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SINGULARITY OF NORMAL COMPLEX ANALYTIC SURFACES ADMITTING NON-ISOMORPHIC FINITE SURJECTIVE ENDOMORPHISMS

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Abstract

For a non-isomorphic finite endomorphism of a germ of a complex analytic normal surface at a point, the pair of the surface and a completely invariant reduced divisor is shown to be log-canonical. It is also shown in many situations that the endomorphism or its square lifts to an endomorphism of another surface by an essential blowing up.

Contents

0. Introduction

We study the singularity of a complex analytic normal surface admitting a non-isomorphic finite surjective endomorphism. More precisely, we consider an endomorphism $\mathfrak{f}$ of the germ $\mathfrak{X} = (X, x)$ of a normal surface X at a point x in which $\mathfrak{f}$ is finite of degree > 1. The singularity of $\mathfrak{X}$ has been shown to be log-canonical by Wahl [62]: In the proof, an invariant $-P \cdot P$ concerning the relative Zariski-decomposition plays an essential role. In [6, Thm. B], Favre proves the log-canonicity by another method applying the theory of valuation spaces, where he proves furthermore that $\mathfrak{X}$ is a quotient singularity when $\mathfrak{f}$ ramifies on $X \setminus \{x\}$. There are also some remarkable results in [6] on the liftability of $\mathfrak{f}$ by bimeromorphic morphisms $Y \to X$ from normal surfaces Y. In this article, we classify the singularity of $\mathfrak{X}$ and check the liftability of $\mathfrak{f}$ by standard arguments of algebraic geometry not using valuation spaces. For the singularity, we consider not only $\mathfrak{X}$ but also the germ at x of the pair (X, S) with a reduced divisor S such that $\mathfrak{f}^{-1}S = S$ set-theoretically; such a divisor S is said to be *completely invariant* under $\mathfrak{f}$. As a generalization of [62] and [6, Thm. B], we can prove:

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Theorem 0.1. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ be a finite surjective endomorphism of a germ $\mathfrak{X} = (X, x)$ of a normal surface X at a point x. Let $\mathfrak{S}$ be the germ (S, x) of a reduced divisor $S \subset X$ at x. Here, S may not contain x. Assume that* $\deg \mathfrak{f} > 1$ *and* $\mathfrak{f}^{-1}\mathfrak{S} = \mathfrak{S}$. *Then (X, S) is log-canonical at x. If $\mathfrak{f}$ is not étale on $\mathfrak{X} \setminus \mathfrak{S}$, then (X, S) is* 1-log-terminal *at x (cf.* Definition 2.1).

The 1-log-terminal is called "purely log terminal" in many articles (see Remark 2.3 below). Note that singularities of 2-dimensional log-canonical pairs with reduced boundary divisors are classified by [30, Thm. 9.6] (cf. [55, App.], [35, Ch. 3]). Theorem 0.1 is a direct consequence of Theorem 3.5 in Section 3 below. On the liftability of $\mathfrak{f}$, [6, Prop. 2.1] is generalized to:

Theorem 0.2. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ be a non-isomorphic finite endomorphism of a germ $\mathfrak{X} = (X, x)$ of a normal surface X at a point x. Let $\varphi\colon Y \to X$ be a bimeromorphic morphism such that $E = \varphi^{-1}(x)$ is a divisor and φ is an isomorphism over $X \setminus \{x\}$. Let $\Phi\colon \mathfrak{Y} \to \mathfrak{X}$ be the morphism induced by φ for the* germ $\mathfrak{Y} = (Y, E)$ of Y along E (*cf. Notation and conventions, below*). *Then there is an endomorphism $\mathfrak{g}\colon \mathfrak{Y} \to \mathfrak{Y}$ such that $\Phi \circ \mathfrak{g} = \mathfrak{f}^2 \circ \Phi$ for the square $\mathfrak{f}^2 = \mathfrak{f} \circ \mathfrak{f}$ provided that one of the following conditions is satisfied*:

(I) *The endomorphism $\mathfrak{f}$ is étale outside $\{x\}$, φ is an* essential blowing up (*cf.* Definition 4.24 *below*) *of the log-canonical singularity $\mathfrak{X}$, and $\mathfrak{X}$ is not a cusp singularity.*

(II) *There is a reduced divisor $S \ni x$ such that*
- $\mathfrak{f}^*\mathfrak{S} = d\mathfrak{S}$ *for an integer $d > 0$ and $\mathfrak{f}$ is étale on $\mathfrak{X} \setminus \mathfrak{S}$ for the germ $\mathfrak{S} = (S, x)$ of S at x, and*
- φ *is an* essential blowing up *at x with respect to* (X, S).

Remark. If φ is an essential blowing up with respect to a log-canonical pair (X, S) of a normal surface X and a reduced divisor S, then $K_Y + S_Y = \varphi^*(K_X + S)$ for the reduced divisor $S_Y = \varphi^{-1}S$, in which (Y, S_Y) is log-canonical, and moreover, it is 1-log-terminal at any point of $Y \setminus \operatorname{Sing} S_Y$ (cf. Definition 4.24); in particular, Y has only quotient singularities. Since φ is not an isomorphism, the singularity $\mathfrak{X} = (X, x)$ is not log-terminal in (I), and the pair (X, S) is not 1-log-terminal at x in (II). Hence, by the classification of log-canonical singularities (cf. [30, Thm. 9.6]), in case (I), $\mathfrak{X}$ is a simple elliptic singularity or a rational singularity whose index 1 cover is either a simple elliptic singularity or a cusp singularity. In case (II), one of the cases (1) and (3) in Fact 2.5 below occurs for (X, S) at x.

Remark. The case (I) is treated in [6, Prop. 2.1] for a certain partial resolution of singularities of $\mathfrak{X}$ and it is stated that not only $\mathfrak{f}^2$ the endomorphism $\mathfrak{f}$ itself lifts to an endomorphism of $\mathfrak{Y}$: The corresponding result is given in Lemmas 5.23 and 5.24 below. Unfortunately, the proof of [6, Prop. 2.1] seems to omit the case where "$F_\bullet$ permutes two branched points of $\Gamma(\mu)$," and the author could not understand why "F (not only F^2) lifts to a holomorphic endomorphism of $\overline{X}$" as stated in [6, Prop. 2.1]. This question is solved in Lemma 5.24 below, as a consequence of our key theorem, Theorem 5.10. We need to exclude cusp singularities in (I) by the remarkable example constructed in [6, Prop. 2.2].

Theorem 0.2 is a direct consequence of Theorem 5.3 in Section 5 below. In Theorems 3.5 and 5.3, instead of an endomorphism of a germ $\mathfrak{X} = (X, x)$ of normal surface X at a point x, we consider more generally a morphism $f\colon X^\circ \to X$ from an open neighborhood X° of x such that f has only discrete fibers, $f^{-1}(x) = \{x\}$, and $\deg_x f > 1$ (cf. Definition 1.9): A non-isomorphic finite endomorphism of the germ $\mathfrak{X}$ is induced by such a morphism f (cf. Remark 3.2).

Organization of this article. Our methods proving theorems above are based on standard arguments on the following topics:

(1) Some morphisms of complex analytic varieties.
(2) Numerical pullbacks of divisors on normal surfaces by non-generate morphisms.
(3) Logarithmic ramification formula.
(4) Classification of 2-dimensional log-canonical singularities of pairs with reduced boundary divisors.
(5) 2-dimensional relative abundance theorem for log-canonical pairs.
(6) Theory of toric surfaces.
(7) Description of cyclic covers.
(8) Essential blowings up.
(9) Dual $\mathbb{R}$-divisors.

We shall explain the organization of this article by these topics. In Section 1, we shall discuss topics (1), (2), and (3). Concerning (1), in Section 1.1, we consider: *morphisms of maximal rank*, *non-degenerate morphisms*, *fully equi-dimensional morphisms*, and *discretely proper morphisms*. Here, the notion of a morphism of maximal rank (resp. a non-degenerate morphism) of complex analytic varieties is analogous to that of a dominant (resp. generically finite and dominant) morphism of integral algebraic schemes. In the discussion in Section 1.1, we borrow many results from [7]. Some basics on divisors on normal complex analytic varieties are explained in Section 1.2, and the topic (2) on divisors on normal surfaces is treated in Section 1.3. Note that the pullback of a Cartier divisor by a morphism of maximal rank is canonically defined, but the pullback of a (Weil) divisor is not defined in general. We have the numerical pullback of a (Weil) divisor by a non-degenerate morphism of normal surfaces: this is known as the Mumford pullback (cf. [36, II, §(b)]) in the case of bimeromorphic morphisms. In this article, the numerical pullback is regarded as the standard pullback for divisors. Remarks on pullbacks and pushforwards of divisors by meromorphic mappings are studied in Section 1.4, which are used in Section 5.3. For (3), in Section 1.5, the logarithmic ramification formula due to Iitaka (cf. [24, §4, (R)], [25, Prop. 2.1]) and its generalizations are given with explanations of the canonical divisor and the ramification divisor.

In Section 2, we treat topics (4) and (5). The log-canonical, log-terminal, and 1-log-terminal singularities for pairs of normal surfaces and effective $\mathbb{Q}$-divisors are defined in Section 2.1 in a little different style from the popular one (cf. Definition 2.1). See Remarks 2.3 and 2.8 for a difference from similar definitions in other articles. In Section 2.2, we give comparison results on log-canonicity etc. for some non-degenerate morphisms of normal surfaces by applying formulas in Section 1.5. The relative abundance theorem in (5) is treated in Section 2.3. This theorem is known in the algebraic case, but the proof seems to be omitted and not given in the complex analytic case. Our proof is based on ideas of Fujita [11] and Kawamata [30] (cf. Theorem 2.19 below). By (5), we define the *log-canonical modification* (see Lemma-Definition 2.22), which plays an important role in the proof of Theorem 3.5.

Some readers may think Sections 1 and 2 superfluous, as most results there are well known at least in the algebraic case. But, we need to confirm some of them in the complex analytic case, since we can not work in the algebraic category. Not all the results in Sections 1 and 2 are used in the other sections of this article, but it is worthwhile to prove them in a general form by the absence of good references in the complex analytic case on the same topics.

The purpose of Section 3 is to prove Theorem 3.5, from which Theorem 0.1 is deduced directly. In Section 3.1, we give the statement and corollaries, and prove its 1-dimensional analogue as Proposition 3.4 below. Theorem 3.5 is proved in Section 3.2 gradually by applying results in Sections 1.5, 2.1, and 2.3.

In Section 4, we shall discuss topics (6)–(9). For (6), some basics on affine toric surfaces are explained briefly in Section 4.1 with properties of morphisms of toric surfaces. For (7), we review the construction of cyclic covers by Esnault and Viehweg in Section 4.2 in a different way from the original, introduce the notion of an *index* 1*-cover* (cf. Definition 4.18), and give a criterion for endomorphisms to lift to index 1-covers (cf. Lemma 4.21). The essential blowing up in (8) is defined in Section 4.3 for log-canonical pairs (X, B) of normal surfaces with reduced divisors, where we discuss the comparison of two essential blowings up (cf. Lemma 4.32 and Corollary 4.33). The name comes from the "essential divisor" on the resolution of a normal surface singularity (cf. [27, Def. 3.3]). The dual $\mathbb{R}$-divisor in (9) is discussed in Section 4.4; it is defined for a normal surface with a compact connected divisor having negative definite intersection matrix. The notion of dual $\mathbb{R}$-divisors comes from arguments in [6, §1.2], where the duals are considered as projective limits of Weil divisors on resolutions (cf. [6, Def. 1.3]).

Section 5 is devoted to proving Theorem 5.3, from which Theorem 0.2 is deduced directly. In Section 5.1, we give the statement explaining our setting on the lifting property. The proof of Theorem 5.3 in the case (II) is given in Section 5.2 by applying results in Sections 4.1, 4.2, and 4.3. For Theorem 5.3 in the case (I), we prove a key theorem (Theorem 5.10) in Section 5.3, and we complete the proof in Section 5.4.

Background. This article is a revised version of a part of a preprint [40] of the author written in 2008, which deals with the classification of normal Moishezon surfaces X admitting non-isomorphic surjective endomorphisms. Even though [40] is non-public and was sent only to limited persons, it has been distributed more widely than the author thought. A preliminary part of [40] is included in the published article [41], and this article and recent preprints [42] and [43] cover the rest of [40]. As a theorem in [40], the author proved that (X, S) is log-canonical for any completely invariant divisor S. The log-canonicity of (X, S) at a point $x \in S$ was shown by using the log-canonical modification (see Lemma-Definition 2.22 below). The log-canonicity of (X, S) at $x \notin S$ is a consequence of results of Wahl [62] or Favre [6]: The author was informed by Favre of their results when preparing [40], and gave a modified proof in [40]. Theorem 3.5 below gives a further modification. The liftability problem of $\mathfrak{f}$ is treated not in [40] but in some modified versions of [40] around 2010.

Notation and conventions. In this article, any complex analytic space is assumed to be Hausdorff and to have a countable open base.

– A *variety* means a complex analytic variety, i.e., an irreducible and reduced complex analytic space. Note that an open subset of a variety is not necessarily irreducible, but a *Zariski-open subset*, the complement of an analytic subset, is a variety (cf. [15, IX, §1.2]).

– For a variety X, the non-singular (resp. singular) locus is denoted by X_{reg} (resp. $\operatorname{Sing} X$). Note that the dimension of X is defined as that of the complex manifold X_{reg}.

– A local isomorphism of complex analytic spaces is called an *étale* morphism. A morphism $f\colon X \to Y$ of normal complex analytic spaces is said to be *étale in codimension* 1 if $f|_{X\setminus Z}\colon X \setminus Z \to Y$ is étale for an analytic subset Z of codimension ≥ 2.

– For the local ring $\mathcal{O}_{X,x}$ of a point x of a complex analytic space X, the maximal ideal is denoted by $\mathfrak{m}_x$ and the residue field by $\mathbb{C}(x)$. The *local dimension* of X at x denoted by $\dim_x X$ is defined as $\dim \mathcal{O}_{X,x}$ (cf. [7, §3.1]).

– The *germ* $\mathfrak{X} = (X, S)$ of a complex analytic space X *along* a subset S is a *pro-object* (cf. [19, §8.10], [28, Def. 6.1.1]) of the category (An) of complex analytic spaces defined as

$$\text{“}\varprojlim\text{”}_{X'\in \mathsf{U}(S)} X',$$

where $\mathsf{U}(S)$ is the category of open neighborhoods of S whose morphisms are open immersions and where “$\varprojlim$” is the projective limit in the category of presheaves on (An) (cf. [19, (8.5.3.2)], [28, Not. 2.6.2]). For the germ $\mathfrak{Y} = (Y, T)$ of another complex analytic space Y along a subset T, a morphism $\mathfrak{X} = (X, S) \to \mathfrak{Y} = (Y, T)$ of germs is defined as a morphism of pro-objects. Since Y is Hausdorff and since

$$\operatorname{Hom}_{\mathrm{Pro(An)}}(\mathfrak{X}, \mathfrak{Y}) = \varprojlim_{Y'\in \mathsf{U}(T)} \varinjlim_{X'\in \mathsf{U}(S)} \operatorname{Hom}_{(\mathrm{An})}(X', Y')$$

for the category Pro(An) of pro-objects of (An) (cf. [19, (8.2.5.1), (8.10.5)], [28, (2.6.3), (2.6.4)]), a morphism $\mathfrak{X} \to \mathfrak{Y}$ of germs is represented by a morphism $f\colon X' \to Y'$ in (An) for some $X' \in \mathsf{U}(S)$ and $Y' \in \mathsf{U}(T)$ such that $f(S) \subset T$.

1. Preliminaries on complex analytic varieties

We shall discuss some morphisms of complex analytic varieties (Section 1.1), basics on divisors (Section 1.2), numerical pullbacks of divisors on normal surfaces (Section 1.3), pullbacks and pushforwards of divisors by meromorphic maps (Section 1.4), canonical divisors, and the ramification formula (Section 1.5).

1.1. Morphisms of complex analytic varieties. We shall explain basic properties of some morphisms of varieties, which consist of: morphisms of maximal rank, non-degenerate morphisms, fully equi-dimensional morphisms, and discretely proper morphisms. The ambiguous notion of a “generically finite morphism” is replaced by the notion of a non-degenerate morphism. A base change property by a fully equi-dimensional morphism is also given (cf. Lemma 1.13). We refer the readers to [7] for some basics on complex analytic spaces.

Definition 1.1. Let $f\colon X \to Y$ be a morphism of varieties.

(1) If f is smooth at a point of $X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}}) \neq \emptyset$, then f is said to be *of maximal rank*.

(2) If f is of maximal rank and $\dim X = \dim Y$, then f is said to be *non-degenerate*.

(3) If $\dim_x f^{-1}(f(x)) = \dim X - \dim Y$ for any $x \in X$, then f is said to be *fully equi-dimensional*.

Remark 1.2. For a point $x \in X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$, the smoothness of f at x is equivalent to each of the following conditions:

- The tangent map $T_x X \to T_{f(x)} Y$ is surjective, where $T_x X$ denotes the tangent space of X at x.
- The canonical pullback homomorphism $f^* \Omega^1_Y \to \Omega^1_X$ of holomorphic 1-forms is injective at x and its cokernel $\Omega^1_{X/Y}$ is free at x, where $\Omega^1_{X/Y}$ denotes the sheaf of relative 1-forms, and $\Omega^1_X := \Omega^1_{X/\operatorname{Spec}\mathbb{C}}$.
- The morphism f is flat at x and the scheme-theoretic fiber $f^{-1}(f(x))$ over $f(x)$ is non-singular at x.
- The morphism f is a *submersion* at x (cf. [7, §2.18]) in the sense that an open neighborhood $\mathcal{U}$ of x is isomorphic to the product $F \times \mathcal{V}$ of an open neighborhood $\mathcal{V}$ of $f(x)$ in Y and a non-singular variety F such that $f|_{\mathcal{U}}$ is isomorphic to the composite of the projection $F \times \mathcal{V} \to \mathcal{V}$ and the immersion $\mathcal{V} \hookrightarrow Y$.

Remark. Let $\mathsf{f}\colon \mathsf{X} \to \mathsf{Y}$ be a morphism of integral separated algebraic schemes over $\mathbb{C}$ and assume that f is the associated morphism $\mathsf{f}^{\mathrm{an}}\colon \mathsf{X}^{\mathrm{an}} \to \mathsf{Y}^{\mathrm{an}}$ of complex analytic varieties (cf. [18, XII, §1]). Then f is of maximal rank (resp. non-degenerate) if and only if f is dominant (resp. dominant and generically finite). Moreover, f is fully equi-dimensional if and only if f is dominant and *equi-dimensional* in the sense of [16, Déf. (13.2.2), (Err$_{\mathrm{IV}}$, 34)].

Lemma 1.3. *For a morphism $f\colon X \to Y$ of varieties, the following conditions are equivalent*:

(i) *f is of maximal rank*;
(ii) *$f(X)$ contains a non-empty open subset of Y*;
(ii′) *$f(X)$ contains a non-empty open subset of Y which is dense in $f(X)$*;
(iii) $\min_{x \in X} \dim_x f^{-1}(f(x)) = \dim X - \dim Y$;
(iv) *$f|_{X'}\colon X' \to Y$ is smooth for a dense Zariski-open subset X' of X*;
(v) *$f|_{X''}\colon X'' \to Y$ is fully equi-dimensional for a dense Zariski-open subset X'' of X.*

Proof. The implications (iv) ⇒ (i) and (ii′) ⇒ (ii) are trivial. If (i) holds, then

$$\{x \in X_{\mathrm{reg}} \mid f(x) \in Y_{\mathrm{reg}} \quad \text{and} \quad \dim \Omega^1_{X/Y} \otimes \mathbb{C}(x) = \dim X - \dim Y\}$$

is a dense Zariski-open subset by [7, §2.17, Lem.], and it implies (iv) by Remark 1.2. We can prove (iv) ⇒ (iii) and (iii) ⇒ (v) by the upper semi-continuity of the function $x \mapsto \dim_x f^{-1}(f(x))$ with respect to the Zariski topology (cf. [50, §3, Satz 17], [7, §3.6, Thm.]). If (v) holds, then

$$\dim_x X'' \cap f^{-1} \operatorname{Sing} Y \leq \dim_x f^{-1} f(x) + \dim_{f(x)} \operatorname{Sing} Y < \dim X = \dim X''$$

for any $x \in X'' \cap f^{-1} \operatorname{Sing} Y$ (cf. [7, §3.9, Prop.]); hence, $X''' = X'' \cap f^{-1} Y_{\mathrm{reg}}$ is also a dense Zariski-open subset of X, and $f(X''') = f(X'') \cap Y_{\mathrm{reg}}$ is an open subset of Y_{reg} by [7, §3.7, Cor.]. Moreover, $f(X''')$ is dense in $f(X)$. In fact, for any $x \in X$ and for any open neighborhood $\mathcal{V}$ of $f(x)$, we have $X''' \cap f^{-1}\mathcal{V} \neq \emptyset$, since X''' is dense in X, and it implies that $\mathcal{V} \cap f(X''') \neq \emptyset$. Thus, we have proved (v) ⇒ (ii′).

For the rest, it suffices to prove (ii) ⇒ (i). We use an argument in the proof of [8, Lem. (IV, 13)]. Replacing Y with Y_{reg}, we may assume that Y is non-singular. The rank of the tangent map $T_xX \to T_{f(x)}Y$ is lower semi-continuous on $x \in X_{\mathrm{reg}}$ (since it equals $\dim X - \dim \Omega^1_{X/Y} \otimes \mathbb{C}(x)$), and we have a unique maximal Zariski-open subset X_o of X_{reg} on which the rank is constant and attains the maximum. Since X is assumed to have a countable open basis, $X \setminus X_o$ is a locally finite countable union of subvarieties $\overline{X}_i$ of dimension less than $\dim X$. Similarly to the above, for each i, we can find a unique maximal Zariski-open subset X_i of $(\overline{X}_i)_{\mathrm{reg}}$ such that the rank of the tangent map $T_x\overline{X}_i \to T_{f(x)}Y$ of the induced morphism $\overline{X}_i \to Y$ is constant on $x \in X_i$ attaining the maximum. Then the complement of $X_o \cup \bigcup X_i$ in X is also a locally finite countable union of subvarieties of dimension less than $\dim X - 1$. By continuing the process, we have a locally finite countable disjoint union $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ of locally closed non-singular analytic subspaces X_λ of X such that the tangent map $T_xX_\lambda \to T_{f(x)}Y$ of $f|_{X_\lambda}$ has constant rank for $x \in X_\lambda$. By [7, §2.19, Cor. 2], locally on X_λ, the morphism $X_\lambda \to Y$ is isomorphic to a submersion to a locally closed submanifold of Y. Since $f(X)$ contains an open subset, $f(X_\lambda)$ is open for some $\lambda \in \Lambda$. We fix such an index λ. Then, for any $x \in X_\lambda$, the composite

$$\Omega^1_Y \otimes \mathbb{C}(f(x)) \to \Omega^1_X \otimes \mathbb{C}(x) \to \Omega^1_{X_\lambda} \otimes \mathbb{C}(x)$$

of canonical linear maps is injective. It implies that the canonical homomorphism $f^*\Omega^1_Y \to \Omega^1_X$ is injective on an open subset U of X containing X_λ. The cokernel $\Omega^1_{X/Y}$ is locally free on a non-empty Zariski-open subset U' of U, since U is reduced (cf. [7, §2.13, Cor.]). Therefore, $f^*\Omega^1_Y$ is a subbundle of Ω^1_X on U', and $f|_{U'} : U' \to Y$ is smooth by Remark 1.2. This shows (ii) $\Rightarrow$ (i), and we are done. □

Remark. If X and Y are non-singular, then (ii) $\Rightarrow$ (i) is a consequence of Sard's theorem on critical values.

Corollary. *A fully equi-dimensional morphism of varieties is of maximal rank. A surjective morphism of varieties is of maximal rank.*

Corollary 1.4. *For a morphism $f : X \to Y$ of varieties of the same dimension, the following conditions are equivalent*:

(i) *f is non-degenerate*;
(ii) *$f(X)$ contains a non-empty open subset of Y (which is dense in $f(X)$)*;
(iii) *there is a point $x \in X$ such that x is isolated in the fiber $f^{-1}(f(x))$*;
(iv) *$f|_{X'}$ is étale for a dense Zariski-open subset X' of X.*

Definition 1.5 ($\deg f$). Let $f : X \to Y$ be a proper non-degenerate morphism of varieties. The *degree* of f, denoted by $\deg f$, is defined as the rank of the coherent $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$. Hence,

$$\deg f = \dim_{\mathbb{C}(y)} f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathbb{C}(y) = \dim_{\mathbb{C}} H^0(\mathcal{O}_{f^{-1}(y)})$$

for a general point $y \in Y$. By Corollary 1.4, we see that $\deg f$ equals the cardinality of $f^{-1}(y)$ for a general point $y \in Y$.

Definition 1.6. A morphism of complex analytic spaces is said to be *discretely proper* if the connected components of the fibers are compact.

Proper morphisms and morphisms with only discrete fibers are discretely proper. Moreover, we know the following as a strong version of the Stein factorization (cf. [57], [2, Thm. 3]):

Fact. *A morphism $f: X \to Y$ of complex analytic space is discretely proper if and only if $f = g \circ \pi$ for a proper morphism $\pi: X \to Y'$ with an isomorphism $\mathcal{O}_{Y'} \simeq \pi_* \mathcal{O}_X$ and for a morphism $g: Y' \to Y$ with only discrete fibers.*

By [7, §1.10, Lem. 1 and §3.2, Lem.], we have:

Lemma 1.7. *Let $f: X \to Y$ be a morphism of complex analytic spaces. For a point $x \in X$ and a connected component Γ of $f^{-1}(f(x))$, if Γ is compact, then there exist an open neighborhood V of $f(x)$ in Y and an open neighborhood U of Γ in $f^{-1}V$ such that $U \cap f^{-1}(f(x)) = \Gamma$ and $f|_U: U \to V$ is proper. If $\Gamma = \{x\}$, then one can choose U and V so that $f|_U$ is a finite morphism.*

Corollary 1.8. *Let $f: X \to Y$ be a morphism of varieties of the same dimension. If $x \in X$ is isolated in $f^{-1}(f(x))$ and if Y is locally irreducible at $f(x)$, then there is an open neighborhood $\mathcal{U}$ of x such that $\mathcal{U} \cap f^{-1}(f(x)) = \{x\}$, $f(\mathcal{U})$ is open, and $f|_{\mathcal{U}}: \mathcal{U} \to f(\mathcal{U})$ is a finite morphism. In particular, if f has only discrete fibers and Y is locally irreducible, then $f(X)$ is open.*

Proof. By assumption and by Lemma 1.7, we have an irreducible open neighborhood $\mathcal{V}$ of $f(x)$ in Y and an open neighborhood $\mathcal{U}$ of x in $f^{-1}\mathcal{V}$ such that $\mathcal{U} \cap f^{-1}(f(x)) = \{x\}$ and $f|_{\mathcal{U}}: \mathcal{U} \to \mathcal{V}$ is finite. Moreover, $f(\mathcal{U}) = \mathcal{V}$ by $\dim_{f(x)} f(\mathcal{U}) = \dim_x \mathcal{U} = \dim X = \dim Y = \dim_{f(x)} \mathcal{V}$ (cf. [7, §3.2, Thm.]). □

Definition 1.9. In the situation of Corollary 1.8, we define the *local degree* of f at x as the degree of $f|_{\mathcal{U}}: \mathcal{U} \to f(\mathcal{U})$ (cf. Definition 1.5): This is independent of the choice of $\mathcal{U}$ and is denoted by $\deg_x f$. Note that $\deg_x f = 1$ if and only if f is an isomorphism at x.

Lemma 1.10. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of complex analytic spaces.*

(1) *If f is proper and if g is discretely proper, then $g \circ f$ is discretely proper.*
(2) *If $g \circ f$ is discretely proper, then f is discretely proper.*
(3) *Assume that $f: X \to Y$ is a morphism of varieties of maximal rank and that Y is locally irreducible. If g has only connected fibers and if $g \circ f$ is surjective and discretely proper, then f is surjective.*

Proof. (1) and (2): For a point $x \in X$ and $y = f(x)$, let Γ_x (resp. Θ_x) be the connected component of $f^{-1}g^{-1}(g(y))$ (resp. $f^{-1}(y)$) containing x. Then Θ_x is a connected component of a fiber of $\Gamma_x \to g^{-1}(g(y))$. In case (1), $f(\Gamma_x)$ is compact, since it is a closed subset of a connected component of $g^{-1}(g(y))$; thus, Γ_x is also compact as a closed subset of $f^{-1}f(\Gamma_x)$. This shows (1). In case (2), Γ_x is compact, and hence, $\Gamma_x \to f(\Gamma_x)$ is proper and Θ_x is compact. This shows (2).

(3): For a point $x \in X$ and the connected component Γ_x of $f^{-1}g^{-1}(g(f(x)))$ containing x, by Lemma 1.7, we have an open neighborhood $\mathcal{U}_x$ of Γ_x in X and an open neighborhood $\mathcal{W}_x$ of $g(f(x))$ in Z such that $g \circ f$ induces a proper morphism $\mathcal{U}_x \to \mathcal{W}_x$. We may assume that $\mathcal{W}_x$ is connected. Then $g^{-1}\mathcal{W}_x$ is a connected open subset of Y, which is irreducible as Y is locally irreducible. Now, f induces a proper morphism $\mathcal{U}_x \to g^{-1}\mathcal{W}_x$. For an irreducible component $\mathcal{U}'$ of $\mathcal{U}_x$, the induced morphism $f|_{\mathcal{U}'}\colon \mathcal{U}' \to g^{-1}\mathcal{W}_x$ is of maximal rank, and hence, $f(\mathcal{U}')$ contains a non-empty open subset by Lemma 1.3. Thus, $f(\mathcal{U}') = f(\mathcal{U}_x) = g^{-1}\mathcal{W}_x$. Therefore, $f(X) = \bigcup f(\mathcal{U}_x) = \bigcup g^{-1}\mathcal{W}_x = Y$, since $g \circ f$ is surjective. □

Corollary 1.11. *For a surjective morphism $f\colon X \to Y$ of normal varieties and for a proper surjective morphism $\tau\colon Y' \to Y$ of normal varieties with only connected fibers, let*

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

be a commutative diagram of varieties such that the induced morphism $X' \to X \times_Y Y'$ is an isomorphism over a non-empty open subset of Y'. If τ' is proper surjective and f is discretely proper, then f' is surjective and discretely proper.

Proof. The composite $f \circ \tau'$ is surjective and is discretely proper by Lemma 1.10(1). Hence, f' is discretely proper by Lemma 1.10(2) applied to $X' \to Y' \to Y$. The morphism f' is of maximal rank by Lemma 1.3, since $f'(X')$ contains the open subset of Y' over which $X' \to X \times_Y Y'$ is an isomorphism. Thus, f' is surjective by Lemma 1.10(3) applied to $X' \to Y' \to Y$, since the normal variety Y' is locally irreducible. □

The openness property in Corollary 1.8 is generalized to:

Lemma 1.12. *Let $f\colon X \to Y$ be a fully equi-dimensional morphism of varieties and assume that Y is locally irreducible. Then f is* universally open *in the sense that the base change $f'\colon X \times_Y Y' \to Y'$ is an open holomorphic map for any morphism $\tau\colon Y' \to Y$ from a complex analytic space Y'. If Y' is a variety, then $f'|_V\colon V \to Y'$ is fully equi-dimensional and* $\dim V - \dim Y' = \dim X - \dim Y$ *for any irreducible component V of $X \times_Y Y'$.*

Proof. The morphism f is open by [7, §3.10, Thm.]. For any point $y' \in Y'$, we have an open neighborhood $\mathcal{Y}'$ with a closed immersion $\iota\colon \mathcal{Y}' \hookrightarrow \mathcal{U}$ into a connected open subset $\mathcal{U}$ of an affine space $\mathbb{C}^n$. Then the induced morphism $(\iota, \tau|_{\mathcal{Y}'})\colon \mathcal{Y}' \hookrightarrow \mathcal{U} \times Y$ is a closed immersion and $\tau|_{\mathcal{Y}'}\colon \mathcal{Y}' \to Y$ is the composite of $(\iota, \tau|_{\mathcal{Y}'})$ and the second projection $\mathcal{U} \times \mathcal{Y}' \to \mathcal{Y}'$. In order to prove the openness of f', we may replace Y' with $\mathcal{Y}'$. If τ is the second projection $Y' = \mathcal{U} \times Y \to Y$, then Y' is locally irreducible and f' is open by [7, §3.10, Thm.]. Thus, we are reduced to the case where τ is a closed immersion, but in this case, the openness of f' is obvious. This proves the first assertion.

For the second assertion, we set $X' := X \times_Y Y'$. Then the function $x \mapsto \dim_x f'^{-1}(f'(x))$ on X' is constant with value $\dim X - \dim Y$, since f is fully equi-dimensional. The openness of f' implies that

$$\dim_x f'^{-1}(f'(x)) = \dim_x X' - \dim_{f'(x)} Y' = \dim_x X' - \dim Y'$$

for any $x \in X'$ by [7, §3.10, Thm.]. In particular, $x \mapsto \dim_x X'$ is constant. For the morphism $g = f'|_V\colon V \to Y'$ of varieties, we have

$$\dim_v X' - \dim Y' \geq \dim_v g^{-1}g(v) \geq \dim_v V - \dim_{g(v)} Y' = \dim V - \dim Y'$$

for any $v \in V$ by [7, §3.9, Prop.], since $f'^{-1}(f'(v)) \supset g^{-1}(g(v))$. For the open dense subset $V^\circ = V \cap (X'_{\mathrm{red}})_{\mathrm{reg}}$ of V, if $v \in V^\circ$, then $\dim V = \dim_v V = \dim_v X'$. Hence, the upper semi-continuous function $v \mapsto \dim_v g^{-1}(g(v))$ on V attains the maximum at any point of V°. Thus, the function is constant with value $\dim V - \dim Y' = \dim X - \dim Y$. As a consequence, g is fully equi-dimensional. □

Remark. For morphisms of schemes which are locally of finite presentation, we have a result similar to Lemma 1.12 by [16, Prop. (14.3.2), Cor. (14.4.4), ($\mathrm{Err}_{\mathrm{IV}}$, 41)]. Lemma 1.12 is not true in general if we drop the assumption on the local irreducibility of Y. For example, if Y is a nodal cubic plane curve and if $f\colon X \to Y$ and $\tau\colon Y' \to Y$ are the normalization of Y, then $X \times_Y Y'$ contains two isolated points.

