, \{\cdots\}')$ where

(1.5) $$\{xyz\}' = 3(\langle x, y\rangle z+\langle y, z\rangle x+\langle z, x\rangle y) - \{xyz\}\ .$$

By [3], Lemma 1.3, we know that $(V, \langle\cdot, \cdot\rangle, \{\cdots\}')$, usually abbreviated by V', is again an isoparametric triple system with the constants $m_1'=m_2$ and $m_2'=m_1$.

A $c\in V$ with $\langle c, c\rangle=1$ is called *minimal* (resp. *maximal*) tripotent if $\{ccc\} = 6c$(resp. $\{ccc\}=3c$). Let $c\in V$ be a minimal tripotent. Then $T(c)$ has only the eigenvalues 0, 2 and 6 and we have

$$V = V_0(c)\oplus V_2(c)\oplus \boldsymbol{R}c$$

where $V_\chi(c)$ denotes the eigenspace of $T(c)$ for the eigenvalue χ.

Let e be a maximal tripotent. Then $T(e)$ has the eigenvalues 1 and 3 and we have

$$V = V_1(e)\oplus V_3(e)\oplus \boldsymbol{R}e\,,$$

where $V_\chi(e)=\{x\in V;\ T(e)x=\chi x,\ \langle e, x\rangle=0\}$ for $\chi=1, 3$. The minimal and maximal tripotents of V and V' are related in the following manner: a minimal (resp. maximal) tripotent of V is a maximal (resp. minimal) tripotent of V'

and vice versa.

Two minimal tripotents e_1 and e_2 are called *orthogonal* if $\{e_1e_1e_2\}=0$ (which is equivalent to $\{e_2e_2e_1\}=0$). It can be shown that orthogonal tripotents always exist. If (e_1, e_2) are orthogonal, then the selfadjoint operators $T(e_1)$, $T(e_2)$ and $T(e_1, e_2)$ commute. Hence we can define simultaneous eigenspaces of $T(e_1)$, $T(e_2)$ and $T(e_1, e_2)$, called *Peirce spaces*

$$(1.6)\quad \begin{aligned} V_{12}(e_1, e_2) &= V_2(e_1)\cap V_2(e_2)\\ V_{12}^{+}(e_1, e_2) &= \{x\in V_{12}(e_1, e_2);\ T(e_1, e_2)x = x\}\\ V_{12}^{-}(e_1, e_2) &= \{x\in V_{12}(e_1, e_2);\ T(e_1, e_2)x = -x\}\\ V_{11}^{-}(e_1, e_2) &= \{x\in V_2(e_1);\ T(e_1, x)y = 0 \text{ for all } y\in V_0(e_1)\}\\ V_{22}^{-}(e_1, e_2) &= \{x\in V_2(e_2);\ T(e_2, x)y = 0 \text{ for all } y\in V_0(e_2)\}\\ V_{10}(e_1, e_2) &= V_2(e_1)\ominus(V_{12}(e_1, e_2)\oplus V_{11}^{-}(e_1, e_2))\\ V_{20}(e_1, e_2) &= V_2(e_2)\ominus(V_{12}(e_1, e_2)\oplus V_{22}^{-}(e_1, e_2))\\ V_{11}(e_1, e_2) &= \boldsymbol{R}e_1\oplus V_{11}^{-}(e_1, e_2)\\ V_{22}(e_1, e_2) &= \boldsymbol{R}e_2\oplus V_{22}^{-}(e_1, e_2) \end{aligned}$$

where we use the notation $U\ominus W$ to denote the orthogonal complement of W in U. When it is clear which pair of orthogonal tripotents is referred to we will write V_{ij} instead of $V_{ij}(e_1, e_2)$. The spaces $V_{ii}^{-}(e_1, e_2)$ depend only on e_i. We therefore frequently use the abbreviations $V_{ii}^{-}(e_1, e_2)=V_{ii}^{-}(e_i)=V_2^0(e_i)$. We have

$$(1.7)\quad \begin{aligned} &V = V_{11}\oplus V_{12}\oplus V_{22}\oplus V_{10}\oplus V_{20}\,,\\ &V_2(e_1) = V_{12}\oplus V_{11}^{-}\oplus V_{10}\,,\quad V_2(e_2) = V_{12}\oplus V_{22}^{-}\oplus V_{20}\\ &V_0(e_1) = V_{22}\oplus V_{20}\,,\qquad V_0(e_2) = V_{11}\oplus V_{10}\,. \end{aligned}$$

For orthogonal tripotents e_1, e_2 we put

$$(1.8)\quad e = \lambda(e_1+e_2)\,,\quad \hat{e} = \lambda(e_1-e_2)\,,\quad \lambda = 2^{-1/2}\,.$$

Then e and $\hat{e}$ are maximal tripotents (which are orthogonal for $\{\ \}'$) and we have

$$(1.9)\quad \begin{aligned} &V_3(e) = \boldsymbol{R}\hat{e}\oplus V_{12}^{+}\,,\quad V_3(\hat{e}) = \boldsymbol{R}e\oplus V_{12}^{-}\\ &V_1(e) = V_{11}^{-}\oplus V_{12}^{-}\oplus V_{22}^{-}\oplus V_{10}\oplus V_{20}\,,\\ &V_1(\hat{e}) = V_{11}^{-}\oplus V_{12}^{+}\oplus V_{22}^{-}\oplus V_{10}\oplus V_{20}\,. \end{aligned}$$

An isoparametric triple system V is said to be of *algebra type* (*relative to* e_1, e_2) if $V_{10}(e_1, e_2)=0=V_{20}(e_1, e_2)$. The following is known

(1.10) ([3] Corollary 5.12) $V_{10}(e_1, e_2) = 0 \Leftrightarrow V_{20}(e_1, e_2) = 0$.

(1.11) ([5], §6) If V is of algebra type relative to (e_1, e_2), then V is not necessarily of algebra type relative to every pair of orthogonal tripotents. However, we have:

(1.12) ([3] Theorem 5.13) V is of algebra type relative to (e_1, e_2) if and only if V is of algebra type relative to (e_1, x_2) for every minimal tripotent $x_2 \in V_0(e_1)$.

Because of (1.12) we often just say V is of algebra type relative to e_1. We have the following useful characterization of V or V' being of algebra type:

Lemma 1.1. *Let (e_1, e_2) be orthogonal tripotents.* a) *Then V' is of algebra type relative to $(\lambda(e_1+e_2), \lambda(e_1-e_2))$ iff $\{V^+_{12}(e_1, e_2)e_1V^-_{12}(e_1, e_2)\}=0$.* b) *$V$ is of algebra type relative to (e_1, e_2) iff $V_0(e_2)=V_0(f)$ for every $f \in V_0(e_1)$ with $\langle f, f\rangle=1$.*

Proof. a) By [3] Corollary 5.20 the assumption $\{V^+_{12}(e_1, e_2)e_1V^-_{12}(e_1, e_2)\}=0$ is equivalent to $(V')^-_{11}=V^-_{12}$ and $(V')^-_{22}=V^+_{12}$ and thus to $V'_{10}=0=V'_{20}$. b) If V is of algebra type relative to (e_1, e_2), then $V_0(e_1)=V_{22}(e_1, e_2)=V^K_2(e_2)$ and $V_0(f)=V_0(e_2)$ follows from [3] Theorem 5.15. Conversely, if $V_0(e_2)=V_0(f)$ for every $f \in V_0(e_2)$ we have by the same theorem that $f \in V^K_2(e_2)=V_{22}(e_1, e_2)$. Thus $V_0(e_1)=V_{22}(e_1, e_2)$ and $V_{20}(e_1, e_2)=0$.

The following lemma connects isoparametric triple systems of algebra type to the paper [9] of H. Ozeki and M. Takeuchi:

Lemma 1.2. *V is of algebra type relative to e_1 if and only if V' satisfies condition* (A) *of* [9] *relative to e_1.*

Proof. The assertion is obviously equivalent to: V satisfies condition (A) of [9] relative to a maximal tripotent e of V iff V' is of algebra type relative to e. We choose orthogonal tripotents (e_1, e_2) such that $e=\lambda(e_1+e_2)$, $\lambda=2^{-1/2}$ and consider the Peirce spaces V_{ij} relative to (e_1, e_2). Then $V_3(e)=\boldsymbol{R}\hat{e}\oplus V^+_{12}$ where $\hat{e}=\lambda(e_1-e_2)$, $V_1(e)=V^-_{11}\oplus V_{10}\oplus V^-_{22}\oplus V_{20}$ and $\ker(T(e, \hat{e})\,|\,V_1(e))=V^-_{12}$. Hence, using the notation of [9], we have by [3], § 3.1 that $P_{\alpha,1}=-\langle w^\alpha_3 \square x^-_{12}, x^-_{11}+x_{10}+x^-_{22}+x_{20}\rangle$. By definition, V satisfies (A) relative to e_1 iff $P_{\alpha,1}=0$ for all α, which is equivalent to $\langle x^+_{12} \square x^-_{12}, x^-_{11}+x_{10}+x^-_{22}+x_{20}\rangle=0$ for all $x_{ij} \in V_{ij}$. Since $x^+_{12} \square x^-_{12} \in V^-_{11} \oplus V_{10} \oplus V^-_{22} \oplus V_{20}$ by [3] (5.10), this condition is fulfilled iff $V^+_{12} \square V^-_{12}=0$. By [3] Lemma 5.17, this is equivalent to $(V')^0_2(e)=V^-_{12}$, i.e. to $(V')_{10}(e, \hat{e})=0$.

2. The principal construction theorem for isoparametric triple systems of algebra type

2.1 We will characterize what it means for an arbitrary triple system to be an isoparametric triple system of algebra type.

Let $(V, \langle\cdot, \cdot\rangle)$ be a euclidean space and $\{\ \}$ a triple system on V (i.e., $\{\cdots\}: V\times V\times V\to V$ is a trilinear map). As usual, we put $T(x, y)z=\{xyz\}$ and $T(x)=T(x, x)$ and assume

(2.1.a) $V = V_1\oplus V_{12}\oplus V_2$ is an orthogonal sum,

(2.1.b) $\{\cdots\}$ is totally symmetric,

(2.1.c) $T(x)$ is self-adjoint with respect to $\langle\cdot, \cdot\rangle$ for all $x\in V$,

(2.1.d) $T(x_i)x_i = 6\langle x_i, x_i\rangle x_i,\ i = 1, 2,\ x_i\in V_i$

(2.1.e) $T(x_i)x_{12} = 2\langle x_i, x_i\rangle x_{12},\ i = 1, 2,\ x_i\in V_i,\ x_{12}\in V_{12}$

(2.1.f) $T(x_1)x_2 = T(x_2)x_1 = 0,\ x_1\in V_1$ and $x_2\in V_2$,

(2.1.g) $T(x_{12})x_{12}\in V_{12}$.

REMARK. It is easy to check from [3] §§2.5 that an isoparametric triple system which is of algebra type relative to e_1, e_2 satisfies the conditions (2.1.a) to (2.1.g) with $V_i=\boldsymbol{R}e_i\oplus V_{\bar{i}i}$; note $\dim V_i=m_2+1$, $\dim V_{12}=2m_1$.

In the following we denote the j-component of a triple product $\{abc\}$ by $\{abc\}_j$.

Lemma 2.1. *Let* $(V, \langle\cdot, \cdot\rangle, \{\cdots\})$ *satisfy* (2.1.a) *to* (2.1.g)

a) *Then, in addition to* (2.1.a) *to* (2.1.g), *the following multiplication rules hold:*

(2.1.h) $T(x_1, x_2)x_{12}\in V_{12}$

(2.1.i) $T(x_{12})x_1 = 2\langle x_{12}, x_{12}\rangle x_1\oplus[T(x_{12})x_1]_2$

(2.1.k) $T(x_{12})x_2 = [T(x_{12})x_2]_1\oplus 2\langle x_{12}, x_{12}\rangle x_2$.

b) *The entire triple product is determined once* $T(x_1, x_2)x_{12}$ *and* $T(x_{12})x_{12}$ *are given for* $x_i\in V_i$, $x_{12}\in V_{12}$.

c) *For* $x=x_1\oplus x_{12}\oplus x_2$ *we have*

$$
\begin{aligned}
(2.2)\qquad \{xxx\} &= (6(\langle x_1, x_1\rangle+\langle x_{12}, x_{12}\rangle)x_1+3\{x_{12}x_{12}x_2\}_1)\\
&\oplus(\{x_{12}x_{12}x_{12}\}+6(\langle x_1, x_1\rangle+\langle x_2, x_2\rangle)x_{12}+6\{x_1x_2x_{12}\})\\
&\oplus(6(\langle x_2, x_2\rangle+\langle x_{12}, x_{12}\rangle)x_2+3\{x_{12}x_{12}x_1\}_2)\ .
\end{aligned}
$$

Proof. a) We have $\langle T(x_1, x_2)x_{12}, y_i\rangle=\langle x_{12}, T(x_1, x_2)y_i\rangle=0$ for $i=1, 2$. Further, $\langle T(x_{12})x_1, y_1\rangle=\langle x_{12}, T(x_1, y_1)x_{12}\rangle=2\langle x_{12}, x_{12}\rangle\langle x_1, y_1\rangle$ (by linearizing (2.1.e)) and $\langle T(x_{12})x_1, y_{12}\rangle=\langle x_1, T(y_{12})y_{12}\rangle=0$ which implies (2.1.i). The formula (2.1.k) follows similarly.

b) The identities (2.1.d) to (2.1.f) determine $T(x_i)$. If $T(x_1, x_2)x_{12}$ and $T(x_{12})x_{12}$ are known, then $\langle T(x_{12})x_1, x_2\rangle=\langle x_{12}, T(x_1, x_2)x_{12}\rangle=\langle T(x_{12})x_2, x_1\rangle$ shows that $T(x_{12})x$ is known, too. This proves b).

c) $\{xxx\} = \{x_1x_1x_1\} + 3\{x_1x_1, x_{12} + x_2\} + 3\{x_1, x_{12} + x_2, x_{12} + x_2\}$

$+\{x_{12}+x_2, x_{12}+x_2, x_{12}+x_2\}=6\langle x_1, x_1\rangle x_1+6\langle x_1, x_1\rangle x_{12}+3\{x_{12}x_{12}x_1\}+6\{x_1x_2x_{12}\}$ $+\{x_{12}x_{12}x_{12}\}+3\{x_{12}x_{12}x_2\}+6\langle x_2, x_2\rangle x_{12}+6\langle x_2, x_2\rangle x_2$, from which c) easily follows.

Lemma 2.2. *Let* $(V, \langle\cdot,\cdot\rangle, \{\cdots\})$ *satisfy* (2.1.e) *to* (2.1.g). *Then* $\{\cdots\}$ *satisfies* (1.3) *if and only if*

(2.3) $T(x_1, x_2)^2 x_{12}=\langle x_1, x_1\rangle\langle x_2, x_2\rangle x_{12}$

(2.4) $3\langle\{x_{12}x_{12}x_1\}_2, \{x_{12}x_{12}x_1\}_2\rangle+\langle x_1, x_1\rangle\langle T(x_{12})x_{12}, x_{12}\rangle=6\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle^2$

(2.5) $3\langle\{x_{12}x_{12}x_2\}_1, \{x_{12}x_{12}x_2\}_1\rangle+\langle x_2, x_2\rangle\langle T(x_{12})x_{12}, x_{12}\rangle=6\langle x_2, x_2\rangle\langle x_{12}, x_{12}\rangle^2$

(2.6) $\langle T(x_1, x_2)x_{12}, T(x_{12})x_{12}\rangle=3\langle x_{12}, x_{12}\rangle\langle T(x_1, x_2)x_{12}, x_{12}\rangle$

(2.7) $\langle T(x_{12})x_{12}, T(x_{12})x_{12}\rangle-9\langle x_{12}, x_{12}\rangle\langle x_{12}, T(x_{12})x_{12}\rangle+18\langle x_{12}, x_{12}\rangle^3=0$

for all $x_1\in V_1$, $x_{12}\in V_{12}$ *and* $x_2\in V_2$.

