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Osaka University
On Cartesian Product of Compact Spaces

By Hidetaka TERASAKA

While the Cartesian product of any number of compact (= bicomplete) spaces is again compact by Tychonoff's theorem [1], there is an $\kappa_0$-compact (= compact in the sense of Fréchet) space $R$ whose product $R \times R$ is not $\kappa_0$-compact, as will be shown in the present note. These circumstances will be somewhat clarified by the introduction of a concept of $\kappa_0$-ultracompactness.

1. Let $M$ be a given set of points and let $M = \{M_\lambda\}$ be an ultrafilter [2], i.e., a collection of subsets $M_\lambda$ of $M$ such that

(i) $M$ has the finite intersection property, i.e., any finite number of $M_\lambda$'s have a non-void intersection,

(ii) $M$ is maximal with respect to the property (i), i.e., should any subset $M'$ of $M$ distinct from any one of $M_\lambda$ be added to $M$, then the resulting collection $M + M'$ fails to satisfy the condition (i).

If $\kappa_a$ denotes the lowest of the potencies of $M_\lambda$, we say that $M$ is of potency $\kappa_a$. A $T_1$-space will be called $\kappa_a$-ultracompact, if every ultrafilter of potency $\kappa_a$ has a cluster point. Then the proof of C. Chevalley and O. Frink [3] for Tychonoff's theorem yields at once the following

Theorem. The Cartesian product of any number of $\kappa_a$-ultracompact spaces is itself $\kappa_a$-ultracompact.

Here arises the question, whether or not, if $R$ is $\kappa_a$-compact, i.e., if every subset $M \subseteq R$ of potency $\kappa_a$ has a cluster point, but if $R$ is not $\kappa_a$-ultracompact, then the product $R \times R$ is not necessarily $\kappa_a$-compact.

As a partly solution of this question we construct in the following an example of an $\kappa_0$-compact but not $\kappa_0$-ultracompact space $R$, whose product $R \times R$ is not $\kappa_0$-compact.

1) The question whether or not such an $\kappa_0$-compact space exists was raised by M. Ohnishi of Osaka University and answered by me in Sizyo Sugaku Danwakai (June 10, 1947): An example of an $\kappa_0$-compact space $R$ whose product $R \times R$ is not $\kappa_0$-compact (In Japanese). After I had written the present note I have been informed by Ohnishi that the question is originally that of Čech, for which an answer is announced to have been given by Novák in Časopis propest. mat. a fys. 74 (1950).
2. Let
\[ X = (x^1, x^2, ..., x^n, ...) \]
be a sequence of \( x^n \) which is either 0 or 1. The family \( X \) of all such \( X \) becomes a Boolean algebra, if we introduce the following assumptions and definitions:

1) \( X \) and \( Y = (y^1, y^2, ..., y^n, ...) \) are to be regarded as equal if and only if
\[ x^n = y^n \]
for almost all \( n \).

2) If \( \max (x^n, y^n) = u^n, \min (x^n, y^n) = v^n, 1 - x^n = w^n \), then
\[ X \cup Y = (u^1, u^2, ..., u^n, ...) \]
\[ X \cap Y = (v^1, v^2, ..., v^n, ...) \]
\[ X^c = (w^1, w^2, ..., w^n, ...) \]

3) \[ 0 = (0, 0, ..., 0, ...) \]
\[ 1 = (1, 1, ..., 1, ...) \]

A filter is by definition a collection of elements \( A \in X \) with the finite intersection property, and an ultrafilter \( A \) is a filter with maximal property. Clearly

**Lemma 1.** If \( A \) is an ultrafilter and if \( X \) is any element of \( X \), then either \( X \) or \( X^c \) (not both) belongs to \( A \). Conversely if for any \( X \) either \( X \) or \( X^c \) belongs to a filter \( A \), then \( A \) must be an ultrafilter.

