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ON THE CROSSING NUMBER OF 2-BRIDGE KNOT AND THE CANONICAL GENUS OF ITS WHITEHEAD DOUBLE

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Abstract

By using Morton's inequality we study the canonical genus of a Whitehead double of a knot. We show that the crossing number of a 2-bridge knot coincides with the canonical genus of its Whitehead double.

1. Introduction

A *link* is a closed 1-manifold smoothly embedded in the 3-sphere S^3, and a *knot* is a link with one connected component. A *Seifert surface* of a knot K is a compact, connected, orientable surface S in S^3 such that the boundary of S is K. The minimal genus among all Seifert surfaces of K is called the *genus* for K, denoted by $g(K)$. A Seifert surface of K with the minimal genus is called a *minimal genus Seifert surface* of K. A Seifert surface of K is said to be *canonical* if it is obtained from a diagram of K by applying Seifert's algorithm. Then the minimal genus among all canonical Seifert surfaces of K is called the *canonical genus* for K, denoted by $g_c(K)$. A Seifert surface S of K is said to be *free* if the fundamental group of the complement of S, namely, $\pi_1(S^3 - S)$ is a free group. Then the minimal genus among all free Seifert surfaces of K is called the *free genus* for K, denoted by $g_f(K)$. For these "genus" of knots we have the fundamental inequality: $g(K) \leq g_f(K) \leq g_c(K)$, since any canonical Seifert surface is free. There are a lot of works constructing knots which give the above inequality strictly. For the free genus and the genus, in 1972, H.C. Lyon [6] constructed a family of knots without free incompressible Seifert surfaces, hence $g(K) < g_f(K)$. In 1987, Y. Moriah [8] showed that there exists a knot K such that $g_f(K) - g(K) \geq n$ for any positive integer n. Subsequently, a similar result was showed by C. Livingston [7]. On the other hand, H.R. Morton [9] pointed out that a twisted Whitehead double of the trefoil knot has the canonical genus at least three although its genus is one. Later, A. Kawauchi [3] showed that there exists a knot K such that $g_c(K) - g(K) = 2n$ for any positive integer n. After that, M. Kobayashi and T. Kobayashi [5] showed that there exists a knot K such that $g_c(K) - g_f(K) = n$ and

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$g_f(K) - g(K) = n$ for any positive integer n. Knots in these results are satellite knots or composite knots. The author [11] showed that there exists a simple fibered knot K such that $g_c(K) \geq n$ and $g_f(K) = 3$ for any positive integer n (≥ 3). Shortly after, J.J. Tripp [14] showed that the canonical genus of a twisted Whitehead double of a torus knot of type $(2, n)$ is equal to n. Then he has conjectured that *the crossing number of a knot coincides with the canonical genus of its Whitehead double*. We give a partial affirmative answer to this conjecture. In fact, we prove:

Theorem 1. *The crossing number of a* 2*-bridge knot coincides with the canonical genus of its Whitehead double.*

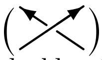
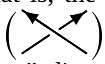
REMARK 2. After having done this work, H. Gruber [2] extended this result in a different way, that is, he showed that the above question is affirmative for all algebraic alternating knots in Conway's sense.

This paper is organized as follows. In Section 2, we will prepare several definitions and notation, *Whitehead doubles*, *doubled links*, *Conway's normal form* and *Morton's inequality* ([9, Theorem 2]). In Sections 3 and 4, we will show that the canonical genus of a Whitehead double of a 2-bridge knot is equal to the crossing number of the 2-bridge knot by using Rudolph's technique in [13, Section 2].

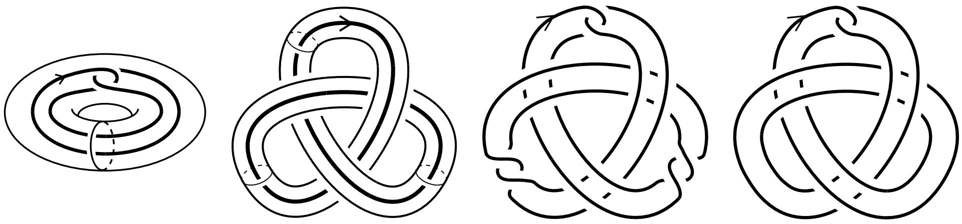
Throughout this paper, all manifolds in S^3 are oriented unless otherwise stated. For the definition of standard terms in knot theory, we refer to [1], [4], [10] and [12].

2. Preliminaries

2.1. Doubles of knots and links. Let C be a knot in an unknotted solid torus $S^1 \times B^2$ as in Fig. 1 (a), called the Whitehead clasp, and $N(K)$ a tubular neighborhood of a nontrivial knot K in S^3 as in Fig. 1 (b). Let $f: S^1 \times B^2 \to N(K)$ be an orientation preserving homeomorphism taking $\{0\} \times B^2$ to the meridian disk of $N(K)$, and $S^1 \times \{0\}$ to K. We call the knot $f(C)$ the *m-twisted Whitehead double* of K, denote by $D_m(K)$, if the linking number of $f(l)$ and K is equal to m, where l is the preferred longitude of $S^1 \times B^2$.