Proof. For $x=x_1+x_{12}+x_2$ we first compute

1) $\langle x, x\rangle^3=(\langle x_1, x_1\rangle+\langle x_{12}, x_{12}\rangle+\langle x_2, x_2\rangle)^3$
$=\langle x_1, x_1\rangle^3+3\langle x_1, x_1\rangle^2\langle x_{12}, x_{12}\rangle+3\langle x_1, x_1\rangle^2\langle x_2, x_2\rangle+3\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle^2$
$+3\langle x_1, x_1\rangle\langle x_2, x_2\rangle^2+6\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle\langle x_2, x_2\rangle+3\langle x_{12}, x_{12}\rangle^2\langle x_2, x_2\rangle$
$+3\langle x_{12}, x_{12}\rangle\langle x_2, x_2\rangle^2+\langle x_{12}, x_{12}\rangle^3+\langle x_2, x_2\rangle^3$.

2) $\langle x, \{xxx\}\rangle=6\langle x_1, x_1\rangle^2+6\langle x_{12}, x_{12}\rangle\langle x_1, x_1\rangle+3\langle\{x_{12}x_{12}x_2\}, x_1\rangle$
$+\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle+6[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]\langle x_{12}, x_{12}\rangle+6\langle\{x_1x_2x_{12}\}, x_{12}\rangle$
$+6\langle x_2, x_2\rangle^2+6\langle x_{12}, x_{12}\rangle\langle x_2, x_2\rangle+3\langle\{x_{12}x_{12}x_1\}, x_2\rangle$
$=6\langle x_1, x_1\rangle^2+12[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]\langle x_{12}, x_{12}\rangle$
$+12\langle\{x_1, x_2, x_{12}\}, x_{12}\rangle+\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle+6\langle x_2, x_2\rangle^2$

where we have used (2.2)

3) $\langle\{xxx\}, \{xxx\}\rangle=\langle 6(\langle x_1, x_1\rangle+\langle x_{12}, x_{12}\rangle)x_1+3\{x_{12}x_{12}x_2\}_1,$
$6(\langle x_1, x_1\rangle+\langle x_{12}, x_{12}\rangle)x_1+3\{x_{12}x_{12}x_2\}_1\rangle$
$+\langle\{x_{12}x_{12}x_{12}\}+6[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]x_{12}+6\{x_1x_2x_{12}\},$
$\{x_{12}x_{12}x_{12}\}+6[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]x_{12}+6\{x_1x_2x_{12}\}\rangle$
$+\langle 6(\langle x_2, x_2\rangle+\langle x_{12}, x_{12}\rangle)x_2+3\{x_{12}x_{12}x_1\}_2, 6(\langle x_2, x_2\rangle+\langle x_{12}, x_{12}\rangle)x_2+3\{x_{12}x_{12}x_1\}_2\rangle$
$=36\langle x_1, x_1\rangle^3+3\cdot 36[\langle x_1, x_1\rangle^2+\langle x_2, x_2\rangle^2]\langle x_{12}, x_{12}\rangle+2\cdot 36\langle x_1, x_1\rangle\langle x_2, x_2\rangle\langle x_{12}, x_{12}\rangle$
$+3\cdot 36[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]\langle\{x_1x_2x_{12}\}, x_{12}\rangle$
$+36\langle x_{12}, x_{12}\rangle^2[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]+72\langle x_{12}, x_{12}\rangle\langle T(x_1, x_2)x_{12}, x_{12}\rangle$
$+9\langle\{x_{12}x_{12}x_1\}_2, \{x_{12}x_{12}x_1\}_2\rangle+9\langle\{x_{12}x_{12}x_2\}_1, \{x_{12}x_{12}x_2\}_1\rangle$
$+\langle\{x_{12}x_{12}x_{12}\}, \{x_{12}x_{12}x_{12}\}\rangle+12[\langle x_1, x_1\rangle+\langle x_2, x_2\rangle]\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle$
$+12\langle\{x_{12}x_{12}x_{12}\}, \{x_1x_2x_{12}\}\rangle+36\langle\{x_1x_2x_{12}\}, \{x_1x_2x_{12}\}\rangle+36\langle x_2, x_2\rangle^2$.

In (1.3) we equate expressions of type (n, m, k), i.e., which are homogeneous of degree n (resp. m, k) in x_1 (resp. x_{12}, x_2). We get

(6,0,0): $0 = 36\langle x_1, x_1\rangle^3 - 9\cdot 6\langle x_1, x_1\rangle\langle x_1, x_1\rangle^2 + 18\langle x_1, x_1\rangle^3$

(5,1,0): does not appear

(5,0,1): does not appear

(4,2,0): $0 = 3\cdot 36\langle x_1, x_1\rangle^2\langle x_{12}, x_{12}\rangle - 9\langle x_1, x_1\rangle\cdot 12\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle$
$-9\langle x_{12}, x_{12}\rangle\cdot 6\langle x_1, x_1\rangle^2 + 18\cdot 3\langle x_1, x_1\rangle^2\langle x_{12}, x_{12}\rangle$

(4,1,1): does not appear

(4,0,2): $0 = -9\cdot\langle x_2, x_2\rangle 6\langle x_1, x_1\rangle^2 + 18\cdot 3\langle x_1, x_1\rangle^2\langle x_2, x_2\rangle$

(3,3,0): does not appear

(3,2,1): $0 = 3\cdot 36\langle x_1, x_1\rangle\langle\{x_1x_2x_{12}\}, x_{12}\rangle - 9\cdot 12\langle x_1, x_1\rangle\langle\{x_1x_2x_{12}\}, x_{12}\rangle$

(3,1,2): does not appear

(3,0,3): does not appear

(2,4,0): $0 = 36\langle x_{12}, x_{12}\rangle^2\langle x_1, x_1\rangle + 9\langle\{x_{12}x_{12}x_1\}_2, \{x_{12}x_{12}x_1\}_2\rangle$
$+12\langle x_1, x_1\rangle\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle$
$-9\cdot 12\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle^2 - 9\langle x_1, x_1\rangle\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle$
$+3\cdot 18\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle^2$
which is equivalent to (2.4)

(2,3,1): does not appear

(2,2,2): $0 = 2\cdot 36\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle\langle x_2, x_2\rangle + 36\langle\{x_1x_2x_{12}\}, \{x_1x_2x_{12}\}\rangle$
$-9\cdot 12\langle x_2, x_2\rangle\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle$
$-9\cdot 12\langle x_1, x_1\rangle\langle x_2, x_2\rangle\langle x_{12}, x_{12}\rangle + 18\cdot 6\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle\langle x_2, x_2\rangle$
which is equivalent to (2.3)

(2,1,3): does not appear

(2,0,4): $0 = -9\langle x_1, x_1\rangle 6\langle x_2, x_2\rangle^2 + 18\cdot 3\langle x_2, x_2\rangle^2\langle x_1, x_1\rangle$

(1,5,0); does not appear

(1,4,1): $0 = 72\langle x_{12}, x_{12}\rangle\langle T(x_1, x_2)x_{12}, x_{12}\rangle + 12\langle\{x_{12}x_{12}x_{12}\}, \{x_1x_2x_{12}\}\rangle$
$-9\langle x_{12}, x_{12}\rangle 12\langle\{x_1x_2x_{12}\}, x_{12}\rangle,$
which is equivalent to (2.6)

(1,3,2): does not appear

(1,2,3): $0 = 3\cdot 36\langle x_2, x_2\rangle\langle\{x_1x_2x_{12}\}, x_{12}\rangle - 9\cdot 12\langle x_2, x_2\rangle\langle\{x_1x_2x_{12}\}, x_{12}\rangle$

(1,1,4): does not appear

(1,0,5): does not appear

(0,6,0): $0 = \langle\{x_{12}x_{12}x_{12}\}, \{x_{12}x_{12}x_{12}\}\rangle - 9\langle x_{12}, x_{12}\rangle\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle$
$+18\langle x_{12}, x_{12}\rangle^3$
which is (2.7)

(0,5,1): does not appear

(0,4,2): $0 = 36\langle x_{12}, x_{12}\rangle^2\langle x_2, x_2\rangle + 9\langle\{x_{12}x_{12}x_2\}_1, \{x_{12}x_{12}x_2\}_1\rangle$
$+12\langle x_2, x_2\rangle\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle - 9\cdot 12\langle x_2, x_2\rangle\langle x_{12}, x_{12}\rangle^2$
$-9\langle x_2, x_2\rangle\langle\{x_{12}x_{12}x_{12}\}, x_{12}\rangle + 3\cdot 18\langle x_1, x_1\rangle\langle x_{12}, x_{12}\rangle^2$
which is equivalent to (2.5).

The remaining identities are trivial.

Lemma 2.3. *Let* { } *be an arbitrary triple system on the finite-dimensional euclidean space* $(V, \langle\cdot, \cdot\rangle)$ *which satisfies* (2.1.a) *to* (2.1.g). *Then*
a) trace $T(x_i, x_{12})=0$ *for* $i=1, 2$, trace $T(x_1, x_2)=\mathrm{trace}(T(x_1, x_2)\,|\,V_{12})$
b) trace $T(x_1, x_1)=\langle x_1, x_1\rangle\, 2(2+\dim V_1+\dim V_{12})$
trace $T(x_2, x_2)=\langle x_2, x_2\rangle 2(2+\dim V_2+\dim V_{12})$
c) trace $T(x_{12}, x_{12})=\langle x_{12}, x_{12}\rangle\cdot 2(\dim V_1+\dim V_2)+\mathrm{trace}\ T(x_{12}, x_{12})\,|\,V_{12}$.

Proof. a) By (2.1) we know $T(x_1, x_2)(V_1+V_2)=0$ and $T(x_1, x_2)V_{12}\subset V_{12}$, hence trace $T(x_1, x_2)=\mathrm{trace}\ (T(x_1, x_2)\,|\,V_{12})$. From (2.1) we get $T(x_1, x_{12})V_1\subset V_{12}$, $T(x_1, x_{12})V_{12}\subset V_1+V_2$ and $T(x_1, x_{12})V_2\subset V_{12}$. Therefore trace $T(x_1, x_{12})=0$. Similarly trace $T(x_2, x_{12})=0$.

b) can be read off from (2.1.d) and (2.1.e).

c) follows from (2.1.g), (2.1.i) and (2.1.k).

Lemma 2.4. *Let* $\{\cdots\}$ *be an arbitrary triple system on the finite-dimensional euclidean vector space* $(V, \langle\cdot, \cdot\rangle)$ *which satisfies* (2.1.a) *to* (2.1.g) *and* (2.3).
a) *Let* $(x_i^{(r)})$ *be an orthonormal basis of* V_i, $i=1, 2$. *Then for every* $x_i\in V_i$ *with* $\langle x_i, x_i\rangle=1$ *we have*

(2.8) $$[T(x_1, x_2^{(j)})T(x_1, x_2^{(k)})+T(x_1, x_2^{(k)})T(x_1, x_2^{(j)})]\,|\,V_{12} = 2\delta_{jk}Id$$

and

(2.8)′ $$[T(x_1^{(j)}, x_2)T(x_1^{(k)}, x_2)+T(x_1^{(k)}, x_2)T(x_1^{(j)}, x_2)]\,|\,V_{12} = 2\delta_{jk}Id\,.$$

b) *If* $\dim V_1\geq 2$ *or* $\dim V_2\geq 2$, *then* trace $(T(x_1, x_2)\,|\,V_{12})=0$.

Proof. a) By linearization we get from (2.3)

$$[T(x_1, x_2)T(x_1, y_2)+T(x_1, y_2)T(x_1, x_2)]\,|\,V_{12} = 2\langle x_1, x_1\rangle\langle x_2, y_2\rangle Id\,,$$

which implies a).

b) It is enough to show trace $T(x_1, x_2^{(j)})=0$ for $\langle x_1, x_1\rangle=1$. From a) we know $[T(x_1, x_2^{(j)})\,|\,V_{12}]^2=Id$, hence V_{12} is the direct sum of the eigenspaces of

$T(x_1, x_2^{(j)})$ for the eigenvalues 1 and -1. By assumption there exists a k different from j. Then (2.8) implies that $T(x_1, x_2^{(k)})$ interchanges the two eigenspaces of $T(x_1, x_2^{(j)})$, which therefore have the same dimension. Thus trace $T(x_1, x_2^{(j)})=0$.

Lemma 2.4.a shows that $(T(x_1, x_2^{(j)})\,|\,V_{12})$ for $\langle x_1, x_1\rangle=1$ and $(T(x_1^{(j)}, x_2)\,|\,V_{12})$ for $\langle x_2, x_2\rangle=1$ are examples of Clifford systems. In general, a *Clifford system* is a tuple $(P_0, \cdots, P_m)$ of symmetric endomorphisms on a finite dimensional Euclidean vector space W such that

$$P_jP_k+P_kP_j = 2\delta_{jk}Id$$

holds. With every Clifford system is associated a totally symmetric triple product

$$\{xyz\} = \langle x, y\rangle z+\langle y, z\rangle x+\langle z, x\rangle y$$
$$+\sum_{r=0}^{m}[\langle P_rx, y\rangle P_rz+\langle P_ry, z\rangle P_rx+\langle P_rz, x\rangle P_ry]$$

which satisfies (1.1) to (1.3). Such triple systems are called *formal FKM-triples*. If $m>0$ and $\frac{1}{2}\dim V-m-1>0$, a formal FKM-triple also satisfies (1.4), i.e., it is isoparametric; in this case it is called an *isoparametric triple system of FKM-type*. These triple systems are studied in [5], the corresponding hypersurfaces in [6].

Theorem 2.5. a) *Let V be an isoparametric triple system of algebra type relative to (e_1, e_2). Put $V_1=\boldsymbol{R}e_1\oplus V_{\bar{1}1}$, $V_{12}=V_{12}(e_1, e_2)$ and $V_2=\boldsymbol{R}e_2\oplus V_{\bar{2}2}$. Then*

(2.9) $\quad \dim V_1 = \dim V_2 = m_2+1\geq 2,\ \dim V_{12} = 2m_1>0$

(2.10) $\quad T(x_1, x_2)^2x_{12} = \langle x_1, x_1\rangle\langle x_2, x_2\rangle x_{12}$

(2.11) $$\langle x_2, x_2\rangle \sum_{r=0}^{m_2}\langle T(x_1, x_2^{(r)})x_{12}, x_{12}\rangle T(x_1, x_2^{(r)})x_{12} = \langle x_1, x_1\rangle \sum_{r=0}^{m_2}\langle T(x_1^{(r)}, x_2)x_{12}, x_{12}\rangle T(x_1^{(r)}, x_2)x_{12}$$

where $x_1^{(r)}$ and $x_2^{(r)}$ are arbitrary orthonormal bases of V_1 and V_2

(2.12) $$\{x_{12}x_{12}x_{12}\} = 9\langle x_{12}, x_{12}\rangle x_{12}-3[\langle x_{12}, x_{12}\rangle x_{12}-\sum_{r=0}^{m_2}\langle P_rx_{12}, x_{12}\rangle P_rx_{12}]$$

where $P_r = T(x_1, x_2^{(r)})\,|\,V_{12}$ with $\langle x_1, x_1\rangle = 1$ or $P_r = T(x_1^{(r)}, x_2)\,|\,V_{12}$ with $\langle x_2, x_2\rangle = 1$.

b) *Conversely, let V_1, V_{12} and V_2 be euclidean vector spaces and T: $V_1\times V_2\rightarrow$* End V_{12} *be a bilinear map such that $T(x_1, x_2)$ is self-adjoint for every $x_i\in V_i$. If, in addition, there exist positive integers m_1, m_2 such that* (2.9) *to* (2.11) *are satisfied, then T can be uniquely extended to a triple system on the orthogonal sum $V=V_1\oplus V_{12}\oplus V_2$ such that V becomes an isoparametric triple system with con-*

stants m_1, m_2 which is of algebra type relative to e_1, e_2 for all $e_i \in V_i$ with $\langle e_i, e_i\rangle=1$.

