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APPROXIMATION OF LINEAR m-ACCRETIVE OPERATORS IN A HILBERT SPACE

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Introduction

Let A be a linear m-accretive operator in a Hilbert space H, and A^* be its adjoint. Then A^*A is a nonnegative selfadjoint operator, and since A is relatively bounded with respect to A^*A with relative bound 0, it follows that for $n=1, 2, \cdots, A_n=\frac{1}{n}A^*A+A$ is also m-acretive in H. Now let $\{U_n(t); t\geqq 0\}$ and $\{U(t); t\geqq 0\}$ be the contraction semigroups generated by $-A_n$ and $-A$, respectively. As is well known, $U_n(t)$ converges strongly to $U(t)$ if and only if $(A_n+\zeta)^{-1}$ converges strongly to $(A+\zeta)^{-1}$ for some ζ with $\operatorname{Re}\zeta>0$. Taking A_n as above, $A_n u\to Au$ for $u\in D(A^*A)$ and $D(A^*A)$ is a core of A so that $(A_n+\zeta)^{-1}$ converges to $(A+\zeta)^{-1}$ strongly (see Kato [2], VIII–§1.1). Consequently, we see that $U_n(t)$ also converges to $U(t)$ strongly.

The purpose of this note is to give some precise estimate showing the convergence rate of sequences $\{(A_n+\zeta)^{-1}\}$ and $\{U_n(t)\}$. As is easily seen, A_n is m-accretive even if A is only accretive and densly defined. Section 1 is concerned with this point. In Section 2 we consider the approximation of the resolvents. It will be shown that $(A_n+\zeta)^{-1}$ converges to $(A+\zeta)^{-1}$ uniformly. The approximation of the semigroups is treated in Section 3. In general, it can not be expected that $U_n(t)$ converges to $U(t)$ uniformly. But the convergence is proved to be uniform if $U(t)$ can be extended holomorphically into a sector $|\arg t|<\omega$ for some $0<\omega\leqq\pi/2$. In this case A is nothing but an m-sectorial operator.

1. Preliminaries

Let H be a Hilbert space. Then a linear operator A with domain $D(A)$ and range $R(A)$ in H is said to be *accretive* if

$$\operatorname{Re}(Au, u)\geqq 0 \quad \text{for every} \quad u\in D(A),$$

or equivalently if

$$\|(A+\xi)u\|\geqq\xi\|u\| \quad \text{for all} \quad u\in D(A) \quad \text{and} \quad \xi>0.$$

It is well known that $R(A+\xi)=H$ either for every $\xi>0$ or for no $\xi>0$; in the former case we say that A is *m-accretive*. A linear m-accretive operator in H is closed and densely defined. Conversely, a densely defined and closed linear accretive operator in H is m-accretive if and only if so is its adjoint.

Now let A be a densely defined and closed accretive operator in H, and A^* be its adjoint. Then A^*A is a nonnegative selfadjoint operator in H and $D(A^*A)$ is a core of A (see Kato [2], Theorem V-3.24). Denoting by $(A^*A)^{1/2}$ the square root of A^*A, we have

$$(A^*A)^{1/2}u=\frac{1}{\pi}\int_0^\infty \xi^{-1/2}(A^*A+\xi)^{-1}A^*Aud\xi\,,\quad u\in D(A^*A)\,.$$

Noting that $||Au||^2=(Au, Au)=(A^*Au, u)=||(A^*A)^{1/2}u||^2$ for $u\in D(A^*A)$, we see that A is relatively bounded with respect to A^*A with relative bound 0. Namely, for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$||Au||\leqq\varepsilon||A^*Au||+C_\varepsilon||u||,\quad u\in D(A^*A)\,.$$

So, it follows that for $n=1, 2, \cdots$, $A_n=\frac{1}{n}A^*A+A$ is m-accretive (see Yosida [7]). Consequently, for every $v\in H$ there exists a unique $u_n\in D(A^*A)$ such that

$$\frac{1}{n}A^*Au_n+Au_n+u_n=v\,,\quad n\geqq1\,. \tag{1.1}$$

Lemma 1.1. *For the* $\{u_n\}$ *we have the estimate*

$$||u_n-u_m||\leqq\frac{1}{\sqrt{2}}\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2}||Au_n||\,,\quad m\leqq n\,. \tag{1.2}$$