Lemma 1.13. *Let $\tau\colon Y' \to Y$ be a proper surjective morphism of normal varieties with connected fibers and let $f\colon X \to Y$ be a fully equi-dimensional morphism of varieties. Then $X \times_Y Y'$ is irreducible and is* generically reduced*, i.e., a dense open subset is reduced.*

Proof. We set $X' = X \times_Y Y'$ and consider the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\tau'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\tau"] & Y. \end{tikzcd}$$

By assumption, there exist non-singular Zariski-open dense subsets $X^\circ \subset X$ and $Y'^\circ \subset Y'$, and a non-singular open dense subset $Y^\circ \subset Y$ such that f is smooth on X°, τ is smooth on Y'°, and $Y^\circ \supset f(X^\circ) \cup \tau(Y'^\circ)$. We set $U_1 := \tau'^{-1}(X^\circ) = X^\circ \times_Y Y'$, $U_2 := f'^{-1}(Y'^\circ) = X \times_Y Y'^\circ$, and $U_3 := U_1 \cap U_2 = X^\circ \times_{Y^\circ} Y'^\circ$. Then U_1 is normal, U_2 is reduced, and U_3 is non-singular, since $U_1 \to Y'$ and $U_2 \to X$ are smooth. Here, U_3 is Zariski-open and dense in U_1 and also in U_2. Since $\tau'|_{U_1}\colon U_1 \to X^\circ$ is a proper surjective morphism with connected fibers to a non-singular variety, we see that U_1 is a normal variety. Thus, U_3 and U_2 are also irreducible. For any irreducible component Z of X', the morphism $f'|_Z\colon Z \to Y'$ is fully equi-dimensional by Lemma 1.12. In particular, $f'|_Z$ is of maximal rank and $Z \cap U_2 \neq \emptyset$. Since $Z \cap U_2$ is a closed analytic subset of the variety U_2 of the same dimension, $Z \supset U_2$, and moreover, Z is the closure of U_2 in $X \times_Y Y'$. Therefore, $X \times_Y Y'$ is irreducible. It is generically reduced, since U_3 is non-singular. □

Corollary 1.14. *Let $\pi_1\colon X_1 \to Y_1$ and $\pi_2\colon X_2 \to Y_2$ be proper surjective morphisms of normal varieties with connected fibers. If $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ are finite surjective morphisms such that $\pi_2 \circ f = g \circ \pi_1$, then* $\deg g \mid \deg f$.

Proof. By Lemma 1.13, $X_2 \times_{Y_2} Y_1$ is irreducible and generically reduced. For the normalization X'_1 of $X_2 \times_{Y_2} Y_1$, we can consider a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\tau"] \arrow[rr, bend left, "f"] \arrow[dr, "\pi_1"'] & X_1' \arrow[r, "p_1"] \arrow[d, "p_2"] & X_2 \arrow[d, "\pi_2"] \\ & Y_1 \arrow[r, "g"] & Y_2. \end{tikzcd}$$

Here, p_1 and τ are finite surjective morphisms, and $\deg p_1 = \deg g$. Therefore, $\deg f / \deg g = \deg f / \deg p_1 = \deg \tau \in \mathbb{Z}$. □

1.2. Glossaries on divisors. We recall basic properties of divisors on normal complex analytic spaces fixing some notation used in this article. Especially, pullbacks of divisors by morphisms of maximal rank are explained in detail. Some of properties are explained also in [39, II, §2].

Convention (Divisor). Let X be a *normal* complex analytic space. A *divisor* on X always means a *Weil divisor*, i.e., a locally finite $\mathbb{Z}$-linear combination of closed subvarieties of codimension 1. A *prime divisor* means a closed subvariety of codimension 1. The *divisor group* of X, i.e., the group of divisors on X, is denoted by $\mathrm{Div}(X)$. We use the following conventions for a divisor D on X:

– The *prime decomposition* of D is the expression $D = \sum_{i \in I} m_i \Gamma_i$ as a locally finite $\mathbb{Z}$-linear combination, where $m_i \in \mathbb{Z}$ and Γ_i are prime divisors and where the set $I_x = \{i \in I \mid m_i \neq 0 \text{ and } x \in \Gamma_i\}$ is finite for any $x \in X$, by the local finiteness. The integer m_i is called the *multiplicity* of D along Γ_i and denoted by $\mathrm{mult}_{\Gamma_i} D$. If $m_i \neq 0$, then Γ_i is called a *prime component* of D.

– We say that D is *effective* (resp. *reduced*) if $\mathrm{mult}_\Gamma D \geq 0$ (resp. $\mathrm{mult}_\Gamma D \in \{0, 1\}$) for any prime divisor Γ on X. For another divisor D', we write $D \geq D'$ or $D' \leq D$ if $D - D'$ is effective.

– The *support* of D, $\mathrm{Supp}\, D$, is the union of prime components of D: This is identified with the reduced divisor $D_{\mathrm{red}} := \sum_{m_i \neq 0} \Gamma_i$ for the prime decomposition of D above. For a closed subset T, $\mathrm{Div}_T(X)$ denotes the group of divisors on X whose supports are contained in T.

– For an open subset U of X, the restriction $D|_U$ is defined as follows: Let Θ be a prime divisor on U such that $\Theta \subset \mathrm{Supp}\, D$. Then $\Theta \subset \Gamma$ for a unique prime component Γ of D. We set $m_\Theta := \mathrm{mult}_\Gamma D$. Then the divisor $D|_U$ on U is defined by $\mathrm{mult}_\Theta(D|_U) = m_\Theta$ for any prime divisor Θ on U.

Remark. The restriction $D \mapsto D|_U$ gives rise to a group homomorphism $\mathrm{Div}(X) \to \mathrm{Div}(U)$ for any open subset U. The correspondence $U \mapsto \mathrm{Div}(U)$ gives rise to a sheaf $\mathcal{D}iv_X$ of abelian groups. In particular, $\mathrm{Div}(X) = H^0(X, \mathcal{D}iv_X)$. If $Z \subset X$ is a closed analytic subset of codimension ≥ 2, then $\mathrm{Div}(X) \to \mathrm{Div}(X \setminus Z)$ is bijective, and hence, $\mathcal{D}iv_X \simeq j_* \mathcal{D}iv_{X \setminus Z}$ for the open immersion $j \colon X \setminus Z \hookrightarrow X$. In particular, $\mathrm{Div}(X) \simeq \mathrm{Div}(X_{\mathrm{reg}})$ for the non-singular locus X_{reg}.

Definition 1.15. For a divisor D, there exist effective divisors D_+ and D_- uniquely such that D_+ and D_- have no common prime component and $D_+ - D_- = D$. In fact, $D_+ =$

$\sum_{i\in I_+} m_i\Gamma_i$ and $D_- = \sum_{i\in I_-}(-m_i)\Gamma_i$ for the prime decomposition $D = \sum_{i\in I} m_i\Gamma_i$ and for $I_\pm = \{i \in I \mid \pm m_i > 0\}$. We call D_+ (resp. D_-) the *positive* (resp. *negative*) *part* of the *prime decomposition* of D.

Convention (Cartier divisor). A *Cartier divisor* on a complex analytic space Y is defined as a divisor on the ringed space $(Y, \mathcal{O}_Y)$ in the sense of [16, §21.1]. This is an element of $H^0(Y, \mathfrak{M}_Y^\star/\mathcal{O}_Y^\star)$ for the sheaf $\mathfrak{M}_Y^\star$ (resp. $\mathcal{O}_Y^\star$) of invertible meromorphic (resp. holomorphic) functions on Y. We set $\mathcal{CD}iv_Y := \mathfrak{M}_Y^\star/\mathcal{O}_Y^\star$ and set $\mathrm{CDiv}(Y) := H^0(Y, \mathcal{CD}iv_Y)$ as the Cartier divisor group. A *principal* divisor is a Cartier divisor belonging to the image of the homomorphism $H^0(Y, \mathfrak{M}_Y^\star) \to \mathrm{CDiv}(Y)$ induced by the surjection $\mathfrak{M}_Y^\star \to \mathcal{CD}iv_Y$. For an invertible meromorphic function φ, we consider the $\mathcal{O}_Y$-module $\mathcal{O}_Y\varphi^{-1}$ generated by φ^{-1} in the sheaf $\mathfrak{M}_Y$ of meromorphic functions on Y. Then $\mathcal{O}_Y\varphi^{-1} \simeq \mathcal{O}_Y$. The correspondence $\varphi \mapsto \mathcal{O}_X\varphi^{-1}$ for "local" invertible meromorphic functions φ defines a bijection between $\mathrm{CDiv}(Y)$ and the set of invertible sheaves contained in $\mathfrak{M}_Y$ as $\mathcal{O}_Y$-submodules. For a Cartier divisor D, the associated invertible sheaf is denoted by $\mathcal{O}_Y(D)$ (cf. [16, (21.2.8)]).

Remark. The correspondence $D \mapsto \mathcal{O}_Y(D)$ defines a homomorphism $\mathrm{CDiv}(Y) \to \mathrm{Pic}(Y) = H^1(Y, \mathcal{O}_Y^\star)$, which is isomorphic to a connecting homomorphism of the exact sequence $0 = \{1\} \to \mathcal{O}_Y^\star \to \mathfrak{M}_Y^\star \to \mathcal{CD}iv_Y \to 0$. Here,

$$\mathcal{O}_Y(-D) \simeq \mathcal{O}_Y(D)^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y(D), \mathcal{O}_Y) \quad \text{and}$$
$$\mathcal{O}_Y(D_1 + D_2) \simeq \mathcal{O}_Y(D_1) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_2)$$

for any D, D_1, $D_2 \in \mathrm{CDiv}(Y)$. A Cartier divisor D is principal if and only if $\mathcal{O}_Y(D) \simeq \mathcal{O}_Y$, by the exactness of $H^0(Y, \mathfrak{M}_Y^\star) \to \mathrm{CDiv}(Y) \to \mathrm{Pic}(Y)$.

Convention 1.16. Let $\mathcal{L}$ be an invertible sheaf on Y. A holomorphic section σ of $\mathcal{L}$ is said to be *nowhere vanishing* if σ induces an isomorphism $\mathcal{O}_Y \xrightarrow{\simeq} \mathcal{L}$, or equivalently, if

$$\sigma(y) := \sigma_y \bmod \mathfrak{m}_y \in \mathcal{L}_y \otimes \mathbb{C}(y)$$

is not zero for any $y \in Y$. A *meromorphic section* φ of $\mathcal{L}$ is by definition a global section of $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathfrak{M}_Y$. We say that φ is *regular* if φ induces an isomorphism $\mathfrak{M}_Y \xrightarrow{\simeq} \mathcal{L} \otimes_{\mathcal{O}_Y} \mathfrak{M}_Y$ (cf. [16, (20.1.8)]). We note the following on the regularity:

- When $\mathcal{L} \simeq \mathcal{O}_Y$, φ is regular if and only if φ is invertible as a meromorphic function.
- When Y is a locally irreducible variety, φ is regular if and only if $\varphi \neq 0$.
- Even if φ is regular, it is not necessarily a holomorphic section of $\mathcal{L}$.

Remark. A Cartier divisor D on Y is in one-to-one correspondence with a pair $(\mathcal{L}, \varphi)$ of an invertible sheaf $\mathcal{L}$ and a regular meromorphic section φ of $\mathcal{L}$. In fact, the inclusion $\mathcal{O}_Y(D) \hookrightarrow \mathfrak{M}_Y$ defines an isomorphism $\mathcal{O}_Y(D) \otimes \mathfrak{M}_Y \xrightarrow{\simeq} \mathfrak{M}_Y$, and we have φ for $\mathcal{L} = \mathcal{O}_Y(D)$ as the inverse of the isomorphism. Conversely, φ^{-1} induces an injection $\mathcal{L} \hookrightarrow \mathfrak{M}_Y$.

Lemma 1.17. *Let $f\colon X \to Y$ be a morphism of varieties* of maximal rank (*cf.* Definition 1.1). *Then there exist a canonical morphism*

$$\begin{tikzcd} H^0(Y, \mathfrak{M}_Y^\star) \arrow[r] \arrow[d, "f^*"'] & \mathrm{CDiv}(Y) \arrow[r] \arrow[d, "f^*"'] & \mathrm{Pic}(Y) \arrow[d, "f^*"] \\ H^0(X, \mathfrak{M}_X^\star) \arrow[r] & \mathrm{CDiv}(X) \arrow[r] & \mathrm{Pic}(X) \end{tikzcd}$$

of exact sequences of abelian groups, where f^* denote pullback homomorphisms of meromorphic functions, Cartier divisors, and invertible sheaves, respectively. In particular, $f^*\mathcal{O}_Y(D) \simeq \mathcal{O}_X(f^*D)$ for any Cartier divisor D on Y.

Proof. Let φ be a holomorphic function defined on an open subset $\mathcal{V}$ of Y. Then φ is invertible as a meromorphic function on $\mathcal{V}$ if and only if it is not identically zero on any connected component of $\mathcal{V}_{\mathrm{reg}}$. By Lemma 1.3, there is a dense Zariski-open subset X' of X such that $f|_{X'} : X' \to Y$ is smooth, where we may assume that $X' \subset X_{\mathrm{reg}} \cap f^{-1}Y_{\mathrm{reg}}$. If $f(X) \cap \mathcal{V} \neq \emptyset$, then $X' \cap f^{-1}\mathcal{V} \neq \emptyset$, and the holomorphic function $f^*\varphi = \varphi \circ f$ defined on $f^{-1}\mathcal{V}$ is not identically zero on each connected component of $X' \cap f^{-1}\mathcal{V}$; thus, $f^*\varphi$ is invertible as a meromorphic function on $f^{-1}\mathcal{V}$. By the observation, we have a group homomorphism $f^{-1}\mathfrak{M}_Y^\star \to \mathfrak{M}_X^\star$ extending $f^{-1}\mathcal{O}_Y^\star \to \mathcal{O}_X^\star$ and compatible with $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. It induces a morphism

$$\begin{tikzcd} 0 \arrow[r] & f^{-1}\mathcal{O}_Y^\star \arrow[r] \arrow[d] & f^{-1}\mathfrak{M}_Y^\star \arrow[r] \arrow[d] & f^{-1}\mathcal{CDiv}_Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_X^\star \arrow[r] & \mathfrak{M}_X^\star \arrow[r] & \mathcal{CDiv}_X \arrow[r] & 0 \end{tikzcd}$$

of exact sequences of sheaves on X. By taking cohomologies, we are done. □

CONVENTION ($\mathrm{div}(\varphi)$). Let X be a *normal* complex analytic space and let φ be a meromorphic section of an invertible sheaf $\mathcal{L}$ on X. Assume that φ is *regular*, i.e., φ is not zero on each connected component of X (cf. Convention 1.16). Then the divisor $\mathrm{div}(\varphi) = \mathrm{div}_{\mathcal{L}}(\varphi)$ on X associated with $(\mathcal{L}, \varphi)$ is defined by the property that $\mathrm{mult}_\Gamma \mathrm{div}(\varphi)$ equals the order of zeros or the minus of the order of poles of φ along Γ for any prime divisor Γ on X. If $\mathcal{L} = \mathcal{O}_X$, then $\mathrm{div}(\varphi)$ is just the *principal divisor* associated with an invertible meromorphic function φ.

REMARK. For a Cartier divisor D on X, if a holomorphic section σ of $\mathcal{O}_X(D)$ is not zero on each connected component of X, then σ is regular as a meromorphic section, and $\mathrm{div}(\varphi) + D = \mathrm{div}(\sigma) = \mathrm{div}_{\mathcal{O}_X(D)}(\sigma) \geq 0$ for the meromorphic function φ defined as the image of σ under the inclusion $\mathcal{O}_X(D) \subset \mathfrak{M}_X$.

REMARK. The correspondence $(\mathcal{L}, \varphi) \mapsto \mathrm{div}_{\mathcal{L}}(\varphi)$ defines an injection $\mathcal{CDiv}_X \hookrightarrow \mathcal{Div}_X$, which is an isomorphism on X_{reg}. Hence, $\mathrm{CDiv}(X)$ is regarded as a subgroup of $\mathrm{Div}(X)$, and we have $\mathrm{Div}(X) \simeq \mathrm{Div}(X_{\mathrm{reg}}) \simeq \mathrm{CDiv}(X_{\mathrm{reg}})$.

DEFINITION ($\mathcal{O}_X(D)$). Let X be a *normal* complex analytic space. For a divisor D on X, we set $\mathcal{O}_X(D) := j_*\mathcal{O}_{X_{\mathrm{reg}}}(D|_{X_{\mathrm{reg}}})$ for the open immersion $j : X_{\mathrm{reg}} \hookrightarrow X$. The sheaf $\mathcal{O}_X(D)$ is regarded as an $\mathcal{O}_X$-submodule of $\mathfrak{M}_X$ and it is a coherent *reflexive* sheaf of rank 1 (cf. [49, App. to §1]). Here, a coherent sheaf $\mathcal{F}$ on X is said to be reflexive if it is isomorphic to the double dual $\mathcal{F}^{\vee\vee} = (\mathcal{F}^\vee)^\vee$, where $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. See [46, II, §1.1] and [22, §1] for details on reflexive sheaves.

Remark 1.18. An effective divisor D is identified with a closed analytic subspace of X defined by the ideal sheaf $\mathcal{O}_X(-D)$; the structure sheaf $\mathcal{O}_D$ is the cokernel of the canonical injection $\mathcal{O}_X(-D) \to \mathcal{O}_X$. Hence, $\operatorname{Supp} D$ is the underlying set of D_{red} for any divisor D. As a property of a divisor D, we consider a property of the complex analytic space D when D is effective. For example, a divisor D is said to be non-singular if D is effective and the complex analytic space D is non-singular. Thus, a non-singular divisor is reduced, and the zero divisor is non-singular by considering it as the empty set.

Convention ($\mathbb{Q}$-divisors and $\mathbb{R}$-divisors). A $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*) on a normal complex analytic space X is a locally finite $\mathbb{Q}$ (resp. $\mathbb{R}$)-linear combination of prime divisors. For an $\mathbb{R}$-divisor D, the prime decomposition $D = \sum_{i \in I} r_i \Gamma_i$ and the multiplicity $\operatorname{mult}_\Gamma D$ along a prime divisor Γ are defined similarly to the case of divisor. Hence, we can speak of effective $\mathbb{R}$-divisors, the support of an $\mathbb{R}$-divisor, prime components of an $\mathbb{R}$-divisor, and the positive and negative parts of the prime decomposition of an $\mathbb{R}$-divisor (cf. Definition 1.15). The group of $\mathbb{Q}$ (resp. $\mathbb{R}$)-divisors on X is denoted by $\operatorname{Div}(X, \mathbb{Q})$ (resp. $\operatorname{Div}(X, \mathbb{R})$), and the group of $\mathbb{Q}$ (resp. $\mathbb{R}$)-divisors on X whose supports are contained in a closed subset T is denoted by $\operatorname{Div}_T(X, \mathbb{Q})$ (resp. $\operatorname{Div}_T(X, \mathbb{R})$) (cf. [39, II, §2.d]); these are $\mathbb{Q}$ (resp. $\mathbb{R}$)-vector spaces. For the prime decomposition of D above, the *round-up* $\ulcorner D \urcorner$, the *round-down* $\llcorner D \lrcorner$, and the *fractional part* $\langle D \rangle$ are defined by

$$\ulcorner D \urcorner := \sum\nolimits_{i \in I} \ulcorner r_i \urcorner \Gamma_i, \quad \llcorner D \lrcorner := \sum\nolimits_{i \in I} \llcorner r_i \lrcorner \Gamma_i, \quad \text{and} \quad \langle D \rangle := D - \llcorner D \lrcorner,$$

where $\llcorner r \lrcorner = \max\{i \in \mathbb{Z} \mid i \leq r\}$ and $\ulcorner r \urcorner = \min\{i \in \mathbb{Z} \mid i \geq r\} = -\llcorner -r \lrcorner$ for $r \in \mathbb{R}$.

Remark. For $\mathfrak{K} = \mathbb{Q}$ or $\mathbb{R}$, we have $\operatorname{Div}(X, \mathfrak{K}) = H^0(X, \mathcal{D}iv_X \otimes \mathfrak{K})$, but $\operatorname{Div}(X, \mathfrak{K})$ is not necessarily isomorphic to $\operatorname{Div}(X) \otimes \mathfrak{K}$. The fractional part of D is written as $\{D\}$ in many articles, but we write $\langle D \rangle$ as in [32] avoiding a confusion with the single set $\{D\}$ consisting of D.

Convention (Linear equivalence). Let X be a normal variety. For two $\mathbb{R}$-divisors D and D' on X, if $D - D'$ is a principal divisor, i.e., $D - D' = \operatorname{div}(\varphi)$ for a non-zero meromorphic function φ on X, then D is said to be *linearly equivalent* to D', and we write $D \sim D'$ for the linear equivalence. If $m(D - D') \sim 0$ for a positive integer m, then D is said to be $\mathbb{Q}$-*linearly equivalent* to D', and we write $D \sim_{\mathbb{Q}} D'$ for the $\mathbb{Q}$-linear equivalence.

Definition ($\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier). Let X be a normal complex analytic space. A $\mathbb{Q}$-divisor D on X is said to be $\mathbb{Q}$-*Cartier* if there is a positive integer m locally on X such that mD is a Cartier divisor. The group of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X is denoted by $\operatorname{CDiv}(X, \mathbb{Q})$. Then we have $\operatorname{CDiv}(X, \mathbb{Q}) = H^0(X, \mathcal{CD}iv_X \otimes \mathbb{Q})$. An $\mathbb{R}$-divisor E on X is said to be $\mathbb{R}$-*Cartier* if it is locally expressed as a finite $\mathbb{R}$-linear combination of Cartier divisors. The group of $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X is denoted by $\operatorname{CDiv}(X, \mathbb{R})$. Then we have $\operatorname{CDiv}(X, \mathbb{R}) = H^0(X, \mathcal{CD}iv_X \otimes \mathbb{R})$.

Lemma 1.19. *Let $f \colon X \to Y$ be a morphism of maximal rank of normal varieties. Then the pullback homomorphism* $\operatorname{CDiv}(Y) \xrightarrow{f^*} \operatorname{CDiv}(X)$ *in* Lemma 1.17 *extends to homomorphisms*

$$\mathrm{CDiv}(Y,\mathbb{Q}) \xrightarrow{f^*} \mathrm{CDiv}(X,\mathbb{Q}) \quad \textit{and} \quad \mathrm{CDiv}(Y,\mathbb{R}) \xrightarrow{f^*} \mathrm{CDiv}(X,\mathbb{R}).$$

Moreover, when $\operatorname{codim}(f^{-1}\operatorname{Sing} Y, X) \geq 2$*, these* f^* *extend to homomorphisms*

$$\mathrm{Div}(Y) \xrightarrow{f^*} \mathrm{Div}(X), \quad \mathrm{Div}(Y,\mathbb{Q}) \xrightarrow{f^*} \mathrm{Div}(X,\mathbb{Q}), \quad \textit{and} \quad \mathrm{Div}(Y,\mathbb{R}) \xrightarrow{f^*} \mathrm{Div}(X,\mathbb{R}),$$

and the following hold on the pullback f^*D *of an* $\mathbb{R}$*-divisor* D *on* Y:

(1) *If* D *is a divisor, then* $(f^*\mathcal{O}_Y(D))^{\vee\vee} \simeq \mathcal{O}_X(f^*D)$*.*
(2) *If* D *is effective, then* f^*D *is also effective and* $\operatorname{Supp} f^*D \subset f^{-1}\operatorname{Supp} D$*. If* D *is* $\mathbb{R}$*-Cartier in addition, then* $\operatorname{Supp} f^*D = f^{-1}\operatorname{Supp} D$*.*
(3) *The equality* $\operatorname{Supp} f^*D = f^{-1}\operatorname{Supp} D$ *holds if* f *is* fully equi-dimensional (*cf.* Definition 1.1).

Proof. We set $\mathfrak{R}$ to be $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. By the proof of Lemma 1.17, we have a homomorphism $f^{-1}(\mathcal{CDiv}_Y \otimes \mathfrak{R}) \to \mathcal{CDiv}_X \otimes \mathfrak{R}$, and a homomorphism $\mathcal{CDiv}_Y \otimes \mathfrak{R} \to f_*(\mathcal{CDiv}_X \otimes \mathfrak{R})$ by adjunction. It defines the expected pullback homomorphism $f^*\colon \mathrm{CDiv}(Y,\mathfrak{R}) \to \mathrm{CDiv}(X,\mathfrak{R})$. We set $X' := f^{-1}(Y_{\mathrm{reg}})$ and $f' := f|_{X'}\colon X' \to Y_{\mathrm{reg}}$. If $\operatorname{codim}(f^{-1}\operatorname{Sing} Y, X) = \operatorname{codim}(X \setminus X', X) \geq 2$, then we have

$$\begin{aligned} &\mathcal{Div}_Y \otimes \mathfrak{R} \simeq i_*(\mathcal{Div}_{Y_{\mathrm{reg}}} \otimes \mathfrak{R}) \simeq i_*(\mathcal{CDiv}_{Y_{\mathrm{reg}}} \otimes \mathfrak{R}) \quad \text{and} \\ &\mathcal{Div}_X \otimes \mathfrak{R} \simeq j_*(\mathcal{Div}_{X'} \otimes \mathfrak{R}) \supset j_*(\mathcal{CDiv}_{X'} \otimes \mathfrak{R}) \end{aligned}$$

for open immersions $i\colon Y_{\mathrm{reg}} \hookrightarrow Y$ and $j\colon X' \hookrightarrow X$, and hence, the homomorphism $(f')^{-1}\mathcal{CDiv}_{Y_{\mathrm{reg}}} \to \mathcal{CDiv}_{X'}$ defines a homomorphism $\mathcal{Div}_Y \otimes \mathfrak{R} \to f_*(\mathcal{Div}_X \otimes \mathfrak{R})$: It induces the expected pullback homomorphisms $\mathrm{Div}(Y) \to \mathrm{Div}(X)$, $\mathrm{Div}(Y,\mathbb{Q}) \to \mathrm{Div}(X,\mathbb{Q})$, and $\mathrm{Div}(Y,\mathbb{R}) \to \mathrm{Div}(X,\mathbb{R})$.

We shall show assertions (1)–(3) on f^*D. We have isomorphisms

$$(f^*\mathcal{O}_Y(D))|_{X'} \simeq f'^*\mathcal{O}_{Y_{\mathrm{reg}}}(D|_{Y_{\mathrm{reg}}}) \simeq \mathcal{O}_{X'}(f'^*(D|_{Y_{\mathrm{reg}}})) \simeq \mathcal{O}_X(f^*D)|_{X'}$$

for any divisor D on Y, since $D|_{Y_{\mathrm{reg}}}$ is Cartier. When $\operatorname{codim}(X \setminus X', X) \geq 2$, by applying j_* to these isomorphisms, we have the isomorphism in (1) (cf. [46, II, Lem. 1.1.12], [22, Prop. 1.6]). In the situation of (2), assume first that D is an effective divisor. By (1) and by pulling back $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$ to X, we have a commutative diagram

$$\begin{tikzcd} & f^*\mathcal{O}_Y(-D) \arrow[r] \arrow[d, "\text{double dual}"'] & f^*\mathcal{O}_Y \arrow[r] \arrow[d, "\simeq"] & f^*\mathcal{O}_D \simeq \mathcal{O}_{X \times_Y D} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_Y(-f^*D) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_{f^*D} \arrow[r] & 0 \end{tikzcd}$$

of exact sequences. Thus, f^*D is effective and $\operatorname{Supp} \mathcal{C}_D \cup \operatorname{Supp} f^*D = f^{-1}\operatorname{Supp} D$ for the cokernel $\mathcal{C}_D$ of the double dual homomorphism $f^*\mathcal{O}_Y(-D) \to \mathcal{O}_Y(-f^*D)$. In particular, if D is Cartier, then $\mathcal{C}_D = 0$ and $\operatorname{Supp} f^*D = f^{-1}\operatorname{Supp} D$. Even in case D is only an $\mathbb{R}$-divisor, for each prime component Γ of D, $f^*\Gamma$ is effective and $\operatorname{Supp} f^*\Gamma \subset f^{-1}\Gamma$; thus, f^*D is also effective and $\operatorname{Supp} f^*D \subset f^{-1}\operatorname{Supp} D$ by linearity. Thus, we have shown the first assertion of (2) and the second assertion in case D is Cartier.

The proof of the second assertion of (2) is reduced to the case of Cartier divisors as follows: Since D is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, in order to prove $\operatorname{Supp} f^*D = f^{-1}\operatorname{Supp} D$, by replacing Y with an open subset, we may assume that

- $D = \sum r_j D_j$ for finitely many Cartier divisors D_j with real numbers r_j,
- each D_j has only finitely many prime components,
- $\sum r_j \operatorname{mult}_\Gamma D_j \ge 0$ for any prime divisor Γ contained in $\bigcup \operatorname{Supp} D_j$, and
- $\sum r_j \operatorname{mult}_\Gamma D_j > 0$ if $\Gamma \subset \operatorname{Supp} D$.

Let L be a finite-dimensional $\mathbb{Q}$-vector space consisting of collections (x_j) of rational numbers x_j such that $\sum x_j \operatorname{mult}_\Gamma D_j = 0$ for any prime divisor $\Gamma \subset \bigcup \operatorname{Supp} D_j$ satisfying $\sum r_j \operatorname{mult}_\Gamma D_j = 0$. Then $(r_j) \in L \otimes_{\mathbb{Q}} \mathbb{R}$ and there is a collection $(r'_j) \in L$ such that $\sum r'_j \operatorname{mult}_\Gamma D_j > 0$ for any prime component Γ of D. We set $D' := \sum r'_j D_j$. Then D' is an effective $\mathbb{Q}$-Cartier divisor on Y such that $\operatorname{Supp} D' = \operatorname{Supp} D$. In particular, $aD \le D' \le bD$ for some positive numbers $a < b$, and we have $af^*D \le f^*D' \le bf^*D$. It implies that $\operatorname{Supp} f^*D' = \operatorname{Supp} f^*D$. Thus, by replacing D with D', we may assume that $r_j \in \mathbb{Q}$ for any j. Moreover, by replacing D with its multiple mD, we may assume that D is Cartier. Thus, we are reduced to the case of Cartier divisors and (2) has been proved.

Finally, we shall show (3). Assume that f is fully equi-dimensional. Then, if B is a subvariety of Y, then $\dim A - \dim B = \dim X - \dim Y$ for any irreducible component A of $f^{-1}B$, by Lemma 1.12. It implies that $\operatorname{codim}(f^{-1} \operatorname{Sing} Y, X) \ge 2$ and that every irreducible component of $f^{-1} \operatorname{Supp} D$ is a prime divisor. If D is effective, then $\operatorname{Supp} f^*D = f^{-1} \operatorname{Supp} D$ by the proof of (2), since $Z \cup \operatorname{Supp} f^*D = f^{-1} \operatorname{Supp} D$ for a closed subset Z with $\operatorname{codim}(Z, X) \ge 2$. When D is not effective, for the decomposition $D = D_+ - D_-$ in Definition 1.15, we have $\operatorname{Supp} D = \operatorname{Supp} D_+ \cup \operatorname{Supp} D_-$. Here, no prime divisor on X is contained in $f^{-1} \operatorname{Supp} D_+ \cap f^{-1} \operatorname{Supp} D_-$. Therefore, $\operatorname{Supp} f^*D = \operatorname{Supp} f^*D_+ \cup \operatorname{Supp} f^*D_- = f^{-1} \operatorname{Supp} D_+ \cup f^{-1} \operatorname{Supp} D_- = f^{-1} \operatorname{Supp} D$. Thus, we are done. □

DEFINITION (PUSHFORWARD). Let $f\colon X \to Y$ be a *non-degenerate morphism* (cf. Definition 1.1) of normal varieties. Let B be an $\mathbb{R}$-divisor on X such that $f|_\Gamma\colon \Gamma \to Y$ is proper for any prime component Γ of B. Then the *pushforward* f_*B is defined as an $\mathbb{R}$-divisor on Y such that

$$\operatorname{mult}_\Theta f_*B = \sum\nolimits_{\Gamma \in C(B;\Theta)} d_{\Gamma/\Theta} \operatorname{mult}_\Gamma B$$

for any prime divisor Θ on Y, where $C(B; \Theta)$ is the set of prime components Γ of B such that $f(\Gamma) = \Theta$ and where $d_{\Gamma/\Theta}$ is the degree of $f|_\Gamma\colon \Gamma \to \Theta$ (cf. Definition 1.5). Note that if B is a divisor (resp. $\mathbb{Q}$-divisor), then f_*B is so.

REMARK. Assume that f is proper. Then f_* gives rise to homomorphisms $\operatorname{Div}(X) \to \operatorname{Div}(Y)$, $\operatorname{Div}(X, \mathbb{Q}) \to \operatorname{Div}(Y, \mathbb{Q})$, and $\operatorname{Div}(X, \mathbb{R}) \to \operatorname{Div}(Y, \mathbb{R})$. If $B \in \operatorname{Div}(X)$, then $\mathcal{O}_Y(f_*B)$ is isomorphic to the double dual of

$$\Bigl(\bigwedge^{\deg f} f_*\mathcal{O}_X(B)\Bigr) \otimes_{\mathcal{O}_Y} \Bigl(\bigwedge^{\deg f} f_*\mathcal{O}_X\Bigr)^\vee$$

(cf. [39, II, §2.e]). Moreover, $f_*(f^*D) = (\deg f)D$ for any $D \in \operatorname{CDiv}(Y, \mathbb{R})$.