Proof. a) As already mentioned, V satisfies (2.1.a) to (2.1.g) and (2.9). Hence, by Lemma 2.2., it also satisfies (2.3) to (2.7). Obviously (2.3) and (2.10) are identical. By multiplying (2.4) with $\langle x_2, x_2\rangle$ and (2.5) with $\langle x_1, x_1\rangle$ we get $\langle x_2, x_2\rangle\langle\{x_{12}x_{12}x_1\}_2, \{x_{12}x_{12}x_1\}_2\rangle=\langle x_2, x_2\rangle\langle\{x_{12}x_{12}x_2\}_1\rangle, \{x_{12}x_{12}x_2\}_1\rangle$. Now $\{x_{12}x_{12}x_1\}_2=\sum_{r=0}^{m_2}\langle T(x_{12})x_1, x_2^{(r)}\rangle x_2^{(r)}$ and the analogous expression for $\{x_{12}x_{12}x_2\}_1$ imply

$$\langle x_2, x_2\rangle \sum_{r=0}^{m_2}\langle T(x_1, x_2^{(r)})x_{12}, x_{12}\rangle^2 = \langle x_1, x_1\rangle \sum_{r=0}^{m_2}\langle T(x_1^{(r)}, x_2)x_{21}, x_{12}\rangle^2$$

from which we obtain (2.11) by differentiating with respect to x_{12}. To derive (2.12) we note that (2.4) with $\langle x_1, x_1\rangle=1$ is equivalent to

$$\langle T(x_{12})x_{12}, x_{12}\rangle = 6\langle x_{12}, x_{12}\rangle^2-3\sum_{r=0}^{m_2}\langle P_r x_{12}, x_{12}\rangle^2$$

where $P_r=T(x_1, x_2^{(r)})\,|\,V_{12}$. Another differentiation with respect to x_{12} gives (2.12). Using (2.5) instead of (2.4) we get the same expression for $\{x_{12}x_{12}x_{12}\}$ with $P_r=T(x_1^{(r)}, x_2)\,|\,V_{12}$.

b) We define $\{x_{12}x_{12}x_{12}\}$ by (2.12) and remark that this makes sense because of (2.11). The remaining triple products are defined by (2.1.b) to (2.1.g) and Lemma 2.1. To prove (1.3) it suffices to show (2.4) to (2.7). Assume $\langle x_1, x_1\rangle=1$; then we have $6\langle x_{12}, x_{12}\rangle^2-\langle T(x_{12})x_{12}, x_{12}\rangle=3\sum_r\langle T(x_1, x_2^{(r)})x_{12}, x_{12}\rangle^2=3\sum_r\langle T(x_{12})x_1, x_2^{(r)}\rangle^2=3\langle\{x_{12}x_{12}x_1\}_2, \{x_{12}x_{12}x_1\}_2\rangle$, which shows (2.4). By a similar computation (2.5) follows. To prove (2.6) we may again assume $\langle x_1, x_1\rangle=1$. We get $\langle T(x_1, x_2)x_{12}, T(x_1, x_2^{(r)})x_{12}\rangle=\tfrac{1}{2}\langle[T(x_1, x_2)T(x_1, x_2^{(r)})+T(x_1, x_2^{(r)})T(x_1, x_2)]\cdot x_{12}, x_{12}\rangle=\langle x_2, x_2^{(r)}\rangle\langle x_{12}, x_{12}\rangle$ by Lemma 2.4.a. Hence $\langle T(x_1, x_2)x_{12}, T(x_{12})x_{12}\rangle-6\langle x_{12}, x_{12}\rangle\langle T(x_1, x_2)x_{12}, x_{12}\rangle=-3\sum_r\langle T(x_1, x_2^{(r)})x_{12}, x_{12}\rangle\langle T(x_1, x_2)x_{12}, T(x_1, x_2^{(r)})x_{12}\rangle$ $=-3\langle x_{12}, x_{12}\rangle\sum_r\langle T(x_1, x_2^{(r)})x_{12}, x_{12}\rangle\langle x_2, x_2^{(r)}\rangle=-3\langle x_{12}, x_{12}\rangle\langle T(x_1, x_2)x_{12}, x_{12}\rangle$, which implies (2.6). We note that (2.7) is satisfied if and only if the restriction of the triple product to V_{12} satisfies (1.3). But this follows from the definition of the triple product on V_{12}: it is the dual triple (see 1.5) of a formal FKM-triple as defined above. Since the latter satisfies (1.3), the former satisfies (1.3) too. Therefore our triple product on V satisfies (1.3).

Obviously, $\dim V=2(m_2+1+m_1)$. To prove the second equation of (1.4) we apply Lemma 2.3. We get trace $T(x_i, x_j)=0$ for $i\neq j$, $x_k\in V_k$ because of Lemma 2.4.b. Further, $\operatorname{trace} T(x_i)=2\langle x_i, x_i\rangle(2+m_2+1+2m_1)=\langle 2x_i, x_i\rangle\cdot(3+2m_1+m_2)$. By definition, we have

$$T(x_{12})y_{12} = 2\langle x_{12}, x_{12}\rangle y_{12}+4\langle x_{12}, y_{12}\rangle x_{12}-\sum_r[\langle P_r x_{12}, x_{12}\rangle P_r y_{12}+2\langle P_r x_{12}, y_{12}\rangle P_r x_{12}]$$

and therefore $\operatorname{trace}(T(x_{12})\,|\,V_{12})=\langle x_{12}, x_{12}\rangle(4m_1+4-2(m_2+1))$ because $\operatorname{trace} P_r=0$

and $\langle P_r x_{12}, P_r x_{12}\rangle = \langle x_{12}, x_{12}\rangle$. Thus trace $T(x_{12}) = 2\langle x_{12}, x_{12}\rangle(2(m_2+1)+2m_1+2-(m_2+1)) = 2\langle x_{12}, x_{12}\rangle(3+2m_1+m_2)$. This shows that the triple product on V also satisfies (1.4). Hence V is an isoparametric triple system.

Finally, for $e_i \in V_i$ with $\langle e_i, e_i\rangle = 1$ we conclude from (2.1) that (e_1, e_2) are orthogonal tripotents with $V_1 = V_{11}(e_1, e_2)$ and $V_{10}(e_1, e_2) = 0 = V_{20}(e_1, e_2)$.

2.2 Let $V = V_1 \oplus V_{12} \oplus V_2$ be an isoparametric triple system of algebra type. From (2.1.g) we derive that $(V_{12}, \{\cdots\})$ is a subtriple of $(V, \{\cdots\})$ which we abbreviate by $\tilde{V}$. We also put $\tilde{T}(x_{12}, y_{12}) = T(x_{12}, y_{12}) \mid V_{12}$. In this section we study $\tilde{V}$ more closely.

Theorem 2.6. *The triple system $\tilde{V}$ is the dual triple of a formal FKM-triple. In particular, it satisfies* (1.1) *to* (1.3). *Put $\tilde{m}_1 := m_1 - (m_2+1)$ and $\tilde{m}_2 := m_2$, then*

(2.13) $\dim \tilde{V} = 2(\tilde{m}_1 + \tilde{m}_2 + 1)$

(2.14) $\operatorname{trace} \tilde{T}(x_{12}, y_{12}) = 2(3 + 2\tilde{m}_1 + \tilde{m}_2)\langle x_{12}, y_{12}\rangle$.

Proof. The first assertion follows from (2.12) and the definition of a dual triple in (1.5). Further, by (2.9), we have $\dim V_{12} = 2m_1 = 2(m_1 - (m_2+1) + m_2 + 1)$ and by (1.4) and Lemma 2.3.c we get trace $\tilde{T}(x_{12}, y_{12}) = 2\langle x_{12}, y_{12}\rangle(3+2m_1+m_2-2(m_2+1)) = 2\langle x_{12}, y_{12}\rangle(1+2m_1-m_2) = 2\langle x_{12}, y_{12}\rangle(3+2\tilde{m}_1+\tilde{m}_2)$.

Corollary 2.7. *$\tilde{V}$ is an isoparametric triple system if and only if $m_1 > m_2 + 1$.*

We will see later that $\tilde{V}$ is not always an isoparametric triple system, i.e., there are examples with $m_1 \le m_2 + 1$. However, we have

Lemma 2.8. *Let V be an isoparametric triple system of algebra type. Then $m_2 \le m_1$.*

Proof. By (2.3) we know that $V_1 \to \operatorname{End} V_{12}$, $x_1 \to T(x_1, e_2)$ induces a representation of the Clifford algebra of $(V_1, \langle\cdot, \cdot\rangle)$. Hence the assertion follows from the table of the degrees of the irrducible representations of these Clifford algebras (see [1] or [5] 2.2).

Another proof of Lemma 2.8 runs as follows. Let V be of algebra type relative to (e_1, e_2). Then $e_{12} \in V_{12}^+$ with $\langle e_{12}, e_{12}\rangle$ is a maximal tripotent by [3] (2.13) and Lemma 5.4 and has the following Peirce spaces (see [4])

$$V_3(e_{12}) = (V_{11} \oplus V_{22}) \cap V_3(e_{12}) \oplus V_{12}^- \cap V_3(e_{12})$$
$$V_1(e_{12}) = (V_{11} \oplus V_{22}) \cap V_1(e_{12}) \oplus V_{12}^- \cap V_1(e_{12}) \oplus (V_{12}^+ \ominus \boldsymbol{R} e_{12}).$$

Moreover, $\dim(V_{11} \oplus V_{22}) \cap V_3(e_{12}) = \dim(V_{11} \oplus V_{22}) \cap V_1(e_{12})$. We put $n := \dim(V_{12}^- \cap V_3(e_{12}))$ and get trace $T(e_{12}) = 3(\dim V_{11} + n + 1) + \dim V_{11} + (\dim V_{12}^- - n) + \dim V_{12}^+ - 1 = 4(m_2+1) + 2m_1 + 2n + 2 = 2(3 + m_1 + 2m_2 + n)$ which, by (1.4), equals $2(3+2m_1+m_2)$ and therefore $m_2 + n = m_1$. This proves $m_2 \le m_1$.

REMARK. By Lemma 2.8 we know $m_2 \leq m_1$. In sections 3, 4 and 5 we will discuss the following three cases separately:

a) $\tilde{m}_1 > 0$, i.e., $m_1 > m_2 + 1$,

b) $\tilde{m}_1 = 0$, i.e., $m_1 = m_2 + 1$, and

c) $\tilde{m}_1 = 1$, i.e., $m_1 = m_2$.

2.3. An *isomorphism* between isoparametric triple systems $(V, \{\cdots\}_V)$ and $(W, \{\cdots\}_W)$ is an orthogonal map $\phi\colon V \to W$ such that $\phi\{xxx\}_V = \{\phi x, \phi x, \phi x\}_W$ holds for every $x \in V$. One says that V and W are *equivalent* if V is isomorphic to W or to W', i.e., if there exists an orthogonal map $\phi\colon V \to W$ such that $\phi\{xxx\}_V = \{\phi x, \phi x, \phi x\}_W$ or $\phi\{xxx\}_V = 9\langle x, x\rangle \phi x - \{\phi x, \phi x, \phi x\}_W$.

Lemma 2.9. *Let V and W be isoparametric triple systems of algebra type and assume $m_2(V) < m_1(V)$. Then V and W are equivalent if and only if V and W are isomorphic.*

Proof. Assume V and W' are isomorphic. Then $m_2(W') = m_2(V) < m_1(V) = m_1(W')$ and since $m_2(W') = m_1(W)$, $m_1(W') = m_2(W)$ we have $m_1(W) < m_2(W)$, which contradicts Lemma 2.8. The lemma now follows easily.

REMARK. If we assume that V and W are isoparametric triples of algebra type such that V and W' are isomorphic we get, by the same argument as in the proof above, that $m_1(W) = m_2(W)$. Theorem 5.17 shows that in this case W is homogeneous, in particular, W is of algebra type relative to every pair of orthogonal tripotents, hence [3] Corollary 5.19 implies that W' cannot be of algebra type. This proves that the assumption $m_2(V) < m_1(V)$ in the lemma above is not necessary.

We have the following characterization of isomorphisms leaving invariant corresponding Peirce spaces.

Theorem 2.10. *Let $V = V_1 \oplus V_{12} \oplus V_2$ be an isoparametric triple system of algebra type and $\phi_j\colon V_j \to W_j$, $j = 1$, 12 and 2, orthogonal maps from V_j onto some euclidean vector spaces W_j.*

a) *For $x_i \in W_i$, $i = 1, 2$, we define*

$$T_W(x_1, x_2) \mid W_{12} = \phi_{12} T_V(\phi_1^{-1} x_1, \phi_2^{-1} x_2) \phi_{12}^{-1}. \tag{2.15}$$

Then there exists a unique extension of $T_W(x_1, x_2) x_{12}$ to a triple product on $W = W_1 \oplus W_{12} \oplus W_2$ such that W becomes an isoparametric triple system of algebra type with $m_i(W) = m_i(V)$ and $\phi = \phi_1 \oplus \phi_{12} \oplus \phi_2$ an isomorphism from V to W.

b) *If $W = W_1 \oplus W_{12} \oplus W_2$ is already an isoparametric triple system of algebra type, then $\phi = \phi_1 \oplus \phi_{12} \oplus \phi_2$ is an isomorphism if and only if* (2.15) *is satisfied.*

Proof. a) It is easy to check that (2.9) to (2.11) are satisfied with $m_i(V) =$

$m_i(W)$. Hence the first part of a) follows from Theorem 2.5.b. Further, define on W an isoparametric triple system $\{\cdots\}^{\sim}$ by $\{xyz\}^{\sim}=\phi(\{\phi^{-1}x, \phi^{-1}y, \phi^{-1}z\}_V)$; then $\{\cdots\}^{\sim}$ is again of algebra type and we have $\tilde{W}_i=W_i$. Obviously, $\{x_1x_2x_{12}\}^{\sim}=\{x_1x_2x_{12}\}_W$; therefore the uniqueness statement of Theorem 2.5 implies $\{\cdots\}^{\sim}=\{\cdots\}_W$, i.e., ϕ is an isomorphism.

b) If ϕ is an isomorphism, then, obviously, (2.15) is satisfied. If (2.15) is satisfied, the assertion follows from a).

3. The case $m_1>m_2+1$

In this section we consider isoparametric triple systems of algebra type with $m_1>m_2+1$. We already know that in this case the subsystem V_{12} is the dual of an FKM-triple and we will show that even V is the dual of an FKM-triple. The proof makes use of the following theorem which characterizes when an isoparametric triple of algebra type is the dual of an FKM-triple.