Let $w(P)$ be the writhe of a diagram P of a knot K, that is, the sum of the signs of all crossings in P, defined as *sgn*  $= 1$ and *sgn*  $= -1$. Then we see that the $w(P)$-twisted Whitehead double of K has a "nice" diagram, which is the 2-parallel diagram for P with a clasp. See Fig. 1 (d). We denote by $D(P)$ this diagram of the $w(P)$-twisted Whitehead double of K.

Lemma 3. *Let P be a knot diagram on S^2 with n crossings. Then the genus of a canonical Seifert surface obtained from $D(P)$ is equal to n.*



(a) (b) (c) (d)

Fig. 1.

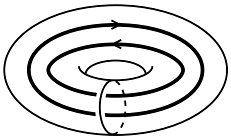


Fig. 2.

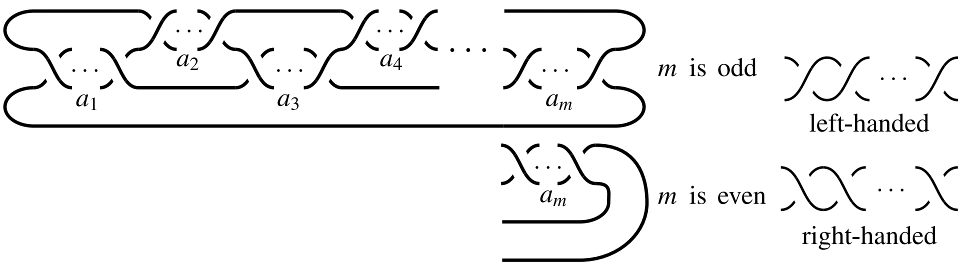
Proof. We see that there exist $2n+3$ Seifert circles in $D(P)$ by applying Seifert's algorithm. Since the number of crossings in $D(P)$ is $4n+2$, the genus of a canonical Seifert surface obtained from $D(P)$ is $\{1+(4n+2)-(2n+3)\}/2 = n$. □

Lemma 3 gives an upper bound on the canonical genus of $D_m(K)$ for a knot K. We give a lower bound in order to prove Theorem 1 by using the HOMFLY polynomial in the next section.

Let L be a link with μ components $K_1, K_2, \ldots, K_\mu$ in S^3, and V_i $(i = 1, 2, \ldots, \mu)$ an unknotted solid torus $S^1 \times B^2$ containing a 2-component parallel link L_i with the opposite orientation as in Fig. 2. Let $f_i\colon V_i \to N(K_i)$ be an orientation preserving homeomorphism taking the meridian disk of V_i to the meridian disk of $N(K_i)$, and the core of V_i, namely, $S^1 \times \{0\}$, to K_i. We call the link $f_1(L_1) \cup \cdots \cup f_\mu(L_\mu)$ the $(m_1, \ldots, m_\mu)$-*twisted doubled link* of L, if the linking number of $f_i(l_i)$ and K_i is equal to m_i, where l_i is the preferred longitude of V_i for each i.

Let P be a diagram of L, and P_i the subdiagram of P corresponding to K_i for $i = 1, 2, \ldots, \mu$. Let w_i be the writhe of P_i. Similarly to the case of the Whitehead doubles of knots, we see that the $(w_1, \ldots, w_\mu)$-twisted doubled link of L has a "nice" diagram, which is the 2-parallel diagram for P. We denote by $D_L(P)$ this diagram of the $(w_1, \ldots, w_\mu)$-twisted doubled link of L.

2.2. 2-bridge links and Conway's normal forms. A link L is said to be a 2-*bridge link* if L has a diagram as in Fig. 3, called *Conway's normal form*. For a link diagram as in Fig. 3, each $|a_i|$ presents the number of half-twists for integers $a_1, a_2, \ldots, a_m$. In this paper, for the sign of a_i, we assume that a right-handed half-



(a) (b)

Fig. 3.

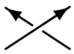
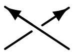
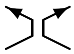
twist is positive if i is odd, and a left-handed half-twist is positive if i is even. See Fig. 3 (b). We denote this link diagram by $C(a_1, a_2, \dots, a_m)$.

A diagram $C(a_1, a_2, \dots, a_m)$ is a 2-bridge link diagram. Conversely, any 2-bridge link L has a diagram of type $C(a_1, a_2, \dots, a_m)$. A diagram $C(a_1, a_2, \dots, a_m)$ is called Conway's normal form of a 2-bridge link L. (For more information, see [1] or [4].) It is well known that a 2-bridge link is an alternating link and an alternating diagram is realized by Conway's normal form $C(a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $i = 1, 2, \dots, m$.

2.3. Morton's inequality and Canonical genus. Let $P_L(v, z)$ be the HOMFLY polynomial of a link L calculated by the following recursive relations.

(1) $P_O(v, z) = 1$,

(2) $v^{-1} P_{L_+}(v, z) - v P_{L_-}(v, z) = z P_{L_0}(v, z)$,

where O is the trivial knot and L_+, L_- and L_0 are three links that are identical except near one point ,  and  respectively.

In [9], Morton showed the following inequality, called *Morton's inequality*. We denote the maximal degree in z of $P_L(v, z)$ by $\operatorname{maxdeg}_z P_L(v, z)$.