Now let
\[ E = (\varepsilon^1, \varepsilon^2, ..., \varepsilon^n, ...) \]
and let
\[ A_i = (a_i^1, a_i^2, ..., a_i^n, ...) \quad (i = 1, 2, ...) \]
be a sequence of \( X \). We denote by \( \varepsilon^n A_n \) the element \( A_n \) itself if \( \varepsilon^n = 1 \) and the null element if \( \varepsilon^n = 0 \) and denote further by
\[ \sum \varepsilon^n A_n \]
any one of the elements \( A \) of \( X \) which are \( \subseteq \varepsilon^n A_n \) for all \( n \), i.e. a superior of the elements \( \varepsilon^n A_n \) \( (n = 1, 2, ...) \). Then we have the following useful

**Lemma 2 [4].** If \( A_n = \{ A^1_n \} (n = 1, 2, ...) \) and \( E = \{ E_\lambda = (\varepsilon_1^\lambda, \varepsilon_2^\lambda, ..., \varepsilon_\infty^\lambda, ...) \} \) are ultrafilters, so is \( A = \{ \sum \varepsilon^\lambda A^\lambda_n \} \).

**Proof.**
(i) First we prove that \( A \) is a filter. In fact, if
\[ A_1 = \sum \varepsilon^\lambda A^\lambda_n, A_2 = \sum \varepsilon^\lambda A^\lambda_n, ..., A_m = \sum \varepsilon^\lambda A^\lambda_n \]

are a finite number of elements of \( A \), we have by our definition
\[
A_1 \cap A_2 \cap \ldots \cap A_m \supseteq \varepsilon_{A_1}^n \cdot \varepsilon_{A_2}^n \cdot \ldots \cdot \varepsilon_{A_m}^n \cdot A_{v_1}^n \cdot A_{v_2}^n \cdot \ldots \cdot A_{v_m}^n.
\]
Since \( E \) and \( \Lambda \) have the finite intersection property, \( \varepsilon_{A_1}^n \cdot \varepsilon_{A_2}^n \cdot \ldots \cdot \varepsilon_{A_m}^n = 1 \)
for some \( n \) and \( A_{v_1}^n \cdot A_{v_2}^n \cdot \ldots \cdot A_{v_m}^n = 0 \) for every \( n \), and consequently we have
\[
A_1 \cap A_2 \cap \ldots \cap A_m = 0.
\]

(ii) To prove that \( \Lambda \) is an ultrafilter, let \( B \) be an element of \( X \)
not contained in \( A \). For each \( n \) let \( \eta^n = 1 \) or \( = 0 \) according as \( B \) belongs to or not to \( A_n \). Then \( B \) can be written in the form
\[
B = \sum \eta^n A_{A_n}^n,
\]
where \( A_{A_n}^n = B \) in case \( \eta^n = 1 \). Since by the assumption on \( B \) \( H = (\eta^1, \eta^2, \ldots, \eta^n, \ldots) \) is non \( \in E \), we have \( H' = E = (\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n, \ldots) \in E \), where \( \varepsilon^n = 1 - \eta^n \), and consequently \( B' \) must be of the form \( \sum \varepsilon^n A_{A_n}^n \in A \). Thus we have shown that for every element \( X \) of \( X \) either \( X \) or \( X' \) belongs to the filter \( A \), whence we conclude by Lemma 1 that \( A \) must be an ultrafilter, and our lemma is proved.

Corresponding to
\[
A = (a^1, a^2, \ldots, a^n, \ldots)
\]
let
\[
A' = (a^0, a^1, \ldots, a^{n-1}, \ldots),
\]
where \( a^0 \) stands either for 0 or for 1. Evidently

**Lemma 3.** If \( A = \{A_\lambda\} \) is an ultrafilter, so is \( A' = \{A_\lambda'\} \).

We call \( A' \) the first transposed ultrafilter of \( A \). In general we can speak of the \( n \)-th transposed ultrafilter of \( A \) for any given integer \( n (-\infty < n < +\infty) \), provided that the 0-th transposed ultrafilter is \( A \) itself and the \( n \)-th transposed ultrafilter of \( A \) is the first transposed ultrafilter of the \( (n-1) \)-th transposed ultrafilter of \( A \).