DEFINITION (EXCEPTIONAL DIVISOR). Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties. A prime divisor Γ on X is said to be *f-exceptional*, or *exceptional for f*, if $\dim_x \Gamma \cap f^{-1}(f(x)) > 0$ for any $x \in \Gamma$. An $\mathbb{R}$-divisor on X is said to be *f-exceptional* if its prime components are all f-exceptional. Note that when f is proper, an $\mathbb{R}$-divisor D on X is

f-exceptional if and only if $f_*D = 0$.

REMARK 1.20. Let Γ be a prime divisor on X which is not f-exceptional. Then $\Gamma \cap X' \neq \emptyset$ for $X' := f^{-1}(Y_{\mathrm{reg}})$, and $\Gamma|_{X'}$ is also a prime divisor on X', since X' is a Zariski-open subset of X (cf. [15, IX, §1.2]). Hence, we can consider the multiplicity of $f'^*(D|_{Y_{\mathrm{reg}}})$ along $\Gamma|_{X'}$ for the morphism $f' = f|_{X'} : X' \to Y_{\mathrm{reg}}$. If f has no exceptional divisor, then $\operatorname{codim}(X \setminus X', X) = \operatorname{codim}(f^{-1} \operatorname{Sing} Y, X) \geq 2$.

REMARK 1.21. If a non-degenerate morphism of normal *surfaces* has no exceptional divisor, then it has only discrete fibers. Conversely, any morphism $f: X \to Y$ of normal surfaces with only discrete fibers is non-degenerate by Corollary 1.4. In this case, f is open and is locally a finite morphism by Corollary 1.8, i.e., for any $x \in X$, there exists an open neighborhood $\mathcal{U}$ of x in X such that $\mathcal{U} \cap f^{-1}(f(x)) = \{x\}$, $f(\mathcal{U})$ is open in Y, $f|_{\mathcal{U}} : \mathcal{U} \to f(\mathcal{U})$ is finite.

DEFINITION 1.22 (STRICT PULLBACK). Let $f: X \to Y$ be a non-degenerate morphism of normal varieties. For an $\mathbb{R}$-divisor D on Y, let $\mathcal{S}_f(D)$ be the set of non-f-exceptional prime divisors on X contained in $f^{-1} \operatorname{Supp} D$. The *strict pullback* $f^{[*]}D$ of D is a $\mathbb{Q}$-divisor on X defined by

$$\operatorname{mult}_\Gamma f^{[*]}D = \begin{cases} \operatorname{mult}_{\Gamma|_{X'}} f'^*(D|_{Y_{\mathrm{reg}}}), & \text{if} \quad \Gamma \in \mathcal{S}_f(D), \\ 0, & \text{if} \quad \Gamma \notin \mathcal{S}_f(D), \end{cases}$$

for prime divisors Γ on X, where $X' = f^{-1}(Y_{\mathrm{reg}})$ and $f' = f|_{X'} : X' \to Y_{\mathrm{reg}}$ (cf. Remark 1.20, [39, II, §2.e]). If f is a *bimeromorphic* morphism, i.e., a proper surjective morphism such that $f^{-1}U \to U$ is an isomorphism for a non-empty open subset $U \subset Y$, then $f^{[*]}D$ is called the *proper transform* of D in X. In this case, $f_*(f^{[*]}D) = D$.

1.3. Numerical pullbacks of a divisor on a normal surface. For a bimeromorphic morphism $f: X \to Y$ of normal surfaces and a divisor D on Y, we have the *numerical pullback* f^*D as a $\mathbb{Q}$-divisor on X, which is introduced by Mumford [36, II, §(b)]. The pullback defines intersection numbers of two divisors on normal surfaces which are not necessarily Cartier. We can extend the definition of numerical pullback to the case of non-generate morphisms of normal surfaces. We shall explain some elementary properties of numerical pullbacks. The following is proved by the same method as in [36], [52, §1], or [41, §2.1].

Lemma-Definition 1.23 (Numerical pullback)**.** *For a non-degenerate morphism $f: X \to Y$ of normal surfaces, there is a functorial linear map $f^*: \operatorname{Div}(Y, \mathbb{Q}) \to \operatorname{Div}(X, \mathbb{Q})$ of $\mathbb{Q}$-vector spaces satisfying the following conditions*:

(1) *For a further non-degenerate morphism $g: Y \to Z$ of normal surfaces, one has $f^* \circ g^* = (g \circ f)^*$.*
(2) *If f is an open immersion, then f^* is the restriction map*: $D \mapsto D|_X$.
(3) *The map f^* extends the pullback homomorphism* $\operatorname{CDiv}(Y) \to \operatorname{CDiv}(X)$ *for Cartier divisors* (*cf.* Lemma 1.17).
(4) *In case X is non-singular and f is proper, the intersection number $(f^*D)E$ is zero for any $\mathbb{Q}$-divisor D on Y and any f-exceptional $\mathbb{Q}$-divisor E.*

The $\mathbb{Q}$-divisor f^*D is called the numerical pullback *of D by f.*

Remark. When X is non-singular and f is a bimeromorphic morphism, the numerical pullback f^*D is expressed as the sum $f^{[*]}D + E$ of the proper transform $f^{[*]}D$ and an f-exceptional $\mathbb{Q}$-divisor E such that $(f^{[*]}D + E)\Gamma = 0$ for any f-exceptional prime divisor Γ. Here, E is uniquely determined, since the intersection matrix $(\Gamma_i\Gamma_j)$ of f-exceptional prime divisors Γ_i contracted to a fixed point of Y is negative definite (cf. [36, p. 6]).

Remark. By resolution of singularities and indeterminacy of meromorphic maps, for the morphism f, we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\nu} & Y \end{array}$$

of normal surfaces such that M and N are non-singular and that μ and ν are bimeromorphic morphisms. Then the numerical pullback is given by $f^*D = \mu_*(g^*(\nu^*D))$ for a divisor D, where g^* and ν^* indicate pullbacks of Cartier divisors, and μ_* indicates the pushforward of a divisor by the proper morphism μ.

Definition (Intersection number). Let D and E be $\mathbb{Q}$-divisors on a normal surface X such that $\operatorname{Supp} D \cap \operatorname{Supp} E$ is compact. Let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface M. Here, $\operatorname{Supp}\mu^*D \cap \operatorname{Supp}\mu^*E$ is also compact, and one can consider the intersection number $DE := (\mu^*D)\mu^*E$: This is independent of the choice of μ, and is called the *intersection number* of D and E.

Remark 1.24. The numerical pullback f^* in Lemma-Definition 1.23 and the intersection number above are defined also for $\mathbb{R}$-divisors by linearity. The following properties are known or shown easily for $f\colon X \to Y$ and an $\mathbb{R}$-divisor D on Y:

(1) If D is effective, then f^*D is so and $\operatorname{Supp} f^*D = f^{-1}(\operatorname{Supp} D)$ (cf. [41, Rem. (4) of Def. 2.4] and Lemma 1.19).
(2) For an $\mathbb{R}$-divisor E on X, if $f^{-1}(\operatorname{Supp} D) \cap \operatorname{Supp} E$ is compact, then the projection formula: $(f^*D)E = D(f_*E)$ holds.
(3) If f is proper, then $(\deg f)D = f_*(f^*D)$.
(4) If an $\mathbb{R}$-divisor D' on Y has no common prime component with D and if $DD' = 0$, then $\operatorname{Supp} D \cap \operatorname{Supp} D' = \emptyset$.
(5) If $\operatorname{codim}(f^{-1}\operatorname{Sing} Y, X) \geq 2$, then the pullback f^*D given in Lemma 1.19 coincides with the numerical pullback, since $(f^*D)|_{X'} = f'^*(D|_{Y_{\mathrm{reg}}})$ for $X' = f^{-1}(Y_{\mathrm{reg}})$ and $f' = f|_{X'}\colon X' \to Y_{\mathrm{reg}}$.

Remark 1.25. Let S be a non-zero reduced compact divisor on a normal surface X such that the intersection matrix $(\Gamma_i\Gamma_j)$ of prime components Γ_i of S is negative definite. Let D is an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} D \subset S$ and that D is *nef on* S (cf. [41, Def. 2.14(ii)]), i.e., $D\Gamma \geq 0$ for any prime component Γ of S. Then $-D$ is effective by [64, Lem. 7.1]. If S is connected in addition, then either $D = 0$ or $\operatorname{Supp} D = S$. In fact, if $\Gamma_i \not\subset \operatorname{Supp} D$ for a prime component Γ_i of S, then $D\Gamma_i = 0$, and hence, $\Gamma_i \cap D = \emptyset$ and $\Gamma_j \cap D = \emptyset$ for any other prime component Γ_j such that $\Gamma_i \cap \Gamma_j \neq \emptyset$; this implies that $D = 0$.

Definition 1.26. Let X be a normal surface and let $\mu\colon M \to X$ be the minimal resolution of singularity. A divisor D on X is said to be *numerically Cartier* if the numerical pullback μ^*D is Cartier (cf. "numerically $\mathbb{Q}$-Cartier" in [39, II, §2.e]). We say that D is numerically Cartier at a point $P \in X$ if D is numerically Cartier on an open neighborhood of P. The *numerical factorial index* $\mathrm{nf}(X, P)$ at $P \in X$ is defined as the smallest positive integer r such that rD is numerically Cartier at P for any divisor D defined on any open neighborhood of P. The *numerical factorial index* $\mathrm{nf}(X)$ of X is defined as $\mathrm{lcm}_{P\in X}\,\mathrm{nf}(X, P)$.

The numerical factorial index $\mathrm{nf}(X, P)$ is calculated by an intersection matrix:

Lemma 1.27. *Let X be a normal surface and let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular surface Y. Let P be a point on X such that $f^{-1}(P)$ is a divisor, and let $\Gamma_1, \dots, \Gamma_k$ be the prime components of $f^{-1}(P)$. Then* $\mathrm{nf}(X, P)$ *equals the smallest positive integer r such that* $r\mathsf{M}^{-1}$ *is integral for the intersection matrix* $\mathsf{M} = \left(\Gamma_i\Gamma_j\right)_{1\le i,j\le k}$.

Proof. We can find an open neighborhood $\mathcal{U}$ of P and prime divisors $B_1, B_2, \dots, B_k$ on $f^{-1}\mathcal{U}$ such that $B_i\Gamma_j = \delta_{i,j}$ for any $1 \le i, j \le k$. We set $D_i := f_*B_i$ as a prime divisor on $\mathcal{U}$. Then $f^*D_i = B_i + \sum_{j=1}^{k} a_{i,j}\Gamma_j$ for non-negative rational numbers $a_{i,j}$ such that $(a_{i,j})_{1\le i,j\le k} = -\mathsf{M}^{-1}$. For a positive integer m, if $f^*(mD_i)$ is Cartier along $f^{-1}(P)$ for any i, then $m(a_{i,j}) = -m\mathsf{M}^{-1}$ is integral. Thus, $r \mid \mathrm{nf}(X, P)$. For a divisor D on an open neighborhood of P, we write $f^*D = f^{[*]}D + \sum_{i=1}^{k} c_i\Gamma_i$ for rational numbers c_i. Since $f^{[*]}D$ is Cartier, we have $d_j := (f^{[*]}D)\Gamma_j \in \mathbb{Z}$ and

$$(f^{[*]}D - \sum\nolimits_{i=1}^{k} d_iB_i)\Gamma_j = 0$$

for any $1 \le j \le k$. This implies that $(c_1, c_2, \dots, c_k) = -(d_1, d_2, \dots, d_k)\mathsf{M}^{-1}$. Then $rc_i \in \mathbb{Z}$ for any $1 \le i \le n$, and $f^*(rD)$ is Cartier. Therefore, $\mathrm{nf}(X, P) = r$. □

The following is a generalization of [52, Thm. (2.1)] and is shown by properties of relative Zariski-decomposition (cf. [39, III, Lem. 5.10(2)]); here, we shall give a direct proof.

Lemma 1.28. *Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular surface Y to a normal surface X. Let D be a divisor on X and let B be a $\mathbb{Q}$-divisor on Y such that $f_*B = D$. Then the canonical injection*

$$\lambda_m\colon f_*\mathcal{O}_Y(\llcorner mB\lrcorner) \to (f_*\mathcal{O}_Y(\llcorner mB\lrcorner))^{\vee\vee} \simeq \mathcal{O}_X(mD)$$

is an isomorphism for any integer $m > 0$ if and only if $B \ge f^*D$.

Proof. Since the assertion is local on X, we may assume that f is an isomorphism over $X \setminus \{x\}$ for a point $x \in X$. For any integer $m > 0$, we have an f-exceptional $\mathbb{Q}$-divisor F_m on Y such that $mf^*D - F_m$ is Cartier and

$$(f^*\mathcal{O}_X(mD))^{\vee\vee} \simeq \mathcal{O}_Y(mf^*D - F_m).$$

Since the support of the cokernel of $f^*\mathcal{O}_X(mD) \to (f^*\mathcal{O}_X(mD))^{\vee\vee}$ is a finite subset of $f^{-1}(x)$, the intersection number $(mf^*D - F_m)\Gamma = -F_m\Gamma$ is non-negative for any f-exceptional prime divisor Γ. Hence, F_m is effective by Remark 1.25.

Assume that $B \ge f^*D$. Then $mB \ge \llcorner mB\lrcorner \ge mf^*D - F_m$ for any $m > 0$. Hence, we have an injection $\mathcal{O}_X(mD) \simeq f_*\mathcal{O}_Y(mf^*D - F_m) \to f_*\mathcal{O}_Y(\llcorner mB\lrcorner)$ giving the inverse of

λ_m. This shows the "if" part. The "only if" part is shown as follows: Suppose that λ_m is an isomorphism for any $m > 0$. Then $f^* f_* \mathcal{O}_Y(\llcorner mB \lrcorner) \to \mathcal{O}_Y(\llcorner mB \lrcorner)$ induces an injection $\mathcal{O}_Y(mf^*D - F_m) \to \mathcal{O}_Y(\llcorner mB \lrcorner)$, which corresponds to an inequality $f^*D - (1/m)F_m \le B$ of $\mathbb{Q}$-divisors. Hence, we are reduced to proving that $F_\infty := \lim_{m\to\infty}(1/m)F_m = 0$. Note that the $\mathbb{R}$-divisor F_∞ exists, since $F_m + F_n \ge F_{m+n}$ for any positive integers m and n (cf. [39, III, Lem. 1.3]).

Let $\Gamma_1, \ldots, \Gamma_l$ be the f-exceptional prime divisors. Then there exist positive integers a_1, $\ldots$, a_l such that $A\Gamma_i > 0$ for any $1 \le i \le l$ for the divisor $A = -\sum a_i\Gamma_i$. In particular, f is a projective morphism and A is f-ample (cf. [37, Prop. 1.4]). Hence, $mf^*D + A$ is also f-ample for any $m > 0$. For any positive integer b such that bf^*D is Cartier, we can find a positive integer $k = k(b)$ such that

$$f^* f_* \mathcal{O}_Y(k(bf^*D + A)) \to \mathcal{O}_Y(k(bf^*D + A))$$

is surjective. Hence, $k(bf^*D + A) \le kbf^*D - F_{kb}$; equivalently, $\mathrm{mult}_{\Gamma_i} F_{kb} \le ka_i$ for any $1 \le i \le l$. By taking $b \to \infty$, we have

$$\mathrm{mult}_{\Gamma_i} F_\infty = \lim_{b\to\infty}(1/k(b)b)\,\mathrm{mult}_{\Gamma_i} F_{k(b)b} \le \lim_{b\to\infty} a_i/b = 0.$$

Therefore, $F_\infty = 0$, and we are done. □

1.4. Pullback and pushforward by meromorphic maps. We shall define pullbacks and pushforwards of $\mathbb{R}$-divisors by "non-degenerate meromorphic maps" under certain conditions, and give some of their properties.

Definition 1.29. Let $f\colon X \cdots\to Y$ be a meromorphic map of normal varieties, and let V be the normalization of the graph of f. Then $f = \pi \circ \mu^{-1}$ for the bimeromorphic morphism $\mu = \mu_f\colon V \to X$ and the morphism $\pi = \pi_f\colon V \to Y$ induced by projections (cf. [50, §6, Def. 15], [60, I, §2, Def. 2.2]). We say that f is proper (resp. of maximal rank, resp. non-degenerate) when π is so.

Definition 1.30. In the situation of Definition 1.29 above, assume that f is non-degenerate. We set $n := \dim X = \dim Y$. Let B and D be $\mathbb{R}$-divisors on X and Y, respectively.

(1) The *strict pullback* $f^{[*]}D$ is defined as the $\mathbb{R}$-divisor $\mu_*(\pi^{[*]}D)$ on X, where $\pi^{[*]}D$ is defined in Definition 1.22.

(2) When D is $\mathbb{R}$-Cartier or when $n = 2$, the (*total*) *pullback* f^*D is defined as the $\mathbb{R}$-divisor $\mu_*(\pi^*D)$ on X.

(3) When $\mathrm{Supp}\,B$ is compact or when f is proper, the *strict pushforward* $f_{[*]}B$ is defined as $\pi_*(\mu^{[*]}B)$.

(4) Assume that B is $\mathbb{R}$-Cartier or $n = 2$. When $\mathrm{Supp}\,B$ is compact or when f is proper, the (*total*) *pushforward* f_*B is defined as $\pi_*(\mu^*B)$.

Remark. (1) When B and D are $\mathbb{R}$-Cartier, we have pullbacks μ^*B and π^*D by Lemma 1.19. When $n = 2$, we have μ^*B and π^*D as the numerical pullbacks (cf. Lemma-Definition 1.23).

(2) If f is holomorphic, then $f^{[*]}D$, f^*D, and f_*B above, respectively, are equal to the same ones defined for the morphism f, since μ_f is an isomorphism. Moreover, in

this case, we have $f_{[*]}B = f_*B$.

(3) When f is a bimeromorphic map, the strict pullback $f^{[*]}D$ is called also the *proper transform* of D. When f is a bimeromorphic morphism, this is expressed as $f_*^{-1}D$ in some articles (e.g. [35]), but this is not equal to the total pushforward $(f^{-1})_*D$ for $f^{-1}\colon Y \dashrightarrow X$.

Lemma 1.31. *Let $f\colon X \dashrightarrow Y$ be a non-degenerate meromorphic map of varieties of dimension n and let $\nu\colon W \to X$ be a bimeromorphic morphism from a normal variety W such that $\varpi = f \circ \nu\colon W \to Y$ is holomorphic. Let B and D be $\mathbb{R}$-divisors on X and Y, respectively.*

(1) *The strict pullback $f^{[*]}D$ equals $\nu_*(\varpi^{[*]}D)$.*
(2) *If D is $\mathbb{R}$-Cartier or $n = 2$, then $f^*D = \nu_*(\varpi^*D)$.*
(3) *If* Supp *B is compact or if f is proper, then $f_{[*]}B = \varpi_*(\nu^{[*]}B)$.*
(4) *Assume that B is $\mathbb{R}$-Cartier or $n = 2$. If* Supp *B is compact or f is proper, then $f_*B = \varpi_*(\nu^*B)$.*

Proof. For the normalization V of the graph of f, there is a bimeromorphic morphism $\sigma\colon W \to V$ such that $\nu = \mu \circ \sigma$ and $\varpi = \pi \circ \sigma$ for morphisms $\mu = \mu_f$ and $\pi = \pi_f$ in Definition 1.29. Then $\varpi^{[*]}D = \sigma^{[*]}(\pi^{[*]}D)$ and $\nu^{[*]}B = \sigma^{[*]}(\mu^{[*]}B)$. Hence, we have (1) and (3) by using $\sigma_*(\varpi^{[*]}D) = \pi^{[*]}D$ and $\sigma_*(\nu^{[*]}B) = \mu^{[*]}B$. Similarly, we can prove (2) and (4), respectively, by $\varpi^*D = \sigma^*(\pi^*D)$ and $\sigma_*(\varpi^*D) = \pi^*D$ and by $\nu^*B = \sigma^*(\mu^*B)$ and $\sigma_*(\nu^*B) = \mu^*B$. □

Lemma 1.32. *Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ be non-degenerate meromorphic maps of normal varieties of dimension n. Then we have a commutative diagram*

$$\begin{tikzcd}
 & & U \arrow[dl, "\mu_h"'] \arrow[dr, "\pi_h"] & & \\
 & V \arrow[rr, dashed, "h"] \arrow[dl, "\mu_f"'] \arrow[dr, "\pi_f"] & & W \arrow[dl, "\mu_g"'] \arrow[dr, "\pi_g"] & \\
X \arrow[rr, dashed, "f"] & & Y \arrow[rr, dashed, "g"] & & Z
\end{tikzcd}$$

of meromorphic maps of normal varieties, where V (resp. W) is the normalization of the graph of f (resp. g), morphisms μ_f (resp. μ_g) and π_f (resp. π_g) are as in Definition 1.29, *U is the normalization of the graph of the meromorphic map $h := \mu_g^{-1} \circ \pi_f\colon V \dashrightarrow W$, and morphisms μ_h and π_h are as in* Definition 1.29. *We consider two conditions*:

(a) *every π_f-exceptional divisor is μ_f-exceptional*;
(b) *every μ_g-exceptional divisor is π_g-exceptional.*

Then the following hold for any $\mathbb{R}$-divisors B and D on X and Z, respectively:

(1) *If* (a) *or* (b) *holds, then $(g \circ f)^{[*]}D = f^{[*]}(g^{[*]}D)$.*
(2) *Assume either that* Supp *B is compact or that f and g are proper. If* (a) *or* (b) *holds, then $(g \circ f)_{[*]}B = g_{[*]}(f_{[*]}B)$.*
(3) *Assume either that $n = 2$ or that D and g^*D are $\mathbb{R}$-Cartier. If* (a) *holds, then $(g \circ f)^*D = f^*(g^*D)$.*

(4) *Assume either that* Supp B *is compact or that* f *and* g *are proper. Moreover, assume either that* $n = 2$ *or that* B *and* g_*B *are* $\mathbb{R}$*-Cartier. If* (b) *holds, then* $(g \circ f)_*B = g_*(f_*B)$.

Proof. We consider $\mathbb{R}$-divisors

$$E = \pi_g^{[*]}D - \mu_g^{[*]}(\mu_{g*}(\pi_g^{[*]}D)) \quad \text{and} \quad \widetilde{E} = \pi_g^*D - \mu_g^*(\mu_{g*}(\pi_g^*D))$$

on W in the cases (1) and (3), respectively, and $\mathbb{R}$-divisors

$$C = \pi_{h*}(\mu_h^{[*]}\mu_f^{[*]}B) - \mu_g^{[*]}(\pi_{f*}(\mu_f^{[*]}B)) \quad \text{and} \quad \widetilde{C} = \pi_{h*}(\mu_h^*\mu_f^*B) - \mu_g^*(\pi_{f*}(\mu_f^*B))$$

on W in the cases (2) and (4), respectively. Here, we have

$$h^{[*]}E = \mu_{h*}(\pi_h^{[*]}\pi_g^{[*]}D) - \pi_f^{[*]}(\mu_{g*}(\pi_g^{[*]}D)), \quad h^*\widetilde{E} = \mu_{h*}(\pi_h^*\pi_g^*D) - \pi_f^*(\mu_{g*}(\pi_g^*D))$$

by $\mu_h^{[*]} \circ \pi_f^{[*]} = \pi_h^{[*]} \circ \mu_g^{[*]}$, $\mu_h^* \circ \pi_f^* = \pi_h^* \circ \mu_g^*$, and $\mu_{f*} \circ \mu_f^{[*]} = \mu_{f*} \circ \mu_f^* = \mathrm{id}$. For these $\mathbb{R}$-divisors, we can prove:

(i) E and $\widetilde{E}$ are μ_g-exceptional;
(ii) if every prime component of $\pi_g^{[*]}D$ is not μ_g-exceptional, then $E = 0$;
(iii) $h^{[*]}E$ and $h^*\widetilde{E}$ are π_f-exceptional;
(iv) C and $\widetilde{C}$ are μ_g-exceptional;
(v) if every prime component of $\mu_f^{[*]}B$ is not π_f-exceptional, then $C = 0$.

In fact, by linearity, we may assume that D and B are prime divisors for proving (i)–(v), and we have

$$\mu_{g*}E = \mu_{g*}\widetilde{E} = \mu_{g*}C = \mu_{g*}\widetilde{C} = 0$$

by $\mu_{g*} \circ \mu_g^{[*]} = \mu_{g*} \circ \mu_g^* = \mathrm{id}$, $\mu_{g*} \circ \pi_{h*} = \pi_{f*} \circ \mu_{h*}$, and $\mu_{h*} \circ \mu_h^{[*]} = \mu_{h*} \circ \mu_h^* = \mathrm{id}$. This shows (i) and (iv), and we have (iii) as a consequence of (i). Moreover, in case (ii), E has no μ_g-exceptional prime component but $\mu_{g*}E = 0$; hence, $E = 0$, and (ii) holds. In case (v), $\pi_{f*}(\mu_f^{[*]}B) = m\Theta$ for a prime divisor Θ on Y and a positive integer m, and $\pi_{h*}(\mu_h^{[*]}\mu_f^{[*]}B) = m\mu_g^{[*]}\Theta$, since μ_h and μ_g are bimeromorphic morphisms; thus, $C = 0$, and we have proved (v).

By Lemma 1.31, we have four equalities

$$(g \circ f)^{[*]}D - f^{[*]}(g^{[*]}D) = \mu_{f*}(h^{[*]}E), \qquad (g \circ f)^*D - f^*(g^*D) = \mu_{f*}(h^*\widetilde{E}),$$
$$(g \circ f)_{[*]}B - g_{[*]}(f_{[*]}B) = \pi_{g*}C, \qquad (g \circ f)_*B - g_*(f_*B) = \pi_{g*}\widetilde{C}.$$

For example, we have

$$(g \circ f)^{[*]}D = (\mu_f \circ \mu_h)_*((\pi_g \circ \pi_h)^{[*]}D) = \mu_{f*}(\mu_{h*}(\pi_h^{[*]}(\pi_g^{[*]}D)))$$

by Lemma 1.31(1), and this implies the first equality. Hence, for the proof of (1)–(4), it suffices to verify:

(I) $h^{[*]}E$ and $h^*\widetilde{E}$ are μ_f-exceptional, and
(II) $h_{[*]}C$ and $h_*\widetilde{C}$ are π_g-exceptional.

If (a) holds, then we have (I) and $C = 0$ by (iii) and (v). It implies (1) in the case (a), (2) in the case (a), and (3). If (b) holds, then we have (II) and $E = 0$ by (ii) and (iv). It implies (1) in the case (b), (2) in the case (b), and (4). Thus, we are done. □

Corollary 1.33. *In the situation of* Lemma 1.32*, assume that $n = 2$ and that $\pi_g^* D$ is μ_g-*nef (*cf.* Convention 2.14(1) *below), i.e., $(\pi_g^* D)\Gamma \geq 0$ for any μ_g-exceptional prime divisor* Γ*. Then $(g \circ f)^* D \leq f^*(g^* D)$.*

Proof. The $\mathbb{R}$-divisor $\widetilde{E}$ in the proof of Lemma 1.32 is μ_g-exceptional and μ_g-nef. Then $-\widetilde{E}$ is effective by Remark 1.25, since the intersection matrix of prime components of any connected non-zero μ_g-exceptional divisor is negative definite (cf. [36, p. 6]). Hence, $(g \circ f)^* D - f^*(g^* D) = \mu_{f*}(h^* \widetilde{E}) \leq 0$. □

Remark. An inequality of currents similar to the above is noticed in the study of dynamical systems (cf. [4, Prop. 1.13] and (†) in the proof of [20, Prop. 1.2]).

1.5. Canonical divisors and ramification formulas for normal varieties. In the first half of Section 1.5, we shall explain the *canonical divisor* K_Y of a normal variety Y and the *ramification formula* $K_X = f^* K_Y + R_f$ for a non-degenerate morphism $f \colon X \to Y$ of normal varieties in some special cases (cf. Situation 1.36), which include the case where $\dim X = \dim Y = 2$. Especially, we want to emphasize that K_Y is unique up to linear equivalence but the ramification formula is regarded as an equality not only as a linear equivalence. In the last half, we shall give some variants of the ramification formula including the logarithmic ramification formula due to Iitaka (cf. (I-2) in Proposition 1.40 below).

Convention (Canonical divisor). The *canonical divisor* K_Y of a normal variety Y is regarded as the following object: We set $n = \dim Y$. In case Y is non-singular, the *canonical sheaf* ω_Y is defined as the sheaf $\Omega_Y^n = \Omega_{Y/\operatorname{Spec}\mathbb{C}}^n$ of germs of holomorphic n-forms on Y. In general, the *canonical sheaf* ω_Y is a coherent reflexive sheaf of rank 1 on Y defined as $j_* \omega_{Y_{\mathrm{reg}}}$ for the open immersion $j \colon Y_{\mathrm{reg}} \hookrightarrow Y$ (cf. [49, App. of §1, Cor. (8)]); this is isomorphic to the $(-n)$-th cohomology sheaf $\mathcal{H}^{-n}(\omega_Y^\bullet)$ of the dualizing complex $\omega_Y^\bullet$ (cf. [21], [48]). If ω_Y has a non-zero meromorphic section η, then $\eta|_{Y_{\mathrm{reg}}}$ is a meromorphic n-form on Y_{reg}, and there is a unique divisor $\operatorname{div}(\eta)$ on Y satisfying $\operatorname{div}(\eta)|_{Y_{\mathrm{reg}}} = \operatorname{div}(\eta|_{Y_{\mathrm{reg}}})$, since $\operatorname{codim}(Y \setminus Y_{\mathrm{reg}}) \geq 2$. The divisor $\operatorname{div}(\eta)$ is called the *canonical divisor* and is denoted by K_Y, even though it depends on the choice of η. Hence, $\mathcal{O}_Y(K_Y) \simeq \omega_Y$, and K_Y is unique up to linear equivalence. Even if ω_Y has no non-zero meromorphic section, the symbol K_Y is used virtually, which means just the canonical sheaf ω_Y.

Remark. If Y is Stein, or more generally, if Y is *weakly 1-complete* with a positive line bundle, then every non-zero reflexive sheaf on Y admits a non-zero meromorphic section (cf. [9, Lem. 3]); thus, we can consider K_Y as a divisor. Even if Y is a reducible normal complex analytic space, one can consider K_Y as the union of canonical divisors of connected components of Y.

Definition 1.34 ($f^{\circledast}\eta$). Let $f \colon X \to Y$ be a non-degenerate morphism of non-singular varieties of dimension $n \geq 1$. For a holomorphic n-form η on Y, we write $f^{\circledast}\eta$ for the pullback of η by f as a holomorphic n-form on X. This is given by the canonical homomorphism $\phi \colon f^* \omega_Y = f^* \Omega_Y^n \to \omega_X = \Omega_X^n$. Even for a meromorphic n-form η on Y, we have the pullback $f^{\circledast}\eta$ as a meromorphic n-form on X by

$$f^*(\mathfrak{M}_Y \otimes \omega_Y) \simeq f^*\mathfrak{M}_Y \otimes f^*\omega_Y \xrightarrow{\psi\otimes\mathrm{id}} \mathfrak{M}_X \otimes f^*\omega_Y \xrightarrow{\mathrm{id}\otimes\phi} \mathfrak{M}_X \otimes \omega_X,$$

where $\psi\colon f^*\mathfrak{M}_Y \to \mathfrak{M}_X$ is the pullback homomorphism of meromorphic functions, which exists as f is non-degenerate (cf. the proof of Lemma 1.17).

Remark. The pullback $f^{\circledast}\eta$ is usually denoted by $f^*\eta$, but here, we use $f^{\circledast}$ for avoiding confusions with other f^*.

Lemma-Definition 1.35. *Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties of dimension $n \geq 1$ and let η be a non-zero meromorphic section of ω_Y. For the open subset $X_\diamond = X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$ and for the induced morphism $f_\diamond = f|_{X_\diamond}\colon X_\diamond \to Y_{\mathrm{reg}}$, the pullback $f_\diamond^{\circledast}(\eta|_{Y_{\mathrm{reg}}})$ as a meromorphic n-form on $X_\diamond$ extends to a unique meromorphic section of ω_X. This section is denoted by $f^{\circledast}\eta$.*

Proof. The uniqueness of $f^{\circledast}\eta$ is obvious. Thus, we can replace Y with any open subset. By the local theory of complex analytic spaces, we may assume that there is a finite surjective morphism $\tau\colon Y \to \Omega$ to a domain Ω of the affine space $\mathbb{C}^n$ (cf. [7, §3.1, Thm. 1]). Let ζ be the standard holomorphic n-form on Ω, i.e., $\zeta = d\mathsf{z}_1 \wedge d\mathsf{z}_2 \wedge \cdots \wedge d\mathsf{z}_n$ for a coordinate $(\mathsf{z}_1, \mathsf{z}_2, \ldots, \mathsf{z}_n)$ of $\mathbb{C}^n$. For the induced morphism $\tau_{\mathrm{reg}}\colon Y_{\mathrm{reg}} \to \Omega$ of non-singular varieties, we have a meromorphic function φ on Y such that

$$\tau_{\mathrm{reg}}^{\circledast}\zeta = \varphi\eta|_{Y_{\mathrm{reg}}}.$$

Let ξ be a meromorphic section of ω_X such that the restriction $\xi|_{X_{\mathrm{reg}}}$ equals the pullback $(\tau \circ f_{\mathrm{reg}})^{\circledast}\zeta$ as a holomorphic n-form on X_{reg} for the induced morphism $f_{\mathrm{reg}} := f|_{X_{\mathrm{reg}}}\colon X_{\mathrm{reg}} \to Y$. Then

$$\xi|_{X_\diamond} = (f^*\varphi)f_\diamond^{\circledast}(\eta|_{Y_{\mathrm{reg}}}).$$

Thus, it is enough to set $f^{\circledast}\eta := (f^*\varphi)^{-1}\xi$. □

Remark. If $\operatorname{codim}(f^{-1}\operatorname{Sing} Y, X) \geq 2$, then $\operatorname{codim}(X \setminus X_\diamond, X) \geq 2$. In this case, for any holomorphic section η of ω_Y, the pullback $f^{\circledast}\eta$ is also a holomorphic section of ω_X. In fact, the section $f^{\circledast}\eta$ is holomorphic if and only if the restriction $f^{\circledast}\eta|_{X_\diamond}$ is so by $\operatorname{codim}(X\setminus X_\diamond, X) \geq 2$, and now $f_\diamond^{\circledast}(\eta|_{Y_{\mathrm{reg}}})$ is holomorphic.