Theorem 4 ([9, Theorem 2]). *For a diagram D of a link L,*

$$\operatorname{maxdeg}_z P_L(v, z) \leq c(D) - s(D) + 1,$$

where $c(D)$ is the number of crossings and $s(D)$ is the number of Seifert circles in D, respectively.

The equality holds for alternating links, positive links and many other links. The right-hand side of Morton's inequality is the first Betti number of a canonical Seifert surface obtained from D. Thus the half of the maximal degree in z of $P_K(v, z)$ gives a lower bound for the canonical genus for a knot K, that is, $\operatorname{maxdeg}_z P_K(v, z) \leq 2g_c(K)$.

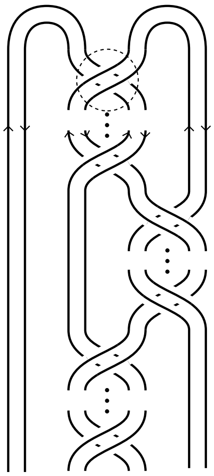


Fig. 4.

3. Lemmas

In the next section, we will prove the following proposition. Let $L_{m,n}$ be the doubled link of a 2-bridge link L, which has the diagram $D_L(P)$, where P is Conway's normal form $C(a_1, a_2, \dots, a_m)$ of L such that $a_1 + a_2 + \cdots + a_m = n$ and $a_i > 0$ for $i = 1, 2, \dots, m$. See Fig. 4.

Proposition 5. *For a link $L_{m,n}$, we have*:

$$\operatorname{maxdeg}_z P_{L_{m,n}}(v, z) = 2n - 1.$$

Note that $C(1, a_2, a_3, \dots, a_m)$ is equivalent to $C(-a_2-1, -a_3, \dots, -a_m)$. Hence we may assume, without loss of generality, that $a_1 \geq 2$. We note that the crossing number, the canonical genus and the maximal degree in z of $P_L(v, z)$ of a link L are the same as those of the mirror image of L. Similarly, we may assume that $a_m \geq 2$.

Let $k := n - m$ (> 1) be a positive integer. In order to prove Proposition 5 by induction on the lexicographic order of a pair (m, k) of positive integers m, k, we first prove the cases $(1, k), (2, k)$ and $(m, 2)$ respectively. The first case, $(1, k)$, has been proved by Tripp in [14, Proposition 1]. Thus we show the second case, $(m, 2)$, that is, $a_1 = a_m = 2$ and $a_i = 1$ $(2 \leq i \leq m-1)$ as follows. Hereafter, we denote by $d(L_{m,n})$ the maximal degree in z for $P_{L_{m,n}}(v, z)$ for short.

Lemma 6. *For the doubled link $L_{n-2,n}$ $(n \geq 3)$ of a 2-bridge link $C(2, 1, \dots, 1, 2)$, we have*:

$$d(L_{n-2,n}) = 2n - 1.$$

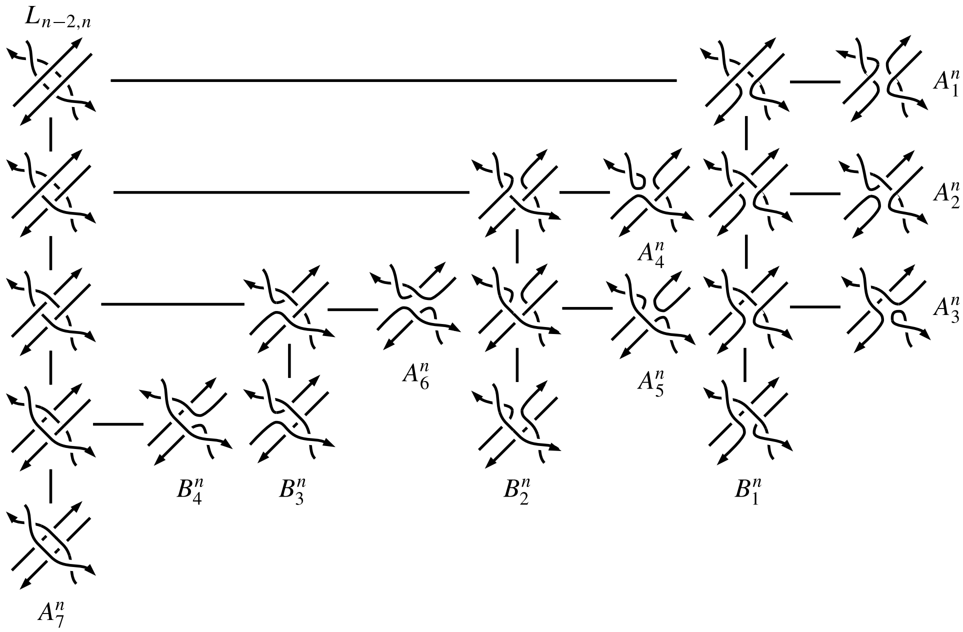


Fig. 5.

Proof of Lemma 6. We prove Lemma 6 by induction on n. By direct calculations, we have $d(L_{1,3}) = 5$, $d(L_{2,4}) = 7$ and $d(L_{3,5}) = 9$.