Theorem 3.1. *Let $V=V_1\oplus V_{12}\oplus V_2$ be an isoparametric triple system of algebra type relative to (e_1, e_2). Then V is the dual of an* FKM-*triple if and only if there exists a bilinear map h: $V_2\times V_2\rightarrow V_1$ which satisfies for all $x_2, y_2\in V_2$*

(3.1) $\quad h(x_2, x_2)=\langle x_2, x_2\rangle e_1$

(3.2) $\quad \langle h(x_2, y_2), h(x_2, y_2)\rangle=\langle x_2, x_2\rangle\langle y_2, y_2\rangle$

(3.3) $\quad T(h(x_2, y_2), y_2)u_{12}=\langle y_2, y_2\rangle T(e_1, x_2)u_{12}$ *for all* $u_{12}\in V_{12}$.

In this case, let $x^0, \cdots, x^m$, $m=m_2$, be an orthonormal basis of V_2, then V' is an FKM-triple relative to $(P_0, \cdots, P_m)$ where

$$P_j=-T(e_1, x^j)+2x^je_1^*+2e_1(x^j)^*-\sum_{r=0}^{m}[h(x^r, x^j)(x^r)^*+x^rh(x^r, x^j)^*]\,.$$

Proof. We apply [5] Theorem 5.4 for $c=e_1$, $g=Id$ and conclude that V' is an FKM-triple iff there exist a bilinear map h: $V_2\times V_2\rightarrow V_1\oplus V_{12}$ such that the following conditions hold

a) $h(x_2, x_2)=\langle x_2, x_2\rangle e_1$, $\quad x_2\in V_2$

b) $\langle h(x_2, y_2), h(x_2, y_2)\rangle=\langle x_2, x_2\rangle\langle y_2, y_2\rangle$, $\quad x_2, y_2\in V_2$

c) $y_2\circ h(x_2, y_2)=0$, $\quad x_2, y_2\in V_2$

d) $\langle\{y_2, h(x_2, y_2), u_{12}\}, v_{12}\rangle=\langle x_2\circ u_{12}, v_{12}\rangle$ for $y_2\in V_2$, $\langle y_2, y_2\rangle=1$ and $u_{12}, v_{12}\in V_{12}=V_2(e_1)\cap V_2(y_2)$.

Obviously, (3.1) and (3.2) are identical with a) and b). By (2.10) the condition c) is equivalent to $h(x_2, y_2)\in V_1$, i.e., h: $V_2\times V_2\rightarrow V_1$. Finally, d) is satisfied iff (3.3) is satisfied since $T(e_1, x_2)u_{12}\in V_{12}$ by (2.1.h).

Theorem 3.2. *Let V be an isoparametric triple system with $m_1>m_2+1$. Then there are equivalent:*

(1) *V is of algebra type.*

(2) *V is the dual of an FKM-triple and* $m_2=1$, 3 *or* 7.

Proof. The implication (2)$\Rightarrow$(1) follows from [5] Theorem 7.4. (Note that $m_2(V)=m_1(V')$.)

We assume now (1) and choose an isometry f_2: $V_2\to V_1$. Then $U=V_1$ and $P(u, v)=T(u, f_2^{-1}v)\mid V_{12}$, u, $v\in V_1$, fulfill the assumptions of [5] Theorem 8.8. Hence, by [5] Corollary 8.9, there exist a composition algebra $(\mathcal{A}, \cdot)$ with $\dim_{\mathbf{R}}\mathcal{A}\geq 2$, i.e., $\mathcal{A}=\mathbf{C}$, $\mathbf{H}$, or $\mathbf{O}$, and isometries F_j: $V_1\to\mathcal{A}$, $j=0, 1, 2$, such that $T(x_1, x_2)\mid V_{12}=T(e_1, F_0^{-1}(F_1(x_1)\cdot F_2\circ f_2(x_2)))$, $x_1\in V_1$, $x_2\in V_2$.

We put $\mathcal{A}=\mathcal{A}_1=\mathcal{A}_2$, $\phi_1=F_1$, $\phi_2=F_2\circ f_2$ and T_0: $\mathcal{A}\to\operatorname{End} V_{12}$: $a\to T(e_1, F_0^{-1}a)$. Then $T(\phi_1^{-1}a, \phi_2^{-1}b)=T_0(a\cdot b)$, a, $b\in\mathcal{A}$ and Theorem 2.10 shows that we may replace V by the isomorphic triple system $W=\mathcal{A}_1\oplus V_{12}\oplus\mathcal{A}_2$ which has the property $\{a_1b_2x_{12}\}=T_0(a\cdot b)x_{12}$.

It is now easy to prove that W (and hence V) is the dual of an FKM-triple. We consider the bilinear map h: $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$: $(a, b)\to a\bar{b}$ where $\bar{b}$ denotes the canonical involution of $\mathcal{A}$. Well-known properties of $\mathcal{A}$ show that h satisfies

a) $h(a, a)=\langle a, a\rangle\, 1$, where 1 is the unit element of $\mathcal{A}$

b) $\langle h(a, b), h(a, b)\rangle=\langle a, a\rangle\langle b, b\rangle$

c) $T(h(a, b)_1, b_2)u_{12}=T_0(a\bar{b}\cdot b)u_{12}=\langle b, b\rangle T(a)u_{12}=\langle b, b\rangle T(1, a_2)u_{12}$.

Hence (3.1) to (3.3) of Theorem 3.1 are fulfilled and the theorem follows.

REMARK. Let V be an isoparametric triple of algebra type with $m_1>m_2+1$. Then V is the dual of an FKM-triple and $m_2=1$, 3 or 7, but not $(m_1, m_2)=(1, 1)$, $(2, 1)$, $(4, 3)$, or $(8, 7)$.

4. The case $m_1=m_2+1$

In this section we classify isoparametric triple systems of algebra type with $m_1=m_2+1$. We will see that such triples are built up from composition triples where in this paper (in contrast to [8]!) a *composition triple* is a triple system $(\cdots)$: $X\times X\times X\to X$ on a finite-dimensional euclidean vector space $(X, \langle\cdot, \cdot\rangle)$ which permits composition, i.e., $\langle(x, y, z), (x, y, z)\rangle=\langle x, x\rangle\langle y, y\rangle\langle z, z\rangle$ holds for every x, y, $z\in X$. Let $L(x, y)\in\operatorname{End} X$ be defined by $L(x, y)z=(x, y, z)$ and let $L(x, y)^*$ denote the adjoint of $L(x, y)$. Then $(\cdots)$ is a composition triple if and only if $L(x, y)^*L(x, y)=\langle x, x\rangle\langle y, y\rangle Id$ which is equivalent to $L(x, y)L(x, y)^*=\langle x, x\rangle\langle y, y\rangle Id$. Hence $(\cdots)$ is a composition triple if and only if $(\cdots)^*$, where $(x, y, z)^*=L(x, y)^*z$, is again a composition triple. We call $(\cdots)^*$ the *dual of* $(\cdots)$.

In the following lemma we construct an isoparametric triple system on the orthogonal sum of four copies of X. To distinguish them, the summands are written as Xe_1, Xe_2, Xe_{12} and $X\bar{e}_{12}$.

Theorem 4.1. *Let* $(\cdots)$ *be a composition triple on* X *with* $\dim X \geq 2$. *Define* $V_1 = Xe_1$, $V_2 = Xe_2$, $V_{12} = Xe_{12} \oplus X\bar{e}_{12}$ *and*

$$(4.1) \qquad T(xe_1, ye_2)(ze_{12} \oplus w\bar{e}_{12}) = (x, y, w)^* e_{12} \oplus (x, y, z)\bar{e}_{12} .$$

Then T *can be uniquely extended to an isoparametric triple system on* $V = V_1 \oplus V_{12} \oplus V_2$ *which is of algebra type and has* $(m_1(V), m_2(V)) = (2, 1), (4, 3)$ *or* $(8, 7)$.

Proof. For every $y \in X$ with $\langle y, y \rangle = 1$ we can define an algebra "$\perp$" on X by $x \perp z = (x, y, z)$. From the defining identities of a composition triple it follows that this algebra permits composition: $\langle x \perp z, x \perp z \rangle = \langle x, x \rangle \langle z, z \rangle$. It is therefore well-known (see e.g., [7]) that $\dim_{\boldsymbol{R}} X = 1, 2, 4$ or 8 where we have ruled out the first case by the assumption $\dim X \geq 2$.

We are going to apply part b) of Theorem 2.5. First note that, by definition, $T(xe_1, ye_2)$ is a self-adjoint endomorphism. Using the notation of Theorem 2.5 we get $(m_1, m_2) = (2, 1), (4, 3)$ or $(8, 7)$, thus, in particular, (2.9) holds. The theorem will follow if we can verify (2.10) and (2.11). To prove (2.10) we have $\langle T(xe_1, ye_2)(ze_{12} + w\bar{e}_{12}), T(xe_1, ye_2)(ze_{12} + w\bar{e}_{12} \rangle = \langle (x, y, z), (x, y, z) \rangle + \langle L(x, y)^* w, L(x, y)^* w \rangle = \langle x, x \rangle \langle y, y \rangle (\langle z, z \rangle + \langle w, w \rangle)$, since $L(x, y)^* L(x, y) = \langle x, x \rangle \langle y, y \rangle Id$ implies $L(x, y) L(x, y)^* = \langle x, x \rangle \langle y, y \rangle Id$. Hence (2.10) follows. To verify (2.11) we may assume $\langle x_1, x_1 \rangle = 1 = \langle x_2, x_2 \rangle$. Let $y^{(r)} e_2$ be an orthonormal basis of $V_2 = Xe_2$. Then $\langle z, z \rangle^{-1/2} T(xe_1, y^{(r)} e_2) ze_{12}$ (resp. $\langle w, w \rangle^{-1/2} \cdot T(xe_1, y^{(r)} e_2) w\bar{e}_{12}$) is an orthonormal basis for Xe_{12} (resp. $X\bar{e}_{12}$) by (2.10) for $z \neq 0$ (resp. $w \neq 0$) and we get

$$\begin{aligned} &\textstyle\sum_r \langle T(xe_1, y^{(r)} e_2)(ze_{12} + w\bar{e}_{12}), ze_{12} + w\bar{e}_{12} \rangle T(xe_1, y^{(r)} e_2)(ze_{12} + w\bar{e}_{12}) \\ &= 2 \textstyle\sum_r \langle T(xe_1, y^{(r)} e_2) ze_{12}, w\bar{e}_{12} \rangle T(xe_1, y^{(r)} e_2) ze_{12} \\ &+ 2 \textstyle\sum_r \langle T(xe_1, y^{(r)} e_2) w\bar{e}_{12}, ze_{12} \rangle T(xe_1, y^{(r)} e_2) w\bar{e}_{12} \\ &= 2 \langle z, z \rangle w\bar{e}_{12} + 2 \langle u, w \rangle ze_{12} . \end{aligned}$$

Because we get the same result if we start with an orthonormal basis in V_1, the formula (2.11) follows.

The isoparametric triple system constructed in Lemma 4.1 will be called the *isoparametric triple system associated with the composition triple system* $(\cdots)$.

Theorem 4.2. *Let* V *be an isoparametric triple system of algebra type with* $m_1 = m_2 + 1$. *Then* V *is isomorphic to an isoparametric triple system associated with a composition triple.*

Proof. We choose $a_{12}^+ \in V_{12}^+$ with $\langle a_{12}^+, a_{12}^+ \rangle = 1$. We may apply [4] and have $m_1 + 1 = \dim V_3(a_{12}^+) = \dim V_1 + \dim (V_3(a_{12}^+) \cap V_{12}^-)$. But $\dim V_1 = m_2 + 1 = m$, whence $\dim V_3(a_{12}^+) \cap V_{12}^- = 1$. We therefore can find an $a_{12}^- \in V_{12}^-$, $\langle a_{12}^-, a_{12}^- \rangle = 1$ which satisfies $\{a_{12}^+ a_{12}^+ a_{12}^-\} = 3a_{12}^-$. We apply [3] Lemma 4.5.b and get

$\{a_{12}^- a_{12}^- a_{12}^+\} = 3a_{12}^+$. As an easy consequence we see that $e_{12} = \lambda(a_{12}^+ + a_{12}^-)$ and $\bar{e}_{12} = \lambda(a_{12}^+ - a_{12}^-)$ are orthogonal minimal tripotents of V.

We form the Peirce decomposition of V relative to $(e_{12}, \bar{e}_{12})$ and denote the corresponding Peirce spaces by $\hat{V}_{ij}$. Then $V_1 \oplus V_2 \subset \hat{V}_{12}$ by [4]. But $\dim(V_1 \oplus V_2) = 2(m_2+1) = 2m_1 = \dim \hat{V}_{12}$ by [3] Corollary 5.5. Hence $V_1 \oplus V_2 = \hat{V}_{12}$ and $V_{12} = V_{12} \cap V_0(e_{12}) \oplus V_{12} \cap V_0(\bar{e}_{12})$, where $V_{12} \cap V_0(\bar{e}_{12}) = \boldsymbol{R} e_{12} \oplus \hat{V}_{1\bar{1}} \oplus \hat{V}_{10}$ and $V_{12} \cap V_0(e_{12}) = \boldsymbol{R} \bar{e}_{12} \oplus \hat{V}_{2\bar{2}} + \hat{V}_{20}$. Because $\{\hat{V}_{10} e_{12} \hat{V}_{20}\} \subset V_{12} \cap \hat{V}_{12} = 0$ we conclude $\hat{V}_{10} = 0 = \hat{V}_{20}$ by [3] Corollary 5.12. We now choose $x_1 \in V_1$, $x_2 \in V_2$ and $x_{12} \in \hat{V}_{11} = \boldsymbol{R} e_{12} \oplus \hat{V}_{1\bar{1}}$, $\langle x_{12}, x_{12} \rangle = 1$. Then $x_1, x_2 \in V_2(x_{12}) = [V_2(e_{12}) \oplus \boldsymbol{R} e_{12}] \ominus \boldsymbol{R} x_{12}$ by [3] Lemma 2.12.d. But then $\{x_1 x_2 x_{12}\} \in V_0(x_{12}) = V_0(e_{12})$ by [3] (2.5) and [3] Theorem 5.15. We have thus proved $T(x_1, x_2)\hat{V}_{11} = \hat{V}_{22}$; hence, by symmetry, $T(x_1, x_2)\hat{V}_{22} \subset \hat{V}_{11}$. In particular, we have $\dim \hat{V}_{11} = \dim \hat{V}_{22} = m_1 = m_2 + 1 = \dim V_i$. We may therefore choose isometries $\phi_i \colon \hat{V}_{22} \to V_i$, $i = 1, 2$ and $\phi_{12} \colon \hat{V}_{22} \to \hat{V}_{11}$ and define a triple system $(\cdots)$ on $\hat{V}_{22}$ by $(x, y, z) = \{\phi_1 x, \phi_2 y, \phi_{12} z\}$, which permits composition because of (2.10). Using Theorem 2.10 it is now easy to check that V is isomorphic to the triple system associated with $(\cdots)$.

We will now investigate the isomorphism problem for isoparametric triple systems associated with composition triples. Two composition triples $(\tilde{X}, (\cdots)^\sim)$ and $(X, (\cdots))$ are said to be *isotopic* if there exist orthogonal maps $F_i \colon \tilde{X} \to X$, $i = 0, 1, 2, 3$, such that $F_0(x, y, z)^\sim = (F_1 x, F_2 y, F_3 z)$. They are called *equivalent* if $(\cdots)^\sim$ is isotopic to $(\cdots)$ or to $(\cdots)^*$.