Proof. It follows from (1.1) that

$$||u_n-u_m||^2=-\mathrm{Re}\left(A(u_n-n_m)+A^*A\left(\frac{1}{n}u_n-\frac{1}{m}u_m\right),\ u_n-u_m\right).$$

Since A is accretive, we have

$$\begin{aligned}||u_n-u_m||^2&\leqq-\mathrm{Re}\left(\frac{1}{n}Au_n-\frac{1}{m}Au_m,\ Au_n-Au_m\right)\\&=\left(\frac{1}{n}+\frac{1}{m}\right)\mathrm{Re}\,(Au_n,\ Au_m)-\frac{1}{n}||Au_n||^2-\frac{1}{m}||Au_m||^2\\&\leqq\frac{1}{2}\left(\frac{1}{m}-\frac{1}{n}\right)||Au_n||^2+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{m}\right)||Au_m||^2\,.\end{aligned}$$

Assuming that $m\leqq n$, we obatin (1.2). Q.E.D.

Proposition 1.2. *Let* A *be a densely defined and closed linear accretive* *operator in H, and u_n be defined by* (1.1). *Then A is m-accretive if and only if for any $v\in H$, $||Au_n||$ is bounded as n tends to infinity.*

Proof. First suppose that $||Au_n||$ is bounded in n. Since A is accretive, it suffices to show that $R(A+1)=H$. We see from Lemma 1.1 that $u_n\rightarrow u\in H$. Again from the boundedness of $||Au_n||$ it follows that $u\in D(A)$ and $Au_n\rightharpoonup Au$ (we denote by $\rightharpoonup$ weak convergence; cf. Okazawa [5], Lemma 2.1). Now let $w\in D(A)$. Then $\left(\frac{1}{n}A^*Au_n, w\right)=\frac{1}{n}(Au_n, Aw)\rightarrow 0\ (n\rightarrow\infty)$. Since $\left\|\frac{1}{n}A^*Au_n\right\|$ is also bounded by assumption, and since $D(A)$ is dense in H, it follows that $\frac{1}{n}A^*Au_n\rightharpoonup 0$. So, we obtain from (1.1), $v=(A+1)u$, which implies $R(A+1)=H$.

Conversely, suppose that A is m-accretive. Then A^* is also m-accretive and

$$\mathrm{Re}\left(\frac{1}{n}A^*Au, Au\right)\geqq 0 \quad \text{for} \quad Au\in D(A^*)\,.$$

Therefore, $\left\|\frac{1}{n}A^*Au_n\right\|^2+||Au_n||^2\leqq\left\|\frac{1}{n}A^*Au_n+Au_n\right\|^2\leqq||v||^2$. So, we obtain $||Au_n||\leqq||v||$, $n\geqq 1$. Q.E.D.

Theorem 1.3. *Let A be a densely defined and closed linear accretive operator in H, and B be a linear accretive operator in H. Assume that $D(A)\subset D(B)$ and there exist nonnegative constants a and $b<1$ such that*

$$(1.3)\qquad ||Bu||\leqq a||u||+b||Au||\,,\quad u\in D(A)\,.$$

Then the following three conditions are equivalent:

(i) For any $v\in H$, $||Au_n||$ *is bounded as n tends to infinity, where u_n is defined by the equation*

$$(1.4)\qquad \frac{1}{n}A^*Au_n+(A+B)u_n+u_n=v\,,\quad v\in H\,,\quad n\geqq 1\,;$$

(ii) *$A+B$ is m-accretive*;

(iii) *A is m-accretive.*

Proof. First we note that $A+B$ is a closed linear accretive operator in H and is relatively bounded with respect to A^*A with relative bound 0. So, (1.4) makes sense and the proofs from (i) to (ii) and (iii) to (i) are almost the same as in Proposition 1.2.