Assume that Lemma 6 holds for every positive integer less than n (≥ 6). We use a technique in [13, Section 2] to compute $P_{L_{n-2,n}}(v, z)$. By constructing a resolution tree with respect to the local diagram in the dotted circle depicted in Fig. 4, we have eleven links $A_1^n, A_2^n, \dots, A_7^n$ and $B_1^n, \dots, B_4^n$ with diagrams identical to the diagram as in Fig. 4 except as indicated in Fig. 5. (The local diagram of $L_{n-2,n}$ in Fig. 5 is added two crossings by Reidemeister move II.) We use this to compute $P_{L_{n-2,n}}(v, z)$ in a standard way. In the partial resolution tree as in Fig. 5, the horizontal lines (resp. the vertical lines) are labeled vz or $-v^{-1}z$ (resp. v^2 or v^{-2}) according to the sign of the crossing which will be altered by a smoothing (resp. crossing change).

Then we have:

$$(*)\quad \begin{aligned} P_{L_{n-2,n}} &= v^2z^2\left(P_{A_1^n} - P_{A_2^n} - P_{A_4^n} + P_{A_6^n}\right) - za^2\left(P_{A_3^n} + P_{A_5^n}\right) + P_{A_7^n} \\ &\quad + v^{-1}z\left(P_{B_1^n} + P_{B_2^n}\right) - vz\left(P_{B_3^n} + P_{B_4^n}\right). \end{aligned}$$

Claim 7. $d(A_i^n) \leq 1$ *for* $i = 2, 3, 4, 5$.

Proof of Claim 7. We can deform each A_i^n into a diagram of a 2-component link which is the boundary of an unknotted, twisted annulus for $i = 2, 3, 4, 5$. Since the

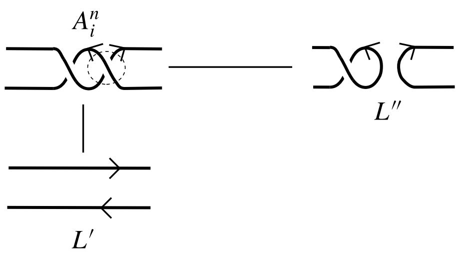


Fig. 6.

canonical Seifert surface obtained from the diagram, namely, the annulus has the first Betti number one, the conclusion follows from Morton's inequality. $\square$

Hence we see that none of $A_2^n, \ldots, A_5^n$ contributes anything to $d(L_{n-2,n})$ by induction hypothesis and the equality $(*)$ when $n \geq 6$. For $i = 1, 2, 3, 4$, we have then

$$d(L_{n-2,n}) \leq \max\{d(A_1^n) + 2, d(A_6^n) + 2, d(A_7^n), d(B_i^n) + 1\}. \tag{1}$$

By some deformations, it is easily seen that A_i^n $(i = 1, 6, 7)$ is the t_i-twisted doubled link of a 2-bridge knot K or the (t_i, t_i')-twisted doubled link of a 2-bridge link L for some integers t_i and t_i'.

Claim 8. *For any integers t_i and t_i', we have $d(A_1^n) = d(L_{n-3,n-1})$, $d(A_6^n) = d(L_{n-4,n-2})$ and $d(A_7^n) = d(L_{n-5,n-3})$.*

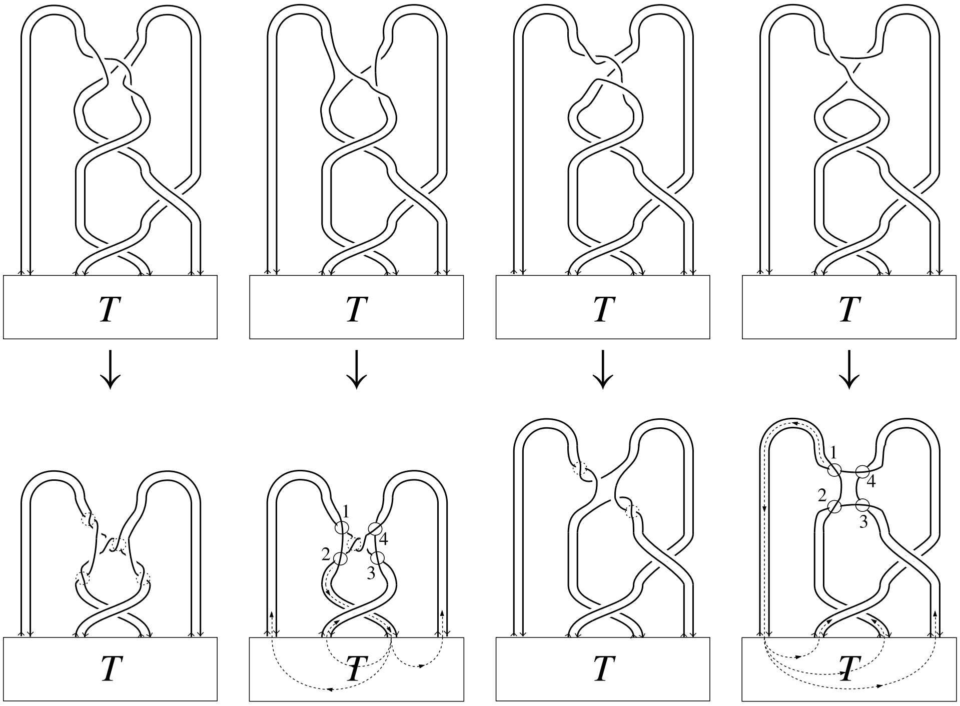
Proof of Claim 8. We prove Claim 8 only for the case where A_i^n is the t_i-twisted doubled link of a 2-bridge knot K_i. The other case can be proved similarly.