Theorem 4.3. *Isoparametric triple systems associated with equivalent composition triples are isomorphic.*

Proof. Let $(\tilde{X}, (\cdots)^\sim)$ and $(X, (\cdots))$ be isotopic and define

$$\phi_i \colon \tilde{V}_i = \tilde{X} e_i \to V_i = X e_i \colon x e_i \to (F_i x) e_i \quad \text{for } i = 1, 2 \text{ and}$$
$$\phi_{12} \colon \tilde{V}_{12} \to V_{12} \colon z e_{12} \oplus w \bar{e}_{12} \to (F_3 z) e_{12} \oplus (F_0 \bar{w}) \bar{e}_{12}\,.$$

Then $\phi_{12}\{x e_1, y e_2, z e_{12} \oplus w \bar{e}_{12}\} = \{\phi_1(x e_{12}), \phi_2(y e_2), \phi_{12}(z e_{12} \oplus w \bar{e}_{12})\}$ follows. Therefore $\phi = \phi_1 \oplus \phi_{12} \oplus \phi_2$ is an isomorphism by Theorem 2.10. To prove the theorem it now suffices to show that the isoparametric triples V and V^* associated with $(\cdots)$ and $(\cdots)^*$ are isomorphic. We define $\phi_{12} \colon X e_{12} \oplus X \bar{e}_{12} \to X e_{12} \oplus X \bar{e}_{12} \colon z e_{12} \oplus w \bar{e}_{12} \to w e_{12} \oplus z \bar{e}_{12}$. Using the definition (4.1) of the triple system associated with $(\cdots)$ resp. $(\cdots)^*$, a trivial verification shows $\phi_{12} T(x e_1, y e_2) \mid V_{12} = T^*(x e_1, y e_2) \phi_{12}$. Hence, again by Theorem 2.10, the triple systems V and V^* are isomorphic.

REMARK. The results below show that isoparametric triple systems associated with composition triples are equivalent (which, by Lemma 2.9, is the same as isomorphic) if and only if the composition triples are equivalent.

The classification of isotopy classes of composition triples (over arbitrary fields) was carried out by K. McCrimmon. As a special case of [8] Theorem 7.6 we get

Theorem 4.4. *Every composition triple is isotopic to one of the following triples on a real composition division algebra* $\mathcal{A}$ *with unit* (i.e., $\mathcal{A}=\boldsymbol{R}$, $\boldsymbol{C}$, *the quaternions* $\boldsymbol{H}$, *or the octonions* $\boldsymbol{O}$):

a) $(a, b, c)=abc$ *for* $\mathcal{A}=\boldsymbol{R}, \boldsymbol{C}$

b) $(a, b, c)=abc$ *or* acb *or* bac *for* $\mathcal{A}=\boldsymbol{H}$

c) $(a, b, c)=(ab)c$, $a(bc)$, $(ac)b$, $a(cb)$, $(ba)c$ *or* $b(ac)$ *for* $\mathcal{A}=\boldsymbol{O}$.

These triples are pairwise nonisotopic.

Corollary 4.5. *Every composition triple is equivalent to exactly one of the following triples defined on* $\mathcal{A}$:

a) $(a, b, c)=abc$ *for* $\mathcal{A}=\boldsymbol{R}, \boldsymbol{C}$

b) $(a, b, c)=abc$ *or* acb *for* $\mathcal{A}=\boldsymbol{H}$

c) $(a, b, c)=(ab)c$, $a(bc)$ *or* $(ac)b$ *for* $\mathcal{A}=\boldsymbol{O}$.

Proof. Since equivalence is a weaker equivalence relation than isotopy it remains to consider the composition triples of Theorem 4.4. Let (a, b, c) be respectively $(ab)c$ or $a(bc)$ or $(ac)b$. Then it is easy to show that $(a, b, c)^*$ is $(\bar{b}\bar{a})c$, $\bar{b}(\bar{a}c)$, $\bar{a}(c\bar{b})$ respectively hence is isotopic to $(ba)c$, resp. $b(ac)$, resp. $a(cb)$. This implies the corollary.

Up to now we have proved that each isoparametric triple system of algebra type with $m_1=m_2+1$ is isomorphic to a triple system associated with one of the following composition triples defined on a real composition division algebra $\mathcal{A}$:

$$abc \text{ for } \mathcal{A}=\boldsymbol{C}$$
$$abc \text{ or } acb \text{ for } \mathcal{A}=\boldsymbol{H}$$
$$(ab)c,\ a(bc) \text{ or } (ac)b \text{ for } \mathcal{A}=\boldsymbol{O}\,.$$

In the sequel we will show that these isoparametric triples are pairwise nonisomorphic.

Lemma 4.6. *Let* V *be the isoparametric triple system associated with the composition triple* $(\cdots)$ *on* $\mathcal{A}=\boldsymbol{C}$, $\boldsymbol{H}$, *or* $\boldsymbol{O}$.

a) *If* $(a, b, c)=(ab)c$, *then* V' *is an FKM-triple.*

b) *If* $(a, b, c)=(ac)b$ *and* $\mathcal{A}=\boldsymbol{H}$ *or* $\boldsymbol{O}$, *then* V' *is not an FKM-triple.*

c) *If* $(a, b, c)=a(bc)$ *and* $\mathcal{A}=\boldsymbol{O}$, *then* V' *is not an FKM-triple.*

Proof. By Theorem 3.1, V' is an FKM-triple if and only if there exists a bilinear map h: $\mathcal{A}e_2\times\mathcal{A}e_2\to\mathcal{A}e_1$: $(ae_2, be_2)\to h(a, b)e_1$ satisfying (3.1) to (3.3), where (3.3) in the case under consideration is equivalent to

$$(*) \qquad (h(x, y), y, z) = \langle y, y\rangle(1, x, z) \qquad \text{for all } x, y, z\in\mathcal{A}\,.$$

a) We put $h(x, y)=x\bar{y}$. Then (3.1) and (3.2) follow and (*) is easily verified. This proves a).
In the cases b) and c) we show that (*) yields a contradiction: In the case b) (*) is equivalent to $(h(x, y)z)y=\langle y, y\rangle zx$ which holds if and only if $h(x, y)z=(zx)\bar{y}$. For $z=1$ this implies $h(x, y)=x\bar{y}$ and thus we have $(x\bar{y})z=(zx)\bar{y}$ which gives $\bar{y}z=z\bar{y}$ for all $y, z\in\boldsymbol{H}$ or $\boldsymbol{O}$, a contradiction. In the case c) we conclude similarly $h(x, y)(yz)=\langle y, y\rangle xz$, hence $h(x, y)=x\bar{y}$ and $(x\bar{y})(yz)=\langle y, y\rangle xz$. Substituting $x=wy$ shows $\langle y, y\rangle w(yz)=\langle y, y\rangle(wy)z$, i.e., $\mathcal{A}$ is associative, a contradiction.

In [5] we introduced the following special Clifford systems. For $\mathcal{A}=\boldsymbol{H}, \boldsymbol{O}$ let $(x_1=1, x_2, \cdots, x_m)$ be an orthonormal basis of $\mathcal{A}$. We identify $\boldsymbol{R}^{4m}$ with the orthogonal sum $V=\mathcal{A}\oplus\mathcal{A}\oplus\mathcal{A}\oplus\mathcal{A}$ and define the *definite* $(m, m-1)$ family on V by

$$P_0(a\oplus b\oplus c\oplus d) = a\oplus -b\oplus c\oplus -d$$
$$P_j(a\oplus b\oplus c\oplus d) = x_jb\oplus\bar{x}_ja\oplus x_jd\oplus\bar{x}_jc \qquad \text{for } 1\le j\le m\,.$$

The *indefinite* $(m, m-1)$-family $(Q_0, \cdots, Q_m)$ is given by

$$Q_0 = P_0$$
$$Q_j(a\oplus b\oplus c\oplus d) = x_jb\oplus\bar{x}_ja\oplus\bar{x}_jd\oplus x_jc \qquad \text{for } 1\le j\le m\,.$$

It was proved in [5] § 6:

Theorem 4.7. *The FKM-triples corresponding to the definite and indefinite* (4, 3)- *resp.,* (8, 7)-*family are of algebra type, but not isomorphic* (*and hence not equivalent*).

Since the isoparametric triples of Theorem 4.7 satisfy the assumption of this section they are isomorphic to a triple system associated with a composition triple:

Lemma 4.8. a) *The FKM-triple corresponding to the indefinite* $(m, m-1)$-*family is isomorphic to the triple system associated with the composition triple* $(a, b, c)=a(bc)$ *for* $\mathcal{A}=\boldsymbol{H}$, *resp.* $\boldsymbol{O}$.

b) *The FKM-triple corresponding to the definite* (4, 3)- *resp.,* (8, 7)-, *family is isomorphic to the triple system associated with the composition triple* $(a, b, c)=(ac)b$ *for* $\mathcal{A}=\boldsymbol{H}$, *resp.* $\boldsymbol{O}$.

Proof. a) The results of [5] § 6 imply that the decomposition of V constructed in the proof of Theorem 4.2 can be realized as $V_1=\{a\oplus0\oplus0\oplus0;\, a\in\mathcal{A}\}$, $V_2=\{0\oplus0\oplus0\oplus b;\, b\in\mathcal{A}\}$ and $V_{12}=\{0\oplus c\oplus0\oplus0;\, c\in\mathcal{A}\}\oplus\{0\oplus0\oplus d\oplus0;\, d\in\mathcal{A}\}$. By definition of the triple product of an FKM-triple we get $\{x\oplus0\oplus0\oplus0,$

$0\oplus 0\oplus 0\oplus y,\ 0\oplus 0\oplus z\oplus 0\} = 0\oplus w\oplus 0\oplus 0$ with $w = (\sum_{r=1}^{m} \langle x_r, y\bar{z}\rangle \bar{x}_r)x = (z\bar{y})x = \overline{\bar{x}(y\bar{z})}$. Hence the corresponding composition triple is isotopic to $(a, b, c)=a(bc)$. The case b) follows similarly.

Finally, we want to recall the following special examples of homogeneous isoparametric triple systems, defined in [3] § 1.5: $V=\mathrm{Mat}(2, 2; \mathcal{A})$ for $\mathcal{A}=\boldsymbol{C}, \boldsymbol{H}$ with $\langle x, y\rangle = \frac{1}{2}\,\mathrm{trace}(x\bar{y}^t + \bar{x}^t y)$ and $\{xxx\} = 6x\bar{x}^t x$. The corresponding constants (m_1, m_2) are (2,1) for $\mathcal{A}=\boldsymbol{C}$ and (4, 3) for $\mathcal{A}=\boldsymbol{H}$. We can now state the main theorem of this section:

Theorem 4.9. *Let V be an isoparametric triple system. Then V is of algebra type with $m_1=m_2+1$ if and only if V is isomorphic to exactly one of the triple systems associated with the following composition triples:*

a) $\mathcal{A}=\boldsymbol{C}$, $(a, b, c)=abc$, $(m_1, m_2)=(2, 1)$.
In this case V is isomorphic to Mat (2, 2; $\boldsymbol{C}$) *which is a realization of theFKM-triple* (2, 1) *and V' is isomorphic to the FKM-triple* (1, 2).

b) $\mathcal{A}=\boldsymbol{H}$, $(a, b, c)=abc$, $(m_1, m_2)=(4, 3)$.
In this case V is isomorphic to the FKM-triple corresponding to the indefinite (4, 3)*-family. Further, V' is isomorphic to the FKM-triple* (3, 4).

c) $\mathcal{A}=\boldsymbol{H}$, $(a, b, c)=acb$, $(m_1, m_2)=(4, 3)$.
In this case V is isomorphic to the FKM-triple corresponding to the definite (4, 3)*-family. Moreover, V is also isomorphic to* Mat (2, 2; $\boldsymbol{H}$).

d) $\mathcal{A}=\boldsymbol{O}$, $(a, b, c)=(ab)c$, $(m_1, m_2)=(8, 7)$.
In this case V' is isomorphic to the FKM-triple (7, 8).

e) $\mathcal{A}=\boldsymbol{O}$, $(a, b, c)=a(bc)$, $(m_1, m_2)=(8, 7)$.
In this case V is isomorphic to the FKM-triple corresponding to the indefinite (8,7)*-family.*

f) $\mathcal{A}=\boldsymbol{O}$, $(a, b, c)=(ac)b$, $(m_1, m_2)=(8, 7)$.
In this case V is isomorphic to the FKM-triple corresponding to the definite (8, 7)*-family.*

Proof. If V is of algebra type with $m_1=m_2+1$, then we already know that V is isomorphic to a triple system associated with one of the composition triples in a)–f). Hence it remains to show that these triples are pairwise nonisomorphic and to prove the various realizations.

From Lemma 4.6.a. we derive the statement about V' in the cases a), b) and d) since there is only one FKM-triple of type (1, 2), (3, 4) and (7, 8). Theorem 4.7 and Lemma 4.8 imply that, in the cases b), c), e) and f), V is an FKM-triple as stated. They also show that b) and c) and e) and f) are pairwise nonisomorphic. Further, e) and f) are not isomorphic to d), because of Lemma 4.6.b) and c).

Finally, Mat(2, 2; $\mathcal{A}$), $\mathcal{A}=\boldsymbol{C}, \boldsymbol{H}$, is of algebra type and hence isomorphic to the case a) for $\mathcal{A}=\boldsymbol{C}$. In the case $\mathcal{A}=\boldsymbol{H}$ it is easy to compute the corresponding composition triple; we thus get c).

REMARKS. 1) It has been shown in [5] §6 that the FKM-triple corresponding to the indefinite (4, 3)-family is equivalent to the FKM-triple (3, 4) but is not isomorphic to the FKM-triple corresponding to the definite (4, 3)-family. Moreover, it has been proved that the FKM-triple (7, 8) and the two (8, 7)-families are pairwise inequivalent.