Situation 1.36. Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties. As a pullback homomorphism f^* for certain $\mathbb{R}$-divisors, we consider one of the following:

(I) The homomorphism $f^*\colon \operatorname{CDiv}(Y, \mathbb{R}) \to \operatorname{CDiv}(X, \mathbb{R})$ in Lemma 1.19.

(II) The homomorphism $f^*\colon \operatorname{Div}(Y, \mathbb{R}) \to \operatorname{Div}(X, \mathbb{R})$ in Lemma 1.19, which is defined only when $\operatorname{codim}(f^{-1}\operatorname{Sing} Y, X) \geq 2$.

(III) The numerical pullback homomorphism $f^*\colon \operatorname{Div}(Y, \mathbb{R}) \to \operatorname{Div}(X, \mathbb{R})$ in Lemma-Definition 1.23, which is defined only when $\dim X = \dim Y = 2$. This f^* extends the homomorphisms f^* in (I) and (II), but does not induce $\operatorname{Div}(Y) \to \operatorname{Div}(X)$ in general.

Lemma 1.37. *Let D be an $\mathbb{R}$-divisor on Y such that the pullback $f^*(K_Y + D)$ is defined in one of cases in* Situation 1.36. *Then $K_X - f^*(K_Y + D)$ is uniquely determined as an $\mathbb{R}$-*

divisor on X when ω_Y has a non-zero meromorphic section η, by setting $K_X = \operatorname{div}(f^{\circledast}\eta)$ and $K_Y = \operatorname{div}(\eta)$.

Proof. For non-zero meromorphic sections η_1 and η_2 of ω_Y, there is a non-zero meromorphic function φ on Y such that $\eta_1 = \varphi\eta_2$. Then $f^{\circledast}\eta_1 = (f^*\varphi)f^{\circledast}\eta_2$, and we have

$$\operatorname{div}(\eta_1) + D = \operatorname{div}(\eta_2) + D + \operatorname{div}(\varphi) \quad \text{and} \quad \operatorname{div}(f^{\circledast}\eta_1) = \operatorname{div}(f^{\circledast}\eta_2) + \operatorname{div}(f^*\varphi).$$

Since $f^* \operatorname{div}(\varphi) = \operatorname{div}(f^*\varphi)$ (cf. Lemma 1.17), we have

$$\operatorname{div}(f^{\circledast}\eta_1) - f^*(\operatorname{div}(\eta_1) + D) = \operatorname{div}(f^{\circledast}\eta_2) - f^*(\operatorname{div}(\eta_2) + D).$$

Thus, $K_X - f^*(K_Y + D)$ is uniquely determined. □

Convention. Let $f : X \to Y$ be a non-degenerate morphism of normal varieties and let B and D be $\mathbb{R}$-divisors on X and Y, respectively. By an equality $K_X + B = f^*(K_Y + D)$, we mean the following:

(1) Assume that ω_Y admits a non-zero meromorphic section η. Then the pullback $f^*(\operatorname{div}(\eta)+D)$ exists in one of cases in Situation 1.36 and $\operatorname{div}(f^{\circledast}\eta)+B = f^*(\operatorname{div}(\eta)+D)$ as an $\mathbb{R}$-divisors on X.

(2) If $Y = \bigcup_\lambda Y_\lambda$ for open subsets Y_λ such that each ω_{Y_λ} admits a non-zero meromorphic section on Y_λ, then

$$K_{X_\lambda} + B|_{X_\lambda} = f_\lambda^*(K_{Y_\lambda} + D|_{Y_\lambda})$$

for any λ, where $X_\lambda = f^{-1}Y_\lambda$ and $f_\lambda = f|_{X_\lambda} : X_\lambda \to Y_\lambda$.

Note that (1) is independent of the choice of η by Lemma 1.37.

Definition (Ramification divisor (cf. [25, §5.6])). In Situation 1.36, we define the ramification divisor of f as a $\mathbb{Q}$-divisor R_f on X such that $K_X = f^*K_Y + R_f$.

Remark. If X and Y are non-singular, then R_f is the usual ramification divisor in the sense that R_f is an effective divisor and that the canonical injection $f^*\omega_Y \to \omega_X$ induces an isomorphism $f^*\omega_Y \simeq \omega_X \otimes \mathcal{O}_X(-R_f)$ (cf. [25, §5.6]). In Situation 1.36(I), R_f exists when K_Y is $\mathbb{Q}$-Cartier, but R_f is not necessarily effective. In fact, when f is a resolution of singularities, R_f is effective if and only if Y has only canonical singularities (cf. [49, Def. (1.1)], [32, Def. 0-2-6]). In Situation 1.36(II), R_f exists always as an effective divisor as the closure of the ramification divisor $R_{f_\diamond}$ of the induced morphism $f_\diamond = f|_{X_\diamond} : X_\diamond \to Y_{\mathrm{reg}}$ for $X_\diamond = X_{\mathrm{reg}} \cap f^{-1}Y_{\mathrm{reg}}$. In Situation 1.36(III), R_f exists always, but it is not necessarily effective.

Now, we shall present some variations of ramification formula for non-degenerate morphisms. We begin with:

Lemma 1.38. *Let $f : X \to Y$ be a non-degenerate morphism of non-singular varieties of dimension $n \geq 1$ and let B and D be non-singular prime divisors on X and Y, respectively, such that $B = f^{-1}D$.*

(1) *If B is not f-exceptional, then $1 + \operatorname{mult}_B R_f = \operatorname{mult}_B f^*D$.*

(2) *If B is f-exceptional, then the image of the pullback homomorphism*

$$f^*\Omega^n_Y(\log D) \to \Omega^n_X(\log B)$$

of logarithmic n-forms is contained in the subsheaf Ω^n_X.

Proof. We shall give a sheaf-theoretic proof even though (1) is obvious by a local description of f. For each $1 \le p \le n$, there is a commutative diagram

$$\text{(I-1)} \qquad \begin{tikzcd} 0 \arrow[r] & f^*\Omega^p_Y \arrow[r] \arrow[d, "\psi^p"] & f^*\Omega^p_Y(\log D) \arrow[r] \arrow[d, "\phi^p"] & f^*\Omega^{p-1}_D \arrow[r] \arrow[d, "\varphi^{p-1}"] & 0 \\ 0 \arrow[r] & \Omega^p_X \arrow[r] & \Omega^p_X(\log B) \arrow[r, "r^p"] & \Omega^{p-1}_B \arrow[r] & 0 \end{tikzcd}$$

of exact sequences on sheaves of holomorphic and logarithmic p-forms, where the pullback homomorphisms $\psi^p = \wedge^p\psi^1$ and $\phi^p = \wedge^p\phi^1$ are injective as f is non-degenerate. Moreover, r^1 is induced by the residue isomorphism $\Omega^1_X(\log B) \otimes \mathcal{O}_B \simeq \mathcal{O}_B$, and φ^{p-1} is expressed as the composite homomorphism

$$f^*\Omega^{p-1}_D \xrightarrow{\pi^{p-1}} g^*\Omega^{p-1}_D \xrightarrow{\psi^{p-1}_g} \Omega^{p-1}_B$$

for $g := f|_B : B \to D$, where ψ^{p-1}_g is the pullback homomorphism of holomorphic $(p-1)$-forms, and π^{p-1} is a surjection induced by $f^*\mathcal{O}_D \to \mathcal{O}_B$.

Assume that B is not f-exceptional. Then g is non-degenerate and ψ^{n-1}_g is injective. We set $m = \mathrm{mult}_B f^*D$. Then $f^*D = mB$, φ^{n-1} is generically surjective on B, and the kernel of φ^{n-1} is isomorphic to

$$\mathcal{O}_{(m-1)B} \otimes \mathcal{O}_X(-B) \otimes f^*\Omega^n_Y(\log Y)$$

if $m > 1$, and is zero if $m = 1$. In particular, ϕ^n is surjective on a dense open subset of B. By applying the snake lemma to (I-1) for $p = n$, we have $\mathrm{mult}_B R_f = m - 1$, since the cokernel of ψ^n is isomorphic to $\omega_X \otimes \mathcal{O}_{R_f}$. This shows (1).

Assume next that B is f-exceptional. Then $n \ge 2$, and $\psi^{n-1}_g = 0$ as g is degenerate. Hence, the image of ϕ^n is contained in Ω^n_X. This shows (2). □

Lemma 1.39. *Let $f : X \to Y$ be a non-degenerate morphism of normal varieties without exceptional divisors and let $B \subset X$ and $D \subset Y$ be reduced divisors such that $B = f^{-1}D$. Then $K_X + B = f^*(K_Y + D) + \Delta$ for an effective divisor Δ having no common prime component with B. In particular, the induced morphism $X \setminus B \to Y \setminus D$ is étale in codimension* 1 *if and only if $\Delta = 0$.*

Proof. We can consider the pullback homomorphism $f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ in Situation 1.36(II), since $\mathrm{codim}(f^{-1}\,\mathrm{Sing}\, Y, X) \ge 2$. Thus, we may assume that X and Y are non-singular by replacing Y and X with Y_{reg} and $X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$, respectively. For the ramification divisor $R_f = K_X - f^*K_Y$, we have $\Delta = R_f + B - f^*D$. Let Γ be a prime divisor on X. If $\Gamma \not\subset B = f^{-1}D$, then $\mathrm{mult}_\Gamma \Delta = \mathrm{mult}_\Gamma R_f \ge 0$. If $\Gamma \subset B$, then $\Gamma \subset f^{-1}\Theta$ for a prime component Θ of D. In this case, since B is not f-exceptional, we have

$$1 + \mathrm{mult}_\Gamma R_f = \mathrm{mult}_\Gamma f^*\Theta = \mathrm{mult}_\Gamma f^*D$$

by applying Lemma 1.38(1) to suitable open subsets $U \subset X$ and $V \subset Y$ such that $U \subset f^{-1}V$ and that $\Gamma|_U = B|_U$ and $\Theta|_U = D|_U$ are non-singular prime divisors; hence, $\mathrm{mult}_\Gamma \Delta = \mathrm{mult}_\Gamma(R_f + B - f^*D) = 0$. Thus, Δ is effective and has no common prime component with B. □

The equality (I-2) below is known as the logarithmic ramification formula due to Iitaka (cf. [24, §4, (R)], [25, Thm. 11.5]). The generalization (I-3) is obtained by an argument of Suzuki in the proof of [58, Prop. 2.1] in the case of bimeromorphic morphisms and by Iitaka [26, Part 2, Prop. 1] in the general case.

Proposition 1.40. *Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties and let B and D be reduced divisors on X and Y, respectively, such that Y is non-singular, D is normal crossing, and $f^{-1}D \subset B$.*

(1) *There is an effective divisor $\overline{R}$ on X such that*

$$K_X + B = f^*(K_Y + D) + \overline{R} \tag{I-2}$$

and that any common prime component of $f^{-1}D$ and $\overline{R}$ is f-exceptional.

(2) *Let C be a non-singular divisor on Y and A a reduced divisor on X such that $(f^{[*]}C)_{\mathrm{red}} \leq A$, $A + B$ is reduced, and $C + D$ is reduced and normal crossing. Then there is an effective divisor $R^{\&}$ on X such that*

$$K_X + A + B = f^*(K_Y + C + D) + R^{\&}. \tag{I-3}$$

Proof. By replacing X with a Zariski-open subset whose complement has codimension at least 2, we may assume that X and B are non-singular and that $\widetilde{B} = (f^*C + A + B)_{\mathrm{red}}$ is also non-singular in the situation of (2).

(1): The pullback homomorphism

$$f^*\Omega^n_Y(\log D) \simeq f^*(\omega_Y \otimes \mathcal{O}_Y(D)) \to \Omega^n_X(\log B) \simeq \omega_X \otimes \mathcal{O}_X(B)$$

of logarithmic n-forms is injective as f is non-degenerate, and it implies that $\overline{R} \geq 0$. It is enough to prove that $\Gamma \not\subset \operatorname{Supp} \overline{R}$ for any non-f-exceptional prime component Γ of $f^{-1}D$. For this, by replacing X and Y with suitable open subsets, we may assume that $\Gamma = B = f^{-1}D$. Then $\Gamma = B \not\subset \operatorname{Supp} \overline{R}$ by Lemma 1.39.

(2): By (1), we have $K_X + \widetilde{B} = f^*(K_Y + C + D) + \widehat{R}$ for an effective divisor $\widehat{R}$. It is enough to prove that $\widehat{R} \geq \widetilde{B} - (A + B)$, or equivalently that $\widehat{R} \geq \Gamma$ for any prime component Γ of $\widetilde{B} - (A + B)$. By assumption, Γ is f-exceptional, $\Gamma \subset f^{-1}C$, and $\Gamma \not\subset B$. By replacing X and Y with open subsets, we may assume that $B = 0$, $\widetilde{B} - (A + B) = (f^*C + A)_{\mathrm{red}} - A = f^{-1}C$, and $\Gamma = f^{-1}C$. Then the image of

$$f^*\Omega^n_Y(\log C) \simeq f^*(\omega_Y \otimes \mathcal{O}_Y(C)) \to \Omega^n_X(\log \Gamma) \simeq \omega_X \otimes \mathcal{O}_X(\Gamma)$$

is contained in ω_X by Lemma 1.38(2). It implies that $\widehat{R} \geq \Gamma$, and we are done. □

Remark. We have a little generalization of [26, Part 2, Prop. 1] in [39, II, Thm. 4.2]. But the assumption $\rho^{[*]}X \leq Y$ in the statement is stronger than what we expect. The correct assumption is $(\rho^{[*]}X)_{\mathrm{red}} \leq Y$. This correct case has been treated in the proof of [26, Part 2, Prop. 1], where $(f^{[*]}C)_{\mathrm{red}}$ is written as $f^{-1}[C]$. The stronger assumption affects [39, II, Lem. 4.4] given as an application of [39, II, Thm. 4.2].

The following lemma is borrowed from [39, II, Lems. 4.3 and 4.4], which are stated for generically finite morphisms.

Lemma 1.41. *Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties and let D be an effective $\mathbb{Q}$-divisor on Y. Assume that Y is non-singular and $\ulcorner D \urcorner$ is reduced and normal crossing.*

(1) *There is an effective $\mathbb{Q}$-divisor $\overline{R}_D$ on X such that*

$$K_X + (f^*D)_{\mathrm{red}} = f^*(K_Y + D) + \overline{R}_D.$$

(2) *If $\llcorner D \lrcorner = 0$, then there is a $\mathbb{Q}$-divisor R_D on X such that $\ulcorner R_D \urcorner$ is effective and $K_X = f^*(K_Y + D) + R_D$.*

(3) *If $C := \llcorner D \lrcorner$ is non-singular, then there is a $\mathbb{Q}$-divisor $R^{\&}_D$ on X such that $\ulcorner R^{\&}_D \urcorner$ is effective and*

$$K_X + (f^{[*]}C)_{\mathrm{red}} = f^*(K_Y + D) + R^{\&}_D.$$

Proof. We may assume that $D \neq 0$, since the ramification divisor $R_f = K_X - f^*K_Y$ is effective. Hence $D_{\mathrm{red}} = \operatorname{Supp} D = \ulcorner D \urcorner$. By replacing X with a Zariski-open subset whose complement has codimension at least 2, we may assume that X and $(f^*D)_{\mathrm{red}}$ are non-singular.

(1) and (2): By Proposition 1.40(1), $K_X + (f^*D)_{\mathrm{red}} = f^*(K_Y + D_{\mathrm{red}}) + \widetilde{R}$ for an effective divisor $\widetilde{R}$. Then $\overline{R}_D$ is effective by

$$\widetilde{R} = R_f + (f^*D)_{\mathrm{red}} - f^*(D_{\mathrm{red}}) = \overline{R}_D - f^*(D_{\mathrm{red}} - D).$$

This proves (1). Assume that $\llcorner D \lrcorner = 0$. Then $\overline{R}_D = R_D + (f^*D)_{\mathrm{red}} \geq 0$. For any prime component Γ of f^*D, we have $\operatorname{mult}_\Gamma f^*(D_{\mathrm{red}} - D) > 0$, and

$$\operatorname{mult}_\Gamma R_D + 1 = \operatorname{mult}_\Gamma \overline{R}_D = \operatorname{mult}_\Gamma \widetilde{R} + \operatorname{mult}_\Gamma f^*(D_{\mathrm{red}} - D) > 0.$$

Hence, $\ulcorner R_D \urcorner$ is effective, and (2) has been proved.

(3): We set $\Delta := \langle D \rangle = D - C$. By Proposition 1.40(2), we have

$$K_X + (f^{[*]}C)_{\mathrm{red}} + (f^*\Delta)_{\mathrm{red}} = f^*(K_Y + C + \Delta_{\mathrm{red}}) + R^{\&}$$

for an effective divisor $R^{\&}$ on X. Then

$$R^{\&}_D + (f^*\Delta)_{\mathrm{red}} = R^{\&} + f^*(\Delta_{\mathrm{red}} - \Delta)$$

is effective. For any prime component Γ of $f^*\Delta$, we have

$$\operatorname{mult}_\Gamma f^*(\Delta_{\mathrm{red}} - \Delta) > 0 \quad \text{and} \quad 1 + \operatorname{mult}_\Gamma R^{\&}_D = \operatorname{mult}_\Gamma(R^{\&}_D + (f^*\Delta)_{\mathrm{red}}) > 0.$$

Therefore, $\ulcorner R^{\&}_D \urcorner$ is effective, and (3) has been proved. □

2. Log-canonical singularities for complex analytic surfaces

We shall explain basic properties of log-canonical singularities and their variants only in the surface case, in Section 2.1, and give results related to ramification formulas in Section 2.2. The relative abundance theorem and the log-canonical modifications for surfaces are given in Section 2.3.

2.1. Log-canonical singularities.

DEFINITION 2.1. Let X be a normal surface with an effective $\mathbb{Q}$-divisor B and let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface M. We set $\Sigma = \Sigma_\mu(X, B)$ to be the union of $\mu^{-1}\operatorname{Supp} B$ and the μ-exceptional locus. Note that $\Sigma \supset \mu^{-1}\operatorname{Sing} X$. Let $B_\mu = B_\mu(X, B)$ and $T_\mu = T_\mu(X, B)$ be the positive and negative parts, respectively, of the prime decomposition of $\mu^*B - R_\mu$ (cf. Definition 1.15) for the *ramification divisor* R_μ (cf. Section 1.5), i.e., $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Note that $B_\mu \geq \mu^{[*]}B$ for the proper transform $\mu^{[*]}B$ in M (cf. Definition 1.22) and that T_μ is μ-exceptional. When there is such a bimeromorphic morphism μ with Σ being a normal crossing divisor, the pair (X, B) is said to be:

- *log-canonical*, if $\ulcorner B_\mu \urcorner$ is reduced;
- *log-terminal*, if $\llcorner B_\mu \lrcorner = 0$;
- 1-*log-terminal*, if $\ulcorner B_\mu \urcorner$ is reduced and if $\llcorner B_\mu \lrcorner$ is a non-singular divisor identified with the proper transform of $\llcorner B \lrcorner$ in M.

Here, the zero divisor is considered as a reduced and non-singular divisor (cf. Remark 1.18). For a point $P \in X$, the pair (X, B) is said to be log-canonical (resp. log-terminal, resp. 1-log-terminal) at P if $(U, B|_U)$ is so for some open neighborhood U of P.

REMARK 2.2. The conditions above are independent of the choice of $\mu\colon M \to X$. This follows from special cases of Lemma 2.10 below.

REMARK. If (X, B) is log-terminal, then $\operatorname{mult}_\Gamma B_\mu < 1$ for any prime component Γ of Σ. The prefix "1-" of 1-log-terminal comes from a property that we allow $\operatorname{mult}_\Gamma B_\mu = 1$ only for proper transforms Γ of prime components of B.

REMARK 2.3. It is known that $K_X + B$ is $\mathbb{Q}$-Cartier if (X, B) is log-canonical in the sense above (cf. [30, Cor. 9.5], [34, §4.1]). We shall prove it in Corollary 2.21 below by applying the relative abundance theorem, Theorem 2.19. As a consequence, our definitions of log-canonical and log-terminal coincide with those given in [32, Def. 0-2-10]. The log-terminal and 1-log-terminal are called "Kawamata log terminal" (klt) and "purely log terminal" (plt), respectively, in [56] and [35]. As our policy, we do not use the notion of "log terminal" in the sense of [56] and [35], since it is not analytically local (cf. Remark 2.8 below). Accordingly, the use of "purely log terminal" is not allowed, since it is weaker than our log-terminal.

REMARK. The pair (X, B) is 1-log-terminal if and only if $(X\&B, 0)$ is log-terminal for the bimeromorphic pair $X\&B$ in the sense of [39, II, Def. 4.8].

Bimeromorphic contraction morphisms of extremal rays in the minimal model program preserve log-canonical (resp. log-terminal, resp. 1-log-terminal) pairs by:

Lemma 2.4. *Let $\nu\colon X \to X'$ be a bimeromorphic morphism of normal surfaces with a unique exceptional prime divisor Γ. Let B be an effective $\mathbb{Q}$-divisor on X such that $(K_X + B)\Gamma \leq 0$. If (X, B) is log-canonical (resp. log-terminal), then (X', B') is so for $B' := \nu_*B$. If*

$(K_X + B)\Gamma < 0$ *and* (X, B) *is log-canonical, then* (X', B') *is* 1*-log-terminal at* $\nu(\Gamma)$.

Proof. By assumption, there is a rational number $\alpha \geq 0$ such that $K_X + B = \nu^*(K_{X'} + B') + \alpha\Gamma$. Here, $\alpha > 0$ if and only if $(K_X + B)\Gamma < 0$. Let $\mu\colon M \to X$, B_μ, and T_μ be as in Definition 2.1 for (X, B). Here, we may assume that the union of $\mu^{-1}(\Gamma \cup \operatorname{Supp} B)$ and the μ-exceptional locus is normal crossing and that the proper transform of $(\llcorner B \lrcorner + \Gamma)_{\mathrm{red}}$ is non-singular. Then

$$K_M + B_\mu = (\nu \circ \mu)^*(K_{X'} + B') + T_\mu + \alpha\mu^*\Gamma.$$

In particular, the first assertion holds when $\alpha = 0$. Thus, we may assume that $\alpha > 0$, i.e., $(K_X + B)\Gamma < 0$. Let $B_{\nu\circ\mu}$ and $T_{\nu\circ\mu}$ be the positive and negative parts, respectively, of the prime decomposition of $B_\mu - (T_\mu + \alpha\mu^*\Gamma)$. Then the following hold for any prime divisor Θ on M:

- If $\Theta \not\subset \mu^{-1}\Gamma$, then $\operatorname{mult}_\Theta B_\mu = \operatorname{mult}_\Theta B_{\nu\circ\mu}$.
- If $\Theta \subset \mu^{-1}\Gamma$ but $\Theta \not\subset \operatorname{Supp} B_\mu$, then $\operatorname{mult}_\Theta B_\mu = \operatorname{mult}_\Theta B_{\nu\circ\mu} = 0$.
- If $\Theta \subset \mu^{-1}\Gamma \cap \operatorname{Supp} B_\mu$, then $1 \geq \operatorname{mult}_\Theta B_\mu > \operatorname{mult}_\Theta B_{\nu\circ\mu}$.

In particular, if (X, B) is log-terminal, then (X', B') is so, since $\llcorner B_\mu \lrcorner = 0$ implies $\llcorner B_{\nu\circ\mu} \lrcorner = 0$. If (X, B) is log-canonical, then $\ulcorner B_{\nu\circ\mu} \urcorner$ is reduced and $\llcorner B_{\nu\circ\mu} \lrcorner$ is a reduced subdivisor of $\llcorner B_\mu \lrcorner$ having no prime component contracted to $\nu(\Gamma)$ by $\nu \circ \mu$; thus, (X', B') is 1-log-terminal at $\nu(\Gamma)$. Therefore, the first assertion for $\alpha > 0$ and the second assertion have been proved, and we are done. □

Remark. The first assertion is a special case of Proposition 2.12(1) below.

Fact 2.5. *The germs of log-canonical pairs* (X, S) *of a normal surface* X *and a reduced divisor* S *at a point* $x \in S$ *are classified in* [30, Thm. 9.6] *(cf.* [35, Ch. 3]*). In particular, one of the following three cases occurs (cf.* [41, Thm. 3.22]*):*

(1) $x \in \operatorname{Sing} S$ *and* (X, S) *is* toroidal at x*;*

(2) $x \in S_{\mathrm{reg}}$ *and* $(X, S + S')$ *is toroidal at* x *for a non-singular divisor* $S' \not\subset S$ *such that* $x \in S'$*;*

(3) $x \in S_{\mathrm{reg}} \cap \operatorname{Sing} X$ *and there is a double cover* $\tau\colon \widetilde{X} \to X$ *such that*
   - τ *is étale over* $X \setminus \{x\}$*,*
   - $\tau^{-1}(x) = \{\tilde{x}\}$ *for a point* $\tilde{x} \in \operatorname{Sing} \widetilde{S}$*, where* $\widetilde{S} := \tau^*S$*, and*
   - $(\widetilde{X}, \widetilde{S})$ *is toroidal at* $\tilde{x}$*.*

Here, for a reduced divisor D*, the pair* (X, D) *is said to be toroidal at* x*, if* $X \setminus D \hookrightarrow X$ *is a toroidal embedding at* x *(cf.* [33, II, §1]*), or equivalently, there exist an affine toric variety* V *and an open immersion* $\theta\colon \mathcal{U} \hookrightarrow V$ *from an open neighborhood of* $\mathcal{U}$ *of* x *such that* $\theta^{-1}(\mathbb{T}) = \mathcal{U} \setminus D$ *for the open torus* $\mathbb{T}$ *of* V*.*

Moreover, for the minimal resolution $\mu\colon M \to X$ *of singularities, the dual graph of prime components of the union of* $\mu^{-1}S$ *and the* μ*-exceptional locus is completely described (cf.* [30, Thm. 9.6], [41, Thm. 3.22]*). In particular,* (X, x) *is a cyclic quotient singularity in* (1) *and* (2)*, and is a cyclic or dihedral quotient singularity in* (3)*. The pair* (X, S) *is* 1*-log-terminal at* x *if and only if* (2) *occurs. The divisor* $K_X + S$ *is Cartier at* x *if and only if either* (1) *occurs or* $x \in X_{\mathrm{reg}} \cap S_{\mathrm{reg}}$*.*

Lemma 2.6. *Let (X, B) be a log-canonical pair of a normal surface X and an effective $\mathbb{Q}$-divisor B. If (X, B) is not 1-log-terminal at a point $x \in X$, then $(X, B+D)$ is not log-canonical for any effective $\mathbb{Q}$-divisor D such that $x \in \operatorname{Supp} D$. In particular,* $\operatorname{Supp}\langle B\rangle \cap \operatorname{Sing} \llcorner B \lrcorner = \emptyset$.

Proof. The last assertion follows from the first one, since (X, S) is log-canonical for $S := \llcorner B \lrcorner$ and (X, S) is not 1-log-terminal at any point of $\operatorname{Sing} S$.

For the bimeromorphic morphism $\mu\colon M \to X$ in Definition 2.1, we may assume that the union of $\mu^{-1}(\operatorname{Supp} B \cup \operatorname{Supp} D)$ and the μ-exceptional locus is normal crossing. For the $\mathbb{Q}$-divisors B_μ and T_μ above, let B'_μ and T'_μ be the positive and negative parts, respectively, of the prime decomposition of $B_\mu + \mu^* D - T_\mu$. Then

$$K_M + B'_\mu = \mu^*(K_X + B + D) + T'_\mu.$$

The first assertion holds if the following condition $(*)$ is satisfied:

$(*)$ There is a prime component Γ of $\llcorner B_\mu \lrcorner$ such that $\mu(\Gamma) = \{x\}$.

In fact, if $(*)$ holds, then $\ulcorner B'_\mu \urcorner$ is not reduced by

$$\operatorname{mult}_\Gamma B'_\mu = \operatorname{mult}_\Gamma B_\mu + \operatorname{mult}_\Gamma \mu^* D = 1 + \operatorname{mult}_\Gamma \mu^* D > 1,$$

and $(X, B + D)$ is not log-canonical (cf. Remark 2.2).

For the rest, we shall check $(*)$. If $x \in S_{\mathrm{reg}}$ for $S = \llcorner B \lrcorner$, then $(*)$ holds, since (X, B) is not 1-log-terminal at x. Thus, we may assume that $x \in \operatorname{Sing} S$. Then (X, S) is toroidal at x by Fact 2.5. Let U be an open neighborhood of x in X such that $\operatorname{Sing} U \subset \{x\}$ and $U \cap \operatorname{Sing} S = \{x\}$. When $x \in \operatorname{Sing} X$, let $\eta\colon Y \to U$ be the minimal resolution of singularity. When $x \in X_{\mathrm{reg}}$, let $\eta\colon Y \to U$ be the blowing up at x. Then

$$K_Y + S_Y = \eta^*(K_U + S|_U) \tag{II-1}$$

for the reduced divisor $S_Y = \eta^{-1}(S|_U)$. In fact, if $x \in \operatorname{Sing} X$, then η is a *toroidal blowing up* with respect to $(U, S|_U)$ (cf. [41, Exam. 3.2, §4.3]), which induces (II-1); if $x \in X_{\mathrm{reg}}$, then we have (II-1) by a direct calculation. Since $\mu^{-1}U \to U$ factors through η, an η-exceptional component of S_Y gives a prime component Γ of $\llcorner B_\mu \lrcorner$ lying over x. Thus $(*)$ is satisfied also in case $x \in \operatorname{Sing} S$, and we are done. □

Corollary 2.7. *For a normal surface X and an effective $\mathbb{Q}$-divisor B, the pair (X, B) is* weak log-terminal *in the sense of* [32, Def. 0-2-10] *if and only if*

(a) *(X, B) is 1-log-terminal at any point of $X \setminus \operatorname{Sing} \llcorner B \lrcorner$,*

(b) *$\operatorname{Sing} \llcorner B \lrcorner \subset X_{\mathrm{reg}} \setminus \operatorname{Supp}\langle B\rangle$, and*

(c) *$\llcorner B \lrcorner|_{X_{\mathrm{reg}}}$ is a normal crossing divisor.*

Proof. Assume that (X, B) is weak log-terminal. Then we have (a) by [32, Def. 0-2-10, (ii′) and (iii)]. By Fact 2.5 and Lemma 2.6, we see that $\operatorname{Sing} \llcorner B \lrcorner \cap \operatorname{Supp}\langle B\rangle = \emptyset$, and (X, B) is toroidal at any point of $\operatorname{Sing} \llcorner B \lrcorner$. Moreover, X is non-singular along $\operatorname{Sing} \llcorner B \lrcorner$ by [32, Def. 0-2-10(iii)]. This shows (b) and (c).

Conversely, assume (a), (b), and (c). Then we can find a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface M such that

- the union of the μ-exceptional locus and $\mu^{-1} \operatorname{Supp} B$ is a normal crossing divisor, and

- μ is an isomorphism over an open neighborhood of $\operatorname{Sing} \llcorner B \lrcorner$.

Moreover, as in the proof of Lemma 1.28, we can find a μ-exceptional effective divisor E such that $-E$ is μ-ample: This implies [32, Def. 0-2-10(iv)]. For the effective $\mathbb{Q}$-divisors B_μ and T_μ in Definition 2.1, $\ulcorner B_\mu \urcorner$ is reduced as (X, B) is log-canonical (cf. Remark 2.2), and moreover, $\llcorner B_\mu \lrcorner$ is the proper transform of $\llcorner B \lrcorner$ in M by (a). Thus, (X, B) is weak log-terminal. □

REMARK 2.8. By the proof above, we see that (X, B) is "log terminal" in the sense of [56] and [35] if and only if (a), (b), and the following stronger version (c′) of (c) are satisfied:

(c′) $\llcorner B \lrcorner|_{X_{\mathrm{reg}}}$ is a *simple* normal crossing divisor.

Note that the condition (c′) is not analytically local. When B is reduced, the "log terminal" condition for (X, B) is equivalent to the condition that (X, B) has only "Kawamata singularities" in the sense of Tsunoda–Miyanishi (cf. [59, 1.1]).

2.2. Relations with ramification formulas. We shall show that singularities on (X, B) such as log-canonical, log-terminal, and 1-log-terminal are preserved by a non-degenerate morphism under certain conditions. The results here give refinements of a similar result [41, Lem. 3.19] in the case of schemes.