Now we show that (ii) is equivalent to (iii). By (1.3) we can find the constants a' and $b'<1$ such that $||Bu||^2\leqq a'||u||^2+b'||Au||^2$. Adding $2\mathrm{Re}(Au, Bu)$

to the both sides of

$$||Au||^2+||Bu||^2 \leqq a'||u||^2+(b'+1)||Au||^2 ,$$

we obtain

$$0 \leqq ||(A+B)u||^2 \leqq 2[\mathrm{Re}(Au, Bu)+a_1||u||^2+b_1||Au||]^2 ,$$

where $a_1=a'/2$, $b_1=(b'+1)/2<1$. So, we can write

$$0 \leqq \mathrm{Re}(Au, (A+B)u)+a_1||u||^2 , \quad u \in D(A+B) = D(A) .$$

This implies that $A+B$ is m-accretive if and only if A is m-accretive (cf. [6], Theorem 1.4). Q.E.D.

The following corollary is a slight generalization of the well known Kato-Rellich theorem (see e.g. Kato [2], V–§4.1).

Corollary 1.4. *Let A, B be two symmetric operators in H, with $D(A) \subset D(B)$. Assume that there are nonnegative constants a and $b<1$ such that for all $u \in D(A)$, $||Bu|| \leqq a||u||+b||Au||$. Then $A+B$ is selfadjoint if and only if A is selfadjoint.*

Note that by relative boundedness assumption $A+B$ is closed if and only if A is closed (see Kato [2], Theorem IV–1.1).

2. Approximation of the resolvents

This section is divided into two parts. In the first part, we consider the approximation of the resolvent for a linear m-accretive operator in a Hilbert space. In the second part, the same problem is considered for a linear m-sectorial operator, the definition of which will be given there.

2.1. m-accretive case

Let A be a linear m-accretive operator in a Hilbert space H. Then A^* is m-accretive and we have

$$\mathrm{Re}(A^*Au, Au) \geqq 0 \quad \text{for} \quad u \in D(A^*A) \subset D(A) .$$

This implies that $A_n=\frac{1}{n}A^*A+A$ is also m-accretive (see e.g. [4] or [5]). Let ζ be a complex number with $\mathrm{Re}\,\zeta>0$. Then for every $v \in H$ there exist $u_n(\zeta) \in D(A^*A)$ and $u(\zeta) \in D(A)$ such that

$$A_n u_n(\zeta)+\zeta u_n(\zeta) = v , \quad n \geqq 1 , \tag{2.1}$$

$$Au(\zeta)+\zeta u(\zeta) = v . \tag{2.2}$$

Lemma 2.1. *Let $u_n(\zeta)$, $u(\zeta)$ be as above. Then*

$$\mathrm{Re}\,\zeta||u(\zeta)-u_n(\zeta)||^2 \leqq \frac{1}{2n}||Au(\zeta)||^2 , \tag{2.3}$$

$$||Au_n(\zeta)|| \leqq ||Au(\zeta)||, \quad n \geqq 1 .$$

Proof. It follows from (2.1) and (2.2) that

$$\zeta[u(\zeta)-u_n(\zeta)] = -[Au(\zeta)-Au_n(\zeta)]+\frac{1}{n}A^*Au_n(\zeta)\,.$$

So, we obtain

$$\begin{aligned}\operatorname{Re}\zeta\|u(\zeta)-u_n(\zeta)\|^2 &\leqq \frac{1}{n}\operatorname{Re}(A^*Au_n(\zeta),\, u(\zeta)-u_n(\zeta))\\ &\leqq \frac{1}{2n}\|Au(\zeta)\|^2-\frac{1}{2n}\|Au_n(\zeta)\|^2\,.\end{aligned}$$ Q.E.D.

