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On Cartesian Product of Compact Spaces

By Hidetaka TERASAKA

While the Cartesian product of any number of compact (=bcompact) spaces is again compact by Tychonoff's theorem [1], there is an \aleph_0 -compact (=compact in the sense of Fréchet) space R whose product $R \times R$ is not \aleph_0 -compact,¹⁾ as will be shown in the present note. These circumstances will be somewhat clarified by the introduction of a concept of \aleph_α -ultracompactness.

1. Let M be a given set of points and let $\mathcal{M} = \{M_\lambda\}$ be an ultrafilter [2], i. e., a collection of subsets M_λ of M such that

(i) \mathcal{M} has the finite intersection property, i. e., any finite number of M_λ 's have a non-void intersection,

(ii) \mathcal{M} is maximal with respect to the property (i), i. e., should any subset M' of M distinct from any one of M_λ be added to \mathcal{M} , then the resulting collection $\mathcal{M} + M'$ fails to satisfy the condition (i).

If \aleph_α denotes the lowest of the potencies of M_λ , we say that \mathcal{M} is of *potency* \aleph_α . A T_1 -space will be called \aleph_α -ultracompact, if every ultrafilter of potency \aleph_α has a cluster point. Then the proof of C. Chevalley and O. Frink [3] for Tychonoff's theorem yields at once the following

Theorem. *The Cartesian product of any number of \aleph_α -ultracompact spaces is itself \aleph_α -ultracompact.*

Here arises the question, *whether or not, if R is \aleph_α -compact, i. e., if every subset $M \subset R$ of potency \aleph_α has a cluster point, but if R is not \aleph_α -ultracompact, then the product $R \times R$ is not necessarily \aleph_α -compact.* As a partly solution of this question we construct in the following an example of an \aleph_0 -compact but not \aleph_0 -ultracompact space R , whose product $R \times R$ is not \aleph_0 -compact.

1) The question whether or not such an \aleph_0 -compact space exists was raised by M. Ohnishi of Osaka University and answered by me in Sizo Sugaku Danwakai (June 10, 1947): An example of an \aleph_0 -compact space R whose product $R \times R$ is not \aleph_0 -compact (In Japanese). After I had written the present note I have been informed by Ohnishi that the question is originally that of Čech, for which an answer is announced to have been given by Novák in Časopis pro pěst. mat. a fys. 74 (1950).

2. Let

$$X = (x^1, x^2, \dots, x^n, \dots)$$

be a sequence of x^n which is either 0 or 1. The family X of all such X becomes a Boolean algebra, if we introduce the following assumptions and definitions:

1) X and $Y = (y^1, y^2, \dots, y^n, \dots)$ are to be regarded as equal if and only if

$$x^n = y^n$$

for almost all n .

2) If $\max(x^n, y^n) = u^n$, $\min(x^n, y^n) = v^n$, $1 - x^n = w^n$, then

$$X \cup Y = (u^1, u^2, \dots, u^n, \dots)$$

$$X \cap Y = (v^1, v^2, \dots, v^n, \dots)$$

$$X^c = (w^1, w^2, \dots, w^n, \dots)$$

$$3) \quad 0 = (0, 0, \dots, 0, \dots)$$

$$1 = (1, 1, \dots, 1, \dots)$$

A *filter* is by definition a collection of elements $A \in X$ with the finite intersection property, and an *ultrafilter* A is a filter with maximal property. Clearly

Lemma 1. *If A is an ultrafilter and if X is any element of X , then either X or X^c (not both) belongs to A . Conversely if for any X either X or X^c belongs to a filter A , then A must be an ultrafilter.*

Now let

$$E = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n, \dots)$$

and let

$$A_i = (a_i^1, a_i^2, \dots, a_i^n, \dots) \quad (i = 1, 2, \dots)$$

be a sequence of X . We denote by $\varepsilon^n A_n$ the element A_n itself if $\varepsilon^n = 1$ and the null element if $\varepsilon^n = 0$ and denote further by

$$\sum \varepsilon^n A_n$$

any one of the elements A of X which are $\leq \varepsilon^n A_n$ for all n , i.e. a superior of the elements $\varepsilon^n A_n$ ($n = 1, 2, \dots$). Then we have the following useful

Lemma 2 [4]. *If $A_n = \{A_\lambda^n\} (n = 1, 2, \dots)$ and $E = \{E_\lambda = (\varepsilon_\lambda^1, \varepsilon_\lambda^2, \dots, \varepsilon_\lambda^n, \dots)\}$ are ultrafilters, so is $A = \{\sum \varepsilon_\lambda^n A_\lambda^n\}$.*

Proof.

(i) First we prove that A is a filter. In fact, if

$$A_1 = \sum \varepsilon_{\lambda_1}^n A_{\lambda_1}^n, A_2 = \sum \varepsilon_{\lambda_2}^n A_{\lambda_2}^n, \dots, A_m = \sum \varepsilon_{\lambda_m}^n A_{\lambda_m}^n$$

are a finite number of elements of \mathcal{A} , we have by our definition

$$A_1 \cap A_2 \cap \dots \cap A_m \supset \varepsilon_{\lambda_1}^n \cdot \varepsilon_{\lambda_2}^n \cdot \dots \cdot \varepsilon_{\lambda_m}^n \cdot A_{\mu_1}^n \cdot A_{\mu_2}^n \cdot \dots \cdot A_{\mu_m}^n.$$