Lemma 2.9. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B and let $f\colon Y \to X$ be a non-degenerate morphism from another normal surface Y. Then there exist bimeromorphic morphisms $\mu\colon M \to X$ and $\nu\colon N \to Y$ from non-singular surfaces M and N with a commutative diagram*

(II-2)
$$\begin{array}{ccc} N & \xrightarrow{\nu} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\mu} & X \end{array}$$

for a non-degenerate morphism g which satisfy the following conditions:

1. *For the μ-exceptional locus E_μ, the union $E = E_\mu \cup \mu^{-1} \operatorname{Supp} B$ is a normal crossing divisor.*
2. *For the ν-exceptional locus E_ν and for*
$$\widetilde{\Sigma}_f := f^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B) \cup \operatorname{Supp} R_f,$$
the union $F = E_\nu \cup \nu^{-1}\widetilde{\Sigma}_f$ is a normal crossing divisor.
3. *The equality $F = g^{-1}E \cup \operatorname{Supp} R_g$ holds for the divisors E and F above.*

Here, R_f and R_g denote the ramification divisors of f and g, respectively. Moreover, there is an effective divisor $\overline{R}_g$ in N such that $K_N + F = g^*(K_M + E) + \overline{R}_g$ and that any common prime component of $\overline{R}_g$ and g^*E is g-exceptional.

Proof. By resolutions of singularity and indeterminacy of meromorphic maps, we have such a commutative diagram satisfying the conditions except (3). The last assertion on $\overline{R}_g$ follows from (3) and Proposition 1.40(1), since $g^{-1}E \subset F$. Thus, it suffices to prove (3): We set $F' = g^{-1}E \cup \operatorname{Supp} R_g$. Then $N \setminus F'$ is the maximum among open subsets of $N \setminus g^{-1}(\mu^{-1} \operatorname{Supp} B)$ étale over $X_{\mathrm{reg}} \setminus \operatorname{Supp} B$. Since f induces an étale morphism $Y \setminus \widetilde{\Sigma}_f \to$

$X_{\mathrm{reg}} \setminus \operatorname{Supp} B$, the complement $N \setminus F$ is étale over $X_{\mathrm{reg}} \setminus \operatorname{Supp} B$. Hence, $F \supset F'$. If a prime divisor Γ on N is not contained in F', then $f \circ \nu \colon N \to X$ is étale along a non-empty open subset of Γ, and hence, Γ is not ν-exceptional and $\nu(\Gamma) \not\subset \widetilde{\Sigma}_f$. Thus, $F \subset F'$, and (3) has been proved. □

Lemma 2.10. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B and let $f \colon Y \to X$ be a non-degenerate morphism from another normal surface Y. Let B_f and T_f be the positive and negative parts, respectively, of the prime decomposition of $f^*B - R_f$ for the ramification divisor R_f, i.e., $K_Y + B_f = f^*(K_X + B) + T_f$.*

(1) *If (X, B) is log-canonical (resp. log-terminal), then $\ulcorner B_f \urcorner$ is reduced (resp. $\llcorner B_f \lrcorner = 0$). If $T_f = 0$ in addition, then (Y, B_f) is log-canonical (resp. log-terminal).*
(2) *If (X, B) is 1-log-terminal, then $\llcorner B_f \lrcorner$ has no f-exceptional prime component. If $T_f = 0$ in addition, then (Y, B_f) is 1-log-terminal.*

Proof. We use the commutative diagram (II-2) in Lemma 2.9. When we consider (2), we may assume that

($\diamond$) the proper transform of $\operatorname{Supp} \llcorner B \lrcorner = (\llcorner B \lrcorner)_{\mathrm{red}}$ in M and the proper transform of $\operatorname{Supp} \llcorner B_f \lrcorner = (\llcorner B_f \lrcorner)_{\mathrm{red}}$ in N are both non-singular,

by taking further blowings up. We may assume that conditions for (X, B) to be log-canonical, log-terminal, and 1-log-terminal, are checked on the bimeromorphic morphism $\mu \colon M \to X$ in (II-2) with $\mathbb{Q}$-divisors B_μ and T_μ defined in Definition 2.1, where $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$.

First, we shall prove the first half of (1): Assume that (X, B) is log-canonical. Then $\ulcorner B_\mu \urcorner$ is reduced, and

$$K_N + (g^* B_\mu)_{\mathrm{red}} = g^*(K_M + B_\mu) + R'$$

for an effective $\mathbb{Q}$-divisor R' by Lemma 1.41(1). By applying ν_*, we have

$$K_Y + \nu_*((g^* B_\mu)_{\mathrm{red}}) = f^*(K_X + B) + \nu_*(g^* T_\mu + R').$$

Then $B_f \le \nu_*((g^* B_\mu)_{\mathrm{red}})$, and $\ulcorner B_f \urcorner$ is reduced. Assume next that (X, B) is log-terminal, i.e., $\llcorner B_\mu \lrcorner = 0$. Then

$$K_N = g^*(K_M + B_\mu) + R''$$

for a $\mathbb{Q}$-divisor R'' such that $\ulcorner R'' \urcorner$ is effective, by Lemma 1.41(2). Hence,

$$K_Y = f^*(K_X + B) + \nu_*(g^* T_\mu + R''),$$

and $\llcorner B_f \lrcorner = 0$ by $T_f - B_f = \nu_*(g^* T_\mu + R'')$. This shows the first half of (1).

Next, we shall prove the first half of (2): Assume that (X, B) is 1-log-terminal and $\llcorner B \lrcorner \neq 0$. We set $C := \llcorner B_\mu \lrcorner$. Then C is just the proper transform of $\llcorner B \lrcorner$ in M, and it is reduced and non-singular by ($\diamond$). By Lemma 1.41(3),

$$K_N + g^{[*]} C = g^*(K_M + B_\mu) + R'''$$

for a $\mathbb{Q}$-divisor R''' such that $\ulcorner R''' \urcorner$ is effective. Applying ν_*, we have

$$K_Y + \nu_*(g^{[*]} C) = f^*(K_X + B) + \nu_*(g^* T_\mu + R''') \quad \text{and}$$

$$T_f - B_f = \nu_*(g^*T_\mu + R''') - \nu_*(g^{[*]}C).$$

Hence, $\llcorner B_f \lrcorner \leq \nu_*(g^{[*]}C)$, and every prime component of $\nu_*(g^{[*]}C)$ is not exceptional for f. This proves the first half of (2).

Finally, we shall prove the remaining parts of (1) and (2): Assume that $T_f = 0$. Let B_ν and T_ν, respectively, be the positive and negative parts of the prime decomposition of $\nu^*B_f - R_\nu$. Then

$$K_N + B_\nu = \mu^*(K_Y + B_f) + T_\nu = \mu^*(f^*(K_X + B)) + T_\nu.$$

Moreover, we have $B_f = \nu_*B_\nu$ and $T_f = \nu_*T_\nu = 0$ by applying ν_*. In the situation of (1), $\ulcorner B_\nu \urcorner$ is reduced (resp. $\llcorner B_\nu \lrcorner = 0$) by the first half of (1) applied to $f \circ \nu \colon N \to X$ and (X, B); hence, (Y, B_f) is log-canonical (resp. log-terminal).

In the situation of (2), $\llcorner B_\nu \lrcorner$ has no $f \circ \nu$-exceptional prime component by the first half of (2) applied to $f \circ \nu$ and (X, B). Hence, $\llcorner B_\nu \lrcorner$ equals the proper transform of $\llcorner B_f \lrcorner$ in N, and it is reduced and non-singular by (1) and ($\diamond$). Therefore (Y, B_f) is 1-log-terminal by (1). Thus, we are done. $\square$

Remark. The proof above does not use any result in Section 2.1 (cf. Remark 2.2). Some reader may think that Lemma 2.10 can be proved by the same argument as in the proof of [34, Prop. 5.20]. But there is a difficulty in constructing the "fiber product diagram" in the proof, since the non-degenerate morphism f is not necessarily proper (cf. [41, Rem. of Cor. 3.20]).

Lemma 2.11. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B and let $f \colon Y \to X$ be a surjective and* discretely proper *morphism* (*cf.* Definition 1.6) *from another normal surface Y with effective $\mathbb{Q}$-divisors B_Y and Δ such that $R_f = f^*B + \Delta - B_Y$, i.e., $K_Y + B_Y = f^*(K_X + B) + \Delta$. For the diagram* (II-2) *of* Lemma 2.9*, let B_ν, T_ν, C_ν, and S_ν be effective $\mathbb{Q}$-divisors on N such that*

- *B_ν and T_ν are the positive and negative parts, respectively, of the prime decomposition of $\nu^*B_Y - R_\nu$, and*
- *C_ν and S_ν are the positive and negative parts, respectively, of the prime decomposition of $B_\nu - \nu^*\Delta$.*

In particular, one has

$$K_N + B_\nu = \nu^*(K_Y + B_Y) + T_\nu \quad \text{and} \quad K_N + C_\nu = \nu^*(f^*(K_X + B)) + S_\nu + T_\nu.$$

In this situation, the following hold:

1. *If $\ulcorner C_\nu \urcorner$ is reduced* (*resp.* $\llcorner C_\nu \lrcorner = 0$)*, then (X, B) is log-canonical* (*resp. log-terminal*).
2. *If $\ulcorner C_\nu \urcorner$ is reduced and if $\llcorner C_\nu \lrcorner$ is a non-singular divisor having no $f \circ \nu$-exceptional prime component, then (X, B) is* 1*-log-terminal.*
3. *Suppose that* $\operatorname{Supp} B_Y \subset \widetilde{\Sigma}_f$ (*cf.* Lemma 2.9(2))*. If $\ulcorner B \urcorner$ and $\ulcorner B_Y \urcorner$ are reduced, then there is an effective $\mathbb{Q}$-divisor $\overline{\Delta}$ such that*

$$K_N + \nu^{[*]}B_Y + E_\nu = g^*(K_M + \mu^{[*]}B + E_\mu) + \overline{\Delta} \tag{II-3}$$

and that any ν-exceptional prime component of $\overline{\Delta}$ is g-exceptional.

Proof. Note that g is surjective and discretely proper by Corollary 1.11. Since $C_\nu \le B_\nu$, effective $\mathbb{Q}$-divisors C_ν and $S_\nu + T_\nu$ have no common prime component, and these are the positive and negative parts, respectively, of the prime decomposition of $(f \circ \nu)^* B - R_{f\circ\nu}$. In particular, $\nu_* C_\nu = B_f$ and $\nu_*(S_\nu + T_\nu) = \nu_* S_\nu = T_f$ for divisors B_f and T_f in Lemma 2.10. Note that $\operatorname{Supp} C_\nu \subset F$ by

$$\operatorname{Supp} C_\nu \subset \nu^{-1}(\operatorname{Supp} B_f) \cup E_\nu \quad \text{and} \quad \operatorname{Supp} B_f \subset \widetilde{\Sigma}_f.$$

By equalities $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$ and $K_N + F = g^*(K_M + E) + \overline{R}_g$ (cf. Lemma 2.9), we have

$$K_N + F = g^*(\mu^*(K_X + B)) + g^*(E + T_\mu - B_\mu) + \overline{R}_g,$$

and by comparing with $K_N + C_\nu = \nu^*(f^*(K_X + B)) + S_\nu + T_\nu$, we have

$$g^*(E + T_\mu - B_\mu) + \overline{R}_g = F - C_\nu + S_\nu + T_\nu. \tag{II-4}$$

We shall prove (1) and (2). Assume that $\ulcorner C_\nu \urcorner$ is reduced. Then $F \ge C_\nu$, and we have $E \ge B_\mu$ by (II-4), since any common prime component of $\overline{R}_g$ and $g^* B_\mu$ is g-exceptional (cf. Lemma 2.9) and since B_μ and T_μ have no common prime component. Hence, (X, B) is log-canonical, and we have proved (1) in the log-canonical case.

For the proof of (1) in the log-terminal case and for that of (2), we consider a prime component Γ of B_μ. We can take a non-g-exceptional prime component Θ of $f^*\Gamma$, since g is surjective. Then $\Theta \not\subset \operatorname{Supp} \overline{R}_g$ by $\operatorname{Supp} B_\mu \subset E$ and by the last assertion of Lemma 2.9. Moreover, the following equalities hold by (II-4):

$$\begin{aligned}(\mathrm{mult}_\Theta\, g^*\Gamma)\, \mathrm{mult}_\Gamma(E - B_\mu) &= \mathrm{mult}_\Theta\, g^*(E - B_\mu) \\ &= \mathrm{mult}_\Theta\, g^*(E - B_\mu + T_\mu) = \mathrm{mult}_\Theta(F - C_\nu) + \mathrm{mult}_\Theta(S_\nu + T_\nu).\end{aligned} \tag{II-5}$$

Assume that $\llcorner C_\nu \lrcorner = 0$. Then $F \ge C_\nu$ and $\operatorname{Supp} F = \operatorname{Supp}(F - C_\nu)$. Thus, $\mathrm{mult}_\Gamma(E - B_\mu) > 0$ for any prime component Γ of B_μ by (II-5). In other words, $E \ge B_\mu$ and $\operatorname{Supp} E = \operatorname{Supp}(E - B_\mu)$. Hence, $\llcorner B_\mu \lrcorner = 0$ and (X, B) is log-terminal. Thus, (1) has been proved.

Next, assume the condition for C_ν in (2). Then $F \ge C_\nu$ and $E \ge B_\mu$ by the proof above for (1) in the log-canonical case. Assume that Γ is a prime component of $\llcorner B_\mu \lrcorner$. Then $\Gamma \not\subset \operatorname{Supp}(E - B_\mu)$, and we have $\Theta \not\subset \operatorname{Supp}(F - C_\nu)$ by (II-5). Thus, Θ is a prime component of $\llcorner C_\nu \lrcorner$, which is not exceptional for $f \circ \nu\colon N \to X$. Hence, Γ is not μ-exceptional. This implies that (X, B) is 1-log-terminal, and we have proved (2).

Finally, we shall prove (3). Note that $E = \operatorname{Supp}(\mu^{[*]} B + E_\mu)$. By the assumption on B_Y, we have

$$\operatorname{Supp}(\nu^{[*]} B_Y + E_\nu) \subset \nu^{-1}\widetilde{\Sigma}_f \cup E_\nu = F.$$

Since $\ulcorner B \urcorner$ and $\ulcorner B_Y \urcorner$ are reduced, there exist effective $\mathbb{Q}$-divisors D_M and D_N on M and N, respectively, such that

$$E = \mu^{[*]} B + E_\mu + D_M \quad \text{and} \quad F = \nu^{[*]} B_Y + E_\nu + D_N.$$

Then the equality (II-3) holds for

$$\overline{\Delta} := g^* D_M - D_N + \overline{R}_g. \tag{II-6}$$

Here, any prime component of D_M (resp. D_N) is not exceptional for μ (resp. ν), and $\operatorname{mult}_\Xi \overline{\Delta} \geq 0$ for any ν-exceptional prime divisor Ξ. On the other hand, we have $\nu_*\overline{\Delta} = \Delta$ by applying ν_* to (II-3). Thus, $\overline{\Delta}$ is effective. It remains to prove that any ν-exceptional prime component Ξ of $\overline{\Delta}$ is g-exceptional. Assume that Ξ is not g-exceptional. Then $\Xi \subset g^{-1}\Gamma$ for a prime divisor Γ on M, and $g|_\Xi \colon \Xi \to \Gamma$ is non-degenerate. Here, Γ is μ-exceptional as Ξ is ν-exceptional. Thus, $\Gamma \subset E_\mu$ and $\Gamma \not\subset \operatorname{Supp} D_M$. Hence, $\Xi \subset \operatorname{Supp} \overline{R}_g$ by (II-6). This contradicts the last assertion of Lemma 2.9, since Ξ is a common prime component of g^*E and $\overline{R}_g$. Therefore, Ξ is g-exceptional. Thus, we are done. □

Proposition 2.12. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B and let $f \colon Y \to X$ be a non-degenerate morphism from another normal surface Y with effective $\mathbb{Q}$-divisors B_Y and Δ such that $R_f = f^*B + \Delta - B_Y$, i.e., $K_Y + B_Y = f^*(K_X + B) + \Delta$. Then the following hold for any $x \in f(Y)$:*

(1) *If (Y, B_Y) is log-canonical (resp. log-terminal) along a non-empty compact connected component of $f^{-1}(x)$, then (X, B) is log-canonical (resp. log-terminal) at x.*
(2) *If (Y, B_Y) is 1-log-terminal along a non-empty compact connected component Λ of $f^{-1}(x)$ such that $\Lambda \cap \operatorname{Supp} \llcorner B_Y \lrcorner$ is finite, then (X, B) is 1-log-terminal at x.*

Proof. For a non-empty compact connected component Λ of $f^{-1}(x)$, there exist an open neighborhood U of x and an open neighborhood V of Λ such that $V \subset f^{-1}U$, $V \cap f^{-1}(x) = \Lambda$, and $f|_V \colon V \to U$ is proper and surjective, by Lemma 1.7. Hence, by replacing X and Y with U and V, respectively, we may assume that f is proper and surjective, (Y, B_Y) is log-canonical (resp. log-terminal) in case (1), and (Y, B_Y) is 1-log-terminal in case (2). Moreover, in case (2), we may assume that

($\natural$) $f|_{\llcorner B_Y \lrcorner} \colon \llcorner B_Y \lrcorner \to X$ is a finite morphism

by Lemma 1.7. We consider the commutative diagram (II-2) in Lemma 2.9 and divisors B_ν and C_ν in Lemma 2.11.

We shall show (1). In this case, $\ulcorner B_\nu \urcorner$ is reduced (resp. $\llcorner B_\nu \lrcorner = 0$) as (Y, B_Y) is log-canonical (resp. log-terminal). Hence, $\ulcorner C_\nu \urcorner$ is reduced (resp. $\llcorner C_\nu \lrcorner = 0$), by $C_\nu \leq B_\nu$. Thus, (X, B) is log-canonical (resp. log-terminal) by Lemma 2.11(1).

Finally, we shall show (2). In this case, $\llcorner B_\nu \lrcorner$ is a non-singular divisor having no ν-exceptional component as (Y, B_Y) is 1-log-terminal. Since $\nu_*B_\nu = B_Y$, $\llcorner B_\nu \lrcorner$ has no $f \circ \nu$-exceptional component by ($\natural$). Hence, $\llcorner C_\nu \lrcorner$ is also a non-singular divisor having no $f \circ \nu$-exceptional component by $C_\nu \leq B_\nu$. Thus, (X, B) is 1-log-terminal by Lemma 2.11(2), and we are done. □

2.3. Relative abundance theorem. The abundance theorem is one of the main results in the theory of open algebraic surfaces (or logarithmic algebraic surfaces), which is proved in several versions in [29], [51], [59], and [11]. Theorem 2.19 below is a relative version of the abundance theorem, and Lemma 2.18 below is its special case. We shall prove them for the sake of completeness not using the classification of log-canonical singularities but using Fujita's argument in [11] and Kawamata's argument in the proof of [30, Lem. 9.3] with some modifications.

Let us consider a proper surjective morphism $\pi \colon X \to Y$ of normal varieties such that $\dim X = 2$, and assume either that $\dim Y > 0$ or that X is a normal Moishezon surface with $\dim Y = 0$. Before Lemma 2.18, we fix the morphism π. We shall explain relative versions of the Kawamata–Viehweg vanishing theorem (cf. Proposition 2.15) and Zariski-decompositions (cf. Lemma-Definition 2.16) for the morphism π. The relative abundance theorem (cf. Theorem 2.19) is stated in the case where X is non-singular, but it is applied to log-canonical pairs by taking resolutions. As an application of the relative abundance theorem, we shall define the log-canonical modification for pairs (X, B) of a normal surface X and an effective $\mathbb{Q}$-divisor B such that $\ulcorner B \urcorner$ is reduced (cf. Lemma-Definition 2.22), and show a compatibility for certain morphisms with only discrete fibers (cf. Proposition 2.23).

Lemma 2.13. *If* $\dim Y > 0$*, then* π *is a projective morphism locally over* Y*, i.e., for any point* $y \in Y$*, there exist an open neighborhood* $\mathcal{Y} \subset Y$ *and an invertible sheaf on* $\pi^{-1}(\mathcal{Y})$ *which are relatively ample over* $\mathcal{Y}$ (*cf.* [37, Prop. 1.4]).

Proof. Since finite morphisms are projective locally over the base varieties, we may assume that every fiber of π is connected by considering Stein factorization. If $\dim Y = 2$, then π is a bimeromorphic morphism and is projective locally over Y by an argument in the last paragraph of the proof of Lemma 1.28. Thus, we may assume that $\dim Y = 1$. Then Y is a non-singular curve and every fiber is 1-dimensional. We fix a point $y \in Y$ and consider an irreducible component Γ of $\pi^{-1}(y)$. For a point $x \in \Gamma_{\mathrm{reg}} \cap X_{\mathrm{reg}}$, there is an open neighborhood $\mathcal{U}$ of x with a coordinate system $(\mathtt{z}_1, \mathtt{z}_2)$ such that $\Gamma|_{\mathcal{U}} = \mathrm{div}(\mathtt{z}_2)$ and that $\pi|_{\mathcal{U}} \colon \mathcal{U} \to Y$ is defined by the function $u(\mathtt{z}_1, \mathtt{z}_2)\mathtt{z}_2^m$ on $\mathcal{U}$ for a positive integer m and a nowhere vanishing function $u(\mathtt{z}_1, \mathtt{z}_2)$. Then $\pi^{-1}(y) \cap \Theta = \{x\}$ for the non-singular divisor $\Theta = \mathrm{div}(\mathtt{z}_1)$ on $\mathcal{U}$. Hence, $\pi|_{\Theta} \colon \Theta \to Y$ is a finite morphism over an open neighborhood of y by Corollary 1.8. By considering divisors Θ for all irreducible components Γ of $\pi^{-1}(y)$, we can find an open neighborhood $\mathcal{Y}$ of y and a non-singular divisor D on $\pi^{-1}(\mathcal{Y})$ such that $D\Gamma > 0$ for any irreducible component Γ of $\pi^{-1}(y)$. Then, by [37, Prop. 1.4], $\pi^{-1}(\mathcal{Y}) \to \mathcal{Y}$ is a projective morphism over an open neighborhood of y in which D is relatively ample. □

CONVENTION 2.14. For the morphism $\pi \colon X \to Y$ with $\dim Y > 0$, a $\mathbb{Q}$-divisor D on X is said to be:

(1) *π-nef* (resp. *π-numerically trivial*), if $DC \geq 0$ (resp. $DC = 0$) for any prime divisor $C \subset X$ such that $\dim \pi(C) = 0$ (cf. [39, II, Def. 5.14], [41, Def. 2.14(i)]);
(2) *π-semi-ample*, if there is a positive integer m locally over Y such that mD is Cartier and the canonical homomorphism $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective (cf. [39, II, Def. 1.9(4)]);
(3) *π-pseudo-effective*, if $D|_C$ is pseudo-effective for any irreducible component C of a sufficiently general fiber of π (cf. [39, II, Cor. 5.17]);
(4) *π-big*, if $D|_C$ is big for any irreducible component C of a general fiber of π (cf. [39, II, Cor. 5.17]).

Note that if $\dim Y = 2$, then any D is π-big. Similarly, if $\dim Y = 1$, then D is π-pseudo-effective (resp. π-big) if and only if $DC \geq 0$ (resp. $DC > 0$) for any irreducible component C of a general fiber of π. For the morphism π with $\dim Y = 0$, i.e., for a normal Moishezon surface X, we use the same notions of nef, numerically trivial, semi-ample, pseudo-effective, and big, respectively, as in [41, Def. 2.11] for $\mathbb{Q}$-divisors on X. Sometimes we add the prefix "π-" even when $\dim Y = 0$.

The Kawamata–Viehweg vanishing theorem for non-singular projective surfaces is generalized to the relative situation as follows (cf. [52, Thms. (2.2) and (5.1)]):

Proposition 2.15. *For any π-nef and π-big $\mathbb{Q}$-divisor D on X and for any $i > 0$, one has* $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$.

Proof. Our proof is slightly different from Sakai's one in [52, Thm. 5.1]. Since the assertion is local on Y, we may assume the existence of a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface M such that the union of the μ-exceptional locus and $\mu^{-1}\operatorname{Supp} D$ is a normal crossing divisor and that $\pi \circ \mu\colon M \to Y$ is a projective morphism. In fact, if $\dim Y = 0$, then M is projective as X is Moishezon, and if $\dim Y > 0$, then π is locally projective by Lemma 2.13. Hence,

$$R^i(\pi \circ \mu)_*\mathcal{O}_M(K_M + \ulcorner \mu^*D \urcorner) = 0 \quad \text{and} \quad R^i\mu_*\mathcal{O}_M(K_M + \ulcorner \mu^*D \urcorner) = 0$$

for any $i > 0$ by a relative version of Kawamata–Viehweg's vanishing theorem on M (cf. [37, Thm. 3.7]). Let $\mathcal{F}$ be the direct image sheaf $\mu_*\mathcal{O}_M(K_M + \ulcorner \mu^*D \urcorner)$. Then $R^i\pi_*\mathcal{F} = 0$ for any $i > 0$ by a standard argument on Leray's spectral sequence. Since $\mathcal{F}$ is a subsheaf of the double dual $\mathcal{F}^{\vee\vee} = \mathcal{O}_X(K_X + \ulcorner D \urcorner)$ with $\dim \operatorname{Supp} \mathcal{F}^{\vee\vee}/\mathcal{F} \le 0$, we have $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) \simeq R^i\pi_*\mathcal{F} = 0$ for any $i > 0$. □

We have a relative version of the notion of *Zariski-decomposition* (cf. [64], [10], [52, §7], [54, App.], [39]) as follows:

Lemma-Definition 2.16. *Let D be a π-pseudo-effective $\mathbb{Q}$-divisor on X. Then there exists a unique effective $\mathbb{Q}$-divisor N satisfying the following conditions*:

- *Every prime component of N is contained in a fiber of π.*
- *The difference $P := D - N$ is π-nef and satisfies $PN = 0$.*
- *If $N \ne 0$, then the intersection matrix $(N_iN_j)_{i,j}$ of any finitely many prime components N_i of N is negative definite.*

The equality $D = P + N$ is called the relative Zariski-decomposition *of D* with respect to *π, where P (resp. N) is called the* positive (*resp.* negative) *part.*

Proof. First assume that $\dim Y = 0$. For the minimal resolution $\mu\colon M \to X$ of singularities, we have the unique Zariski-decomposition $\mu^*D = P^\sim + N^\sim$ on the non-singular projective surface M by [10], since μ^*D is pseudo-effective, where $P^\sim$ (resp. $N^\sim$) is the positive (resp. negative) part. Here, $P^\sim$ is μ-numerically trivial. In fact, for a μ-exceptional prime divisor Γ, if $\Gamma \subset \operatorname{Supp} N^\sim$, then $P^\sim\Gamma = 0$ by $P^\sim N^\sim = 0$, and if $\Gamma \not\subset \operatorname{Supp} N^\sim$, then $P^\sim\Gamma = 0$ by $(\mu^*D)\Gamma = 0$, $P^\sim\Gamma \ge 0$, and $N^\sim\Gamma \ge 0$. Thus, $P^\sim = \mu^*P$ and $N^\sim = \mu^*N$ for $P := \mu_*P^\sim$ and $N := \mu_*N^\sim$, and $D = P + N$ is the Zariski-decomposition of D.

Second, assume that $\dim Y > 0$. Our proof in this case is based on Sakai's argument in [52, §7] and [54, App.]. By the uniqueness of the decomposition, we can localize Y. Thus, we may assume the finiteness of the set $\mathcal{S}(X/Y)$ of prime divisors Γ on X such that $\Gamma^2 < 0$ and $\dim \pi(\Gamma) = 0$. Note that

- if $\dim Y = 2$, then $\mathcal{S}(X/Y)$ is the set of π-exceptional prime divisors;
- if $\dim Y = 1$, then $\mathcal{S}(X/Y)$ is the set of irreducible components of reducible fibers of π.

We shall prove the existence and the uniqueness of relative Zariski-decomposition by induction on $s(X/Y) := \#\mathcal{S}(X/Y)$. We may assume that D is not π-nef; for, otherwise, $N = 0$ satisfies the condition and it is unique. Then $D\Gamma < 0$ for an irreducible component Γ of a fiber of π. If $\Gamma^2 \geq 0$, then $\dim Y = 1$, $\Gamma^2 = 0$, and Γ is a connected component of a fiber of π; this implies $D\Gamma \geq 0$, a contradiction. Hence, $\Gamma \in \mathcal{S}(X/Y)$ and $s(X/Y) > 0$. Let $\nu\colon X \to X'$ be the contraction morphism of Γ, i.e., a bimeromorphic morphism to a normal surface X' with a point x' such that $\nu^{-1}(x') = \Gamma$ and ν is an isomorphism outside Γ: The existence of ν follows from a generalization [53, Thm. 1.2] of the Grauert contraction criterion [13, (e), pp. 366–367] (cf. [41, Thm. 2.6]). Let $\pi'\colon X' \to Y$ be the induced morphism such that $\pi' \circ \nu = \pi$. Then $s(X'/Y) = s(X/Y) - 1$. We have $D = \nu^*(\nu_* D) + \alpha\Gamma$ for $\alpha := D\Gamma/\Gamma^2 > 0$. By induction, the π'-pseudo-effective $\mathbb{Q}$-divisor $\nu_* D$ admits a relative Zariski-decomposition over Y. For the negative part N' of $\nu_* D$, the $\mathbb{Q}$-divisor $N := \nu^* N' + \alpha\Gamma$ satisfies the condition of the negative part of the relative Zariski-decomposition of D over Y. In order to prove the uniqueness, assume that another effective $\mathbb{Q}$-divisor $\widetilde{N}$ satisfies the condition of negative part. Then $D\Gamma < 0$ implies that $\widetilde{N}\Gamma < 0$ and $(D - \widetilde{N})\Gamma = 0$. Thus, $\widetilde{N} = \nu^*(\nu_*\widetilde{N}) + \alpha\Gamma$, and $\nu_*\widetilde{N}$ equals the negative part N' of the relative Zariski-decomposition of $\nu_* D$. Hence, $\widetilde{N} = N$. Therefore, D admits a unique relative Zariski-decomposition. □

The following is well known in the absolute case.

Lemma 2.17. *In the situation of* Lemma-Definition 2.16*, let E be an effective $\mathbb{Q}$-divisor on X such that $D - E$ is π-nef. Then $E \geq N$. In particular, for any rational number $t \geq 0$,*

$$\pi_*\mathcal{O}_X(\llcorner tP \lrcorner) \simeq \pi_*\mathcal{O}_X(\llcorner tD \lrcorner).$$

Proof. For the first assertion, we may assume that $N \neq 0$. Let B_+ and B_- be the positive and negative parts, respectively, of the prime decomposition of $E - N$. Then $\operatorname{Supp} B_- \subset \operatorname{Supp} N$, and

$$(B_+ - B_-)B_- = (E - N)B_- \leq (D - N)B_- = PB_- = 0.$$

Hence, $B_-^2 \geq B_+B_- \geq 0$, and we have $B_- = 0$, since the intersection matrix of finitely many prime components of N is negative definite. Thus, $E \geq N$. For the last assertion, let $\mathcal{F}$ be the image of the canonical homomorphism

$$\pi^*\pi_*\mathcal{O}_X(\llcorner tD \lrcorner) \to \mathcal{O}_X(\llcorner tD \lrcorner).$$

Then the double dual $\mathcal{F}^{\vee\vee}$ is expressed as $\mathcal{O}_X(\llcorner tD \lrcorner - F)$ for an effective divisor F, and $\llcorner tD \lrcorner - F$ is π-nef, since the support of $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is at most 0-dimensional. Hence, we can apply the first assertion to $E = (1/t)(\langle tD \rangle + F)$, where $\langle tD \rangle = tD - \llcorner tD \lrcorner$, since $D - E = (1/t)(\llcorner tD \lrcorner - F)$. As a consequence, $\langle tD \rangle + F \geq tN$, or equivalently, $\llcorner tP \lrcorner \geq \llcorner tD \lrcorner - F$. Therefore, $\pi_*\mathcal{O}_X(\llcorner tD \lrcorner - F) = \pi_*\mathcal{O}_X(\llcorner tP \lrcorner) = \pi_*\mathcal{O}_X(\llcorner tD \lrcorner)$. □

The following is a special case of the relative abundance theorem.

Lemma 2.18. *For a normal surface X, let $\mu\colon M \to X$ be the minimal resolution of singularities. Let B be an effective $\mathbb{Q}$-divisor on M such that $\ulcorner B \urcorner$ is reduced and that $K_M + B$ is μ-numerically trivial. If mB is Cartier for a positive integer m, then $m(K_X + \mu_* B)$ is Cartier and $m(K_M + B) \sim \mu^*(m(K_X + \mu_* B))$.*

Proof. We borrow an argument in the proof of [30, Lem. 9.3]. Since the assertion is local on X, we may assume that X is Stein and $\operatorname{Sing} X$ consists of one point x. Then $\Sigma := \mu^{-1}(x)$ is the μ-exceptional locus, which is considered as a compact connected reduced divisor on M. First, we treat the case where (X, x) is a rational singularity, i.e., $R^1\mu_*\mathcal{O}_M = 0$. Then the element of the Picard group $\operatorname{Pic}(M) = H^1(M, \mathcal{O}_M^\star)$ corresponding to the invertible sheaf $\mathcal{O}_X(m(K_M + B))$ is sent to zero by the canonical homomorphism

$$\begin{aligned}\operatorname{Pic}(M) \to H^0(X, R^1\mu_*\mathcal{O}_M^\star) \simeq (R^1\mu_*\mathcal{O}_M^\star)_x \simeq (R^2\mu_*\mathbb{Z}_M)_x \\ \simeq H^2(\Sigma, \mathbb{Z}) \simeq \bigoplus\nolimits_{\Gamma\subset\Sigma} H^2(\Gamma, \mathbb{Z}) \simeq \bigoplus\nolimits_{\Gamma\subset\Sigma} \mathbb{Z},\end{aligned}$$

since $(K_M + B)\Gamma = 0$ for any prime component Γ of Σ. The kernel of the homomorphism is $\mu^* \operatorname{Pic}(X)$. Hence, $m(K_M + B) \sim \mu^*L$ for a Cartier divisor L on X, and $L \sim \mu_*(m(K_M + B)) = m(K_X + \mu_*B)$. This proves the assertion for rational singularities (X, x).