Now, $\|Au(\zeta)\|$ is estimated as follows:

$$\begin{aligned}\|Au(\zeta)\| &\leqq \|(A+i\operatorname{Im}\zeta)u(\zeta)\|+|\operatorname{Im}\zeta|\,\|u(\zeta)\|\\ &\leqq \|(A+\zeta)u(\zeta)\|+\frac{|\operatorname{Im}\zeta|}{\operatorname{Re}\zeta}\|v\|\\ &= \left(1+\frac{|\operatorname{Im}\zeta|}{\operatorname{Re}\zeta}\right)\|v\|\,.\end{aligned}$$

Thus, we have proved the following

Theorem 2.2. *Let A be m-accretive in H, and set $A_n=\frac{1}{n}A^*A+A$, $n\geqq 1$. Then for every ζ with $\operatorname{Re}\zeta>0$, $(A_n+\zeta)^{-1}$ converges uniformly to $(A+\zeta)^{-1}$:*

$$\|(A+\zeta)^{-1}-(A_n+\zeta)^{-1}\|\leqq\frac{1}{\sqrt{2n}}\frac{\operatorname{Re}\zeta+|\operatorname{Im}\zeta|}{(\operatorname{Re}\zeta)^{3/2}}\,,\quad n\geqq 1\,.$$

Now let $0<\alpha<1$, and A^α be the fractional power of A. Using the inequality of moments (see Krein [3], I–§5), we obtain

Corollary 2.3. *Let A, A_n be as in Theorem 2.2. Then*

$$\|A^\alpha[(A+\zeta)^{-1}-(A_n+\zeta)^{-1}]\| = O(n^{-(1-\alpha)/2}),\quad n\to\infty\,.$$

Assuming further that $1/2<\alpha<1$, we have an estimate for the fractional powers of the resolvents:

$$\|(A+1)^{-\alpha}-(A_n+1)^{-\alpha}\|\leqq\frac{\sin\pi\alpha}{\pi\sqrt{2n}}B\left(1-\alpha,\,\alpha-\frac{1}{2}\right),\quad n\geqq 1\,.$$

2.2. m-sectorial case

A linear accretive operator A in H is said to be *sectorial* with a vertex 0 and a semi-angle $\pi/2-\omega$, where $0<\omega\leqq\pi/2$, if $e^{i\theta}A$ is also accretive for $-\omega\leqq\theta\leqq\omega$. Namely, A is sectorial if the numerical range of A is a subset of the sector $|\arg\zeta|\leqq\pi/2-\omega$:

$$|\operatorname{Im}(Au,\,u)|\tan\omega\leqq\operatorname{Re}(Au,\,u)\,,\quad u\in D(A)\,.$$

If $e^{i\theta}A$ is m-accretive for all $|\theta|\leqq\omega$, then A is said to be *m-sectorial* with a vertex 0 and a semi-angle $\pi/2-\omega$, and then the contraction semigroup $\{U(t)\}$ generated by $-A$ can be extended holomorphically into the sector $|\arg t|<\omega$. In this connection, we note that for any $\varepsilon>0$,

$$\|(A+\zeta)^{-1}\|\leqq\frac{1}{|\zeta|\sin\varepsilon}, \quad |\arg\zeta|\leqq\frac{\pi}{2}+\omega-\varepsilon. \tag{2.4}$$

Now let A be a linear m-sectorial operator with a vertex 0 and a semi-angle $\pi/2-\omega$. Then $A_n=\frac{1}{n}A^*A+A$ is also m-sectorial with the same vertex and semi-angle (uniformly in n), and satisfies the inequality (2.4) with A replaced by A_n. Let ζ be a complex number with $|\arg\zeta|\leqq\frac{\pi}{2}+\omega-\varepsilon$. Then for any $v\in H$ there exist $u_n(\zeta)\in D(A^*A)$ and $u(\zeta)\in D(A)$ such that (2.1) and (2.2) hold.