2) As a corollary of Theorem 4.9 we get that a triple system associated with a composition triple on $\mathcal{A}$ with $(a, b, c)=(ac)b$ or $(a, b, c)=a(bc)$ is of FKM-type. This can also be shown directly, as indicated by the following. We use our standard representation of $V=\mathcal{A}e_1\oplus\mathcal{A}e_{12}\oplus\mathcal{A}\bar{e}_{12}\oplus\mathcal{A}e_2$ as introduced above. Let $(x_1, \cdots, x_m)$ be an orthonormal basis of $\mathcal{A}$ with $x_1=1$. In case $T(ae_1, be_2)ce_{12}=(ac)b\bar{e}_{12}$ we define

$$\begin{aligned}
h_{j0} &= \lambda(x_je_{12}\oplus -x_j\bar{e}_{12}) = -h_{0j}, && 1\leq j\leq m,\\
h_{jk} &= \lambda(x_j\bar{x}_ke_1\oplus\bar{x}_jx_ke_2), && 1\leq j,\ k\leq m,\\
h_{jj} &= e=\lambda(e_1\oplus e_2), && 0\leq j\leq m,
\end{aligned}$$

and in case $T(ae_1, be_2)ce_{12}=a(bc)\bar{e}_{12}$ we put

$$\begin{aligned}
h_{j0} &= \lambda(x_je_{12}\oplus -x_j\bar{e}_{12}) = -h_{0j}, && 1\leq j\leq m,\\
h_{jk} &= \lambda(x_k\bar{x}_je_1\oplus x_j\bar{x}_ke_2), && 1\leq j,\ k\leq m,\\
h_{jj} &= e=\lambda(e_1\oplus e_2), && 0\leq j\leq m.
\end{aligned}$$

Then (h_{jk}) is an FKM-family (see [5] §4) relative to $y_0=e=\lambda(e_1\oplus(-e_2))$, $y_j=\lambda(x_je_{12}\oplus x_j\bar{e}_{12})$, $1\leq j\leq m$. This is seen by a straightforward but lengthy computation using standard facts about composition algebras and the following description of the Peirce spaces of y_j (see [3] §5, [4]):

$$V_3(\hat{e}) = \boldsymbol{R}e\oplus V_{12}^{-},\quad V_1(\hat{e}) = V_{11}^{-}\oplus V_{12}^{+}\oplus V_{22}^{-},$$

and in the first case

$$\begin{aligned}
V_3(y_j) &= \boldsymbol{R}h_{0j}\oplus\{ae_1\oplus\bar{x}_j\bar{a}x_je_2;\ a\in\mathcal{A}\}\\
V_1(y_j) &= (V_{12}\ominus(\boldsymbol{R}y_j\oplus\boldsymbol{R}h_{0j}))\oplus\{ae_1\oplus(-x_j\bar{a}x_j)e_2;\ a\in\mathcal{A}\}
\end{aligned}$$

and in the second case

$$\begin{aligned}
V_3(y_j) &= \boldsymbol{R}h_{0j}\oplus\{ae_1\oplus\bar{a}e_2;\ a\in\mathcal{A}\}\\
V_1(y_j) &= (V_{12}\ominus(\boldsymbol{R}y_j\oplus\boldsymbol{R}h_{0j}))\oplus\{ae_1\oplus -\bar{a}e_2;\ a\in\mathcal{A}\}.
\end{aligned}$$

5. The case $m_1=m_2$

5.1. We first prove some elementary results and reduce the classification problem to a problem for the real division algebras $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ or $\boldsymbol{O}$.

Lemma 5.1. *Let $(V, \{\cdots\})$ be an isoparametric triple which is of algebra type relative to e_1, e_2. Then the following are equivalent*

1) $m_1 = m_2$,
2) *each element of $V_{12}(e_1, e_2)$ is a scalar multiple of maximal tripotent,*
3) $\{x_{12}x_{12}x_{12}\} = 3\langle x_{12}, x_{12}\rangle x_{12}$ *for all* $x_{12} \in V_{12}$.

Proof. (1)$\Leftrightarrow$(3): Choose $e^{\varepsilon}_{12} \in V^{\varepsilon}_{12}$ with $\langle e^{\varepsilon}_{12}, e^{\varepsilon}_{12}\rangle = 1$. From [4] we get $m_1 + 1 = \dim V_3(e^{\varepsilon}_{12}) = \dim V_{11} + \dim V_3(e^{\varepsilon}_{12}) \cap V^{-\varepsilon}_{12}$. But $\dim V_{11} = m_2 + 1$ whence $V_3(e^{\varepsilon}_{12}) \cap V^{-}_{12} = 0$ if and only if $m_1 = m_2$. Since we always have $V^{\varepsilon}_{12} \ominus \boldsymbol{R}e_{12} \subset V_1(e^{\varepsilon}_{12})$ we see that $m_1 = m_2$ is equivalent to $\{x^{\varepsilon}_{12}x^{\varepsilon}_{12}y_{12}\} = \langle x^{\varepsilon}_{12}, y_{12}\rangle x^{\varepsilon}_{12} + 2\langle x^{\varepsilon}_{12}, x^{\varepsilon}_{12}\rangle y_{12}$ for every $x^{\varepsilon}_{12} \in V^{\varepsilon}_{12}$. This is just another formulation of (3). Obviously (2) and (3) are equivalent.

As in [3] we put $\{xe_1y\} = x \circ y$ and $\{xe_2y\} = x * y$.

Corollary 5.2. *For $x^{\varepsilon}_{12} \in V^{\varepsilon}_{12}$ and $m_1 = m_2$ we have*

$$x^{\varepsilon}_{12} \circ (x^{\varepsilon}_{12} \circ x^{-\varepsilon}_{12}) = \langle x^{\varepsilon}_{12}, x^{\varepsilon}_{12}\rangle x^{\varepsilon}_{12} = x^{\varepsilon}_{12} * (x^{\varepsilon}_{12} * x^{-\varepsilon}_{12}) .$$

Proof. From Lemma 5.1 it follows that $\{x^{\varepsilon}_{12}x^{\varepsilon}_{12}x^{-\varepsilon}_{12}\} = \langle x^{\varepsilon}_{12}, x^{\varepsilon}_{12}\rangle x^{-\varepsilon}_{12}$. On the other hand we get $\{x^{\varepsilon}_{12}x^{\varepsilon}_{12}x^{-\varepsilon}_{12}\} = 3\langle x^{\varepsilon}_{12}, x^{\varepsilon}_{12}\rangle x^{-\varepsilon}_{12} - (x^{\varepsilon}_{12} \circ (x^{\varepsilon}_{12} \circ x^{-\varepsilon}_{12}) + x^{\varepsilon}_{12} * (x^{\varepsilon}_{12} * x^{-\varepsilon}_{12})$ from [4] (2.26)′, (2.27)′. Since $x^{\varepsilon}_{12} * (x^{\varepsilon}_{12} * x^{-\varepsilon}_{12}) = x^{\varepsilon}_{12} \circ (x^{\varepsilon}_{12} \circ x^{-\varepsilon}_{12})$ by [3] (5.20), the corollary easily follows.

Theorem 5.3. *Let $(V, \{\cdots\})$ be an isoparametric triple with $m_1 = m_2$ which is of algebra type relative to e_1, e_2. Then there exist a real composition division algebra $\mathcal{A}$ with unit and isometries $\phi: V^{-}_{11} \to \mathcal{A}$, $\phi_{\varepsilon}: V^{\varepsilon}_{12} \to \mathcal{A}$ such that*

$$T(x^{-}_{11}, e_2)(u^{+}_{12} \oplus u^{-}_{12}) = \phi_{+}^{-1}(\phi(x^{-}_{11})\phi_{-}(u^{-}_{12})) \oplus \phi_{-}^{-1}(\overline{\phi(x^{-}_{11})}\phi_{+}(u^{+}_{12}))$$

where "—" denotes the canonical involution of $\mathcal{A}$.

Proof. We choose $a^{+}_{12} \in V^{+}_{12}$, $q^{-}_{11} \in V^{-}_{11}$, $|a^{+}_{12}| = |q^{-}_{11}| = 1$ arbitrarily and fix it in the sequel. We put $a^{-}_{12} := q^{-}_{11} * a^{+}_{12}$ and define

(1) $\quad y^{-}_{11} \perp z^{-}_{11} := (y^{-}_{11} * a^{+}_{12}) * (z^{-}_{11} * a^{-}_{12})$.

Since $V^{-}_{11} * V^{\varepsilon}_{12} \subset V^{-\varepsilon}_{12}$ and $V^{+}_{12} * V^{-}_{12} \subset V^{-}_{11}$, (1) defines an algebra on V^{-}_{11}. We will show that this algebra is a real composition division algebra with unit q^{-}_{11}. We first prove that q^{-}_{11} is the unit of the algebra defined by $\perp$: we first note that $\langle a^{-}_{12}, a^{-}_{12}\rangle = \langle q^{-}_{11}, q^{-}_{11}\rangle\langle a^{+}_{12}, a^{+}_{12}\rangle = 1$ because of (2.10). We apply this and [4] (2.35) and get $q^{-}_{11} \perp z^{-}_{11} = a^{-}_{12} * (a^{-}_{12} * z^{-}_{11}) = z^{-}_{11}$. Similarly one has $y^{-}_{11} * q^{-}_{11} = (y^{-}_{11} * a^{+}_{12}) * a^{+}_{12} = y^{-}_{11}$. Next we show that $\perp$ admits composition:

$$\begin{aligned} &\langle y^{-}_{11} \perp z^{-}_{11}, y^{-}_{11} \perp z^{-}_{11}\rangle \\ &= \langle (y^{-}_{11} * a^{+}_{12}) * (z^{-}_{11} * a^{-}_{12}), (y^{-}_{11} * a^{+}_{12}) * (z^{-}_{11} * a^{-}_{12})\rangle \end{aligned}$$

$$= \langle (y_{11}^- * a_{12}^+) * ((y_{11}^- * a_{12}^+) * (z_{11}^- * a_{12}^+)),\ z_{11}^- * a_{12}^+ \rangle$$
$$= \langle y_{11}^- * a_{12}^+,\ y_{11}^- * a_{12}^+ \rangle \langle z_{11}^- * a_{12}^+,\ z_{11}^- * a_{12}^+ \rangle$$
$$= \langle y_{11}^-, y_{11}^- \rangle \langle z_{11}^-, z_{11}^- \rangle \quad \text{where we have used Corollary 5.2 and (2.10).}$$

This proves that $\perp$ defines on V_{11}^- the structure of a real composition division algebra with unit q_{11}^-. The opposite of this algebra will be denoted by $\mathcal{A}$, its unit by 1 and its product by $xy = L(x)y$. Then $\overline{xy} = \bar{x} \perp \bar{y}$ where $\bar{x} := 2\langle x, 1 \rangle \cdot 1 - x =: Ix$, i.e., I is an isomorphism of the two algebras. Note that $I^2 = id$.

We define now $\phi = id$, $\phi_+(u^+) := u^+ * a^-$, $\phi_-(v^-) := I(v^- * a^+)$, denoting $a^\pm = a_{12}^\pm$, and have to prove

(2) $\phi_-(x * u^+) = \bar{x}\phi_+(u^+)$ for $x \in V_{11}^-$, $u^+ \in V_{12}^+$

(3) $\phi_+(x * v^-) = x\phi_-(v^-)$ for $x \in V_{11}^-$, $v^- \in V_{12}^-$.

Obviously, $I\phi_-(x * u^+) = (x * u^+) * a^+$ and $I(\bar{x}\phi_+(u^+)) = x \perp I(u^+ * a^-) = (x * a^+) * [(2\langle u^+, a^+ \rangle q_{11}^- - u^+ * a^-) * a^-] = 2\langle u^+, a^+ \rangle x - (x * a^+) * u^+$ (because of $q_{11}^- * a^- = a^+$ and Corollary 5.2) $= 2\langle u^+, a^+ \rangle x - 2\langle a^+, u^+ \rangle x + (x * u^+) * a^+$ by linearization of Corollary 5.2. This proves (2).

Finally,

$$I\phi_+(x * v^-) = I((x * v^-) * a^-) \text{ and } I(x\phi_-(v^-))$$
$$= \bar{x} \perp (v^- * a^+) = (\bar{x} * a^+) * ((v^- * a^+) * a^-)$$
$$= (\bar{x} * a^+) * (2\langle a^-, v^- \rangle a^+ - q_{11}^- * v^-) = 2\langle a^-, v^- \rangle \bar{x} - (\bar{x} * a^+) * (q_{11}^- * v^-)$$
$$= 2\langle a^-, v^- \rangle \bar{x} - 2\langle a^+, q_{11}^- * v^- \rangle \bar{x} + (\bar{x} * (q_{11}^- * v^-)) * a^+$$
$$= (\bar{x} * (q_{11}^- * v^-)) * a^+ = \langle 2x, q_{11}^- \rangle v^- * a^+ - (x * (q_{11}^- * v^-)) * a^+$$
$$= 2\langle x, q_{11}^- \rangle v^- * a^+ - 2\langle x, q_{11}^- \rangle v^- * a^+ + (q_{11}^- * (x * v^-)) * a^+$$
$$= (q_{11}^- * (x * v^-)) * a^+ = 2\langle x * v^-, a^+ \rangle q_{11}^- - (x * v^-) * a^- = I((x * v^-) * a^-).$$

This proves (3) and hence the lemma.

Corollary 5.4. *We have* $m := m_1 = m_2 = 1, 2, 4,$ *or* 8 *and the Clifford-algebra for* $(V_{11}^-, \langle \cdot, \cdot \rangle)$ *operates irreducibly on* V_{12} *for* $m = 4, 8$.

Proof. Every composition algebra has dimension 1, 2, 4 or 8. The second assertion follows from the theory of representations of a Clifford algebra (see e.g., [1] or [5]).

REMARK 5.5. In what follows we always identify V_{11}^-, V_{12}^+, V_{12}^- and V_{22}^- with the same real composition division algebra $\mathcal{A}$ (i.e., $\mathcal{A} = \boldsymbol{R}$, $\boldsymbol{C}$, the quaternions $\boldsymbol{H}$ or the octonians $\boldsymbol{O}$) such that $T(x, e_2)(u^+ \oplus v^-) = xv^- \oplus \bar{x}u^+$ for all $x \in V_{11}^- = \mathcal{A}$, $u^+ \in V_{12}^+ = \mathcal{A}$ and $v^- \in V_{12}^- = \mathcal{A}$. In this realization we always have $T(x, e_2)^2 | V_{12} = \langle x, x \rangle Id$. Moreover, we know that $T(e_1, y)$, $y \in V_{22}^- = \mathcal{A}$, interchanges V_{12}^+ and V_{12}^- and from [4] (2.16.a) we get $(T(x, y)V_{12}^\varepsilon)_{12\varepsilon} = -\frac{\varepsilon}{2}[T(x, e_2)T(e_1, y)V_{12}^\varepsilon + T(e_1, y)T(x, e_2)v_{12}^\varepsilon]$, $\varepsilon = \pm$, $x \in V_{11}^-$, $y \in V_{22}^-$. We thus may write (for $a\in V_{12}^{+}$, $b\in V_{12}^{-}$): $T(e_1, y)(a\oplus b)=f(y)b\oplus f(y)^*a$ and $T(x,y)(a\oplus b)=[R(x,y)b-\frac{1}{2}(L(x)f(y)^*a+f(y)L(\bar{x})a)]\oplus[R(x,y)^*a+\frac{1}{2}(L(\bar{x})f(y)b+f(y)^*L(x)b]$ with endomorphisms $f(y)$, $R(x,y)$ of the real vector space $\mathcal{A}$. Obviously, $f(y)$ is linear in y and $R(x,y)$ is linear in x and in y.

We next express the property $T(x_1, x_2)^2=\langle x_1, x_1\rangle\langle x_2, x_2\rangle Id$ in terms of $L(x)$, $f(y)$ and $R(x,y)$.

Lemma 5.6. *The property* $T(x_1, x_2)^2=\langle x_1, x_1\rangle\langle x_2, x_2\rangle Id$ *is equivalent to*
(1) $f(y)f(y)^*=\langle y, y\rangle Id$,
(2) $0=L(x)R(x,y)^*+R(x,y)L(\bar{x})$
(3) $0=f(y)R(x,y)^*+R(x,y)f(y)^*$
(4) $R(x,y)^*R(x,y)=-\frac{1}{4}(f(y)^*L(x)-L(\bar{x})f(y))^2$
for all $x, y\in\mathcal{A}$.