We see that K_i has a diagram D_i of Conway's normal form $C(2, a_1, a_2, \ldots, a_{m_i}, 2)$ or $C(-2, -a_1, -a_2, \ldots, -a_{m_i}, -2)$, where $a_j = 1$ $(1 \leq j \leq m_i)$ and $m_1 = n - 5$, $m_6 = n - 6$ and $m_7 = n - 7$. Let w_i be the writhe of D_i. If $t_i = w_i$, we have the conclusion obviously. Suppose $t_i - w_i > 0$. (The case $t_i - w_i < 0$ can be proved similarly.) For A_i^n $(i = 1, 6, 7)$ by a skein relation for the crossing in the dotted circle in Fig. 6, we have:

$$P_{A_i^n} = v^{-2} P_{L'} - v^{-1} z P_{L''}$$

for certain links L' and L''.

Then we see that L'' is equivalent to the trivial knot. (In the case where A_i^n is a (t_i, t_i')-twisted doubled link of a 2-bridge link, L'' is the 3-component trivial link or a 3-component link which is the split union of the trivial knot and the boundary of an unknotted, twisted annulus.) On the other hand, L' is $(t_i - 1)$-twisted doubled link of K and $d(L') = d(A_i^n)$. By repeating this procedure if necessary, we obtain the mirror image of $L_{n-3,n-1}$ from A_1^n, $L_{n-4,n-2}$ from A_6^n and the mirror image $L_{n-5,n-3}$ from A_7^n,



(a) B_1^n (b) B_2^n (c) B_3^n (d) B_4^n

Fig. 7.

respectively as the result of crossing changes. Hence we have the conclusion by induction hypothesis. □

Claim 9. $d(B_i^n) \leq 2n - 6$ *for* $i = 1, 2, 3, 4$.

Proof of Claim 9. First we deform the upper diagram B_i^n $(i = 1, 2, 3, 4)$ into the lower diagram as in Fig. 7, where each rectangle contains the same tangle T.

Now we consider the lower diagrams of B_1^n and B_3^n. Then by performing crossing changes at the crossings in the dotted circles in Fig. 7 (a) (resp. Fig. 7 (c)), we obtain a diagram with $2n - 3$ Seifert circles and $4n - 12$ crossings from the lower diagram of B_1^n (resp. a diagram with $2n - 1$ Seifert circles and $4n - 8$ crossings from the lower diagram of B_3^n). On the other hand, smoothing at each crossing yields a 2-component link which is the boundary of an unknotted, twisted annulus (or a link with $d(L_{n-5,n-3})$ for B_1^n).

For the lower diagram of B_2^n, we consider the crossings labeled 1, 2, 3 and 4 as in Fig. 7 (b). Then by moving the crossing 2 along the dotted line as in Fig. 7 (b), we see that the crossing 2 and one of the other labeled crossings are cancelled wherever the crossing 2 reaches. Hence we see that the other pair of crossings a, b say, forms a right-handed full-twist. Then by performing crossing changes at either a or b and the crossing in the dotted circle in Fig. 7 (b), we obtain a diagram with $2n-3$ Seifert circles and $4n-12$ crossings from the lower diagram of B_2^n. On the other hand, smoothing at each crossing yields also a 2-components link which is the boundary of an unknotted, twisted annulus, or a link with $d(L_{n-5,n-3})$.

For the lower diagram of B_4^n, we also consider the crossings labeled 1, 2, 3 and 4 as in Fig. 7 (d). Note that the crossings labeled 1 and 4 are positive, and the crossings labeled 2 and 3 are negative. Hence there are two cases to be considered.

CASE 1. The crossing 1 and the crossing 2 or 3 are cancelled by moving the crossing 1 along the dotted line as in Fig. 7 (d). Then we see that the other crossings are also cancelled.

CASE 2. The crossing 1 and the crossing 4 form a right-handed full-twist by moving the crossing 1 along the dotted line as in Fig. 7 (d). Then the other pair of crossings 2 and 3 forms a left-handed full-twist. In this case, we perform crossing changes at the crossings 1 and 2.

In both cases we obtain a diagram with $2n-1$ Seifert circles and $4n-8$ crossings. Smoothing at each crossing yields also a 2-components link which is the boundary of an unknotted, twisted annulus.