Lemma 2.4. *Let $u_n(\zeta)$, $u(\zeta)$ be as above. Then we have*

$$(|\zeta|\sin\varepsilon)\|u(\zeta)-u_n(\zeta)\|^2\leqq\frac{1}{2n}(\|Au(\zeta)\|^2+\|Au_n(\zeta)\|^2). \tag{2.5}$$

Proof. It suffices by (2.3) to show that (2.5) holds for $\pi/2-\varepsilon\leqq|\arg\zeta|\leqq\pi/2+\omega-\varepsilon$. It follows from (2.1) and (2.2) that $\zeta[u(\zeta)-u_n(\zeta)]=-[Au(\zeta)-A_nu_n(\zeta)]$. Let θ be a real number such that $0\leqq\theta\leqq\omega$, and set $\zeta=|\zeta|e^{i(\pi/2+\theta-\varepsilon)}$. Then we can write

$$|\zeta|e^{i(\pi/2-\varepsilon)}[u(\zeta)-u_n(\zeta)]=-[e^{-i\theta}Au(\zeta)-e^{-i\theta}A_nu_n(\zeta)].$$

Since $e^{-i\theta}A$ is accretive for $|\theta|\leqq\omega$, we see that

$$\begin{aligned}(|\zeta|\sin\varepsilon)\|u(\zeta)-u_n(\zeta)\|^2&\leqq\frac{1}{n}\operatorname{Re}(e^{-i\theta}A^*Au_n(\zeta),\ u(\zeta)-u_n(\zeta))\\&\leqq\frac{1}{2n}\|Au(\zeta)\|^2+\frac{1-2\cos\theta}{2n}\|Au_n(\zeta)\|^2.\end{aligned}$$

Q.E.D.

Lemma 2.5. *Let $u_n(\zeta)$, $u(\zeta)$ be as in Lemma 2.4. Then*

$$\|Au_n(\zeta)\|\leqq\left(1+\frac{1}{\sin\varepsilon}\right)\|v\|, \quad n\geqq1, \tag{2.6}$$

$$\|Au(\zeta)\|\leqq\left(1+\frac{1}{\sin\varepsilon}\right)\|v\|, \quad |\arg\zeta|\leqq\frac{\pi}{2}+\omega-\varepsilon. \tag{2.7}$$

Proof. Since $\|Au_n(\zeta)\|\leqq\|A_nu_n(\zeta)\|$, we obtain

$$\|Au_n(\zeta)\| \leqq \|(A_n+\zeta)u_n(\zeta)\| + |\zeta|\,\|u_n(\zeta)\|$$
$$\leqq \|v\| + \frac{1}{\sin\varepsilon}\|v\|,$$

where we have used (2.4) with A replaced by A_n. In the same way we can prove (2.7). Q.E.D.

Combining (2.5) with (2.6) and (2.7), we obtain

Theorem 2.6. *Let A be m-sectorial with a vertex 0 and a semi-angle $\pi/2-\omega$, and set $A_n=\frac{1}{n}A^*A+A$, $n\geqq 1$. Then for any $\varepsilon>0$ there exists a constant $N_\varepsilon>0$ such that*

$$\|(A+\zeta)^{-1}-(A_n+\zeta)^{-1}\| \leqq \frac{1}{\sqrt{n}}\frac{N_\varepsilon}{|\zeta|^{1/2}}, \quad |\arg\zeta| \leqq \frac{\pi}{2}+\omega-\varepsilon. \tag{2.8}$$

REMARK 2.7. The estimate (2.8) is an abstract version of some particular case of the result obtained by Friedman (see [1], Theorem 2.2 and the note after it).

The following example is extracted from [1].

EXAMPLE 2.8. Let Ω be a bounded domain of class C^4 in R^N. Let $A=-\Delta$ ($\Delta=$ Laplacian) with $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$. Then A is a positive definite selfadjoint operator in $L^2(\Omega)$, and $A_n=\frac{1}{n}\Delta^2-\Delta$ with $D(A_n)=D(A^2)=\{u\in D(A);\ Au\in D(A)\}$. Consequently, (2.8) holds with $\omega=\pi/2$.

3. Approximation of the semigroups

3.1. m-accretive case

Let A be a linear m-accretive operator in a Hilbert space H, and $\{U(t)\}$ be the contraction semigroup generated by $-A$. Then for every $u_0\in D(A)$, $u(t)=U(t)u_0$ is a unique solution of the Cauchy problem for the equation

$$du/dt+Au(t)=0, \quad t\geqq 0, \tag{3.1}$$

with the initial condition $u(0)=u_0$.

