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# On Cartesian Product of Compact Spaces

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While the Cartesian product of any number of compact (=bicompact) spaces is again compact by Tychonoff's theorem [1], there is an $\aleph_0$-compact (=compact in the sense of Fréchet) space $R$ whose product $R\times R$ is not $\aleph_0$-compact,[1] as will be shown in the present note. These circumstances will be somewhat clarified by the introduction of a concept of $\aleph_\alpha$-ultracompactness.

1. Let $M$ be a given set of points and let $\boldsymbol{M}=\{M_\lambda\}$ be an ultrafilter [2], i.e., a collection of subsets $M_\lambda$ of $M$ such that

(i) $\boldsymbol{M}$ has the finite intersection property, i.e., any finite number of $M_\lambda$'s have a non-void intersection,

(ii) $\boldsymbol{M}$ is maximal with respect to the property (i), i.e., should any subset $M'$ of $M$ distinct from any one of $M_\lambda$ be added to $\boldsymbol{M}$, then the resulting collection $\boldsymbol{M}+M'$ fails to satisfy the condition (i).

If $\aleph_\alpha$ denotes the lowest of the potencies of $M_\lambda$, we say that $\boldsymbol{M}$ is of *potency* $\aleph_\alpha$. A $T_1$-space will be called *$\aleph_\alpha$-ultracompact*, if every ultrafilter of potency $\aleph_\alpha$ has a cluster point. Then the proof of C. Chevalley and O. Frink [3] for Tychonoff's theorem yields at once the following

**Theorem.** *The Cartesian product of any nnmber of $\aleph_\alpha$-ultracompact spaces is itself $\aleph_\alpha$-ultracompact.*

Here arises the question, *whether or not, if $R$ is $\aleph_\alpha$-compact, i.e., if every subset $M\subset R$ of potency $\aleph_\alpha$ has a cluster point, but if $R$ is not $\aleph_\alpha$-ultracompact, then the product $\Pi R$ is not necessarily $\aleph_\alpha$-compact.* As a partly solution of this question we construct in the following an example of an $\aleph_0$-compact but not $\aleph_0$-ultracompact space $R$, whose product $R\times R$ is not $\aleph_0$-compact.

1) The question whether or not such an $\aleph_0$-compact space exists was raised by M. Ohnishi of Osaka University and answered by me in Sizyo Sugaku Danwakai (June 10, 1947): An example of an $\aleph_0$-compact space $R$ whose product $R\times R$ is not $\aleph_0$-compact (In Japanese). After I had written the present note I have been informed by Ohnishi that the question is originally that of Čech, for which an answer is announced to have been given by Novák in Časopis propěst. mat. a fys. 74 (1950).

2. Let

$$X = (x^1,\ x^2,\ \dots,\ x^n,\ \dots)$$

be a sequence of $x^n$ which is either 0 or 1. The family $\boldsymbol{X}$ of all such $X$ becomes a Boolean algebra, if we introduce the following assumptions and definitions:

1) $X$ and $Y=(y^1, y^2, \dots, y^n, \dots)$ are to be regarded as equal if and only if

$$x^n = y^n$$

for almost all $n$.

2) If $\max(x^n, y^n)=u^n$, $\min(x^n, y^n) = v^n$, $1-x^n = w^n$, then

$$X \cup Y = (u^1,\ u^2,\ \dots,\ u^n,\ \dots)$$
$$X \cap Y = (v^1,\ v^2,\ \dots,\ v^n,\ \dots)$$
$$X^c = (w^1,\ w^2,\ \dots,\ w^n,\ \dots)$$

3)
$$0 = (0,\ 0,\ \dots,\ 0,\ \dots)$$
$$1 = (1,\ 1,\ \dots,\ 1,\ \dots)$$

A *filter* is by definition a collection of elements $A \in \boldsymbol{X}$ with the finite intersection property, and an *ultrafilter* $\boldsymbol{A}$ is a filter with maximal property. Clearly

**Lemma 1.** *If* $\boldsymbol{A}$ *is an ultrafilter and if* $X$ *is any element of* $\boldsymbol{X}$*, then either* $X$ *or* $X^c$ (not both) *belongs to* $\boldsymbol{A}$*. Conversely if for any* $X$ *either* $X$ *or* $X^c$ *belongs to a filter* $\boldsymbol{A}$*, then* $\boldsymbol{A}$ *must be an ultrafilter.*