Next, we treat the case where (X, x) is not a rational singularity. We set

$$B^\dagger := \sum\nolimits_{\Gamma\subset\Sigma} (\operatorname{mult}_\Gamma B)\Gamma \quad \text{and} \quad D := \llcorner B^\dagger \lrcorner.$$

Then $B - B^\dagger$ is μ-nef, and $-B^\dagger - K_M = (B - B^\dagger) - (K_M + B)$ is also μ-nef. Hence, $R^1\mu_*\mathcal{O}_M(-D) = 0$ by Proposition 2.15, since $\ulcorner -B^\dagger \urcorner = -D$. Thus,

$$0 \neq (R^1\mu_*\mathcal{O}_M)_x \simeq (R^1\mu_*\mathcal{O}_D)_x \simeq H^1(D, \mathcal{O}_D),$$

and D is connected by the surjection $\mathcal{O}_X \simeq \mu_*\mathcal{O}_M \to \mu_*\mathcal{O}_D$, since $\mu_*\mathcal{O}_D$ is the skyscraper sheaf of the residue field $\mathbb{C}(x)$ at x. In particular, $(K_M + D)D = \deg \omega_D = -2\chi(D, \mathcal{O}_D) \geq 0$ by Riemann–Roch. On the other hand, $(K_M + D)D \leq (K_M + B^\dagger)D \leq 0$, since $-(K_M + B^\dagger)$ is μ-nef. Hence, $(K_M + D)D = 0$ and $H^1(D, \mathcal{O}_D) \simeq H^0(D, \omega_D)^\vee \simeq \mathbb{C}$, which imply $\mathcal{O}_M(K_M + D)|_D \simeq \omega_D \simeq \mathcal{O}_D$. Moreover, $D \cap \operatorname{Supp}(B - D) = \emptyset$ by $0 = (K_M + B)D - (K_M + D)D = (B - D)D$. If $\Sigma \neq D$, then $\Gamma \cap D \neq \emptyset$ for some prime component Γ of $\Sigma - D$, since Σ is connected. In this case, $\Gamma \not\subset \operatorname{Supp} B$ by $D \cap \operatorname{Supp}(B - D) = \emptyset$, but $K_M\Gamma \geq 0$, $B\Gamma \geq 0$, and $(K_M + B)\Gamma = 0$ imply that $\Gamma \cap \operatorname{Supp} B = \emptyset$; this contradicts $\Gamma \cap D \neq \emptyset$. Therefore, $\Sigma = D$. Since $m(K_M + B) - B^\dagger - K_M$ is μ-nef, by Proposition 2.15, we have $R^1\mu_*\mathcal{O}_M(m(K_M + B) - \Sigma) = 0$ and a surjection

$$\mu_*\mathcal{O}_M(m(K_M + B)) \to \mu_*\mathcal{O}_\Sigma(m(K_M + B)|_\Sigma) \simeq \mu_*\mathcal{O}_\Sigma.$$

Hence, a section of $\mathcal{O}_M(m(K_M + B))$ over an open neighborhood of Σ is nowhere vanishing. This means that $m(K_M + B) \sim \mu^*L$ for a Cartier divisor L on X, and $L \sim \mu_*(m(K_M + B)) = m(K_X + \mu_*B)$. Thus, we are done. □

Theorem 2.19 (Relative Abundance Theorem)**.** *Let M be a non-singular surface with an effective $\mathbb{Q}$-divisor B such that $\ulcorner B \urcorner$ is reduced. Let $\pi\colon M \to Y$ be a proper surjective morphism to a normal variety Y such that either* $\dim Y > 0$ *or M is projective with* $\dim Y = 0$*. Assume that $K_M + B$ is π-pseudo-effective. Then the positive part P of the relative Zariski-decomposition $K_M + B = P + N$ with respect to π is π-semi-ample.*

Proof. The assertion is known as [11, Main Thm. (1.4)] in case $\dim Y = 0$. Hence, we may assume that $\dim Y > 0$ and that π is a fibration by taking Stein factorization. Since the assertion is local on Y, we may assume further that Y is Stein, mB is Cartier for a positive integer m, and π is smooth over $Y \setminus \{y\}$ for a point $y \in Y$. For a prime divisor Θ on M is contained in a fiber of π, if $\Theta^2 < 0$, then $\pi(\Theta) = \{y\}$, by the assumption. Thus,

$\operatorname{Supp} N \subset \pi^{-1}(y)$.

First, we reduce the assertion to the case where $K_M + B$ is π-nef (cf. [11, (3.2)–(3.5)]). Assume that $K_M + B$ is not π-nef, i.e., $N \neq 0$. By subtracting some effective $\mathbb{Q}$-divisor from B and N, we may assume that B and N have no common prime component. By the Grauert contraction criterion, we have the contraction morphism $\gamma \colon M \to \overline{M}$ of $\operatorname{Supp} N$, since the intersection matrix of prime components of N is negative definite. Then $\pi = \bar{\pi} \circ \gamma$ for the induced fibration $\bar{\pi} \colon \overline{M} \to Y$, $\overline{P} := \gamma_* P$ is $\bar{\pi}$-nef, and $P = \gamma^* \overline{P}$. It suffices to prove that $\overline{P}$ is $\bar{\pi}$-semi-ample. Let $\nu \colon M' \to \overline{M}$ be the minimal resolution of singularities. Then there is a bimeromorphic morphism $\gamma' \colon M \to M'$ such that $\gamma = \nu \circ \gamma'$. For pushforwards $B' = \gamma'_* B$, $P' = \gamma'_* P$, and $N' = \gamma'_* N$, we have $K_{M'} + B' = P' + N'$ and $P' = \nu^* \overline{P}$, and $\operatorname{Supp} N'$ is just the ν-exceptional locus. Moreover, $K_{M'}\Gamma \geq 0$ and $B'\Gamma \geq 0$ for any prime component Γ of N', since ν is the minimal resolution and since B' and N' have no common prime component. Hence, $(K_{M'} + B')N' = P'N' + N'^2 = N'^2 \geq 0$. Therefore, $N' = 0$, ν is an isomorphism, and $K_{M'} + B' = P' = \nu^* \overline{P}$ is relatively nef over Y. In order to prove the $\bar{\pi}$-semi-ampleness of $\overline{P}$, by replacing (M, B) with (M', B'), we may assume that $K_M + B$ is π-nef.

Second, we consider the case where $K_M + B$ is π-nef and π-big. Let $\mathcal{P}$ be the set of prime divisors Θ on M contained in $\pi^{-1}(y)$ such that $(K_M + B)\Theta = 0$. The intersection matrix of members of $\mathcal{P}$ is negative definite, since $K_M + B$ is π-big. Let $\mu \colon M \to X$ be the contraction morphism of all the members of $\mathcal{P}$ and let $\mu^\dagger \colon M^\dagger \to X$ be the minimal resolution of singularities. Then there is a bimeromorphic morphism $\delta \colon M \to M^\dagger$ such that $\mu = \mu^\dagger \circ \delta$, and we have $K_M + B = \delta^*(K_{M^\dagger} + B^\dagger)$ for $B^\dagger = \delta_* B$. Hence, by replacing (M, B) with $(M^\dagger, B^\dagger)$, we may assume that μ is the minimal resolution of singularities of X. Since mB is Cartier for an integer $m > 0$, there is a Cartier divisor L on X such that $m(K_M + B) \sim \mu^* L$, by Lemma 2.18. By the definition of $\mathcal{P}$, $L\Xi > 0$ for any prime divisor Ξ contained in the fiber over y of the fibration $X \to Y$ induced by π. Thus, L is relatively ample over Y (cf. [37, Prop. 1.4]), and $K_M + B$ is π-semi-ample.

Finally, we consider the case where $K_M + B$ is π-nef but not π-big. Then $\dim Y = 1$ and $(K_M + B)F = 0$ for any smooth fiber F of π. If $BF > 0$, then $F \simeq \mathbb{P}^1$ and $BF = 2$. If $BF = 0$, then F is an elliptic curve and $\operatorname{Supp} B$ is contained in a union of fibers of π. In both cases, $\mathcal{O}_F(m(K_M + B)|_F) \simeq \mathcal{O}_F$ for a positive integer m such that mB is Cartier. In particular, $\pi_* \mathcal{O}_M(m(K_X + B)) \neq 0$. Then there is an effective divisor E on M such that $\operatorname{Supp} E \subset \pi^{-1}(y)$ and

$$\mathcal{O}_X(m(K_M + B)) \simeq \mathcal{O}_X(E) \otimes \pi^* \pi_* \mathcal{O}_X(m(K_M + B)).$$

We may assume that $\pi_* \mathcal{O}_M(m(K_M + B)) \simeq \mathcal{O}_Y$ by replacing Y with an open neighborhood of y. Hence, $m(K_M + B) \sim E$. For the relative Zariski-decomposition $E = P_E + N_E$ with respect to π, it suffices to show that the positive part P_E is π-semi-ample, since $P_E \sim mP$. Now $\operatorname{Supp} P_E \subset \operatorname{Supp} E \subset \pi^{-1}(y)$. As is well known, the intersection matrix of prime components of $\pi^{-1}(y)$ is negative semi-definite with signature $(0, r-1)$ for the number r of prime components of $\pi^{-1}(y)$. Hence, $P_E = q\pi^*(y)$ for a rational number $q \geq 0$, since P_E is π-nef. Therefore, P_E and P are π-semi-ample. Thus, we are done. □

By Lemma 2.17 and Theorem 2.19, we have:

Corollary 2.20. *In the situation of* Theorem 2.19*, the graded $\mathcal{O}_Y$-algebra*

$$\bigoplus\nolimits_{m\geq 0} \pi_* \mathcal{O}_M(\llcorner m(K_M + B)\lrcorner)$$

is finitely generated locally on Y.

Corollary 2.21. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B. If (X, B) is log-canonical at a point $x \in X$ (in the sense of* Definition 2.1*), then $K_X + B$ is $\mathbb{Q}$-Cartier at x.*

Proof. By localizing X, we may assume that X is Stein, $\operatorname{Sing} X = \{x\}$, and (X, B) is log-canonical. Let $\mu\colon M \to X$, B_μ, and T_μ be as in Definition 2.1. Then $\ulcorner B_\mu \urcorner$ is reduced, and $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Hence, $\mu^*(K_X + B)$ is the positive part of the relative Zariski-decomposition of $K_M + B_\mu$ over X and it is μ-semi-ample by Theorem 2.19. Therefore, there is a positive integer m such that mB is a divisor and that $m\mu^*(K_X + B) \sim 0$. It implies that $m(K_X + B)$ is Cartier. □

Lemma-Definition 2.22. *Let X be a normal surface and B an effective $\mathbb{Q}$-divisor on X such that $\ulcorner B \urcorner$ is reduced. Then there exist a bimeromorphic morphism $\rho\colon Y \to X$ from a normal surface Y and an effective $\mathbb{Q}$-divisor B_Y such that*

- *(Y, B_Y) is log-canonical,*
- *$K_Y + B_Y$ is ρ-ample, and*
- *$B_Y = \rho^{[*]}B + E_\rho$ for the ρ-exceptional locus E_ρ.*

The pair (Y, B_Y) is unique up to isomorphism over X, and $\operatorname{Sing} Y \cup \operatorname{Supp} B_Y = \rho^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B)$*. The pair (Y, B_Y) and the morphism $\rho\colon (Y, B_Y) \to (X, B)$ are called the* log-canonical modification *of (X, B).*

Proof. First, we shall show the existence of (Y, B_Y). Let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface M such that the union of $\mu^{-1} \operatorname{Supp} B$ and the μ-exceptional locus E_μ is a normal crossing divisor. We set $B_M := \mu^{[*]}B + E_\mu$. Then $\ulcorner B_M \urcorner$ is reduced, $\operatorname{Supp} B_M$ is normal crossing, and $\mu_* B_M = B$. Let P be the positive part of the relative Zariski-decomposition of $K_M + B_M$ with respect to $\mu\colon M \to X$. Then P is μ-semi-ample by Theorem 2.19. Therefore, there exist bimeromorphic morphisms $\phi\colon M \to Y$, $\rho\colon Y \to X$, and a ρ-ample $\mathbb{Q}$-divisor A such that Y is a normal surface, $\mu = \rho \circ \phi$, and $P \sim_{\mathbb{Q}} \phi^* A$. In particular, $Y \simeq \mathbf{Projan}_X \mathcal{R}$ over X for the graded $\mathcal{O}_X$-algebra

$$\mathcal{R} = \bigoplus\nolimits_{m\geq 0} \mu_* \mathcal{O}_M(\llcorner m(K_M + B_M)\lrcorner) \simeq \bigoplus\nolimits_{m\geq 0} \mu_* \mathcal{O}_M(\llcorner mP \lrcorner),$$

which is finitely generated locally over X (cf. Lemma 2.17 and Corollary 2.20). The negative part N of the relative Zariski-decomposition of $K_M + B_M$ is ϕ-exceptional, since $PN = (\phi^* A)N = 0$. Hence, $\phi_* P = \phi_*(K_M + B_M) = K_Y + B_Y \sim_{\mathbb{Q}} A$ for the $\mathbb{Q}$-divisor $B_Y := \phi_* B_M$. It implies that (Y, B_Y) is log-canonical, $K_Y + B_Y$ is ρ-ample, and $\rho_* B_Y = B$. Moreover, $B_Y = \rho^{[*]}B + E_\rho$ for the ρ-exceptional locus E_ρ by $B_M = \mu^{[*]}B + E_\mu$. Therefore, (Y, B_Y) is a log-canonical modification of (X, B).

Second, we shall show the uniqueness of (Y, B_Y). Let $\rho'\colon (Y', B_{Y'}) \to (X, B)$ be another log-canonical modification. Then we have bimeromorphic morphisms $\phi'\colon M' \to Y'$ and $\theta\colon M' \to M$ from a non-singular surface M' such that the diagram

$$\begin{tikzcd} M' \arrow[r, "\phi'"] \arrow[d, "\theta"'] & Y' \arrow[r, "\rho'"] & X \\ M \arrow[r, "\phi"'] \arrow[urr, "\mu"] & Y \arrow[ur, "\rho"'] & \end{tikzcd}$$

is commutative and that the union of $\theta^{-1}(\mu^{-1} \operatorname{Supp} B)$ and the $\mu \circ \theta$-exceptional locus $E_{\mu\circ\theta}$ is a normal crossing divisor. We set $B_{M'} = (\mu \circ \theta)^{[*]}B + E_{\mu\circ\theta}$ as above. Then $K_{M'} + B_{M'} = \phi'^*(K_{Y'} + B_{Y'}) + R'$ for a ϕ'-exceptional effective $\mathbb{Q}$-divisor R', since $(Y', B_{Y'})$ is log-canonical. Thus, $\phi'^*(K_{Y'} + B_{Y'})$ is the positive part of the relative Zariski-decomposition of $K_{M'} + B_{M'}$ over X. On the other hand, we have $K_{M'} + B_{M'} = \theta^*(K_M + B_M) + R''$ for a θ-exceptional effective $\mathbb{Q}$-divisor R'', since (M, B_M) is log-canonical. Hence, $\theta^*P = \theta^*(\phi^*(K_Y + B_Y))$ is equal to $\phi'^*(K_{Y'} + B_{Y'})$ as the positive part of the relative Zariski-decomposition of $K_{M'} + B_{M'}$ over X. Therefore, $Y \simeq Y'$ over X.

Finally, we shall show the equality on $\operatorname{Sing} Y \cup \operatorname{Supp} B_Y$. By $B_Y = \rho^{[*]}B + E_\rho$ and by the isomorphism $Y \setminus E_\rho \simeq X \setminus \rho(E_\rho)$, we have $\operatorname{Supp} B_Y = E_\rho \cup \rho^{-1} \operatorname{Supp} B$, $\operatorname{Sing} Y \cup E_\rho = (\rho^{-1} \operatorname{Sing} X) \cup E_\rho$, and $E_\rho = \rho^{-1}\rho(E_\rho)$. Moreover, the uniqueness of log-canonical modification of (X, B) over $X \setminus (\operatorname{Sing} X \cup \operatorname{Supp} B)$ implies that $\rho(E_\rho) \subset \operatorname{Sing} X \cup \operatorname{Supp} B$. Therefore,

$$\operatorname{Sing} Y \cup \operatorname{Supp} B_Y = \operatorname{Sing} Y \cup E_\rho \cup \rho^{-1} \operatorname{Supp} B = \rho^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B).$$

Thus, we are done. □

A certain morphism of normal surfaces with only discrete fibers lifts to log-canonical modifications as follows:

Proposition 2.23. *Let $f\colon Y \to X$ be a morphism of normal surfaces with only discrete fibers and let B_X and B_Y be effective $\mathbb{Q}$-divisors on X and Y, respectively, such that $\ulcorner B_X \urcorner$ and $\ulcorner B_Y \urcorner$ are reduced and $K_Y + B_Y = f^*(K_X + B_X)$. Let $\sigma\colon (V, B_V) \to (X, B_X)$ and $\tau\colon (W, B_W) \to (Y, B_Y)$ be the log-canonical modifications. Then there is a morphism $h\colon W \to V$ with only discrete fibers such that $f \circ \tau = \sigma \circ h$ and $K_W + B_W = h^*(K_V + B_V)$.*

Proof. We set $B = B_X$ and apply results in Section 2.2. For the commutative diagram (II-2) of Lemma 2.9 defined for $(X, B) = (X, B_X)$, by the proof of Lemma-Definition 2.22, we can find bimeromorphic morphisms $\phi\colon M \to V$, $\sigma\colon V \to X$, $\psi\colon N \to W$, and $\tau\colon W \to Y$ such that the extended diagram

$$\begin{tikzcd} N \arrow[r, "\psi"'] \arrow[d, "g"'] \arrow[rr, bend left, "\nu"] & W \arrow[r, "\tau"'] & Y \arrow[d, "f"] \\ M \arrow[r, "\phi"] \arrow[rr, bend right, "\mu"'] & V \arrow[r, "\sigma"] & X \end{tikzcd}$$

is commutative and that $\phi^*(K_V + B_V)$ (resp. $\psi^*(K_W + B_W)$) is the positive part of the relative Zariski-decomposition of $K_M + \mu^{[*]}B_X + E_\mu$ (resp. $K_N + \nu^{[*]}B_Y + E_\nu$) over X (resp. Y), where E_μ (resp. E_ν) is the exceptional locus for μ (resp. ν). By assumption, $B_f = B_Y$ and $T_f = \Delta = 0$ for $\mathbb{Q}$-divisors B_f, T_f, and Δ in Lemmas 2.10 and 2.11. Hence, $\operatorname{Supp} B_Y \subset \widetilde{\Sigma}_f$, and

$$K_N + \nu^{[*]}B_Y + E_\nu = g^*(K_M + \mu^{[*]}B_X + E_\mu) + \overline{\Delta}$$

for an effective $\mathbb{Q}$-divisor $\overline{\Delta}$ which is exceptional for both ν and g by Lemma 2.11(3) as $\nu_*\overline{\Delta} = \Delta = 0$. Therefore,

$$K_N + \nu^{[*]}B_Y + E_\nu = g^*(\phi^*(K_V + B_V)) + G \tag{II-7}$$

for an effective $\mathbb{Q}$-divisor G exceptional for $\phi \circ g$. The fiber product $V \times_X Y$ is irreducible and generically reduced by Lemma 1.13. For the normalization V' of $V \times_X Y$, we have a commutative diagram

$$\begin{tikzcd} N \arrow[r, "\phi'"'] \arrow[d, "g"'] \arrow[rr, bend left, "\nu"] & V' \arrow[r, "\sigma'"'] \arrow[d, "p"] & Y \arrow[d, "f"] \\ M \arrow[r, "\phi"] & V \arrow[r, "\sigma"] & X, \end{tikzcd}$$

in which ϕ' and σ' are bimeromorphic morphisms and p is induced by the first projection $V \times_X Y \to V$. Note that p also has only discrete fibers. Then G is exceptional for ϕ', $g^*(\phi^*(K_V + B_V)) = \phi'^*(p^*(K_V + B_V))$, and $p^*(K_V + B_V)$ is σ'-ample. Hence, by (II-7), we have an equality

$$\psi^*(K_W + B_W) = g^*(\phi^*(K_V + B_V))$$

as the positive part of the relative Zariski-decomposition of $K_N + \nu^{[*]}B_Y + E_\nu$ over Y. Consequently, there is an isomorphism $\lambda\colon W \to V'$ such that $\lambda \circ \psi = \phi'$, $\tau = \sigma' \circ \lambda$, and $K_W + B_W = \lambda^*(p^*(K_V + B_V))$. Then the morphism $h = p \circ \lambda$ satisfies the required conditions. □

3. Singularities of pairs for endomorphisms of surfaces

As a generalization of an endomorphism of a normal surface X, we shall consider a morphism $X^\circ \to X$ from an open subset X° of X. The main result in Section 3 is Theorem 3.5 below on the log-canonicity of pairs (X, B) in which X admits a morphism $X^\circ \to X$ with only discrete fibers and B satisfies a special condition. Theorem 0.1 in the introduction is a direct consequence of Theorem 3.5. As a corollary of Theorem 3.5, we can prove results of Wahl [62] and Favre [6] on the log-canonicity of a normal surface singularity which admits a non-isomorphic finite surjective endomorphism (cf. Corollary 3.7). In Section 3.1, we explain the situation, the statement, and corollaries of Theorem 3.5, as well as a 1-dimensional analogue, Proposition 3.4. The proof of Theorem 3.5 is given in Section 3.2.

3.1. Setting and statements.

DEFINITION 3.1. For a normal variety X, let $f\colon X^\circ \to X$ be a morphism from an open subset X° of X. We define open subsets $X^{(k)} = X_f^{(k)}$ for $k \geq 0$ inductively by

$$X^{(0)} := X, \quad X^{(1)} = X^\circ, \quad \text{and} \quad X^{(k+1)} = f^{-1}(X^{(k)}).$$

Composing f and its restrictions to $X^{(i)}$, we have a morphism

$$f^{(k)}\colon X^{(k)} \xrightarrow{f} X^{(k-1)} \xrightarrow{f} \cdots \xrightarrow{f} X^{(0)} = X$$

for any $k \geq 0$, where $f^{(0)} = \mathrm{id}_X$ and $f^{(1)} = f$. Note that $f^{(k)}$ has a meaning when $X^{(k)} \neq \emptyset$. We define $X_{(k)} = X_{f,(k)}$ to be the image $f^{(k)}(X^{(k)})$. Note that $X_{(k)}$ is an open subset of X when f has only discrete fibers (cf. Corollary 1.8). The intersection $\bigcap_{k\geq 1} X_{(k)}$ is called the limit set of f and is denoted by $X_{(\infty)} = X_{f,(\infty)}$.

Remark 3.2. For a germ $\mathfrak{X} = (X, x)$ of a normal variety X at a point x, an endomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ is induced by a morphism $f\colon X^\circ \to X$ from an open neighborhood X° of x such that $f(x) = x$. The k-th power $\mathfrak{f}^k = \mathfrak{f} \circ \cdots \circ \mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ is induced by $f^{(k)}\colon X^{(k)} \to X$. The endomorphism $\mathfrak{f}$ also corresponds to an endomorphism $\mathfrak{f}^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ as a local ring homomorphism. When $\mathfrak{f}^*$ is finite, $\mathfrak{f}$ is said to be finite. In this case, x is an isolated point of $f^{-1}(x)$, and we may assume that $f^{-1}(x) = \{x\}$ and f has only discrete fibers by replacing X° with an open neighborhood of x (cf. Corollaries 1.4 and 1.8).

Remark. For the germ $\mathfrak{X} = (X, x)$ above, assume that x is an isolated singular point. Then we may take X as the complex analytic space X^{an} associated with an algebraic scheme X over $\operatorname{Spec}\mathbb{C}$ by [1, Thm. 3.8]. Hence, the endomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ is induced by a morphism $\mathsf{f}\colon \mathsf{U} \to \mathsf{X}$ of algebraic schemes from an étale neighborhood U of x. It is not clear that one can choose U as a Zariski-open neighborhood of x.

We use the following notation for $\mathbb{Q}$-divisors in Section 3.

Notation 3.3. Let B be a $\mathbb{Q}$-divisor on a normal variety with the prime decomposition $B = \sum b_i\Gamma_i$, where $b_i \in \mathbb{Q}$, and Γ_i are prime divisors. For a rational number c, we define

$$B^{\geq c} := \sum\nolimits_{b_i \geq c} b_i\Gamma_i, \quad B^{\leq c} := \sum\nolimits_{b_i \leq c} b_i\Gamma_i, \quad \text{and} \quad B_{=c} := \sum\nolimits_{b_i = c} \Gamma_i.$$

The following deals with the 1-dimensional case, which improves a part of [40, Lem. 3.5.1].

Proposition 3.4. *Let X be a non-singular curve and B an effective $\mathbb{Q}$-divisor on X such that* $\operatorname{Supp} B^{\geq 1}$ *is a finite set. Let $f\colon X^\circ \to X$ be a non-degenerate morphism from an open subset X° of X such that*

$$K_{X^\circ} + B|_{X^\circ} = f^*(K_X + B) + \Delta$$

for an effective $\mathbb{Q}$-divisor Δ on X°. Then the following hold for any point $P \in X_{f,(\infty)} = X_{(\infty)}$:

(1) *If* $\operatorname{mult}_P B \geq 1$, *then* $(f^{(k)})^{-1}(P) \cap X_{(\infty)} = \{P\}$ *for some* $k > 0$.

(2) *If* $\operatorname{mult}_P B > 1$, *then f is a local isomorphism at P and* $\operatorname{mult}_{f(P)} B = \operatorname{mult}_P B$.

(3) *If* $\operatorname{mult}_P B = 1$, *then* $P \notin \operatorname{Supp}\Delta$ *and* $\operatorname{mult}_{f(P)} B = 1$.

(4) *If $f(P) = P$, then*

$$(d - 1)(\operatorname{mult}_P B - 1) = -\operatorname{mult}_P \Delta$$

for $d := \operatorname{mult}_P f^*P$. *In particular, when f is not an isomorphism at P,* $\operatorname{mult}_P B < 1$ *if and only if* $\operatorname{mult}_P \Delta > 0$.

Proof. For a point $Q \in X^\circ$, we set $d_Q := \operatorname{mult}_Q f^*(f(Q))$. Note that f is a local isomorphism at Q if and only if $d_Q = 1$. We have equalities

$$d_Q - 1 = \operatorname{mult}_Q R_f = d_Q \operatorname{mult}_{f(Q)} B - \operatorname{mult}_Q B + \operatorname{mult}_Q \Delta$$

for the ramification divisor $R_f = K_{X^\circ} - f^*K_X = f^*B - B|_{X^\circ} + \Delta$ of f. Hence,

(III-1) $$\operatorname{mult}_Q B - 1 = d_Q(\operatorname{mult}_{f(Q)} B - 1) + \operatorname{mult}_Q \Delta \geq d_Q(\operatorname{mult}_{f(Q)} B - 1).$$

Then we have (4) by the first equality of (III-1) for $P = Q$. Moreover, (III-1) implies that

$$f^{-1}(\operatorname{Supp} B^{\geq 1}) \subset \operatorname{Supp} B^{\geq 1}.$$

In particular, for any $k \geq 1$, we have

$$X_{(k+1)} \cap \operatorname{Supp} B^{\geq 1} \subset f(X_{(k)}) \cap \operatorname{Supp} B^{\geq 1} \subset f(X_{(k)} \cap \operatorname{Supp} B^{\geq 1}).$$

We set $\mathcal{S} := X_{(\infty)} \cap \operatorname{Supp} B^{\geq 1}$. Then $\mathcal{S} = X_{(k)} \cap \operatorname{Supp} B^{\geq 1}$ for $k \gg 0$, since $\operatorname{Supp} B^{\geq 1}$ is finite, and hence, $f(\mathcal{S}) = \mathcal{S}$, $X_{(\infty)} \cap f^{-1}\mathcal{S} = \mathcal{S}$, and $f|_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$ is bijective. We may assume that $\mathcal{S} \neq \emptyset$ for assertions (1)–(3). We write $\mathcal{S} = \{P_1, P_2, \dots, P_n\}$. Then there is a permutation σ of $\{1, 2, \dots, n\}$ such that $f(P_i) = P_{\sigma(i)}$ for any i. Let k be the order of σ. Then $(f^{(k)})^{-1}(P) \cap X_{(\infty)} = \{P\}$ for any $P \in \mathcal{S}$; this shows (1). We set

$$d_i := d_{P_i} = \operatorname{mult}_{P_i} f^*(f(P_i)), \quad \beta_i := \operatorname{mult}_{P_i} B \geq 1, \quad \text{and} \quad \delta_i := \operatorname{mult}_{P_i} \Delta$$

for $1 \leq i \leq n$. Then

(III-2) $$\beta_i - 1 = d_i(\beta_{\sigma(i)} - 1) + \delta_i \geq d_i(\beta_{\sigma(i)} - 1)$$

by (III-1) for $Q = P_i$, and hence,

(III-3) $$\beta_i - 1 \geq d_i d_{\sigma(i)} \cdots d_{\sigma^{k-1}(i)}(\beta_i - 1)$$

for any $1 \leq i \leq n$. If $\beta_i > 1$, then $d_i = 1$, $\beta_i = \beta_{\sigma(i)}$, and $\delta_i = 0$ by (III-2) and (III-3); this shows (2). If $\beta_i = 1$, then $\beta_{\sigma(i)} = 1$ and $\delta_i = 0$ by (III-2); this shows (3). Thus, we are done. □

Remark. The idea of the proof above is originally in the proof of [40, Lem. 3.5.1]. It is used in the proof of Lemma 5.3 of the preprint version of [44] (= RIMS-**1613**, Kyoto Univ. 2007) and in the proof of [23, Prop. 2.4].

The following is the main result of Section 3, and it is regarded as a 2-dimensional analogue of a part of Proposition 3.4:

Theorem 3.5. *Let X be a normal surface and B an effective $\mathbb{Q}$-divisor on X such that* $\operatorname{Sing} X \cup \operatorname{Sing} B_{\mathrm{red}}$ *is a finite set. Let $f \colon X^\circ \to X$ be a morphism with only discrete fibers from an open subset X° of X such that*

$$K_{X^\circ} + B|_{X^\circ} = f^*(K_X + B) + \Delta$$

for an effective $\mathbb{Q}$-divisor Δ on X°. Then the following hold for the $\mathbb{Q}$-divisor $\widetilde{B} := B^{\leq 1} + \sum_{c>1} B_{=c}$ (cf. Notation 3.3) *and for any point x of the limit set $X_{(\infty)} = X_{f,(\infty)}$ (cf.* Definition 3.1):

1. *If $x \in \operatorname{Supp} \Delta$, then $(X, \widetilde{B})$ is 1-log-terminal at x (cf.* Definition 2.1).
2. *If $(X, \widetilde{B})$ is not log-canonical at x, then f is a local isomorphism at x, and $(f^{(k)})^{-1}(x) \cap X_{(\infty)} = \{x\}$ for some $k \geq 1$.*

By Remark 3.2, we have Theorem 0.1 directly from Theorem 3.5. We have two corollaries of Theorem 3.5. The first corollary below is a generalization of [40, Thm. 4.3.1], where X is assumed to be a normal Moishezon surface:

Corollary 3.6. *Let $f\colon X \to X$ be a non-isomorphic finite surjective endomorphism of a normal surface X and let S be a reduced divisor on X such that* $\operatorname{Sing} X \cup \operatorname{Sing} S$ *is a finite set and that $f^{-1}S = S$. Then (X, S) is log-canonical.*

Proof. There is an effective $\mathbb{Q}$-divisor Δ such that $K_X+S = f^*(K_X+S)+\Delta$ by Lemma 1.39. Thus, we can apply Theorem 3.5 to the situation where $X^\circ = X$ and $B = S$. Here, $X_{f,(\infty)} = X$, since f is surjective. Assume that (X, S) is not log-canonical at a point x. Then f is a local isomorphism at x and $(f^k)^{-1}(x) = \{x\}$ for some k by Theorem 3.5(2). This contradicts: $\deg f > 1$. Thus, (X, S) is log-canonical. □

The second corollary below is well known: The first assertion has been proved by Wahl in [62] by using an invariant $-P \cdot P$, and the second assertion has been proved by Favre in [6, Thm. B(3)] by using the theory of valuation spaces of normal surface singularities.

Corollary 3.7 (Wahl, Favre)**.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ be a non-isomorphic finite surjective endomorphism of a germ $\mathfrak{X} = (X, x)$ of a normal surface X at a point x. Then $\mathfrak{X}$ is log-canonical. If the ramification divisor $R_{\mathfrak{f}}$ is not zero at x, then $\mathfrak{X}$ is log-terminal.*

Proof. By Remark 3.2, we may assume that $\mathfrak{f}$ is induced by a morphism $f\colon X^\circ \to X$ with only discrete fibers from an open neighborhood X° of x such that $f(x) = x$ and f is not a local isomorphism at x. Then $x \in X_{f,(\infty)}$. Moreover, $x \in \operatorname{Supp} R_f$ when $x \in \operatorname{Supp} R_{\mathfrak{f}}$. Obviously, we may assume that $\operatorname{Sing} X$ is finite. Hence, the required assertions are derived from Theorem 3.5 applied to the case where $B = 0$. □

3.2. Proof of Theorem 3.5. We shall prove Theorem 3.5 after proving preliminary results Lemma 3.8, Proposition 3.9, and Lemma 3.10, in which the latter two are special cases of Theorem 3.5.