Then we obtain, by Morton's inequality and induction hypothesis,

$$\begin{aligned} d(B_i^n) &\leq \max\{(4n-12)-(2n-3)+1, (4n-8)-(2n-1)+1, d(L_{n-5,n-3})+1\} \\ &= \max\{2n-8, 2n-6, 2(n-3)-1+1\} \\ &= 2n-6. \end{aligned}$$

The proof of Claim 9 is completed. □

By inequality (1) and Claim 9, we have

$$d(L_{n-2,n}) \leq \max\left\{d\left(A_1^n\right)+2, d\left(A_6^n\right)+2, d\left(A_7^n\right), (2n-6)+1\right\}. \tag{2}$$

Since $d(A_1^n)$, $d(A_6^n)$ and $d(A_7^n)$ are equal to $d(L_{n-3,n-1})$, $d(L_{n-4,n-2})$, $d(L_{n-5,n-3})$, respectively, by Claim 8, it follows from induction hypothesis

$$\begin{aligned} d(L_{n-2,n}) &\leq \max\{d(L_{n-3,n-1})+2, d(L_{n-4,n-2})+2, d(L_{n-5,n-3}), 2n-5\} \\ &= \max\{2(n-1)-1+2, 2(n-2)-1+2, 2(n-3)-1, 2n-5\} \\ &= \max\{2n-1, 2n-3, 2n-7, 2n-5\}. \end{aligned}$$

Since there exist the terms in $P_{L_{n-2,n}}$ whose degree in z is $2n-1$, we obtain $d(L_{n-2,n}) = 2n-1$. This completes the proof of Lemma 6. □

Next, we study the third case $(2, k)$.

Lemma 10. *For the doubled link $L_{2,n}$ of a 2-bridge link $C(a_1, a_2)$, we have*:

$$d(L_{2,n}) = 2n - 1.$$

Proof of Lemma 10. We prove Lemma 10 by induction on n. First, direct calculations show that $d(L_{2,4}) = 7$ and $d(L_{2,5}) = 9$. Now assume that Lemma 10 holds for every positive integer less than n (≥ 6).

We construct a partial resolution tree for $L_{2,n}$ as in the proof of Lemma 6. We also obtain $A_1^n, A_2^n, \dots, A_7^n$ and $B_1^n, \dots, B_4^n$. Note that Claim 7 holds for these $A_2^n, \dots, A_5^n$ and Claim 8 holds for A_1^n, A_6^n and A_7^n.

If $a_1 > 2$, then we see that $d(A_1^n) = d(L_{2,n-1})$, $d(A_6^n) = d(L_{1,n-a_1})$ and $d(A_7^n) = d(L_{2,n-2})$ (or $d(L_{1,n-2})$ if $a_1 = 3$), respectively by Claim 8. If $a_1 = 2$, then we have $d(A_1^n) = d(L_{1,n-1})$, $d(A_6^n) = d(L_{1,n-2})$ and $d(A_7^n) = 1$, since A_7^n is equivalent to the boundary of an unknotted, a_2-twisted annulus. Hence by induction hypothesis, for $i = 1, 2, 3, 4$, we have,

$$\begin{aligned} d(L_{2,n}) &\leq \max\{d(L_{2,n-1}) + 2, d(L_{1,n-a_1}) + 2, d(L_{2,n-2}), d(B_i) + 1\} \\ &= \max\{2(n-1) - 1 + 2, 2(n-a_1) - 1 + 2, 2(n-2) - 1, d(B_i) + 1\} \\ &= \max\{2n - 1, 2n - 2a_1 + 1, 2n - 5, d(B_i) + 1\}. \end{aligned}$$

Since $a_1 \geq 2$, it follows for $i = 1, 2, 3, 4$,

$$d(L_{2,n}) \leq \max\{2n - 1, d(B_i) + 1\}. \tag{3}$$

Claim 11. $d(B_i^n) < 2n - 2$ *for* $i = 1, 2, 3, 4$.

Proof of Claim 11. We consider two cases such as $a_1 = 2$ and $a_1 > 2$.

CASE 1. Suppose $a_1 = 2$. We can deform the diagram of B_i^n into a diagram whose canonical Seifert surface has the first Betti number two. See Fig. 8, which illustrates the case of B_1^n. (The diagram of B_i^n (for $i = 2, 3, 4$) can be deformed similarly.) Therefore we have $d(B_i^n) \leq 2$ by Morton's inequality.

CASE 2. Suppose $a_1 > 2$. We can deform both diagrams of B_3^n and B_4^n into diagrams with $3a_2 + 3$ Seifert circles and $3a_2 + 4$ crossings. Thus the conclusion follows from Morton's inequality.

We deform the diagram B_1^n into the diagram in Fig. 9 (b). Then, we obtain the diagram B_2^{n-1} by applying crossing change at each of the crossings labeled $1, 2, 3$ and 4 as indicated in Fig. 9 (b). In order to calculate $d(B_1^n)$, we construct a partial resolution tree with respect to the crossings $1, 2, 3$ and 4 in this order. Let L_i be the link obtained from B_1^n by a smoothing at the crossing labeled i for $i = 1, 2, 3, 4$ respectively.