Now let

$$E = (\varepsilon^1,\ \varepsilon^2,\ \dots,\ \varepsilon^n,\ \dots)$$

and let

$$A_i = (a_i^1,\ a_i^2,\ \dots,\ a_i^n,\ \dots) \qquad (i = 1, 2, \dots)$$

be a sequence of $\boldsymbol{X}$. We denote by $\varepsilon^n A_n$ the element $A_n$ itself if $\varepsilon^n=1$ and the null element if $\varepsilon^n = 0$ and denote further by

$$\sum \varepsilon^n A_n$$

any one of the elements $A$ of $\boldsymbol{X}$ which are $\subset \varepsilon^n A_n$ for all $n$, i.e. a superior of the elements $\varepsilon^n A_n$ $(n=1, 2, \dots)$. Then we have the following useful

**Lemma 2** [4]. *If* $\boldsymbol{A}_n=\{A_\lambda^n\}(n=1,2,\dots)$ *and* $\boldsymbol{E}=\{E_\lambda=(\varepsilon_\lambda^1, \varepsilon_\lambda^2, \dots, \varepsilon_\lambda^n, \dots)\}$ *are ultrafilters, so is* $\boldsymbol{A}=\{\sum \varepsilon_\lambda^n A_\mu^n\}$.

Proof.

(i) First we prove that $\boldsymbol{A}$ is a filter. In fact, if

$$A_1 = \sum \varepsilon_{\lambda_1}^n A_{\mu_1}^n,\ A_2 = \sum \varepsilon_{\lambda_2}^n A_{\mu_2}^n,\ \dots,\ A_m = \sum \varepsilon_{\lambda_m}^n A_{\mu_m}^n$$

are a finite number of elements of $\boldsymbol{A}$, we have by our definition

$$A_1 \frown A_2 \frown \dots \frown A_m \supset \varepsilon^n_{\lambda_1} \cdot \varepsilon^n_{\lambda_2} \cdot \dots \varepsilon^n_{\lambda_m} \cdot A^n_{\mu_1} \cdot A^n_{\mu_2} \cdot \dots A^n_{\mu_m}.$$

Since $\boldsymbol{E}$ and $\boldsymbol{A}$ have the finite intersection property, $\varepsilon^n_{\lambda_1} \cdot \varepsilon^n_{\lambda_2} \cdot \dots \varepsilon^n_{\lambda_m} = 1$ for some $n$ and $A^n_{\mu_1} \cdot A^n_{\mu_2} \cdot \dots A^n_{\mu_m} \neq 0$ for every $n$, and consequently we have

$$A_1 \frown A_2 \frown \dots \frown A_m \neq 0.$$

(ii) To prove that $\boldsymbol{A}$ is an ultrafilter, let $B$ be an element of $\boldsymbol{X}$ not contained in $\boldsymbol{A}$. For each $n$ let $\eta^n = 1$ or $=0$ according as $B$ belongs to or not to $\boldsymbol{A}_n$. Then $B$ can be written in the form

$$B = \sum \eta^n A^n_{\lambda_n},$$

where $A^n_{\lambda_n} = B$ in case $\eta^n = 1$. Since by the assumption on $B$ $H = (\eta^1, \eta^2, \dots, \eta^n, \dots)$ is non $\in \boldsymbol{E}$, we have $H^c = E = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n, \dots) \in \boldsymbol{E}$, where $\varepsilon^n = 1 - \eta^n$, and consequently $B^c$ must be of the form $\sum \varepsilon^n A^n_{\lambda_n} \in \boldsymbol{A}$. Thus we have shown that for every element $X$ of $\boldsymbol{X}$ either $X$ or $X^c$ belongs to the filter $\boldsymbol{A}$, whence we conclude by Lemma 1 that $\boldsymbol{A}$ must be an ultrafilter, and our lemma is proved.

Corresponding to

$$A = (a^1, a^2, \dots, a^n, \dots)$$

let

$$A' = (a^0, a^1, \dots, a^{n-1}, \dots),$$

where $a^0$ stands either for 0 or for 1. Evidently

**Lemma 3.** *If $\boldsymbol{A} = \{A_\lambda\}$ is an ultrafilter, so is $\boldsymbol{A}' = \{A'_\lambda\}$.*

We call $\boldsymbol{A}'$ the first *transposed ultrafilter of* $\boldsymbol{A}$. In general we can speak of the $n$-th transposed ultrafilter of $\boldsymbol{A}$ for any given integer $n\,(-\infty < n < +\infty)$, provided that the 0-th transposed ultrafilter is $\boldsymbol{A}$ itself and the $n$-th transposed ultrafilter of $\boldsymbol{A}$ is the first transposed ultrafilter of the $(n-1)$-th transposed ultrafilter of $\boldsymbol{A}$.