Proof. Put $x_1=\alpha e_1+x$, $x_2=\beta e_2+y$, $x\in V_{11}^{-}\cong\mathcal{A}$, $y\in V_{22}^{-}\cong\mathcal{A}$, then $T(x_1, x_2)^2=|x_1|^2|x_2|^2Id$ is equivalent to

$$\begin{aligned}&[\alpha\beta T(e_1, e_2)+\alpha T(e_1, y)+\beta T(e_2, x)+T(x,y)]^2\\&=\alpha^2\beta^2Id+\alpha^2T(e_1, y)^2+\beta^2|x|^2Id+T(x,y)^2+\alpha^2\beta[T(e_1, e_2)T(e_1, y)\\&\quad+T(e_1, y)T(e_1, e_2)]+\alpha\beta^2[T(e_1, e_2)T(e_2, x)+T(e_2, x)T(e_1, e_2)]\\&\quad+\alpha\beta[T(e_1, e_2)T(x, y)+T(x,y)T(e_1, e_2)]\\&\quad+\alpha\beta[T(e_1, y)T(e_2, x)+T(e_2, x)T(e_1, y)]+\alpha[T(e_1, y)T(x,y)\\&\quad+T(x,y)T(e_1, y)]+\beta[T(e_2, x)T(x,y)+T(x,y)T(e_2, x)]\\&=(\alpha^2+|x|^2)(\beta^2+|y|^2)Id\,.\end{aligned}$$

This gives the following list of identities

(L.1) $T(e_1, y)^2=|y|^2Id$
(L.2) $T(x,y)^2=|x|^2|y|^2Id$,
(L.3) $T(e_1, e_2)T(e_1, y)+T(e_1, y)T(e_1, e_2)=0$,
(L.4) $T(e_1, e_2)T(e_2, x)+T(e_2, x)T(e_1, e_2)=0$,
(L.5) $T(e_1, e_2)T(x,y)+T(x,y)T(e_1, e_2)+T(e_1, y)T(e_2, x)+T(e_2, x)T(e_1, y)=0$
(L.6) $T(e_1, y)T(x,y)+T(x,y)T(e_1, y)=0$,
(L.7) $T(e_2, x)T(x,y)+T(x,y)T(e_2, x)=0$.

It is easy to see that (L.1) is equivalent to (1) and that (L.3), (L.4) and (L.5) are trivially satisfied. Further, a simple computation shows that (L.6) is equivalent to (3) and (L.7) is equivalent to (2). Thus only (L.2) remains to be translated. But $|x|^2|y|^2(a\oplus b)$

$$=T(x,y)^2(a\oplus b)=\{R(x,y)[R(x,y)^*a+\tfrac{1}{2}(L(\bar{x})f(y)b+f(y)^*L(x)b]$$

$$-\tfrac{1}{2}(L(x)f(y)^*+f(y)L(\bar{x}))[R(x, y)b-\tfrac{1}{2}(L(x)f(y)^*a+f(y)L(\bar{x})a)]\}$$
$$\oplus\{R(x, y)^*[R(x, y)b-(L(x)f(y)^*a+f(y)L(\bar{x})a)]$$
$$+\tfrac{1}{2}(L(\bar{x})f(y)+f(y)^*L(x))[R(x, y)^*a+\tfrac{1}{2}(L(\bar{x})f(y)b+f(y)^*L(x)b)]\}$$

whence

(i) $R(x, y)R(x, y)^*+\tfrac{1}{4}(L(x)f(y)^*+f(y)L(\bar{x}))^2=|x|^2|y|^2 Id$

(ii) $R(x, y)^*R(x, y)+\tfrac{1}{4}(L(\bar{x})f(y)+f(y)^*L(x))^2=|x|^2|y|^2 Id$

and two more equations which are consequences of (2) and (3). It is easy to verify that (ii) is equivalent to (4). Finally, (i) follows from (1), $\cdots$, (4).

REMARK. (i) is equivalent to $R(x, y)R(x, y)^*=-\tfrac{1}{4}(L(x)f(y)^*-f(y)L(\bar{x}))^2$.

We derive some immediate corollaries of Lemma 5.6.

Corollary 5.7. $\ker R(x, y)=\{a\in\mathcal{A};\ f(y)^*L(x)a=L(\bar{x})f(y)a\}$, *in particular*

$$R([f(y)a]\bar{a}, y)a=0.$$

Proof. This is an immediate consequence of (4) of Lemma 5.6.

Corollary 5.8. $[L(\bar{x})R(x, y)][L(\bar{x})f(y)]=[L(\bar{x})f(y)]^*[L(\bar{x})R(x, y)]$ *for all* $x, y\in\mathcal{A}$.

Proof. Follows immediately from (2) and (3) of Lemma 5.6.

Corollary 5.9. *Without loss of generality we may assume* $f(y)1=y$ *for all* $y\in\mathcal{A}$.

Proof. Let $h(y):=f(y)1$. Then h is an isometry of $\mathcal{A}$. Put $\tilde{T}(x_1, x_2):=T(x_1, h^{-1}(x_2))$. By Theorem 2.10, we can pass to an isomorphic triple which still satisfies the assumptions of Remark 5.5, but in addition has $f(y)1=y$ for all $y\in\mathcal{A}$.

5.2. In this section we prove that the cases $m=4$, 8 do not appear. We start with some general results.

Lemma 5.10. *Let* $\mathcal{A}=\boldsymbol{H}$ (*resp.* $\boldsymbol{O}$) *be the real composition division algebra of quaternions* (*resp. octonions*) *and* $A\in\mathrm{End}_{\boldsymbol{R}}\mathcal{A}$. *If* $L(x)A=AL(\bar{x})$ *for all* $x\in\mathcal{A}$ *then* $A=0$.

Proof. Put $a:=A1$, then $Ax=\bar{x}a$ for all $x\in\mathcal{A}$. In particular, we get $A=0$ if $a=0$. Hence without loss of generality we may assume $\langle a, a\rangle=1$. Then $(\bar{y}x)a=(\overline{\bar{x}y})a=A(\bar{x}y)=x(Ay)=x(\bar{y}a)$ for all $x, y\in\mathcal{A}$. For $y:=a$ we get $(\bar{a}x)a=x$, whence $\bar{a}x=x\bar{a}$. Therefore $\bar{a}=\pm 1$ and $\bar{y}x=x\bar{y}$ for all $x, y\in\mathcal{A}$. By assumption, $\mathcal{A}$ is not commutative so $a=0$ and thus $A=0$

We use the notation introduced in the last section and get

Lemma 5.11. a) dim ker $R(x, y)=0, 4$, *or* 8,

b) ker $R(x, y)$ *is invariant under* $L(\bar{x})f(y)$,

c) ker $R(x, y)$ *is spanned by the eigenvectors of* $f(y)^*L(x)$

d) $L(\bar{x})f(y)$ *has no real eigenvalue on the orthogonal complement of* ker $R(x, y)$.

Proof. Let $A :=L(\bar{x})R(x, y)$, and $B :=\frac{1}{2}(L(\bar{x})f(y)-f(y)^*L(x))$. Then $A^*=-A$, $B^*=-B$; further, $AB=-BA$ by Corollary 5.8 and $A^2=B^2$ by (4) of Lemma 5.6. Thus $\ker A=\ker B=: V_0$ and $V_0^{\perp}$ is invariant under A and B. As A is skew-adjoint on $V_0^{\perp}$ there exists a two dimensional subspace $U\subset V_0^{\perp}$ which is left invariant by A. Let $x\in U$, $x\neq 0$. Then U is generated by x and Ax. But $\langle Bx, x\rangle=0$ and $\langle Bx, Ax\rangle=-\langle ABx, x\rangle=\langle BAx, x\rangle=-\langle Ax, Bx\rangle =-\langle Bx, Ax\rangle$ whence $\langle Bx, U\rangle=0$. Also $\langle BAx, x\rangle=0$ and $\langle BAx, Ax\rangle=0$; therefore BU is orthogonal to U and is two dimensional. Hence $U\oplus BU$ is a four dimensional subspace of $V_0^{\perp}$. Repeating this construction (if possible) we see that the dimension of $V_0^{\perp}$ is a multiple of 4. This proves a). b) is an immediate consequence of Corollary 5.8. Since $[f(y)^*L(x)]^*=L(\bar{x})f(y)$ we conclude from b) and Corollary 5.7 that ker $R(x, y)$ is spanned by eigenvectors of $f(y)^*L(x)$. On the other hand $L(\bar{x})f(y)$ is a multiple of an orthogonal map and is therefore self-adjoint on the sum of all eigenvectors. This implies c) and d).

We next investigate the endomorphisms $Q(y, 1)$ where $Q(y, z)x :=R(x, y)z$.

Lemma 5.12. a) $Q(y, 1)$ *is skew-adjoint,*

b) $y\in\ker Q(y, 1)$

c) $\ker Q(y, 1)=\{a\in\mathcal{A}; f(y)^*a=\bar{a}y\}$.

Proof. a) $Q\langle(y, 1)x, x\rangle=\langle R(x, y)1, x\rangle=\langle L(\bar{x})R(x, y)1, 1\rangle=0$.

b) $Q(y, 1)y=R(y, y)1=0$ by Corollary 5.7.

c) $a\in\ker Q(y, 1)$ is equivalent to $R(a, y)1=0$.

By (4) of Lemma 5.6, this is equivalent to $f(y)^*a-\bar{a}\cdot f(y)1=0$. This proves the assertion

Corollary 5.13. *Assume* dim $\mathcal{A}\geq 2$. *Then*

a) $\ker Q(1, 1)$ *has even dimension* ≥ 2,

b) $1\in\ker Q(1, 1)$ *and* $x_0\in\ker Q(1, 1)$ *for some* $x_0\in\mathcal{A}$, $\bar{x}_0=-x$, $|x_0|=1$. *Moreover,* $f(1)1=1$ *and* $f(1)x_0=-x_0$.

c) $\ker Q(1, 1)\subset\ker R(1, 1)$

d) *If* $f(1)\neq f(1)^*$, *then* dim $\mathcal{A}=8$, dim ker $R(1, 1)=4$ *and the multiplicity of the eigenvalues* 1 *and* -1 *of* $f(1)$ *is odd.*

Proof. a) and b) are immediate consequences of Lemma 5.12. We know

$a \in \ker Q(1, 1)$ iff $f(1)^* a = \bar{a}$. As $1 \in \ker Q(1, 1)$ we see that $\ker Q(1, 1)$ is spanned by orthonormal vectors $a_1, \cdots, a_s$, $a_1 = 1$, $\bar{a}_r = -a_r$ for $r \neq 1$ satisfying $f(1)^* 1 = 1$ and $f(1)^* a_r = -a_r$ for $r \neq 1$. Hence, $\ker Q(1, 1)$ is spanned by eigenvectors of $f(1)^*$, whence $\ker Q(1, 1) \subset \ker R(1, 1)$ by c) of Lemma 5.11. Assume $f(1) \neq f(1)^*$; then $\dim \ker R(1, 1) = 0$ or 4 by Corollary 5.7 and Lemma 5.11. But $R(1, 1)1 = 0$ by Corollary 5.7 and Corollary 5.9. Hence $\dim \ker R(1, 1) = 4$. Since $f(1) \neq f(1)^*$ we get $\dim \mathcal{A} = 8$. We know $\ker Q(1, 1) \subset \ker R(1, 1)$, thus $\dim \ker Q(1, 1) = 2$ or 4. From c) of Lemma 5.12 we conclude that the multiplicity of the eigenvalue -1 is one if $\dim \ker Q(1, 1) = 2$ and three if $\dim \ker Q(1, 1) = 4$. This proves the lemma.
We put $\mathcal{A}^{\varepsilon} := \{x \in \mathcal{A}; f(1)x = \varepsilon x\}$, $\varepsilon = \pm$. Then $\ker R(1, 1) = \mathcal{A}^+ \oplus \mathcal{A}^-$ by Lemma 5.11.

Lemma 5.14. *Assume* $R(1, 1) = 0$ *and* $x \in \mathcal{A}$, $\bar{x} = x$.

a) $\ker R(x, 1) = \{a^+ \in \mathcal{A}^+; xa^+ \in \mathcal{A}^-\} \oplus \{a^- \in \mathcal{A}^-; xa^- \in \mathcal{A}^+\}$,
b) $L(x)(\ker R(x, 1) \cap \mathcal{A}^{\varepsilon}) = \ker R(x, 1) \cap \mathcal{A}^{-\varepsilon}$, *if* $x \neq 0$,
c) $1 \in \ker R(x, 1) \Leftrightarrow x \in \mathcal{A}^-$,
d) $\ker Q(1, 1) = \boldsymbol{R}1 \oplus \mathcal{A}^-$,
e) $R(x, 1)$ *is skew-adjoint and commutes with* $f(1)$.

Proof. We note that $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ and $f(1) = f(1)^*$ because $R(1, 1) = 0$. d) follows from c) of Lemma 5.12. To prove c) we use that $1 \in \ker R(x, 1)$ is equivalent to $0 = R(x, 1)1 = Q(1, 1)x$. This implies $x \in \boldsymbol{R}1 \oplus \mathcal{A}^-$ by d). But $\langle x, 1 \rangle = 0$ and c) follows. To verify a) we use (4) of Lemma 5.6 and get

(1) $a \in \ker R(x, 1)$ iff $f(1)(xa) = -xf(1)a$.

We next linearize (2) of Lemma 5.6 and get

(2) $R(x, 1) = -R(x, 1)^*$.

From (3) of Lemma 5.6 we now derive

(3) $f(1)R(x, 1) = R(x, 1)f(1)$.

This implies e) and $f(1) \ker R(x, 1) = \ker R(x, 1)$; therefore $\ker R(x, 1) = (\ker R(x, 1) \cap \mathcal{A}^+) \oplus (\ker R(x, 1) \cap \mathcal{A}^-)$. Applying (1) gives a). Finally, from (2) and (2) of Lemma 5.6 we derive that $\ker R(x, 1)$ is left invariant by $L(x)$; hence b) follows from a).

We are now able to rule out the cases $m = 4, 8$.

Theorem 5.15. $\mathcal{A}$ *is commutative.*

Proof. Assume $\mathcal{A} = \boldsymbol{H}$ or $\mathcal{A} = \boldsymbol{O}$. We distinguish two cases.
1 Case: $R(1, 1) = 0$. We know that $\ker Q(1, 1)$ has even dimension by Corollary 5.13. By our assumption we may apply Corollary 5.13 and Lemma 5.14.d and thus see that $\mathcal{A}^-$ has odd dimension. As $\mathcal{A}$ has even dimension and $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ we conclude that also $\mathcal{A}^+$ has odd dimension. From e) of Lemma

5.14 we get that $R(x, 1)$ leaves invariant $\mathcal{A}^\varepsilon$ and is skew-adjoint on $\mathcal{A}^\varepsilon$. Hence $\mathcal{A}^\varepsilon \cap \ker R(x, 1) \neq 0$ for $\varepsilon = \pm$ and all $x \in \mathcal{A}$. Choose $x \in \mathcal{A}$, $\langle x, 1\rangle = 0$, $|x| = 1$ and let U^ε denote the orthogonal complement of $\ker R(x, 1) \cap \mathcal{A}^\varepsilon$ in $\mathcal{A}^\varepsilon$. Put $d_\varepsilon := \dim \ker R(x, 1) \cap \mathcal{A}^\varepsilon$ then $d = d_+ = d_-$ and $0 \neq \dim \ker R(x, 1) = 2d$ by Lemma 5.14.b. From Lemma 5.11.a we conclude that d is even. Moreover, by construction the skew-adjoint endomorphism $R(x, 1)$ is bijective on U^ε. Hence $\dim U^\varepsilon$ is even. But $\dim U^\varepsilon + d = \dim \mathcal{A}^\varepsilon$ is odd, a contradiction.