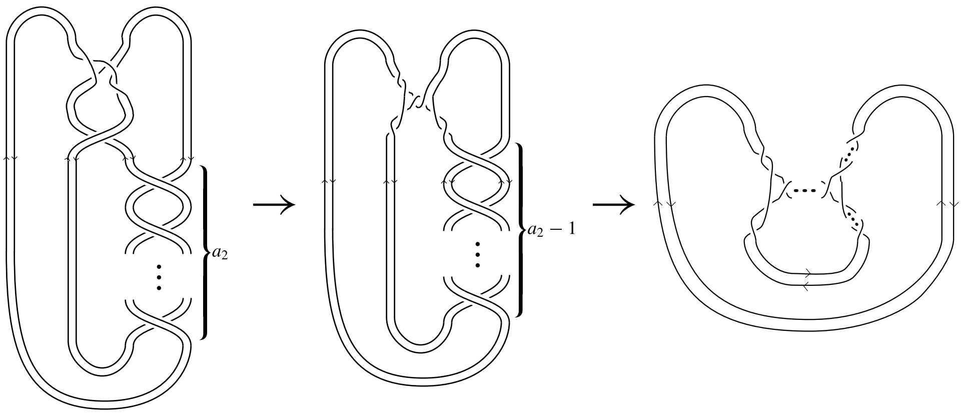
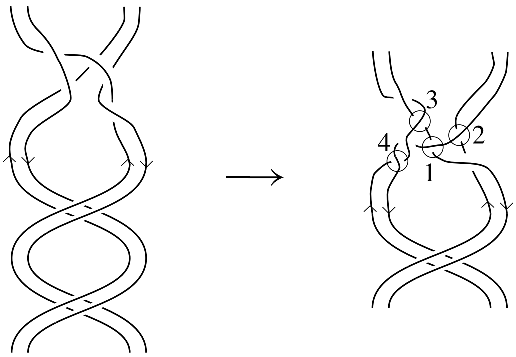


Fig. 8.



(a) B_1^n (b)

Fig. 9.

Then we see that $d(L_1) = d\left(A_1^{n-1}\right)$ by Claim 8. We also see that none of three links L_2, L_3 and L_4 contributes anything to $d(B_1^n)$ by the argument parallel to that in the proof of Claim 7. Thus we obtain

$$d\left(B_1^n\right) \leq \max\left\{d\left(A_1^{n-1}\right) + 1, d\left(B_2^{n-1}\right)\right\}.$$

Furthermore we obtain the following inequality by replacing B_1^n by B_2^{n-1} in the above argument:

$$d\left(B_2^{n-1}\right) \leq \max\left\{d\left(A_1^{n-2}\right) + 1, d\left(B_1^{n-2}\right)\right\}.$$

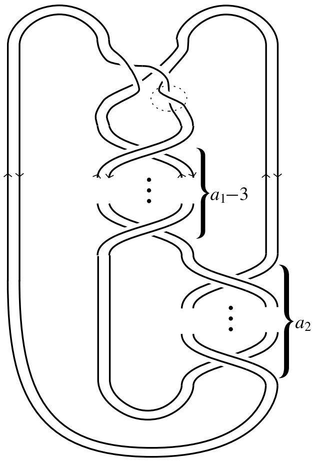


Fig. 10.

Therefore we have by induction hypothesis

$$\begin{aligned} d(B_1^n) &\le \max\left\{d\left(A_1^{n-1}\right)+1,\, d\left(B_1^{n-2}\right)\right\} \\ &= \max\left\{2(n-2)-1+1,\, d\left(B_1^{n-2}\right)\right\}. \end{aligned}$$

Furthermore by performing a crossing change at the crossing in the dotted circle in Fig. 10, we obtain from B_1^{n-2} a new diagram with $2(a_1+a_2)-4$ Seifert circles and $4(a_1+a_2)-9$ crossings. Smoothing at this crossing yields a 2-components link which is the boundary of an unknotted twisted annulus. By a skein relation, we see that it does not contribute anything to $d\left(B_1^{n-2}\right)$.

Since $a_1+a_2=n$, we have $d\left(B_1^{n-2}\right)\le 2n-4$ by Morton's inequality, and hence, $d(B_1^n)\le 2n-4$.

We apply this argument to B_2^n, and we have, by induction hypothesis,

$$\begin{aligned} d(B_2^n) &\le \max\left\{d\left(A_1^{n-1}\right)+1,\, d\left(B_2^{n-2}\right)\right\} \\ &= \max\left\{2(n-2)-1+1,\, d\left(B_2^{n-2}\right)\right\}. \end{aligned}$$

Now since B_2^{n-2} has a diagram with $2(a_1+a_2)-5$ Seifert circles and $4(a_1+a_2)-9$ crossings, we have $d\left(B_2^{n-2}\right)\le 2n-3$ by Morton's inequality, and thus, $d(B_2^n)\le 2n-3$. This completes the proof of Claim 11. $\square$

By the above claims and inequality (3), we have $d(L_{2,n})=2n-1$. The proof of Lemma 10 is completed. $\square$

4. Proof of Theorem 1

In this section, we prove our main theorem, Theorem 1. For this purpose, first we prove Proposition 5 by induction on the lexicographic order of m, k.