3. We now consider the following Hausdorff space $R^*$:

(i) First let

$$q_1, q_2, \dots, q_n, \dots$$

be introduced and defined to be a countable set of *isolated points* of $R^*$ distinct from each other.

(ii) To define the remaining points of $R^*$, first make correspond to every subset $Q$ of $q_1, q_2, \dots$ the element $A = (a^1, a^2, \dots, a^n, \dots)$ of $\boldsymbol{X}$ in such a way that for each $n$ $a^n = 1$ or $=0$ according as $q_n$ belongs to or not to $Q$. Every ultrafilter $\boldsymbol{A} = \{A_\lambda\}$ of $\boldsymbol{X}$ is then defined as a *point* $a$ of $R^*$, the neighbourhood $U_\lambda(a)$ (for each $\lambda$) of $a$ being the subset $Q$ of

$q_1, q_2, \ldots$ corresponding to $A_\lambda$ together with all the ultrafilters $\boldsymbol{B}=\{B_\lambda\}$, $B_\lambda \in \boldsymbol{X}$, which contain $A_\lambda$.

4. Now we proceed to the construction of the desired $\aleph_0$-compact space $R$ on the basis of $R^*$.

Since every cluster point of $q_1, q_2, \ldots$ is by its definition an ultrafilter $\boldsymbol{A}$, the potency of all noints of $R^*$ different from $q_1, q_2, \ldots$ is by Pospisil's theorem [5] equal to $\mathfrak{f}=2^{2^{\aleph_0}}$. Applying our Lemma 2 on a given sequence of distinct points $a_1, a_2, \ldots$ of $R^*$ other than $q_1, q_2, \ldots$, we see immediately that the potency of all cluster points of the sequence $a_1, a_2, \ldots$ is likewise of potency $\mathfrak{f}$.

Following Kuratowski and Sierpiński [6] let

$$(a) \qquad a_0, a_1, \ldots, a_\lambda, \ldots \quad (\lambda < \omega_{\mathfrak{f}})$$

$$(M) \qquad M_0, M_1, \ldots, M_\lambda, \ldots \quad (\lambda < \omega_{\mathfrak{f}})$$

be transfinite sequences of all points of $R^*$ other than $q_1, q_2, \ldots$ and of all countable subsets $M_\lambda$ of $R^*$ respectively, where $\omega_{\mathfrak{f}}$ denotes the first ordinal number of potency $\mathfrak{f}$.

Of all cluster points of $M_0$ let $a_\nu$ be the first one which appears in the transfinite sequence $(a)$ and call $a_\nu$ as well as the $n$-th transposed ultrafilters for all even $n$ *points of class* 1. The rest of all transposed ultrafilters of $a_\nu$ will be called *points of class* 2.

Suppose that for every ordinal number $\mu(\eta < \lambda < \omega_{\mathfrak{f}})$ points of class 1 and class 2 have been suitably defined and consider $M_\lambda$. Of all the cluster points of $M_\lambda$ which have not been previously defined as points of class 1 or class 2, let $a_\rho$ be the first one which appears in the transfinite sequence $(a)$ and define as above points of class 1 and class 2.

Let $R$ be the subspace of $R^*$ consisting of all points of class 1 together with all isolated points $q_1, q_2, \ldots$ of $R^*$. We shall show that $R$ possesses the property we are seeking for.

First $R$ is $\aleph_0$-compact, for if $M$ is a countable subset of $R$, then $M$ is a member of the sequence of $(M)$, say $M_\lambda$, and the cluster point $a_\rho$ considered above is just a cluster point of $M$ in $R$.

To prove that $R \times R$ fails to be $\aleph_0$-compact, let the points of $R \times R$ be represented by $(x, y)$, where $x, y \in R$. Then the sequence of points $Q$:

$$(q_1, q_2),\ (q_3, q_4),\ \ldots,\ (q_{2n-1}, q_{2n}),\ \ldots$$

has no cluster point in $R$. In fact, if $Q$ should have a cluster point $(a, a')$, then $a'$ must be the first transposed ultrafilter of $a$ and consequently $a$ and $a'$ could not be points of $R$ at the same time, which is absurd.

Thus we have proved that $R$ is the required $\aleph_0$-compact space, whose product $R \times R$ is not $\aleph_0$-compact.

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