Sudden change in fractality of basin boundary in passive dynamic walking

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1 Introduction

Passive dynamic walking is a model that walks down a shallow slope without any control or input [6]. This model has been widely used to investigate how stable walking is generated from a dynamic viewpoint, which is useful to provide design principles for developing energy-efficient biped robots [2]. The basin boundary shows fractal even for a single attractor [9]. The fractal basin boundary has final state sensitivity [4, 5], which means that the system has unpredictability even when the system is deterministic.

Our previous work [1] quantitatively showed sudden changes in the fractality of the basin boundary at some slope angles. In addition, another previous works [7,8] showed the formation mechanism of the fractal basin boundary based on the bending and stretching functions in the inverse image of the Poincaré map. In this study, we investigate the mechanism of why the fractality suddenly changes.

2 Model

We used the simplest walking model [3] (Fig. 1), which walks down a slope of angle γ without any control or input. This model has two legs (swing and stance legs), the lengths of which are both l, connected by a frictionless hip joint. The tip of the stance leg is fixed on the slope, and the stance leg rotates around the leg tip without friction. Here, θ is the angle of the stance leg for the slope normal, and φ is the relative angle between the stance and swing legs. The mass is located only at the hip and the leg tips. The hip mass is M, and the leg tip mass is m. We assumed $m/M \to 0$ as in [3]. g is the acceleration due to gravity.

This model is governed by hybrid dynamics composed of the continuous dynamics generated by the equations of motion when the swing leg is swung to move forward and the discontinuous dynamics generated by the impact upon foot contact. The Poincaré section is defined as $[\theta, \dot{\theta}]^\top$ just after the touchdown and the Poincaré map represents one step. There is an attracting fixed point on the section for

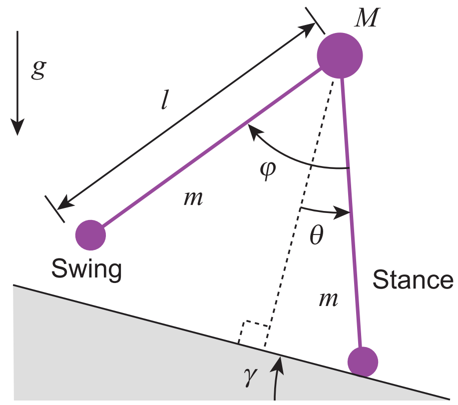

Figure 1: Simplest walking model.

$0 < \gamma < 0.015$ and a period-doubling cascade to chaos for $0.015 < \gamma < 0.019$ [3]. We focus on the basin of attraction on the section. While the basin is not fractal for $\gamma < 0.0075$, it is fractal for $\gamma > 0.0075$ [8].

3 Final state sensitivity of basin boundary

We evaluated the fractality of the basin boundary based on the uncertainty exponent [4, 5], which is defined as follows:

$$\alpha = \dim(B) - \dim(\partial B) \tag{1}$$

where α is the uncertainty exponent, B and ∂B are the basin and basin boundary, respectively, and $\dim(\xi)$ is the dimension of set ξ. If $0 < \alpha < 1$, the basin boundary has a non-integer dimension and is fractal.

We calculated the uncertainty exponent α as described below. First, we put many squares with the length ε, which is sufficiently larger than the bin size of the initial conditions, randomly on a limited range of the Poincaré section. We calculated the proportion f_ε of the squares that involve the basin boundary. When the square is coarse-grained as a single point, it is “uncertain” whether the point is inside or