Proof of Proposition 5. Assume that Proposition 5 holds for a pair of positive integers less than (m, k) as the lexicographic order and $m > 2$, $k > 2$, since the initial cases have been proved in [14, Proposition 1], Lemmas 6 and 10. There are two cases to be considered such as $a_1 = 2$ and $a_1 > 2$. For both cases, we construct a partial resolution tree for $L_{m,n}$ as in the proofs of Lemmas 6 and 10. We also obtain $A_1^n, A_2^n, \dots, A_7^n$ and $B_1^n, \dots, B_4^n$. Note that Claim 7 holds for these $A_2^n, \dots, A_5^n$ and Claim 8 holds for A_1^n, A_6^n and A_7^n. Hence we have the following inequality.

$$d(L_{m,n}) \le \max\{d(A_1^n) + 2, d(A_6^n) + 2, d(A_7^n), d(B_i^n) + 1\}. \tag{4}$$

CASE 1. Suppose $a_1 = 2$. In this case, we see that A_1^n is equivalent to the mirror image of $L_{m-1,n-1}$, A_6^n is equivalent to the mirror image of $L_{m-1,n-2}$ (or to $L_{m-2,n-2}$ if $a_2 = 1$) and A_7^n is equivalent to $L_{m-2,n-(2+a_2)}$ (or to the mirror image of $L_{m-3,n-2}$ if $a_3 = 1$) except the number of twists. By induction hypothesis, we have $d(A_1^n) = 2(n-1) - 1$, $d(A_6^n) = 2(n-2) - 1$ and $d(A_7^n) = 2(n-2-a_2) - 1$. For the evaluation of $d(B_i^n)$, we apply the argument parallel to that in the proof of Claim 9 if $a_2 = 1$, or to that in the proof of Case 1 in Claim 11 if $a_2 > 1$. Then we see that B_i^n cannot contribute anything to $d(L_{m,n})$.

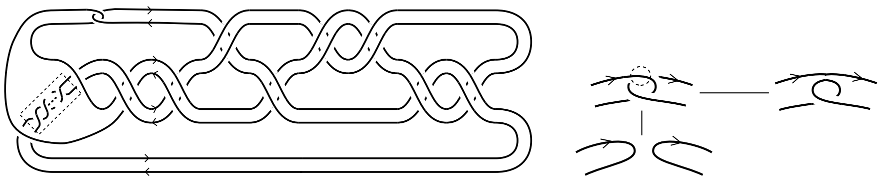
CASE 2. Suppose $a_1 > 2$. We see that A_1^n is equivalent to $L_{m,n-1}$, A_6^n is equivalent to the mirror image of $L_{m-1,n-a_1}$ (or to $L_{m-2,n-a_1}$ if $a_2 = 1$) and A_7^n is equivalent to $L_{m,n-2}$ (or to the mirror image of $L_{m-1,n-2}$ if $a_1 = 3$) except the number of twists. By induction hypothesis, we have $d(A_1^n) = 2(n-1) - 1$, $d(A_6^n) = 2(n-a_1) - 1$ and $d(A_7^n) = 2(n-2) - 1$. By applying the argument similar to that in Case 2 in Claim 11, we see that B_i^n cannot contribute anything to $d(L_{m,n})$.

For both cases, we obtain $d(L_{m,n}) = 2n - 1$ by inequality (4). This completes the proof of Proposition 5. □

Proof of Theorem 1. Let K be a Whitehead double of a 2-bridge knot $C(a_1, \dots, a_m)$ with $a_i > 0$ for any i and $a_1 + \cdots + a_m = n$. We see that the genus of a canonical Seifert surface obtained from the diagram of K as in Fig. 11 (a) is equal to n. (Although the diagram as in Fig. 11 (a) is a diagram of a Whitehead double of $C(3, 1, 1, 2, 2)$, we can easily see the general case. We note that this kind of diagram was appeared in [3] for the trefoil knot and also observed by Tripp for the other torus knots of $(2, n)$ in [14].)

At the crossing in the Whitehead clasp as indicated in Fig. 11 (b), we perform a crossing change and a smoothing. Then by a skein relation, we have

$$P_K(v, z) = v^2 P_O(v, z) + vzP_L,$$



(a) (b)

Fig. 11.

where O is the trivial knot and L is a doubled link of $C(a_1, \dots, a_m)$. Then we see that $d(L) = d(L_{m,n}) = 2n - 1$ by applying the argument similar to that in the proof of Claim 8 to the twists in the dotted rectangle in Fig. 11 (a). Hence we obtain that the maximal degree in z of $P_K(v, z)$ is equal to $2n$ by Proposition 5. Therefore the canonical genus of K is equal to n by Morton's inequality. The proof is now completed. $\square$

5. Concluding remark

Let K be a knot of crossings number $c(K) = n$ (≤ 10) and $D(K)$ a twisted (possibly untwisted) Whitehead double of K. In order to consider Tripp's conjecture in general case, we calculate the maximal degree in z, say $d(D(K))$, of HOMFLY polynomial for $D(K)$ by a computer software. Then we see that $d(D(K)) = 2n$ if K is alternating. Hence Tripp's conjecture is true for alternating knots of ten crossings or less. Then we propose the following conjecture.

Conjecture 12. *For any alternating knot K of crossing number n, we have $d(D(K)) = 2n$. Therefore $g_c(D(K)) = c(K) = n$.*

If K is not alternating, this conjecture is false. For example, let K be a torus knot of type $(4, 3)$. It is known that the crossing number of K is equal to 8 (cf. [10]). However a computer calculation shows $d(D(K)) = 14$.

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