Lemma 3.8. *In the situation of* Theorem 3.5*, there is an inclusion*

$$f^{-1}(\operatorname{Supp} B^{\geq 1}) \subset \operatorname{Supp} B^{\geq 1}, \tag{III-4}$$

and there is an effective $\mathbb{Q}$-divisor $\widetilde{\Delta}$ on X° such that

$$K_{X^\circ} + \widetilde{B}|_{X^\circ} = f^*(K_X + \widetilde{B}) + \widetilde{\Delta}. \tag{III-5}$$

Assume the following three conditions:

(i) *The $\mathbb{Q}$-divisor $B^{\geq 1}$ has only finitely many prime components.*
(ii) *For any prime component Γ of $B^{\geq 1}$, $\Gamma|_{X^\circ}$ is a prime divisor.*
(iii) *For any prime component Γ of $B^{\geq 1}$, $f^{-1}\Gamma$ is not empty.*

Then $f^*(B_{=c}) = B_{=c}|_{X^\circ}$ for any $c > 1$, $f^{-1}(B_{=1}) = B_{=1}|_{X^\circ}$, and $B^{\geq 1}|_{X^\circ}$ has no common prime component with Δ. In particular, in this case, $\widetilde{\Delta} = \Delta$, and

$$K_{X^\circ} + B^{\leq 1}|_{X^\circ} = f^*(K_X + B^{\leq 1}) + \Delta.$$

Proof. Let $\widehat{\mathcal{S}}$ be the set of prime divisors on X and let $\mathcal{T}_f$ be the set of prime divisors Γ° on X° such that Γ° is a prime component of $f^{-1}D$ for an effective divisor D on X. Then, for each $\Gamma^\circ \in \mathcal{T}_f$, there is a unique prime divisor Γ on X such that Γ° is a prime component of $f^{-1}\Gamma$, and we have a map $\psi\colon \mathcal{T}_f \to \widehat{\mathcal{S}}$ by $\Gamma^\circ \mapsto \Gamma$. For $\Gamma^\circ \in \mathcal{T}_f$ and $\Gamma = \psi(\Gamma^\circ)$, the integer $a := \operatorname{mult}_{\Gamma^\circ} f^*\Gamma$ is the ramification index of f along Γ°. Hence,

$$a - 1 = \operatorname{mult}_{\Gamma^\circ} R_f = a \operatorname{mult}_\Gamma B - \operatorname{mult}_{\Gamma^\circ} B|_{X^\circ} + \operatorname{mult}_{\Gamma^\circ} \Delta$$

for the ramification divisor $R_f = K_{X^\circ} - f^*K_X = f^*B - B|_{X^\circ} + \Delta$, and

$$\text{(III-6)} \qquad \operatorname{mult}_{\Gamma^\circ} B|_{X^\circ} - 1 = a(\operatorname{mult}_\Gamma B - 1) + \operatorname{mult}_{\Gamma^\circ} \Delta \geq a(\operatorname{mult}_\Gamma B - 1).$$

If $\Gamma \subset \operatorname{Supp} B^{\geq 1}$, i.e., $\operatorname{mult}_\Gamma B \geq 1$, then $\Gamma^\circ \subset \operatorname{Supp} B^{\geq 1}|_{X^\circ}$ by (III-6). This shows (III-4). Next, we shall prove that the $\mathbb{Q}$-divisor $\widetilde{\Delta}$ defined by (III-5) is effective. The $\mathbb{Q}$-divisor is written as

$$\widetilde{\Delta} = R_f + \widetilde{B}|_{X^\circ} - f^*\widetilde{B} = \Delta - (B - \widetilde{B})|_{X^\circ} + f^*(B - \widetilde{B}),$$

where $B - \widetilde{B} = \sum_{c>1}(c-1)B_{=c}$. It is enough to show that $\operatorname{mult}_{\Gamma^\circ} \widetilde{\Delta} \geq 0$ for any prime divisor Γ° such that $\Gamma^\circ \subset \operatorname{Supp}(B - \widetilde{B})|_{X^\circ} \cap f^{-1} \operatorname{Supp} B$, since $\operatorname{Supp} B = \operatorname{Supp} \widetilde{B}$. Here, $\Gamma^\circ \in \mathcal{T}_f$ and $\Gamma := \psi(\Gamma^\circ) \subset \operatorname{Supp} B$. Hence, $\operatorname{mult}_{\Gamma^\circ} \widetilde{B}|_{X^\circ} = 1$, $\operatorname{mult}_\Gamma \widetilde{B} \leq 1$, and

$$\operatorname{mult}_{\Gamma^\circ} \widetilde{\Delta} = a - 1 + \operatorname{mult}_{\Gamma^\circ} \widetilde{B}|_{X^\circ} - a \operatorname{mult}_\Gamma \widetilde{B} \geq 0$$

for the ramification index a of f along Γ°. Therefore, $\widetilde{\Delta}$ is effective.

For the rest of the proof, we assume three conditions (i)–(iii). Let $\mathcal{S}$ be the set of prime components of $B^{\geq 1}$. Then $\mathcal{S}$ is finite by (i), and $\psi\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ is surjective by (iii) and (III-4). On the other hand, by (ii) and by the inclusion (III-4), we have an injection $i\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ such that $\Gamma^\circ = i(\Gamma^\circ)|_{X^\circ}$ for any $\Gamma^\circ \in \psi^{-1}(\mathcal{S})$. Thus, $i\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ and $\psi\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ are both bijective. Let $\Gamma_1, \ldots, \Gamma_n$ be the elements of $\mathcal{S}$. Then, by maps ψ and i, there is a permutation σ of the set $\{1, \ldots, n\}$ such that

$$f^{-1}(\Gamma_{\sigma(i)}) = \Gamma_i|_{X^\circ}$$

for any $1 \leq i \leq n$. We set

$$a_i = \operatorname{mult}_{\Gamma_i|_{X^\circ}} f^*\Gamma_{\sigma(i)}, \quad \beta_i := \operatorname{mult}_{\Gamma_i} B, \quad \text{and} \quad \delta_i = \operatorname{mult}_{\Gamma_i|_{X^\circ}} \Delta.$$

Here, $a_i \in \mathbb{Z}_{\geq 1}, \beta_i \in \mathbb{Q}_{\geq 1}$, and $\delta_i \in \mathbb{Q}_{\geq 0}$. By (III-6) for $\Gamma_i|_{X^\circ}$, we have

$$\text{(III-7)} \qquad \beta_i - 1 = a_i(\beta_{\sigma(i)} - 1) + \delta_i \geq a_i(\beta_{\sigma(i)} - 1).$$

Let k be the order of the permutation σ. Then

$$\text{(III-8)} \qquad \beta_i - 1 \geq a_i a_{\sigma(i)} \cdots a_{\sigma^{k-1}(i)}(\beta_i - 1)$$

for any $1 \leq i \leq n$ by (III-7). If $\beta_i > 1$, then $a_i = 1, \beta_{\sigma(i)} = \beta_i$, and $\delta_i = 0$ by (III-7) and (III-8). Therefore, for any $c > 1$, the equality $f^*(B_{=c}) = B_{=c}|_{X^\circ}$ holds, and $B_{=c}|_{X^\circ}$ has no common prime component with Δ. Subtracting $f^*(B_{=c}) = B_{=c}|_{X^\circ}$ from $K_{X^\circ} + B|_{X^\circ} = f^*(K_X + B) + \Delta$, we have

$$K_{X^\circ} + B^{\leq 1}|_{X^\circ} = f^*(K_X + B^{\leq 1}) + \Delta \quad \text{and} \quad \Delta = \widetilde{\Delta}.$$

If $\beta_i = 1$, then $\beta_{\sigma(i)} = 1$ and $\delta_i = 0$ by (III-7). Therefore, $f^{-1}(B_{=1}) = B_{=1}|_{X^\circ}$, and $B_{=1}|_{X^\circ}$ has no common prime component with Δ. Thus, we are done. $\square$

We shall prove the following special case of Theorem 3.5(1).

Proposition 3.9. *In the situation of* Theorem 3.5*, assume that* $\ulcorner B \urcorner$ *is reduced, i.e.,* $B = B^{\leq 1}$*. Let* x *be a point of* X° *such that* $f(x) = x$ *and* $x \in \operatorname{Supp} \Delta$*. Then* (X, B) *is* 1*-log-terminal at* x*.*

Proof. There is a positive integer m such that mB is a divisor on an open neighborhood of x in X. Then $m\Delta$ is also a divisor on an open neighborhood $\mathcal{U}$ of x in X° by $\Delta = R_f - f^*B + B|_{X^\circ}$ (cf. Remark 1.24(5)). Here, we may assume that $\operatorname{Sing} \mathcal{U} \subset \{x\}$. Thus, $mr\Delta$ is numerically Cartier on $\mathcal{U}$ for the numerical factorial index $r := \operatorname{nf}(X, x)$ (cf. Definition 1.26). For an integer $k \geq 1$, we set $B^{(k)} := B|_{X^{(k)}}$, $\Delta^{(k)} := \Delta|_{X^{(k)}}$, and

$$\Delta_k := \Delta|_{X^{(k)}} + \sum\nolimits_{i=1}^{k-1} f_{k,i}^*(\Delta^{(i)}) \tag{III-9}$$

for the composite $f_{k,i} \colon X^{(k)} \to X^{(k-1)} \to \cdots \to X^{(i)}$ of morphisms induced by f. Then the ramification formula for $f^{(k)}$ is equivalent to:

$$K_{X^{(k)}} + B^{(k)} = (f^{(k)})^*(K_X + B) + \Delta_k. \tag{III-10}$$

We can take a bimeromorphic morphism $\mu \colon M \to X$ from a non-singular surface M such that

- the union Σ_μ of $\mu^{-1} \operatorname{Supp} B$ and the μ-exceptional locus is a normal crossing divisor,
- the proper transform of $\llcorner B \lrcorner$ in M is non-singular.

Note that $\llcorner B \lrcorner$ is reduced by $B = B^{\leq 1}$. Then $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$ for effective $\mathbb{Q}$-divisors B_μ and T_μ having no common prime components such that $\operatorname{Supp} B_\mu \cup \operatorname{Supp} T_\mu \subset \Sigma_\mu$, $\mu_* B_\mu = B$, and $\mu_* T_\mu = 0$. For an integer $k \geq 0$, we set

$$M^{(k)} := \mu^{-1} X^{(k)}, \quad B_\mu^{(k)} := B_\mu|_{M^{(k)}}, \quad T_\mu^{(k)} := T_\mu|_{M^{(k)}}$$

and let $\mu^{(k)} \colon M^{(k)} \to X^{(k)}$ to be the morphism induced by μ. Let C_k and S_k be the positive and negative parts, respectively, of the prime decomposition of $B_\mu^{(k)} - (\mu^{(k)})^* \Delta_k$. Then

$$\begin{aligned} K_{M^{(k)}} + B_\mu^{(k)} &= (\mu^{(k)})^*(K_{X^{(k)}} + B^{(k)}) + T_\mu^{(k)}, \quad \text{and} \\ K_{M^{(k)}} + C_k &= (g^{(k)})^*(K_X + B) + S_k + T_\mu^{(k)} \end{aligned} \tag{III-11}$$

by (III-10) for the composite $g^{(k)} := f^{(k)} \circ \mu^{(k)} \colon M^{(k)} \to X$. Here, $C_k \leq B_\mu^{(k)}$, and C_k has no common prime component with $S_k + T_\mu^{(k)}$. In particular, $\operatorname{Supp} C_k$ is normal crossing and $\ulcorner C_k \urcorner$ is reduced. Let Γ be a prime divisor on M contained in $\mu^{-1}(x)$. Then Γ is also a divisor on $M^{(k)}$ for any $k \geq 1$, and

$$\operatorname{mult}_\Gamma (\mu^{(k)})^* \Delta_k = \operatorname{mult}_\Gamma \mu^* \Delta + \sum\nolimits_{i=1}^{k-1} \operatorname{mult}_\Gamma (f_{k,i})^* \Delta^{(i)} \geq k/(mr)$$

by (III-9), since $x \in \operatorname{Supp} \Delta$ and since $mr\Delta$ is numerically Cartier on $\mathcal{U}$. Hence, if $k > mr$, then any prime component of C_k is not contained in $\mu^{-1}(x)$, since $\ulcorner C_k \urcorner$ is reduced. For this k, $\mu^{-1}(x) \cap \operatorname{Supp} \llcorner C_k \lrcorner$ is a finite set contained in the proper transform of $\llcorner B \lrcorner$ in M, and hence, $\llcorner C_k \lrcorner$ is non-singular along $\mu^{-1}(x)$. In particular, $(M^{(k)}, C_k)$ is 1-log-terminal along $\mu^{-1}(x)$. Since $\mu^{-1}(x)$ is a compact connected component of $(g^{(k)})^{-1}(x)$, (X, B) is 1-log-terminal at x by (III-11) and by Proposition 2.12(2) applied to $g^{(k)} \colon M^{(k)} \to X$. $\square$

Remark. The iteration $f^{(k)}$ is also considered in the proof of [6, Thm. B(3)].

We shall prove the following special case of Theorem 3.5(2) by applying the *log-canonical modification* (cf. Lemma-Definition 2.22) and Proposition 2.23.

Lemma 3.10. *In the situation of* Theorem 3.5*, assume that* $\ulcorner B \urcorner$ *is reduced, i.e.,* $B = B^{\leq 1}$*. Let* x *be a point of* X° *such that* $f(x) = x$ *and* $x \notin \operatorname{Supp} \Delta$*. If* f *is not a local isomorphism at* x*, then* (X, B) *is log-canonical at* x*.*

Proof. We shall derive a contradiction by assuming that (X, B) is not log-canonical at x. By replacing X° with an open neighborhood of x, we may assume that $\Delta = 0$. Let $\rho\colon (Y, B_Y) \to (X, B)$ be the log-canonical modification. Then $\rho^{-1}(x)$ is a non-zero compact divisor as (X, B) is not log-canonical at x. We set $Y^\circ = \rho^{-1}(X^\circ)$, $B_{Y^\circ} = B_Y|_{Y^\circ}$, and $\rho^\circ := \rho|_{Y^\circ}\colon Y^\circ \to X^\circ$. Since ρ° is the log-canonical modification of $(X^\circ, B|_{X^\circ})$, by Proposition 2.23, there is a morphism $f_Y\colon Y^\circ \to Y$ with only discrete fibers such that $\rho \circ f_Y = f \circ \rho^\circ$ and $K_{Y^\circ} + B_{Y^\circ} = f_Y^*(K_Y + B_Y)$. On the other hand, by Remark 1.21, we can find open neighborhoods V_1 and V_2 of x in X° and X, respectively, such that $f(V_1) = V_2$, $f^{-1}(x) \cap V_1 = \{x\}$, and the induced morphism $\tau := f|_{V_1}\colon V_1 \to V_2$ is finite. Here, $\deg \tau > 1$, since f is not a local isomorphism at x. We set $Y_i := \rho^{-1} V_i$ for $i = 1, 2$. Then τ lifts to a finite surjective morphism $\theta := f_Y|_{Y_1}\colon Y_1 \to Y_2$ such that $\deg \theta = \deg \tau$. In particular, $\theta|_{\rho^{-1}(x)}\colon \rho^{-1}(x) \to \rho^{-1}(x)$ is also finite and surjective. Let $\mathcal{S}$ be the set of prime components of $\rho^{-1}(x)$. Then $\Gamma \mapsto f_Y(\Gamma) = \theta(\Gamma)$ gives rise to a bijection $\mathcal{S} \to \mathcal{S}$. By replacing $f\colon X^\circ \to X$ with the k-th power $f^{(k)}\colon X^{(k)} \to X$ for some $k > 1$, we may assume that $\Gamma = f_Y(\Gamma) = \theta(\Gamma)$ for any $\Gamma \in \mathcal{S}$. Then $\theta^*\Gamma = d\Gamma$ for a positive integer d, where $d^2 = \deg \theta = \deg \tau$ by $\Gamma^2 < 0$ and $(\theta^*\Gamma)^2 = (\deg \theta)\Gamma^2$ (cf. Remark 1.24). Hence, $(K_Y + B_Y)\Gamma = 0$ for any $\Gamma \in \mathcal{S}$ by $d > 1$ and by

$$\begin{aligned} d(K_Y + B_Y)\Gamma &= (K_Y + B_Y)\theta^*\Gamma = (K_{Y^\circ} + B_{Y^\circ})\theta^*\Gamma = (f_Y^*(K_Y + B_Y))\theta^*\Gamma \\ &= (K_Y + B_Y) f_{Y*}(\theta^*\Gamma) = (\deg \theta)(K_Y + B_Y)\Gamma = d^2(K_Y + B_Y)\Gamma. \end{aligned}$$

This contradicts the ρ-ampleness of $K_Y + B_Y$. Thus, we are done. □

Now, we are ready to prove Theorem 3.5:

Proof of Theorem 3.5. Let $\Sigma \subset X$ be the set of points x such that $(X, \widetilde{B})$ is not 1-log-terminal at x. Then $f^{-1}\Sigma \subset \Sigma$ by Proposition 2.12(2) applied to the equality (III-5) in Lemma 3.8. Note that Σ is finite by $\Sigma \subset \operatorname{Sing} X \cup \operatorname{Sing} B_{\mathrm{red}}$. We set $\Sigma_{(\infty)} := \Sigma \cap X_{(\infty)}$. Then $\Sigma_{(\infty)} = \Sigma \cap X_{(k)}$ for $k \gg 0$. Since

$$X_{(k+1)} \cap \Sigma \subset f(X_{(k)}) \cap \Sigma \subset f(X_{(k)} \cap \Sigma)$$

for any $k \geq 1$, we have $f(\Sigma_{(\infty)}) = \Sigma_{(\infty)}$ and $X_{(\infty)} \cap f^{-1}\Sigma_{(\infty)} = \Sigma_{(\infty)}$; hence, $f|_{\Sigma_{(\infty)}}\colon \Sigma_{(\infty)} \to \Sigma_{(\infty)}$ is bijective, and $f^{-1}(x) \cap X_{(\infty)} = (f|_{\Sigma_{(\infty)}})^{-1}(x)$ for any $x \in \Sigma_{(\infty)}$. There is a positive integer k such that $f^k(x) = x$ for any $x \in \Sigma_{(\infty)}$. By replacing f with f^k, we may assume that $f|_{\Sigma_{(\infty)}} = \mathrm{id}$. Then $f^{-1}(x) \cap X_{(\infty)} = \{x\}$ for any $x \in \Sigma_{(\infty)}$.

For the proof of Theorem 3.5, we may assume that $\Sigma_{(\infty)} \neq \emptyset$. For a point $x \in \Sigma_{(\infty)}$, we can choose an open neighborhood U of x in X satisfying the following conditions:

- If $x \notin \operatorname{Supp} B^{\geq 1}$, then $B^{\geq 1}|_U = 0$.
- If $x \in \operatorname{Supp} B^{\geq 1}$, then $B^{\geq 1}|_U$ has only finitely many prime components, and each

component contains x and is locally irreducible at x.

There is an open neighborhood U° of x such that $U^\circ \subset U \cap f^{-1}U$ and that $\Gamma|_{U^\circ}$ is irreducible for any prime component Γ of $B^{\geq 1}|_U$. Then we can apply Lemma 3.8 to the restriction $U^\circ \to U$ of f and to $B^{\geq 1}|_U$. As a consequence,

$$K_{U^\circ} + \widetilde{B}|_{U^\circ} = f^*(K_X + \widetilde{B})|_{U^\circ} + \Delta|_{U^\circ}$$

(cf. (III-5) in Lemma 3.8). Then $x \notin \operatorname{Supp}\Delta$ by Proposition 3.9 applied to $U^\circ \to U$. This proves Theorem 3.5(1). Moreover, if $(X, \widetilde{B})$ is not log-canonical at x, then f is a local isomorphism by Lemma 3.10 applied to $(U^\circ \to U, \widetilde{B}|_U)$ instead of $(X^\circ \xrightarrow{f} X, B)$, since $x \notin \operatorname{Supp}\Delta$. This proves Theorem 3.5(2), and we are done. □

4. Some technical notions for the study of endomorphisms

We prepare some technical results on toric surfaces (Section 4.1) and cyclic covers (Section 4.2), and introduce two notions: *essential blowings up* (Section 4.4) and *dual $\mathbb{R}$-divisors* (Section 4.4) with their properties. These results and properties are applied to discussions in Section 5 on lifts of endomorphisms.

4.1. Endomorphisms of certain affine toric surfaces. We shall explain basic properties of toric surfaces, toric morphisms, and toric endomorphisms, by using the theory of toric varieties (cf. [33], [45], [12], etc.) with some related arguments in [38, §3.1] and [41, §3.1] in addition. An affine toric surface, which is considered as a complex analytic surface, is expressed as

$$\mathbb{T}_{\mathsf{N}}(\sigma) = (\operatorname{Spec}\mathbb{C}[\sigma^\vee \cap \mathsf{M}])^{\mathrm{an}},$$

for a free abelian group N of rank 2, a closed strictly convex rational polyhedral cone σ in $\mathsf{N} \otimes \mathbb{R}$, the dual abelian group $\mathsf{M} := \operatorname{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$, and the dual cone

$$\sigma^\vee = \{m \in \mathsf{M} \otimes \mathbb{R} \mid m(x) \geq 0 \text{ for any } x \in \sigma\}.$$

Here, $^{\mathrm{an}}$ stands for the analytic space associated to an algebraic scheme over $\mathbb{C}$ (cf. [18, XII, §1]), the strict convexity means that $\sigma \cap (-\sigma) = \{0\}$, and $\mathbb{C}[\sigma^\vee \cap \mathsf{M}]$ denotes the semi-group ring over $\mathbb{C}$. We write $\mathbb{T}_{\mathsf{N}} = \mathbb{T}_{\mathsf{N}}(\{0\})$, which is canonically isomorphic to the algebraic torus $\mathsf{N} \otimes_{\mathbb{Z}} \mathbb{C}^\star$, where $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$. The toric surface admits an action of $\mathbb{T}_{\mathsf{N}}$ and an equivariant open immersion $\mathbb{T}_{\mathsf{N}}(\{0\}) \hookrightarrow \mathbb{T}_{\mathsf{N}}(\sigma)$.

Remark. If σ is 1-dimensional, then $\sigma = \mathbb{R}_{\geq 0}e$ for a primitive element e of N and we have an isomorphism $\mathbb{T}_{\mathsf{N}}(\sigma) \simeq \mathbb{C} \times \mathbb{C}^\star$ extending $\mathbb{T}_{\mathsf{N}}(\{0\}) \simeq \mathbb{C}^\star \times \mathbb{C}^\star$.

Fact 4.1 (cf. [41, Exam. 3.2]). *Assume that σ is 2-dimensional. Then N has two primitive elements e_1, e_2 such that (e_1, e_2) is a basis of $\mathsf{N} \otimes \mathbb{R}$ and $\sigma = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$. Let $\mathcal{E}$ be the set of elements $e \in \sigma \cap \mathsf{N}$ such that $\mathsf{N} = \mathbb{Z}e + \mathbb{Z}e_2$, and let $u \in \mathcal{E}$ be the element attaining the minimum of $e_1^\vee(e)$ for $e \in \mathcal{E}$, where $(e_1^\vee, e_2^\vee)$ is the dual basis of (e_1, e_2) in $\mathsf{M} \otimes \mathbb{R}$. Then there exist integers $n > q \geq 0$ such that $\gcd(n, q) = 1$ and $u = (1/n)(e_1 + qe_2)$. The integer n is uniquely determined by (N, σ). But q can be replaced with an integer $0 \leq q^\dagger < n$ by interchanging e_1 and e_2, where $q^\dagger = 0$ if $q = 0$, and $qq^\dagger \equiv 1 \bmod n$ if $q > 0$.*

DEFINITION 4.2. When $\dim \boldsymbol{\sigma} = 2$, the number n above is called the *order* of $(\mathsf{N}, \boldsymbol{\sigma})$, and the pair (n, q) is called the *type* of $(\mathsf{N}, \boldsymbol{\sigma})$.

REMARK (CF. [41, Exam. 3.2]). For $\boldsymbol{\sigma}$ in Fact 4.1, $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ has a unique fixed point $*$ on the action of $\mathbb{T}_{\mathsf{N}}$: For e_1 and e_2 above, the complement of $\mathbb{T}_{\mathsf{N}}(\mathbb{R}_{\geq 0}e_1) \cup \mathbb{T}_{\mathsf{N}}(\mathbb{R}_{\geq 0}e_2)$ in $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is just $\{*\}$. If $q = 0$, then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \simeq \mathbb{C}^2$. If $q > 0$, then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is singular at $*$, and it is a cyclic quotient singularity of type (n, q) (or type $(1/n)(1, q)$ in some literature); in this case, the exceptional locus of the minimal resolution forms a linear chain of rational curves whose self-intersection numbers are calculated by a kind of continued fraction of n/q.

In general, a toric surface is expressed as

$$\mathbb{T}_{\mathsf{N}}(\triangle) = \bigcup\nolimits_{\sigma \in \triangle} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$$

for a free abelian group N of rank 2 and for a *fan* $\triangle$ of N: A finite collection $\triangle$ of closed strictly convex rational polyhedral cones of $\mathsf{N} \otimes \mathbb{R}$ is called a *fan* if each face of a cone in $\triangle$ belongs to $\triangle$ and the intersection of two cones in $\triangle$ is a face of both cones. The open immersion $\mathbb{T}_{\mathsf{N}}(\{0\}) \subset \mathbb{T}_{\mathsf{N}}(\triangle)$ is also $\mathbb{T}_{\mathsf{N}}$-equivariant. The open orbit $\mathbb{T}_{\mathsf{N}}(\{0\})$ or $\mathbb{T}_{\mathsf{N}}$ is called the *open torus* and the complement $\mathbb{T}_{\mathsf{N}}(\triangle) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$ is called the *boundary divisor*. We have the following analogy of [41, Exam. 3.4].

EXAMPLE 4.3. Assume that the union $|\triangle| = \bigcup_{\sigma \in \triangle} \boldsymbol{\sigma}$ is a strictly convex cone of dimension 2. Then $\triangle$ gives a subdivision of $|\triangle|$ and there exist primitive elements v_i of N for $0 \leq i \leq l$ such that $\triangle$ consists of

- 2-dimensional cones $\boldsymbol{\sigma}_i = \mathbb{R}_{\geq 0}v_i + \mathbb{R}_{\geq 0}v_{i+1}$ for $0 \leq i \leq l - 1$,
- 1-dimensional cones $\mathbf{R}_i := \mathbb{R}_{\geq 0}v_i$ for $0 \leq i \leq l$, and
- the 0-dimensional cone $\{0\}$,

where $|\triangle| = \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_l$. The toric surface $\mathbb{T}_{\mathsf{N}}(\triangle)$ is obtained by gluing $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_i)$ for $0 \leq i \leq l - 1$ by open immersions $\mathbb{T}_{\mathsf{N}}(\mathbf{R}_{i+1}) \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_i)$ and $\mathbb{T}_{\mathsf{N}}(\mathbf{R}_{i+1}) \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_{i+1})$. The boundary $\mathbb{T}_{\mathsf{N}}(\triangle) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$ consists of prime divisors $\boldsymbol{\Gamma}(v_i)$ for $0 \leq i \leq l$ which are determined by the property that $\boldsymbol{\Gamma}(v_i) \cap \mathbb{T}_{\mathsf{N}}(\mathbf{R}_i) = \mathbb{T}_{\mathsf{N}}(\mathbf{R}_i) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$.

REMARK 4.4. For $m \in \mathsf{M}$, let $\boldsymbol{e}(m)$ denote the nowhere vanishing function on $\mathbb{T}_{\mathsf{N}} = (\operatorname{Spec} \mathbb{C}[\mathsf{M}])^{\mathrm{an}}$ corresponding to the invertible element m of $\mathbb{C}[\mathsf{M}]$. We regard $\boldsymbol{e}(m)$ as a meromorphic function on a toric surface $\mathbb{T}_{\mathsf{N}}(\triangle)$ for the fan $\triangle$ in Example 4.3. Then the principal divisor $\operatorname{div}(\boldsymbol{e}(m))$ is written as $\sum_{i=0}^{l} m(v_i)\boldsymbol{\Gamma}(v_i)$ for any $m \in \mathsf{M}$ (cf. [12, §3.3, Lem.], [45, Prop. 2.1(ii)]).

REMARK. If $\triangle$ consists of the faces of the cone $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$ in Fact 4.1, then $\mathbb{T}_{\mathsf{N}}(\triangle)$ is just the affine toric surface $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, and $l = 1$ in Example 4.3.

DEFINITION 4.5. For toric varieties $\mathbb{T}_{\mathsf{N}}(\triangle)$ and $\mathbb{T}_{\mathsf{N}'}(\triangle')$, a morphism $f \colon \mathbb{T}_{\mathsf{N}'}(\triangle') \to \mathbb{T}_{\mathsf{N}}(\triangle)$ of varieties is called a *toric morphism* if there is a homomorphism $\phi \colon \mathsf{N}' \to \mathsf{N}$ such that f is equivariant under actions of $\mathbb{T}_{\mathsf{N}'}$ and $\mathbb{T}_{\mathsf{N}}$ with respect to the complex Lie group homomorphism $\phi \otimes \mathbb{C}^{\star} \colon \mathbb{T}_{\mathsf{N}'} = \mathsf{N}' \otimes \mathbb{C}^{\star} \to \mathbb{T}_{\mathsf{N}} = \mathsf{N} \otimes \mathbb{C}^{\star}$.

A homomorphism $\phi \colon \mathsf{N}' \to \mathsf{N}$ is said to be *compatible* with their fans $\triangle'$ and $\triangle$ (or ϕ is called a morphism $(\mathsf{N}', \triangle') \to (\mathsf{N}, \triangle)$ of fans) if, for any $\boldsymbol{\sigma}' \in \triangle'$, there is a cone

$\boldsymbol{\sigma} \in \triangle$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}') \subset \boldsymbol{\sigma}$, where $\phi_{\mathbb{R}}$ denotes the induced linear map $\phi \otimes \mathbb{R} \colon \mathsf{N}' \otimes \mathbb{R} \to \mathsf{N} \otimes \mathbb{R}$ (cf. [45, §1.5]). In this case, the dual homomorphism $\phi^{\vee} \colon \mathsf{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z}) \to \mathsf{M}' = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}', \mathbb{Z})$ induces homomorphisms $\boldsymbol{\sigma}^{\vee} \cap \mathsf{M} \to \boldsymbol{\sigma}'^{\vee} \cap \mathsf{M}'$ of semi-groups, and toric morphisms $\mathbb{T}_{\mathsf{N}'}(\boldsymbol{\sigma}') \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. These are glued to a toric morphism $\mathbb{T}_{\mathsf{N}'}(\triangle') \to \mathbb{T}_{\mathsf{N}}(\triangle)$, which is denoted by $\mathbb{T}(\phi)$. Note that every toric morphism $\mathbb{T}_{\mathsf{N}'}(\triangle') \to \mathbb{T}_{\mathsf{N}}(\triangle)$ is expressed as $\mathbb{T}(\phi)$ for a homomorphism $\phi \colon \mathsf{N}' \to \mathsf{N}$ compatible with $\triangle'$ and $\triangle$ (cf. [45, Thm. 1.13]).

REMARK 4.6. The toric morphism f in Definition 4.5 is proper if, for any $\boldsymbol{\sigma} \in \triangle$, the inverse image $\phi_{\mathbb{R}}^{-1}\boldsymbol{\sigma}$ is the union of some cones $\boldsymbol{\sigma}'$ in $\triangle'$ (cf. [45, Thm. 1.15]). In particular, the fan $\triangle$ in Example 4.3 gives a toric bimeromorphic morphism $\mu \colon \mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}}(|\triangle|)$, where $\boldsymbol{\Gamma}(v_i)$ is μ-exceptional for any $1 \le i \le l-1$. If μ is an isomorphism, then $l = 1$, i.e., $\triangle$ consists of the faces of the cone $|\triangle|$.

REMARK 4.7. The toric morphism $\mu \colon \mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}}(|\triangle|)$ above is expressed as the blowing up along an ideal as follows: Let Γ_1 and Γ_2 be the boundary prime divisors of $\mathbb{T}_{\mathsf{N}}(|\triangle|)$ defined by $\mathbb{R}_{\ge 0}v_0$ and $\mathbb{R}_{\ge 0}v_l$, respectively. We have positive rational numbers a_i and b_i for $1 \le i \le l-1$ such that $v_i = a_i v_0 + b_i v_l$. Then $a_1/b_1 > a_2/b_2 > \cdots > a_{l-1}/b_{l-1}$. Let p_i for $1 \le i \le l-1$ be positive integers such that $-\sum p_i \boldsymbol{\Gamma}(v_i)$ is μ-very ample. Then μ is the blowing up along the ideal sheaf

$$\mathcal{J} := \mu_* \mathcal{O}_{\mathbb{T}_{\mathsf{N}}(\triangle)}(-\sum\nolimits_{i=1}^{l-1} p_i \boldsymbol{\Gamma}(v_i)).$$

For an element $m \in |\triangle|^{\vee} \cap \mathsf{M}$, the holomorphic function $\boldsymbol{e}(m)$ on $\mathbb{T}_{\mathsf{N}}(|\triangle|)$ belongs to $\mathcal{J}$ if and only if

$$\mathrm{div}(\boldsymbol{e}(m)) \ge \sum\nolimits_{i=1}^{l-1} p_i \boldsymbol{\Gamma}(v_i),$$

i.e., $m(v_i) = a_i m(v_0) + b_i m(v_l) \ge p_i$ for any $1 \le i \le l-1$. Since $\mathcal{J}$ is preserved by the action of $\mathbb{T}_{\mathsf{N}}$, $\mathcal{J}$ is generated by such $\boldsymbol{e}(m)$. Hence,

$$\mathcal{J} = \bigcap\nolimits_{i=1}^{l-1} \sum\nolimits_{a_i c + b_i d \ge p_i} \mathcal{O}_{\mathbb{T}_{\mathsf{N}}(|\triangle|)}(-c\Gamma_1 - d\Gamma_2),$$

where c and d are non-negative integers.