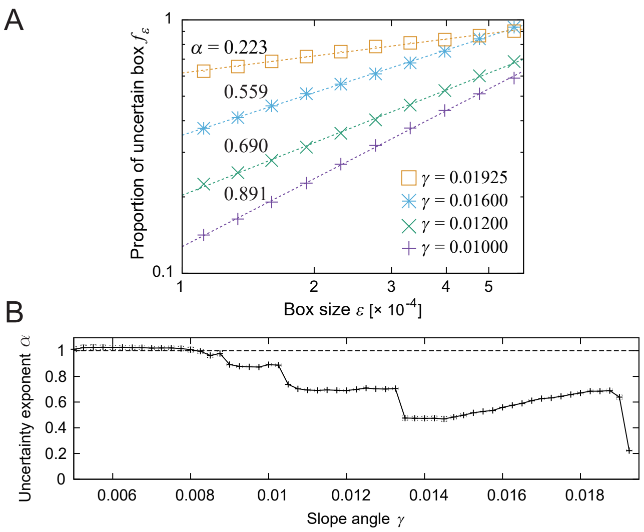

Figure 2: Fractality of basin boundaries. A. Proportion f_ε vs size ε for four values of slope angle γ. The dotted lines represent corresponding linear regression lines. B. Uncertainty exponent α vs γ.

outside of the basin. Therefore, α explains not only the fractality but also the final state sensitivity [4, 5]. The following relationship between α and ε holds:

$$f_\varepsilon \propto \varepsilon^\alpha. \tag{2}$$

Therefore, we can obtain α by calculating the slope of the linear regression line for ε versus f_ε using a log-log plot.

4 Result

Figure 2A shows the results of f_ε for ε for $\gamma = 0.01$, 0.012, 0.016, and 0.01925 and the linear regression lines using a log-log graph. We obtain the uncertainty exponent α from the coefficient of this regression. Figure 2B shows α for γ. When $\gamma < 0.008$, the basin boundary is not fractal because $\alpha \approx 1$. When $\gamma > 0.008$, the basin boundary becomes fractal because $0 < \alpha < 1$. We can find dramatic changes in α at certain values of γ, which includes $\gamma \approx 0.0103$, 0.0135, and 0.019, as shown in [1].

Figures 3A and B show the basin of attraction and range R of the Poincaré map for γ less and larger, respectively, than 0.0135, where the fractality of the basin boundaries suddenly changed in Fig. 2B. Many slits penetrate the lower edge of R by increasing γ. Because of iterative stretching and bending deformations by the inverse image of the Poincaré map, the penetration of R by slits makes a significant effect on the formation process of the basin of attraction [8]. Because the number of slits that penetrate R suddenly increases at $\gamma \approx 0.0135$, the formation process of the basin of attraction discontinuity changes. Specifically, a new infinite number of slits are generated in the basin of attraction after the penetration of slits and the fractality suddenly changes.

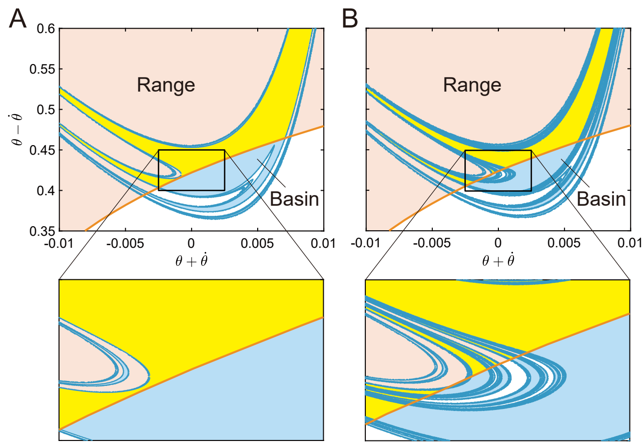

Figure 3: Basin of attraction and range of Poincaré map at $\gamma = 0.0134$ (A) and $\gamma = 0.0136$ (B).

5 Conclusion

In this study, we showed sudden changes in the fractality of the basin boundaries at $\gamma \approx 0.0135$ by calculating the uncertainty exponent. We also showed the mechanism of the sudden changes based on the stretching and bending functions of the inverse image of the Poincaré map. We would like to further investigate the mechanism of the sudden changes at other values of γ.

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