Set $A_n=\frac{1}{n}A^*A+A$, $n\geqq 1$. Then $A_n+\varepsilon$ $\left(\varepsilon\geqq\frac{1}{n}\right)$ is m-sectorial with a vertex 0 and a semi-angle arctan $(n/2)$. In fact, let $u\in D(A^*A)$. Then

$$|\operatorname{Im}(A_nu,u)| = |\operatorname{Im}(Au,u)| \leqq \frac{1}{2}(A^*Au,u)+\frac{1}{2}\|u\|^2$$
$$\leqq \frac{n}{2}\operatorname{Re}(A_nu,u)+\frac{1}{2}\|u\|^2.$$

Denoting by $\{U_n(t)\}$ the semigroup generated by $-A_n$, $U_n(t)$ is holomorphic in some sector containing the positive real axis. Therefore, for every $u\in H$, $u_n(t)=U_n(t)u$ satisfies the equation

$$(d/dt)u_n(t)+A_nu_n(t)=0, \quad t>0. \tag{3.2}$$

Lemma 3.1. *Let $u\in H$ and $t>0$. Then for $u_n(t)=U_n(t)u$ we have*

$$||Au_n(t)||\leqq||Au_n(s)||, \quad 0<s\leqq t. \tag{3.3}$$

Proof. First we note that $||Au_n(s)||\leqq||A_nu_n(s)||=||(d/ds)u_n(s)||$. Since $u_n(s)$ is holomorphic in s, we obtain

$$(d/ds)||Au_n(s)||^2 = -2\mathrm{Re}\,(A_nu_n(s),\ A^*Au_n(s))\leqq 0.$$

Q.E.D.

Theorem 3.2. *Let $u_0\in D(A)$ and $t>0$. Let $u(t)=U(t)u_0$ and $u_n(t)=U_n(t)u_0$. Then $||Au_n(t)||\leqq||Au_0||$, $t>0$, and*

$$||u_n(t)-u(t)||\leqq\left(\frac{t}{n}\right)^{1/2}||Au_0||, \quad t\geqq 0, \quad n\geqq 1.$$

Proof. It follows from (3.1) and (3.2) that

$$\begin{aligned}(d/ds)||u_n(s)-u(s)||^2 &= -2\mathrm{Re}\,(A_nu_n(s)-Au(s),\ u_n(s)-u(s))\\ &\leqq -\frac{2}{n}\,\mathrm{Re}\,(A^*Au_n(s),\ u_n(s)-u(s))\\ &\leqq \frac{1}{n}||Au(s)||^2-\frac{1}{n}||Au_n(s)||^2, \quad s>0.\end{aligned}$$

Since $||Au(s)||=||U(s)Au_0||\leqq||Au_0||$, we see from (3.3) that

$$\frac{d}{ds}||u_n(s)-u(s)||^2\leqq\frac{1}{n}||Au_0||^2-\frac{1}{n}||Au_n(t)||^2, \quad 0<s\leqq t.$$

Hence we obtain $||u_n(t)-u(t)||^2+\frac{t}{n}||Au_n(t)||^2\leqq\frac{t}{n}||Au_0||^2, \quad t>0.$ Q.E.D.

The following corollary corresponds to Corollary 2.3.

Corollary 3.3. *Let $u_n(t)$, $u(t)$ be as in Theorem 3.2. Then*

$$||A^{\alpha}[u_n(t)-u(t)]|| = O(n^{-(1-\alpha)/2}), \quad n\to\infty,$$

where A^{α}, $0<\alpha<1$, is the fractional power of A.

3.2. *m*-sectorial case

Let A be a linear *m*-sectorial operator with a vertex 0 and a semi-angle

$\pi/2-\omega$, and $\{U(t)\}$ be the holomorphic semigroup generated by $-A$. Then $U(t)$ is given by the contour integral:

$$U(t)=\frac{1}{2\pi i}\int_{\Gamma} e^{\zeta t}(A+\zeta)^{-1}d\zeta\,, \tag{3.4}$$

where Γ is a curve running, in the sector $|\arg\zeta|<\pi/2+\omega$, from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$ with $\phi=\pi/2+\omega-\varepsilon$.