3. We now consider the following Hausdorff space \( R^* \):

(i) First let
\[
q_1, q_2, \ldots, q_n, \ldots
\]
be introduced and defined to be a countable set of isolated points of \( R^* \) distinct from each other.

(ii) To define the remaining points of \( R^* \), first make correspond to every subset \( Q \) of \( q_1, q_2, \ldots \) the element \( A = (a^1, a^2, \ldots, a^n, \ldots) \) of \( X \) in such a way that for each \( n \) \( a^n = 1 \) or \( = 0 \) according as \( q_n \) belongs to or not to \( Q \). Every ultrafilter \( A = \{A_\lambda\} \) of \( X \) is then defined as a point \( a \) of \( R^* \), the neighbourhood \( U_\lambda(a) \) (for each \( \lambda \)) of \( a \) being the subset \( Q \) of
$q_1, q_2, \ldots$ corresponding to $A_\lambda$ together with all the ultrafilters $B = \{ B_\lambda \}$, $B_\lambda \in \mathcal{X}$, which contain $A_\lambda$.

4. Now we proceed to the construction of the desired $\aleph_0$-compact space $R$ on the basis of $R^*$.

Since every cluster point of $q_1, q_2, \ldots$ is by its definition an ultrafilter $A$, the potency of all points of $R^*$ different from $q_1, q_2, \ldots$ is by Pospisil's theorem [5] equal to $\mathfrak{c} = 2^{2^{\aleph_0}}$. Applying our Lemma 2 on a given sequence of distinct points $a_1, a_2, \ldots$ of $R^*$ other than $q_1, q_2, \ldots$, we see immediately that the potency of all cluster points of the sequence $a_1, a_2, \ldots$ is likewise of potency $\mathfrak{c}$.

Following Kuratowski and Sierpiński [6] let

\begin{align*}
(a) & & a_0, a_1, \ldots, a_\lambda, \ldots \quad (\lambda < \omega_1) \\
(M) & & M_0, M_1, \ldots, M_\lambda, \ldots \quad (\lambda < \omega_1)
\end{align*}

be transfinite sequences of all points of $R^*$ other than $q_1, q_2, \ldots$ and of all countable subsets $M_\lambda$ of $R^*$ respectively, where $\omega_1$ denotes the first ordinal number of potency $\mathfrak{c}$.

Of all cluster points of $M_0$ let $a_\nu$ be the first one which appears in the transfinite sequence $(a)$ and call $a_\nu$ as well as the $n$-th transposed ultrafilters for all even points of class 1. The rest of all transposed ultrafilters of $a_\nu$ will be called points of class 2.

Suppose that for every ordinal number $\eta < \lambda < \omega_1$ points of class 1 and class 2 have been suitably defined and consider $M_\lambda$. Of all the cluster points of $M_\lambda$ which have not been previously defined as points of class 1 or class 2, let $a_\eta$ be the first one which appears in the transfinite sequence $(a)$ and define as above points of class 1 and class 2.

Let $R$ be the subspace of $R^*$ consisting of all points of class 1 together with all isolated points $q_1, q_2, \ldots$ of $R^*$. We shall show that $R$ possesses the property we are seeking for.

First $R$ is $\aleph_0$-compact, for if $M$ is a countable subset of $R$, then $M$ is a member of the sequence of $(M)$, say $M_\lambda$, and the cluster point $a_\eta$, considered above is just a cluster point of $M$ in $R$.

To prove that $R \times R$ fails to be $\aleph_0$-compact, let the points of $R \times R$ be represented by $(x, y)$, where $x, y \in R$. Then the sequence of points $Q$:

\begin{align*}
(q_1, q_2), (q_3, q_4), \ldots, (q_{2n-1}, q_{2n}), \ldots
\end{align*}

has no cluster point in $R$. In fact, if $Q$ should have a cluster point $(a, a')$, then $a'$ must be the first transposed ultrafilter of $a$ and consequently $a$ and $a'$ could not be points of $R$ at the same time, which is absurd.

Thus we have proved that $R$ is the required $\aleph_0$-compact space, whose product $R \times R$ is not $\aleph_0$-compact.

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Bibliography