Since \mathcal{E} and \mathcal{A} have the finite intersection property, $\varepsilon_{\lambda_1}^n \cdot \varepsilon_{\lambda_2}^n \cdot \dots \cdot \varepsilon_{\lambda_m}^n = 1$ for some n and $A_{\mu_1}^n \cdot A_{\mu_2}^n \cdot \dots \cdot A_{\mu_m}^n \neq 0$ for every n , and consequently we have

$$A_1 \cap A_2 \cap \dots \cap A_m \neq 0.$$

(ii) To prove that \mathcal{A} is an ultrafilter, let B be an element of X not contained in \mathcal{A} . For each n let $\eta^n = 1$ or $= 0$ according as B belongs to or not to \mathcal{A}_n . Then B can be written in the form

$$B = \sum \eta^n A_{\lambda_n}^n,$$

where $A_{\lambda_n}^n = B$ in case $\eta^n = 1$. Since by the assumption on B $H = (\eta^1, \eta^2, \dots, \eta^n, \dots)$ is non $\in \mathcal{E}$, we have $H^c = E = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^n, \dots) \in \mathcal{E}$, where $\varepsilon^n = 1 - \eta^n$, and consequently B^c must be of the form $\sum \varepsilon^n A_{\lambda_n}^n \in \mathcal{A}$. Thus we have shown that for every element X of X either X or X^c belongs to the filter \mathcal{A} , whence we conclude by Lemma 1 that \mathcal{A} must be an ultrafilter, and our lemma is proved.

Corresponding to

$$A = (a^1, a^2, \dots, a^n, \dots)$$

let

$$A' = (a^0, a^1, \dots, a^{n-1}, \dots),$$

where a^0 stands either for 0 or for 1. Evidently

Lemma 3. *If $\mathcal{A} = \{A_\lambda\}$ is an ultrafilter, so is $\mathcal{A}' = \{A'_\lambda\}$.*

We call \mathcal{A}' the first *transposed ultrafilter* of \mathcal{A} . In general we can speak of the n -th transposed ultrafilter of \mathcal{A} for any given integer n ($-\infty < n < +\infty$), provided that the 0-th transposed ultrafilter is \mathcal{A} itself and the n -th transposed ultrafilter of \mathcal{A} is the first transposed ultrafilter of the $(n-1)$ -th transposed ultrafilter of \mathcal{A} .

3. We now consider the following Hausdorff space R^* :

(i) First let

$$q_1, q_2, \dots, q_n, \dots$$

be introduced and defined to be a countable set of *isolated points* of R^* distinct from each other.

(ii) To define the remaining points of R^* , first make correspond to every subset Q of q_1, q_2, \dots the element $A = (a^1, a^2, \dots, a^n, \dots)$ of X in such a way that for each n $a^n = 1$ or $= 0$ according as q_n belongs to or not to Q . Every ultrafilter $\mathcal{A} = \{A_\lambda\}$ of X is then defined as a *point* a of R^* , the neighbourhood $U_\lambda(a)$ (for each λ) of a being the subset Q of

q_1, q_2, \dots corresponding to A_λ together with all the ultrafilters $B = \{B_\lambda\}$, $B_\lambda \in X$, which contain A_λ .

4. Now we proceed to the construction of the desired \aleph_0 -compact space R on the basis of R^* .

Since every cluster point of q_1, q_2, \dots is by its definition an ultrafilter A , the potency of all points of R^* different from q_1, q_2, \dots is by Pospisil's theorem [5] equal to $\mathfrak{f} = 2^{2^{\aleph_0}}$. Applying our Lemma 2 on a given sequence of distinct points a_1, a_2, \dots of R^* other than q_1, q_2, \dots , we see immediately that the potency of all cluster points of the sequence a_1, a_2, \dots is likewise of potency \mathfrak{f} .

Following Kuratowski and Sierpiński [6] let

$$(a) \quad a_0, a_1, \dots, a_\lambda, \dots \quad (\lambda < \omega_{\mathfrak{f}})$$

$$(M) \quad M_0, M_1, \dots, M_\lambda, \dots \quad (\lambda < \omega_{\mathfrak{f}})$$

be transfinite sequences of all points of R^* other than q_1, q_2, \dots and of all countable subsets M_λ of R^* respectively, where $\omega_{\mathfrak{f}}$ denotes the first ordinal number of potency \mathfrak{f} .

Of all cluster points of M_0 let a_n be the first one which appears in the transfinite sequence (a) and call a_n as well as the n -th transposed ultrafilters for all even n points of class 1. The rest of all transposed ultrafilters of a_n will be called points of class 2.

Suppose that for every ordinal number $\mu (\eta < \lambda < \omega_{\mathfrak{f}})$ points of class 1 and class 2 have been suitably defined and consider M_λ . Of all the cluster points of M_λ which have not been previously defined as points of class 1 or class 2, let a_μ be the first one which appears in the transfinite sequence (a) and define as above points of class 1 and class 2.

Let R be the subspace of R^* consisting of all points of class 1 together with all isolated points q_1, q_2, \dots of R^* . We shall show that R possesses the property we are seeking for.

First R is \aleph_0 -compact, for if M is a countable subset of R , then M is a member of the sequence of (M), say M_λ , and the cluster point a_μ considered above is just a cluster point of M in R .

To prove that $R \times R$ fails to be \aleph_0 -compact, let the points of $R \times R$ be represented by (x, y) , where $x, y \in R$. Then the sequence of points Q :

$$(q_1, q_2), (q_3, q_4), \dots, (q_{2n-1}, q_{2n}), \dots$$

has no cluster point in R . In fact, if Q should have a cluster point (a, a') , then a' must be the first transposed ultrafilter of a and consequently a and a' could not be points of R at the same time, which is absurd.

Thus we have proved that R is the required \aleph_0 -compact space, whose product $R \times R$ is not \aleph_0 -compact.

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