Set $A_n=\frac{1}{n}A^*A+A$, $n\geqq 1$, and let $\{U_n(t)\}$ be the holomorphic semigroup generated by $-A_n$. Then $U_n(t)$ is also given by (3.4) with U and A replaced by U_n and A_n, respectively.

Theorem 3.4. *Let A be m-sectorial with a vertex* 0 *and a semi-angle* $\pi/2-\omega$, *and set* $A_n=\frac{1}{n}A^*A+A$, $n\geqq 1$. *Let* $\{U(t)\}$, $\{U_n(t)\}$ *be the semigroup generated by* $-A$, $-A_n$, *respectively. Then* $U_n(t)$ *converges uniformly to* $U(t)$:

$$\|U(t)-U_n(t)\|\leqq\frac{1}{\sqrt{n}}\frac{M_\varepsilon}{\sqrt{|t|}}\,,\quad |\arg t|\leqq\omega-\varepsilon, \tag{3.5}$$

furthermore

$$\left\|\frac{dU(t)}{dt}-\frac{dU_n(t)}{dt}\right\|\leqq\frac{1}{\sqrt{n}}\frac{M_\varepsilon'}{|t|^{3/2}}\,,\quad |\arg t|\leqq\omega-\varepsilon\,. \tag{3.6}$$

Proof. Since we can write

$$U(t)-U_n(t)=\frac{1}{2\pi i}\int_{\Gamma} e^{\zeta t}[(A+\zeta)^{-1}-(A_n+\zeta)^{-1}]d\zeta\,,$$

it follows from (2.8) that

$$\begin{aligned}\|U(t)-U_n(t)\|&\leqq\frac{1}{\sqrt{n}}\frac{N_\varepsilon}{2\pi}\int_{\Gamma}|e^{\zeta t}|\,|\zeta|^{-1/2}|d\zeta|\\&=\frac{1}{\sqrt{n}}\frac{N_\varepsilon'}{\sqrt{|t|}}\int_{\Gamma'}|e^{\zeta'}|\,|\zeta'|^{-1/2}|d\zeta'|.\end{aligned}$$

To obtain (3.6) it suffices to note that

$$\frac{dU(t)}{dt}=\frac{1}{2\pi i}\int_{\Gamma} e^{\zeta t}\zeta(A+\zeta)^{-1}d\zeta\,. \qquad \text{Q.E.D.}$$

Now let $f(t)$ be a H-valued Hoelder continuous function on $[0,\infty)$, and set

$$v(t)=\int_0^t U(t-s)f(s)ds\,.$$

Then $v(t)$ is a unique solution of the Cauchy problem for the equation $dv/dt+$

$Av(t)=f(t)$, $t>0$, with the initial condition $v(0)=0$. The same assertion holds for $v_n(t)=\int_0^t U_n(t-s)f(s)ds$ (see e.g. [2], Theorem IX–1.27).

Corollary 3.5. *Let* $v(t)$, $v_n(t)$ *be as above. Then we have*

$$\|v(t)-v_n(t)\| \leqq 2M_\varepsilon \left(\frac{t}{n}\right)^{1/2} \sup_{0\leqq s\leqq t} \|f(s)\| . \tag{3.7}$$

In fact, it follows from (3.5) that

$$\begin{aligned}\|v(t)-v_n(t)\| &\leqq \int_0^t \|U(t-s)-U_n(t-s)\| \, \|f(s)\| ds \\ &\leqq \frac{M_\varepsilon}{\sqrt{n}} \int_0^t (t-s)^{-1/2} ds \sup_{0\leqq s\leqq t} \|f(s)\| .\end{aligned}$$

REMARK 3.6. (3.7) corresponds to the estimate (4.6) in Theorem 4.1 of Friedman [1].

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