2 Case: $R(1, 1) \neq 0$. Let $N := \{x \in \mathcal{A};\ R(x, 1) = 0\}$. By assumption $1 \notin N$, whence $\dim N \leq \dim \mathcal{A} - 1$. a) $\dim N = \dim \mathcal{A} - 1$. Here we have a linear form $\zeta\colon \mathcal{A} \to \boldsymbol{R}$ such that $R(x, 1) = \zeta(x) R(1, 1)$ for all $x \in \mathcal{A}$. From (2) of Lemma 5.6 we conclude that $R(1, 1)$ and hence $R(x, 1)$ is skew-adjoint. Therefore by applying (2) of Lemma 5.6 once again, we get

$$\begin{aligned}&\zeta(x)[L(\bar{x})R(1, 1) - R(1, 1)L(x)]\\ &\quad = L(\bar{x})\zeta(x)R(1, 1) - \zeta(x)R(1, 1)L(x)\\ &\quad = L(\bar{x})R(x, 1) - R(x, 1)L(x) = 0\,.\end{aligned}$$

But $\zeta \neq 0$, consequently $L(x)R(1, 1) = R(1, 1)L(x)$ for all $x \in \mathcal{A}$. Therefore $R(1, 1) = 0$ by Lemma 5.10, a contradiction. b) $\dim N \leq \dim \mathcal{A} - 2$. In this case we choose a subspace $U \subset \mathcal{A}$ such that $N + U = \mathcal{A}$, $U \cap N = 0$, $1 \in U$. Obviously, $\dim U \geq 2$. We next show $\mathcal{A} = \boldsymbol{O}$ and rank $R(x, 1) = 4$ for all $x \in U$, $x \neq 0$. By assumption $R(1, 1) \neq 0$ and by Corollary 5.7 $R(1, 1)1 = 0$. Lemma 5.11 thus implies rank $R(1, 1) = 4$ and $\dim \mathcal{A} = 8$. Hence the rank of $R(x, 1)$ is 4 or 8 for all x of an open and dense subset of $\mathcal{A}$. It thus suffices to prove $\det R(x, 1) \equiv 0$ for $x \in \mathcal{A}$. To verify this we first note that $f(1) \neq f(1)^*$ by Corollary 5.7 and the assumption. Therefore the multiplicity of the eigenvalue 1 or of the eigenvalue -1 of $f(1)$ is one by Corollary 5.13.d. Assume first that 1 has multiplicity one. We consider the map $h\colon \mathcal{A} \to \mathcal{A}$, $h(a) := f(1)a \cdot \bar{a}$ and compute its defferential $d_a h(u) = f(1)u \cdot \bar{a} + f(1)a \cdot \bar{u}$. Now let $a = 1$ and assume $d_a h(u) = 0$ for $u = u_0 1 + u'$, $\langle u', 1\rangle = 0$. Then $f(1)u = -\bar{u}$ by Corollary 5.9. But $-\bar{u} = -u_0 1 + u'$ and $f(1)u = u_0 1 + f(1)u'$, thus $u_0 1 + f(1)u' = -u_0 1 + u'$, whence $u = u'$ and $f(1)u' = u'$. This implies $u' = 0$. Therefore h is locally invertible near 1 and Corollary 5.7 implies $\det R(x, 1) = 0$ for an open neighborhood of 1. But then $\det R(x, 1) \equiv 0$. Assume now that -1 has multiplicity one. Without restriction we may assume $f(1)i = -i$. We consider again the map $h(a) = f(1)a \cdot \bar{a}$ and compute the kernel of its defferintial at the point $a = i$. We get $d_i h(u) = 0$ iff $(f(1)u)\bar{i} = i\bar{u}$ which is equivalent to $f(1)u = (i\bar{u})i$. Let $u = u_0 1 + u_1 i + u'$ with $\langle 1, u'\rangle = 0 = \langle i, u'\rangle$. Then $\bar{u} = u_0 1 - u_1 i - u'$ and $f(1)u = u_0 1 - u_1 i + f(1)u'$. Moreover $(i\bar{u})i = (u_0 i + u_1 1 + u' i)i = -u_0 1 + u_1 i - u'$ and $u_0 1 - u_1 i + f(1)u' = -u_0 1 + u_1 i - u'$ follows. Hence $u = u'$ and $f(1)u' = -u'$. This implies $u' = 0$. Hence h is locally invertible near i. By Corollary 6.7 we get $\det R(x, 1) = 0$ in an open neighborhood of i. But then also $\det R(x, 1) \equiv 0$ in $\mathcal{A}$. We thus have proved that rank $R(x, 1)\leq 4$ for all $x\in\mathcal{A}$ and consequently rank $R(x, 1)=4$ for all $x\in U$. Now let $x\in U$ with $|x|=1$; then $L(\bar{x})f(1)$ is orthogonal, leaves invariant the four dimensional kernel of $R(x, 1)$, is self-adjoint on $\ker R(x, 1)$ and has no real eigenvalue on the orthogonal complement of $\ker R(x, 1)$ by Corollary 5.7 and Lemma 5.11. We conclude that for $x\in U$, $|x|=1$, the map $L(\bar{x})f(1)$ has only 1 or -1 as real eigenvalues and the sum of the multiplicities of 1 and -1 is 4. The "contituity of eigenvalues" [13] § 14 shows that the multiplicity of 1 and of -1 is a locally constant function on the sphere of U. Hence these multiplicities are the same for all $x\in U$, $|x|=1$, and thus are equal to the corresponding multiplicities of $f(1)=L(\bar{1})f(1)$. Moreover, $L(\bar{x})f(1)$ and $L(-\bar{x})f(1)$ have the same multiplicities. Therefore -1 and 1 have multiplicity 2. This is a contradiction to e) of Corollary 5.13. This proves the theorem.

5.3. In this section we classify all isoparametric triples of algebra type with $m_1=m_2=m$. As shown above we may realize such a triple as described in Remark 5.5. Theorem 5.15 then implies that we only have to consider the cases $\mathcal{A}=\boldsymbol{R}$ or $\boldsymbol{C}$.

Lemma 5.16. *Let $\mathcal{A}=\boldsymbol{R}$ or $\boldsymbol{C}$. Then $R(x, y)=0$ and $f(y)a=y\bar{a}$ for all $a, x, y\in\mathcal{A}$.*

Proof. $R(x, y)=0$: By Corollary 5.7 we have $\ker R(y, y)\neq 0$ for all $y\in\mathcal{A}$. Thus Lemma 5.11 shows $R(y, y)=0$ and hence only $\mathcal{A}=\boldsymbol{C}$ remains to be considered. Let $y=\alpha 1+\beta i$, then $0=R(y, y)=\alpha\beta(R(1, i)+R(i, 1))$. Therefore $R(i, 1)=-R(1, i)$ is skew-adjoint by (2) of Lemma 5.6. Moreover, $Q(1, 1)=0$ by Lemma 5.13, thus $R(i, 1)1=0$ and consequently $R(i, 1)=0$.

$f(y)a=\bar{a}y$: Since $R(x, y)=0$ we have $Q(y, 1)=0$. Therefore Lemma 5.12 implies $f(y)^*a=\bar{a}y$. The assertion follows easily.

The last lemma implies that in the situation we are considering there exist —up to isomorphism— at most one triple for $m=1$ and at most one triple for $m=2$. For each case we give an example and thereby prove Theorem 5.17 below.

Let $V := \mathrm{Mat}(2, 3; \boldsymbol{R})$, $\langle A, B\rangle := \mathrm{trace}\, AB^t$ and $\{AAA\} := 6AA^tA$. Then $(V, \{\cdots\})$ is an isoparametric triple, of FKM-type with $m_1=m_2=1$ (see [3] 1.5). A Peirce decomposition with respect to the pair of orthogonal tripotents e_1, e_2, defined by $e_1=\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}$, $e_2=\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}$, shows $V_{12}^{\varepsilon}=\left\{\begin{pmatrix}0&\alpha&0\\\varepsilon\alpha&0&0\end{pmatrix}, \alpha\in\boldsymbol{R}\right\}$ and $\{V_{12}^{-}, e_1, V_{12}^{+}\}=0$. Using Lemma 1.1 it is easy to check that the dual triple is of algebra type. Thus Mat $(2, 3; \boldsymbol{R})'$ is "the" isoparametric triple of algebra type with $m_1=m_2=1$.

Finally, let $U=u(2; \boldsymbol{H}) := \{A\in\mathrm{Mat}(2, 2; \boldsymbol{H});\ \bar{A}^t=-A\}$, $\langle A, B\rangle = \frac{1}{2}\,Re\,\mathrm{trace}(A\bar{B}^t+\bar{A}^tB)$ and $\{AAA\}=6A\bar{A}^tA$. As a subtriple of Mat$(2, 2; \boldsymbol{H})$ (see [3] **1.5**) U fulfills (1.1) to (1.3) and it is easy to check that (1.4) is also satisfied with $m_1=m_2=2$. A Peirce decomposition with respect to the piar of orthogonal tripotents e_1, e_2, defined by $e_1=\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$, $e_2=\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$, shows that the triple is of algebra type relative to e_1, e_2. Thus $u(2, \boldsymbol{H})$ is "the" isoparametric triple of algebra type with $m_1=m_2=2$.

Theorem 5.17. *Let* $(V, \{\cdots\})$ *be an isoparametric triple with* $m_1=m_2$ *which is of algebra type relative to* e_1, e_2. *Then* $(V, \{\cdots\})$ *is isomorphic to* $\mathrm{Mat}(2, 3; \boldsymbol{R})'$ *—if* $m_1=m_2=1$*—and isomorphic to* $u(2, \boldsymbol{H})$*—if* $m_1=m_2=2$. *In both cases the triple is homogeneous.*

Summing up our results on isoparametric triples of algebra type we have

Theorem 5.18. *Let* $(V, \{\cdots\}$ *be an isoparametric triple. Then* V *is of algebra type iff*

1) V' *is of FKM-type and* $m_2=1$, 3 *or* 7, *in case* $m_1>m_2+1$.

or 2a) V *is isomorphic to an FKM-triple of type* (2, 1), (4, 3) *or* (8, 7).

or 2b) V' *is isomorphic to an FKM-triple of type* (1, 2), (3, 4) *or* (7, 8),

or 3) V *is isomorphic to* $u(2, \boldsymbol{H})$ *or to* $\mathrm{Mat}(2, 3; \boldsymbol{R})'$.

Corollary 5.19. *Let* $(V, \{\cdots\})$ *be an isoparametric triple of algebra type. Then* $(V, \{\cdots\})$ *is either equivalent to an FKM-triple or it is isomorphic to* $u(2, \boldsymbol{H})$.

Finally, we compare our results with the work of H. Ozeki and M. Takeuchi. In our notation they proved the following. (For a definition of condition (A) and (B) see [9].)

Theorem 5.20 ([9] Theorem 2). *Let* V *be an isoparametric triple. Then the following are equivalent*:

a) V' *satisfies condition* (A) *and* (B),

b) V *is the dual of an FKM-triple and* $m_2(V)=1$, 3 *or* 7.

A comparison with our results shows

Theorem 5.21. *Let* V *be an isoparametric triple of algebra type* (*i.e.* V' *satisfies condition* (A).) *Then* V' *fails to satisfy condition* (B) *iff* V *is isomorphic to exactly one of the following* 4 *triples*: Mat (2, 2; $\boldsymbol{H}$), $u(2, \boldsymbol{H})$ *and the two FKM-triples* (8, 7).

6. Isoparametric triples of generic algebra type

We finally consider triples of *generic algebra type*, i.e., isoparametric triples which are of algebra type relative to each pair of orthogoanl tripotents.

Lemma 6.1. *Let* V *be an isoparametric triple of generic algebra type and* e_1, e_2 *any pair of orthogonal tripotents. If* $m_1>m_2+1$, *then the triple* $V=(V_{12}(e_1, e_2), \{\cdots\})$ *is also of generic algebra type with* $m_1(\tilde{V})=m_1-(m_2+1)$ *and* $m_2(\tilde{V})=m_2$.

Proof. By Theorem 2.6 and Corollary 2.7 we know that $\tilde{V}=(V_{12}, \{\cdots\})$ is an isoparametric triple with $\tilde{m}_1=m_1-(m_2+1)$, $\tilde{m}_2=m_2>0$. Hence there exist orthogonal tripotents $c_1, c_2\in\tilde{V}$ by [3] Corollary 4.9. Obviously, c_1, c_2 are also orthogonal tripotents of V. It now suffices to prove $V_2^0(c_j)\subset V_{12}$, $j=1, 2$. By the assumptions we may apply [4] Theorem 4.8 and get $V_0(c_j)\subset V_{12}(e_1, e_2)$. Hence $V_2^0(c_1)\subset V_0(c_2)\subset V_{12}$, and $V_2^0((c_2)\subset V_1(c_0)\subset V_{12}$. The lemma is proved.

We call an isoparametric triple V *homogeneous* if there exists a subgroup $\Gamma\subset\mathrm{Aut}\,V$ which operates transitively on the corresponding hypersurfaces in the sphere of V. We use the notation introduced at the end of section 4.

Lemma 6.2. *Let V be an isoparametric triple of algebra type.*

a) *If V is homogeneous, it is of generic algebra type.*

b) *If $m_1=m_2$, then each triple is of generic algebra triple.*

c) *If $m_1=m_2+1$ then the following triples represent the equivalence classes of triples of generic algebra type*:

$$\mathrm{Mat}(2, 2; \boldsymbol{C}) \quad \textit{and} \quad \mathrm{Mat}(2, 2; \boldsymbol{H})\,.$$

Proof. a) is obvious and it implies b) using Theorem 5.17, since both $u(2, \boldsymbol{H})$ and $\mathrm{Mat}(2, 3; \boldsymbol{R})$ are homogeneous. It also implies c) since the cases b), d), e) and f) of Theorem 4.9 are not of generic algebra type which follows from [5] Theorem 6.19,b, Corollary 6.12, Theorem 6.17 and Theorem 6.15.

Theorem 6.3. *Let $(V, \{\cdots\})$ be an isoparametric triple of algebra type. Then V is of generic algebra type iff V is homogeneous. More precisely, V is isomorphic to exactly one of the following triples*

a) *the dual of an FKM-triple of type* $(1, m_2)$, $m_2>1$,

b) $\mathrm{Mat}(2, 2; \boldsymbol{C})$, *resp.,* $\mathrm{Mat}(2, 2; \boldsymbol{H})$. *These triples are of FKM-type* $(2, 1)$, *resp.* $(4, 3)$.

c) $\mathrm{Mat}(2, 3; \boldsymbol{R})'$, *resp.* $u(2, \boldsymbol{H})$. *Here* $\mathrm{Mat}(2, 3; \boldsymbol{R})'$ *is the dual of the FKM-triple of type* $(1, 1)$. *The triple* $u(2, \boldsymbol{H})$ *and its dual are not of FKM-type.*

Proof. By Lemma 6.1 we may consider a maximal descending chain of isoparametric triples $V_1\supset\cdots\supset V_n$ where V_k is the V_{12}-space of V_{k-1} relative some pair of orthogonal tripotents and where $m_1(V_k)=m_1(V_{k-1})-(m_2+1)$, $m_2(V_k)=m_2$. Applying Lemma 2.8 shows $m_1(V_n)=m_2$ or $m_1(V_n)=m_2+1$. Hence $m_1(V_k)=(n+1-k)(m_2+1)-1$ or $m_1(V_k)=(n+1-k)(m_2+1)$. If $n>1$, i.e., $m_1>m_2+1$, it follows from Theorem 3.2 that V_k, $k<n$, is the dual of an FKM-triple and $m_2=1$, 3 or 7. But the cases $m_2=3$ or 7 are ruled out by Lemma 6.1 and [5] Theorem 7.6 and in the case $m_2=1$ the triple is homogeneous by [12] or [6], § 6. The remaining cases have been settled in Lemma 6.2.

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