Lemma 4.8. *Let $\triangle$ and $\triangle'$ be fans of a free abelian group N of rank 2 such that $\boldsymbol{\tau} = |\triangle|$ and $\boldsymbol{\tau}' = |\triangle'|$ are strictly convex cones of dimension 2 and $\boldsymbol{\tau}' \subset \boldsymbol{\tau}$. Let*

$$\vartheta \colon \mathbb{T}_{\mathsf{N}'}(\triangle') \xrightarrow{\mu'} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}') \xrightarrow{t} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}) \overset{\mu^{-1}}{\dashrightarrow} \mathbb{T}_{\mathsf{N}}(\triangle)$$

be the composite of meromorphic maps, where μ and μ' are canonical bimeromorphic toric morphisms defined as in Remark 4.6*, and t is the toric morphism defined by $\boldsymbol{\tau}' \subset \boldsymbol{\tau}$. Then ϑ is holomorphic if and only if any $\boldsymbol{\sigma}' \in \triangle'$ is contained in some cone $\boldsymbol{\sigma} \in \triangle$. In particular, when $\boldsymbol{\tau} = \boldsymbol{\tau}'$ and $\#\triangle = \#\triangle'$, the map ϑ is holomorphic if and only if $\triangle = \triangle'$, and in this case, ϑ is the identity morphism of $\mathbb{T}_{\mathsf{N}}(\triangle)$.*

Proof. The second assertion follows from the first one, since fans $\triangle$ and $\triangle'$ give polyhedral decompositions of the same cone $\boldsymbol{\tau} = \boldsymbol{\tau}'$. For the first assertion, it suffices to prove the "only if" part, and we may assume that $\triangle'$ consists of the faces of a single 2-dimensional cone. Thus, from the beginning we may assume that $\mathbb{T}_{\mathsf{N}}(\triangle') = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}')$ and μ' is the identity morphism. The normalization of the fiber product of μ and t over $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau})$ is a toric variety expressed as $\mathbb{T}_{\mathsf{N}}(\triangle'')$ for the fan $\triangle'' = \{\boldsymbol{\tau}' \cap \boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \triangle\}$. If ϑ is holomorphic, then $\mathbb{T}_{\mathsf{N}}(\triangle'') \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}')$ is an isomorphism, and it implies that $\triangle''$ consists of the faces of $\boldsymbol{\tau}'$ by Remark 4.6. Hence, $\boldsymbol{\tau}' \subset \boldsymbol{\sigma}$ for some $\boldsymbol{\sigma} \in \triangle$. □

Lemma 4.9. *For* $(\mathsf{N}, \boldsymbol{\sigma})$ *in* Fact 4.1, *let* $\phi \colon \mathsf{N}' \to \mathsf{N}$ *be an injective homomorphism of free abelian groups of rank* 2, *and let* $\boldsymbol{\sigma}'$ *be a* 2*-dimensional strictly convex rational polyhedral cone of* $\mathsf{N}' \otimes \mathbb{R}$ *such that* $\phi_{\mathbb{R}}(\boldsymbol{\sigma}') \subset \boldsymbol{\sigma}$ *for the isomorphism* $\phi_{\mathbb{R}} = \phi \otimes \mathbb{R} \colon \mathsf{N}' \otimes \mathbb{R} \to \mathsf{N} \otimes \mathbb{R}$. *As in Fact* 4.1, *we write* $\boldsymbol{\sigma}' = \mathbb{R}_{\geq 0} e'_1 + \mathbb{R}_{\geq 0} e'_2$ *for two primitive elements* e'_1 *and* e'_2 *of* N' *which form a basis of* $\mathsf{N}' \otimes \mathbb{R}$. *Let* $\pi \colon \mathbb{T}_{\mathsf{N}'}(\boldsymbol{\sigma}') \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ *be the toric morphism* $\mathbb{T}(\phi)$. *Then*

$$\pi^*\boldsymbol{\Gamma}(e_1) = a_{11}\boldsymbol{\Gamma}(e'_1) + a_{12}\boldsymbol{\Gamma}(e'_2) \quad \textit{and} \quad \pi^*\boldsymbol{\Gamma}(e_2) = a_{21}\boldsymbol{\Gamma}(e'_1) + a_{22}\boldsymbol{\Gamma}(e'_2)$$

for non-negative integers a_{ij} *defined by*

$$(\phi(e'_1), \phi(e'_2)) = (e_1, e_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Moreover, $\#\mathsf{N}/\phi(\mathsf{N}') = (n/n')|a_{11}a_{22} - a_{12}a_{21}|$ *for the order* n' *of* $(\mathsf{N}', \boldsymbol{\sigma}')$.

Proof. Let $(e_1^\vee, e_2^\vee)$ be the dual basis of (e_1, e_2) in $\mathsf{M} \otimes \mathbb{R}$ and let $(e_1'^\vee, e_2'^\vee)$ be the dual basis of (e'_1, e'_2) in $\mathsf{M}' \otimes \mathbb{R}$, where $\mathsf{M}' = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}', \mathbb{Z})$. Let $\phi^\vee \colon \mathsf{M} \to \mathsf{M}'$ be the dual homomorphism of ϕ. Then $\phi^\vee_{\mathbb{R}} = \phi^\vee \otimes \mathbb{R}$ is given by

$$(\phi^\vee_{\mathbb{R}}(e_1^\vee), \phi^\vee_{\mathbb{R}}(e_2^\vee)) = (e_1'^\vee, e_2'^\vee)\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Let k be a positive integer such that $ke_1^\vee, ke_2^\vee \in \mathsf{M}$ and $ke_1'^\vee, ke_2'^\vee \in \mathsf{M}'$. Then

$$\pi^*\boldsymbol{e}(ke_i^\vee) = \boldsymbol{e}(\phi^\vee(ke_i^\vee)) = \boldsymbol{e}(ka_{i1}e_1'^\vee)\boldsymbol{e}(ka_{i2}e'^\vee)$$

for $i = 1, 2$. By Remark 4.4, we have $\mathrm{div}(\boldsymbol{e}(ke_i^\vee)) = k\boldsymbol{\Gamma}(e_i)$, and hence,

$$k\pi^*\boldsymbol{\Gamma}(e_i) = \mathrm{div}(\pi^*\boldsymbol{e}(ke_i^\vee)) = ka_{i1}\boldsymbol{\Gamma}(e'_1) + ka_{i2}\boldsymbol{\Gamma}(e'_2)$$

for $i = 1, 2$: this proves the first assertion. For the last assertion, we choose an element of N' of the form $u' = (1/n')(e'_1 + q'e'_2)$ such that $\mathsf{N}' = \mathbb{Z}u' + \mathbb{Z}e'_2$. Then

$$\begin{aligned}(\phi(u'), \phi(e'_2)) = (\phi(e'_1), \phi(e'_2))\begin{pmatrix} 1/n' & 0 \\ q'/n' & 1 \end{pmatrix} &= (e_1, e_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1/n' & 0 \\ q'/n' & 1 \end{pmatrix} \\ &= (u, e_2)\begin{pmatrix} 1/n & 0 \\ q/n & 1 \end{pmatrix}^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1/n' & 0 \\ q'/n' & 1 \end{pmatrix}.\end{aligned}$$

Taking determinants of matrices above, we have the equality for $\#\mathsf{N}/\phi(\mathsf{N}')$. □

Lemma 4.10. *For* $(\mathsf{N}, \boldsymbol{\sigma})$ *in Fact* 4.1, *let* $f \colon \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ *be the finite surjective toric morphism* $\mathbb{T}(\phi)$ *associated with an injective homomorphism* $\phi \colon \mathsf{N} \to \mathsf{N}$ *such that* $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. *Then there exist positive integers* d_1 *and* d_2 *and a permutation* $\iota \colon \{1, 2\} \to \{1, 2\}$ *such that*

$$\deg f = d_1 d_2, \quad f^*\Gamma_1 = d_1\Gamma_{\iota(1)}, \quad \textit{and} \quad f^*\Gamma_2 = d_2\Gamma_{\iota(2)}$$

where $\Gamma_1 = \boldsymbol{\Gamma}(e_1)$ *and* $\Gamma_2 = \boldsymbol{\Gamma}(e_2)$ *are prime components of the boundary divisor of* $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, *and n is the order of* $(\mathsf{N}, \boldsymbol{\sigma})$*. If* $\iota(1) = 1$*, then* $d_1 \equiv d_2 \bmod n$*. If* $\iota(1) = 1$ *and* $d_1 = d_2$*, then* ϕ *is the multiplication map by* d_1*.*

Proof. By Lemma 4.9, there exist positive integers d_1 and d_2 and a permutation ι such that $\phi(e_{\iota(1)}) = d_1 e_1$ and $\phi(e_{\iota(2)}) = d_2 e_2$, since Γ_1 and Γ_2 are not f-exceptional. Thus, $\deg f = d_1 d_2$, $f^*\Gamma_1 = d_1\Gamma_{\iota(1)}$, and $f^*\Gamma_2 = d_1\Gamma_{\iota(2)}$. Assume that $\iota(1) = 1$. Then, for the primitive element $u = (1/n)(e_1 + qe_2)$ in Fact 4.1, we have

$$\phi(u) = (1/n)(d_1 e_1 + q d_2 e_2) = d_1 u + (q/n)(d_2 - d_1)e_2 \in \mathsf{N}.$$

Thus, $d_1 \equiv d_2 \bmod n$. If $d_1 = d_2$, then ϕ is the multiplication map by d_1, since $\phi(u) = d_1 u$ and $\phi(e_2) = d_2 e_2$. □

4.2. Lifting endomorphisms to certain cyclic covers. There is a well-known construction of cyclic covers of normal varieties due to Esnault [5, §1] and Viehweg [61, §1]. A similar construction can be found in [47, §5] and [3]. We shall present another construction of cyclic covers from a $\mathbb{Q}$-divisor whose multiple is principal: This yields the notion of an *index* 1 *cover* (cf. Definition 4.18(2) below), which is a generalization of the same cover considered in [31]. As a byproduct, we shall give a sufficient condition for an endomorphism of a variety to lift to an index 1 cover (cf. Lemma 4.21). In Section 4.2, varieties are not necessarily 2-dimensional.

Definition 4.11. For a normal variety X and a $\mathbb{Q}$-divisor L on X, assume that mL is a principal divisor for a positive integer m; hence, we have an isomorphism $s\colon \mathcal{O}_X(mL) \xrightarrow{\simeq} \mathcal{O}_X$. We consider the $\mathcal{O}_X$-module

$$\mathcal{R}(L, m, s) := \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X(\llcorner iL \lrcorner)$$

and endow it an $\mathcal{O}_X$-algebra structure by homomorphisms

$$\tilde{\mu}_{i,j}\colon \mathcal{O}_X(\llcorner iL \lrcorner) \otimes \mathcal{O}_X(\llcorner jL \lrcorner) \to \mathcal{O}_X(\llcorner m\langle (i+j)/m\rangle L \lrcorner)$$

defined as follows for integers $0 \le i, j < m$: If $i + j < m$, then $\tilde{\mu}_{i,j}$ is just the composite

$$\mu_{i,j}\colon \mathcal{O}_X(\llcorner iL \lrcorner) \otimes \mathcal{O}_X(\llcorner jL \lrcorner) \to \mathcal{O}_X(\llcorner iL \lrcorner + \llcorner jL \lrcorner) \to \mathcal{O}_X(\llcorner (i+j)L \lrcorner),$$

where the first homomorphism is given by taking the double dual and the second one is induced by the inequality $\llcorner iL \lrcorner + \llcorner jL \lrcorner \le \llcorner (i+j)L \lrcorner$ of divisors. If $i + j \ge m$, then $\tilde{\mu}_{i,j}$ is the composite

$$\mathcal{O}_X(\llcorner iL \lrcorner) \otimes \mathcal{O}_X(\llcorner jL \lrcorner) \xrightarrow{\mu_{i,j}} \mathcal{O}_X(\llcorner (i+j)L \lrcorner) \xrightarrow{\otimes s} \mathcal{O}_X(\llcorner (i+j-m)L \lrcorner).$$

The associated finite morphism $\pi\colon \mathbb{V}(L, m, s) := \mathbf{Specan}_X\, \mathcal{R}(L, m, s) \to X$ is called the *cyclic cover* with respect to (L, m, s). For **Specan**, see [7, §1.14]. Note that $\mathcal{R}(L, m, s) = \mathcal{O}_X$ and $\mathbb{V}(L, m, s) = X$ when $m = 1$.

Remark. For the variety X above, let H be a Cartier divisor on X with a non-zero global section σ of $\mathcal{O}_X(mH)$ for an integer $m > 1$. Then the effective divisor $D = \operatorname{div}(\sigma)$, the *divisor of zeros* of σ, is linearly equivalent to mH, and σ induces an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{O}_X(mH)$. We set $L := (1/m)D - H$ as a $\mathbb{Q}$-divisor, and set $s\colon \mathcal{O}_X(mL) = \mathcal{O}_X(D - mH) \to \mathcal{O}_X$

to be the isomorphism induced by σ. Then $\mathbb{V}(L, m, s)$ is the cyclic cover defined in Esnault [5, §1] and Viehweg [61, (1.1)] for (H, m, σ). Conversely, for (L, m, s) in Definition 4.11, if we set $H := -\llcorner L \lrcorner$ and $D := m\langle L\rangle$, then we have a section σ of $\mathcal{O}_X(mH)$ such that $\operatorname{div}(\sigma) = D$ by the isomorphism $s\colon \mathcal{O}_X(mL) = \mathcal{O}_X(D - mH) \to \mathcal{O}_X$. Thus, the notion of cyclic covers in the sense of Esnault and Viehweg is equivalent to our notion.

Remark 4.12. The $\mathcal{O}_X$-algebra $\mathcal{R}(L, m, s)$ is graded by $\mathbb{Z}/m\mathbb{Z}$. Hence, $\mathbb{V}(L, m, s)$ admits an action of the group $\boldsymbol{\mu}_m$ of m-th roots of unity over X. The action of $\zeta \in \boldsymbol{\mu}_m$ is defined by multiplication maps $\mathcal{O}_X(\llcorner iL \lrcorner) \to \mathcal{O}_X(\llcorner iL \lrcorner)$ by ζ^i. For an open subset U such that $L|_U$ is Cartier, we know that $\mathbb{V}(L|_U, m, s) \to U$ is a $\boldsymbol{\mu}_m$-torsor by [17, Prop. 4.1]. For another isomorphism $s'\colon \mathcal{O}_X(mL) \xrightarrow{\simeq} \mathcal{O}_X$, there is a $\boldsymbol{\mu}_m$-equivariant isomorphism $\mathbb{V}(L, m, s') \simeq \mathbb{V}(L, m, s)$ over X if and only if $s' = \varepsilon^m s$ for a nowhere vanishing function ε on X.

Lemma 4.13. *Let X be a non-singular variety with a non-zero holomorphic function t such that the principal divisor $D = \operatorname{div}(t)$ is non-zero and non-singular. For an integer $0 < a < m$, we define $L := (a/m)D$ as a $\mathbb{Q}$-divisor on X, and consider t^a as a nowhere vanishing section of $\mathcal{O}_X(-mL) = \mathcal{O}_X(-aD) = \mathcal{O}_X t^a$. Then*

$$\mathcal{R}(L, m, t^a) \simeq \mathcal{O}_X[\mathsf{u}, \mathsf{y}]/(\mathsf{u}^d - 1, \mathsf{y}^{m'} - t)\mathcal{O}_X[\mathsf{u}, \mathsf{y}] \tag{IV-1}$$

as an $\mathcal{O}_X$-algebra for integers $d := \gcd(a, m)$ and $m' := m/d$, where u and y are variables. In particular, $\mathbb{V}(L, m, t^a)$ is non-singular and is a disjoint union of d copies of $\mathbb{V}((1/m')D, m', t)$.

Proof. Let $\mathcal{B}$ be the $\mathcal{O}_X$-algebra in the right hand side of (IV-1), and let us consider an $\mathcal{O}_X$-algebra

$$\mathcal{A} := \mathcal{O}_X[\mathsf{z}]/(\mathsf{z}^m - t^a)\mathcal{O}_X[\mathsf{z}]$$

for a variable z. Then there an $\mathcal{O}_X$-algebra homomorphism $\mathcal{A} \to \mathcal{B}$ given by $\mathsf{z} \mapsto \mathsf{u}\mathsf{y}^{a'}$ for $a' := a/d$, since $m'a = a'm$. Moreover,

$$(\mathsf{u}\mathsf{y}^{a'})^i = t^{\llcorner ai/m \lrcorner}\mathsf{u}^i\mathsf{y}^{m'\langle ai/m\rangle}$$

in $\mathcal{B}$ for any $i \in \mathbb{Z}$, and the correspondence

$$i \mapsto (i \bmod d,\ m'\langle ai/m\rangle \bmod m')$$

gives rise to a bijection $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/m'\mathbb{Z}$. Hence, $\mathcal{A} \to \mathcal{B}$ is isomorphic to the canonical injection

$$\bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X\mathsf{z}^i \to \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X t^{-\llcorner ia/m \lrcorner}\mathsf{z}^i.$$

As a consequence, we have (IV-1), i.e., $\mathcal{B} \simeq \mathcal{R}(L, m, t^a)$. The last assertion is deduced from the isomorphism

$$\mathbb{V}(L, m, s) = \mathbf{Specan}_X\, \mathcal{B} \simeq \boldsymbol{\mu}_d \times \mathbb{V}((1/m')D, m', t)$$

with a property that $\mathbb{V}((1/m')D, m', t) \simeq \mathbf{Specan}_X\, \mathcal{O}_X[\mathsf{y}]/(\mathsf{y}^{m'} - t')\mathcal{O}_X[\mathsf{y}]$ is non-singular. $\square$

Lemma 4.14. *Let* $\pi\colon \mathbb{V} = \mathbb{V}(L, m, s) \to X$ *be the cyclic cover in* Definition 4.11 *with* $m > 1$. *Then* $\mathbb{V}$ *is normal,* π^*L *is a principal divisor on* $\mathbb{V}$, *and* $\mathcal{O}_{\mathbb{V}}(\pi^*L)$ *has a* $\boldsymbol{\mu}_m$*-linearization such that the associated* $\mathbb{Z}/m\mathbb{Z}$*-graded* $\mathcal{R}(L, m, s)$*-module* $\pi_*\mathcal{O}_{\mathbb{V}}(l\pi^*L)$ *is isomorphic to the twist* $\mathcal{R}(L, m, s)(l)$ *by* l *for any* $l \in \mathbb{Z}$, *i.e.,*

$$\pi_*\mathcal{O}_{\mathbb{V}}(l\pi^*L) \simeq \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X(\llcorner(l + i)L\lrcorner). \tag{IV-2}$$

Here, the image v *of* 1 *under the injection*

$$\mathcal{O}_X = \mathcal{R}(L, m, s)(-1)_1 \subset \mathcal{R}(L, m, s)(-1) \simeq \pi_*\mathcal{O}_{\mathbb{V}}(-\pi^*L)$$

is regarded as a nowhere vanishing section of $\mathcal{O}_{\mathbb{V}}(-\pi^*L)$ *satisfying* $\pi^*s = v^m$. *If* X *and* $\operatorname{Supp}\langle L\rangle$ *are non-singular, then* $\mathbb{V}$ *is also non-singular.*

Proof. We set $X^\circ := X \setminus (\operatorname{Sing} X \cup \operatorname{Sing} \operatorname{Supp}\langle L\rangle)$. For any point $x \in X^\circ \cap \operatorname{Supp}\langle L\rangle$, we can find an open neighborhood U of x and a non-zero holomorphic function t on U such that

- $\operatorname{div}(t)$ is non-singular,
- $\langle L\rangle|_U = (a/m)\operatorname{div}(t)$ for an integer $0 < a < m$, and
- $s|_U = \varepsilon^m t^a$ as a section of $\mathcal{O}_X(-mL)|_U$ for a nowhere vanishing section ε of $\mathcal{O}_X(-\llcorner L\lrcorner)|_U$, where we regard $\mathcal{O}_X(-m\langle L\rangle)|_U$ as an ideal sheaf of $\mathcal{O}_U$ generated by t^a.

In particular, $\mathbb{V}|_U \simeq \mathbb{V}((a/m)\operatorname{div}(t), m, t^a)$ by Remark 4.12 and it is non-singular by Lemma 4.13. Hence, $\mathbb{V}^\circ := \pi^{-1}(X^\circ)$ is non-singular, since $\mathbb{V} \to X$ is a $\boldsymbol{\mu}_m$-torsor over $X^\circ \setminus \operatorname{Supp}\langle L\rangle$ (cf. Remark 4.12). This shows the last assertion. For open immersions $j\colon X^\circ \hookrightarrow X$ and $j'\colon \mathbb{V}^\circ \hookrightarrow \mathbb{V}$, we have isomorphisms $\mathcal{R}(L, m, s) \simeq j_*(\mathcal{R}(L, m, s)|_{X^\circ})$ and $\mathcal{O}_{\mathbb{V}} \simeq j'_*\mathcal{O}_{\mathbb{V}^\circ}$, since $\mathcal{R}(L, m, s)$ is a reflexive $\mathcal{O}_X$-module and $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$ (cf. [46, II, Lem. 1.1.12], [22, Prop. 1.6]). Hence, $\mathbb{V}$ is normal.

For the rest, by the same property of reflexive sheaves, we may assume that X and $\operatorname{Supp}\langle L\rangle$ are non-singular, by replacing X with X°. Let

$$\psi\colon \mathcal{O}_X(\llcorner L\lrcorner) \to \mathcal{R}(L, m, s) = \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X(\llcorner iL\lrcorner) = \pi_*\mathcal{O}_{\mathbb{V}}$$

be the canonical injection from the factor of $i = 1$. For the m-th tensor product $\psi^{\otimes m}$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X(m\llcorner L\lrcorner) \arrow[r, "\delta_m"] & \mathcal{O}_X(mL) \arrow[r, "s", "\simeq"'] & \mathcal{O}_X \arrow[d] \\ \mathcal{O}_X(\llcorner L\lrcorner)^{\otimes m} \arrow[u, "\simeq"] \arrow[r, "\psi^{\otimes m}"] & (\pi_*\mathcal{O}_{\mathbb{V}})^{\otimes m} \arrow[r, "p_m"] & \pi_*\mathcal{O}_{\mathbb{V}}, \end{tikzcd} \tag{IV-3}$$

in which δ_m is the inclusion corresponding to the inequality $m\llcorner L\lrcorner \leq mL$ of divisors, p_m is defined by m-times products in the $\mathcal{O}_X$-algebra $\pi_*\mathcal{O}_{\mathbb{V}}$, and the right vertical arrow indicates the canonical homomorphism of $\mathcal{O}_X$-algebras. Let $\varphi\colon \pi^*\mathcal{O}_{\mathbb{V}}(\llcorner L\lrcorner) \to \mathcal{O}_{\mathbb{V}}$ be an injection corresponding to ψ by adjunction for (π^*, π_*). Then the image of φ is the ideal sheaf $\mathcal{O}_{\mathbb{V}}(-E)$ of an effective Cartier divisor E on $\mathbb{V}$. By (IV-3), the m-th tensor product

$$\varphi^{\otimes m}\colon \pi^*\mathcal{O}_{\mathbb{V}}(\llcorner L\lrcorner)^{\otimes m} \to \mathcal{O}_{\mathbb{V}}^{\otimes m} = \mathcal{O}_{\mathbb{V}}$$

equals the composite $(\pi^* s) \circ \pi^*\delta_m$, and hence, $mE = \pi^*(mL - m\llcorner L \lrcorner) = m\pi^*\langle L\rangle$. Therefore, $E = \pi^*\langle L\rangle$, and $\pi^* L = \pi^*(\llcorner L \lrcorner) + E$ is a principal divisor. For an integer n, let us consider the diagram

$$\text{(IV-4)}\qquad \begin{CD} \pi_*\mathcal{O}_{\mathbb{V}}(-n\pi^* L) @>{\cdots\cdots}>> \bigoplus_{i=0}^{m-1}\mathcal{O}_X(\llcorner (i-n)L \lrcorner) \\ @VVV @VVV \\ \pi_*\mathcal{O}_{\mathbb{V}}(-n\pi^*\llcorner L \lrcorner) @>{\simeq}>> \bigoplus_{i=0}^{m-1}\mathcal{O}_X(-n\llcorner L \lrcorner) \otimes \mathcal{O}_X(\llcorner iL \lrcorner) \end{CD}$$

of $\mathcal{R}(L, m, s)$-modules in which the bottom isomorphism is derived from the projection formula and vertical arrows are injections defined by inequalities $\llcorner (i-n)L \lrcorner \le -n\llcorner L \lrcorner + \llcorner iL \lrcorner$ of divisors for $0 \le i < m$. We shall show that the dotted arrow exists as the isomorphism (IV-2) for $l = -n$ and that it makes the diagram (IV-4) commutative. For the purpose, we can localize X and we may assume that $L = (a/m)D$, $D = \operatorname{div}(t)$, and $s = t^a$ as in Lemma 4.13. In this case, $\llcorner L \lrcorner = 0$, $\pi^* L = aE$, $E = \operatorname{div}(\mathsf{z})$ for $\mathsf{z} = \mathsf{u}\mathsf{y}^{a'}$ in the proof of Lemma 4.13, and the diagram (IV-4) is expressed as

$$\begin{CD} (\mathsf{u}\mathsf{y}^{a'})^n\mathcal{O}_X[\mathsf{u},\mathsf{y}]/(\mathsf{u}^d - 1, \mathsf{y}^{m'} - 1)\mathcal{O}_X[\mathsf{u},\mathsf{y}] @>{\cdots\cdots}>> \bigoplus_{i=0}^{m-1}\mathcal{O}_X t^{-\llcorner (i-n)a/m \lrcorner}\mathsf{z}^i \\ @VVV @VVV \\ \mathcal{O}_X[\mathsf{u},\mathsf{y}]/(\mathsf{u}^d - 1, \mathsf{y}^{m'} - 1)\mathcal{O}_X[\mathsf{u},\mathsf{y}] @>{\simeq}>> \bigoplus_{i=0}^{m-1}\mathcal{O}_X t^{-\llcorner ia/m \lrcorner}\mathsf{z}^i. \end{CD}$$

Thus, we have the dotted arrow as an isomorphism making the diagram commutative. As a consequence, $\pi_*\mathcal{O}_{\mathbb{V}}(l\pi^* L) \simeq \mathcal{R}(L, m, v)(l)$ for any $l \in \mathbb{Z}$.

For the section v of $\mathcal{O}_{\mathbb{V}}(-\pi^* L)$ in the statement, the section v^m of $\mathcal{O}_{\mathbb{V}}(-m\pi^* L)$ corresponds to the section s of $\mathcal{O}_X(-mL)$ by the isomorphism

$$\pi_*\mathcal{O}_{\mathbb{V}}(-m\pi^* L) \simeq \mathcal{R}(L, m, s)(-m) \simeq \mathcal{R}(L, m, s) \otimes \mathcal{O}_X(-mL).$$

Thus, $\pi^* s = v^m$, and we are done. □

Corollary 4.15. *The cyclic cover $\mathbb{V} = \mathbb{V}(L, m, s)$ is reducible if and only if there exist a positive integer k and a nowhere vanishing section w of $\mathcal{O}_X(-kL)$ such that $k < m$, $k \mid m$, kL is Cartier, and $s = w^{m/k}$. If $\mathbb{V}$ is irreducible, then*

$$\text{(IV-5)}\qquad K_{\mathbb{V}} = \pi^*\left(K_X + \sum\nolimits_i (1 - 1/e_i)\Gamma_i\right)$$

for the prime components Γ_i of $\langle L\rangle$ and for the denominator e_i of the rational number $\operatorname{mult}_{\Gamma_i} L$*.*

Proof. We may assume that X and $\operatorname{Supp}\langle L\rangle$ are non-singular as in the proof of Lemma 4.14. The second assertion is reduced to the case where $L = (1/m)D$ for $D = \operatorname{div}(t)$ in Lemma 4.13, and we have (IV-5) from the ramification formula for the cyclic cover

$$\mathbf{Specan}_X\, \mathcal{O}_X[\mathsf{y}]/(\mathsf{y}^m - t)\mathcal{O}_X[\mathsf{y}] \to X.$$

For the first assertion, it is enough to prove the "only if" part, since the "if" part is shown by the isomorphism

$$\mathbb{V}(L, m, s) \simeq \boldsymbol{\mu}_{m/k} \times \mathbb{V}(L, k, w).$$

Assume that $\mathbb{V}$ is reducible, and let Y be an irreducible component of $\mathbb{V}$. Then $Y \cap \pi^{-1}(X^\star)$ is a connected component of the $\boldsymbol{\mu}_m$-torsor $\pi^{-1}(X^\star)$ over $X^\star := X \setminus (\operatorname{Sing} X \cup \operatorname{Supp}\langle L\rangle)$ (cf. Remark 4.12). Let $H \subset \boldsymbol{\mu}_m$ be the subgroup consisting elements $\zeta \in \boldsymbol{\mu}_m$ such that $\zeta(Y) \subset Y$, and set $k := \#H$. Then H is the Galois group of the Galois cover $\pi_Y = \pi|_Y : Y \to X$, $k \mid m$, $k < m$, and $\mathbb{V}$ is a disjoint union of m/k-copies of Y. Let v be the nowhere vanishing section of $\mathcal{O}_{\mathbb{V}}(-\pi^*L)$ in Lemma 4.14. Since $v \in \mathcal{R}(L, m, s)(-1)_1$, for any $\zeta \in \boldsymbol{\mu}_m$, the pullback $\zeta^* v$ by the automorphism $\zeta : \mathbb{V} \to \mathbb{V}$ equals ζv as a section of $\mathcal{O}_{\mathbb{V}}(-\pi^*L)$. Thus,

$$(-1)^{k-1} \prod\nolimits_{\zeta \in H} \zeta^*(v|_Y) = (-1)^{k-1}(\prod\nolimits_{\zeta\in H} \zeta)(v^k|_Y) = v^k|_Y$$

is an H-invariant nowhere vanishing section of $\mathcal{O}_{\mathbb{V}}(-k\pi^*L) \otimes \mathcal{O}_Y \simeq \mathcal{O}_Y(-\pi_Y^*(kL))$. Hence, kL is a principal divisor on X and $\pi_Y^*(w) = v^k|_Y$ for a nowhere vanishing section w of $\mathcal{O}_X(-kL)$. Here, $w^{m/k} = s$ by $v^m = \pi^* s$. Thus, we are done. □

Lemma 4.16. *For the quadruplet* (X, L, m, s) *in* Definition 4.11 *with* $m > 1$, *let* $f : Y \to X$ *be a morphism of* maximal rank (*cf.* Definition 1.1) *from a normal variety* Y *such that* $\operatorname{codim}(f^{-1} \operatorname{Sing} X, Y) \geq 2$. *Then* $\mathbb{V}(f^*L, m, f^*s)$ *is isomorphic to the normalization of* $\mathbb{V}(L, m, s) \times_X Y$ *over* Y.

Proof. For each $i \in \mathbb{Z}$, we have a composite homomorphism

$$\gamma_i : f^*\mathcal{O}_X(\llcorner iL \lrcorner) \xrightarrow{\alpha} \mathcal{O}_Y(f^*\llcorner iL \lrcorner) \xrightarrow{\beta} \mathcal{O}_Y(\llcorner i f^*L \lrcorner),$$

where α is the canonical homomorphism on the pullback (cf. Lemma 1.19(1)) and β corresponds to the inequality $f^*(\llcorner iL \lrcorner) \leq \llcorner i f^*L \lrcorner$. Note that γ_i is an isomorphism over $Y' := Y \setminus f^{-1}(\operatorname{Sing} X \cup \operatorname{Supp}\langle L\rangle)$, which is a non-empty open subset of Y, since f is of maximal rank. The sum of γ_i induces an $\mathcal{O}_Y$-algebra homomorphism $f^*\mathcal{R}(L, m, s) \to \mathcal{R}(f^*L, m, f^*s)$ and the associated finite morphism $\mathbb{V}(f^*L, m, f^*s) \to \mathbb{V}(L, m, s) \times_X Y$ over Y, which is an isomorphism over Y'. Then the assertion is a consequence of a theorem of Grauert–Remmert (cf. [14], [18, XII, Thm. 5.4]), since $\mathbb{V}(f^*L, m, f^*s)$ is normal (cf. Lemma 4.14). □

Proposition 4.17. *For the quadruplet* (X, L, m, s) *in* Definition 4.11 *with* $m > 1$, *let* $f : X' \to X$ *be a morphism of maximal rank from a normal variety* X' *such that* $\operatorname{codim}(f^{-1} \operatorname{Sing} X, X') \geq 2$. *Let* L' *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X' *such that* $mL' \sim 0$ *and* s' *a nowhere vanishing section of* $\mathcal{O}_{X'}(-mL')$. *We set* $\pi : \mathbb{V} := \mathbb{V}(L, m, s) \to X$ *and* $\pi' : \mathbb{V}' := \mathbb{V}(L', m, s') \to X'$ *as the associated cyclic covers. For an integer k, assume that* $f^*L \sim kL'$ *and* $f^*s = \varepsilon^m(s')^k$ *for a nowhere vanishing section* ε *of* $\mathcal{O}_{X'}(kL' - f^*L)$. *Then*:

(1) *There is a morphism* $g : \mathbb{V}' \to \mathbb{V}$ *such that* $\pi \circ g = f \circ \pi'$ *and that it is equivariant under the actions of* $\boldsymbol{\mu}_m$ *on* $\mathbb{V}$ *and* $\mathbb{V}'$ *explained in* Remark 4.12, *with respect to the k-th power map* $\boldsymbol{\mu}_m \to \boldsymbol{\mu}_m$, *i.e.,* $g(\zeta x) = \zeta^k g(x)$ *for any* $x \in \mathbb{V}'$ *and* $\zeta \in \boldsymbol{\mu}_m$.

(2) *If k is coprime to m, then* $\mathbb{V}'$ *is isomorphic to the normalization of* $\mathbb{V} \times_X X'$ *over* X'.

Proof. By Lemma 4.16, it suffices to construct a certain morphism $\mathbb{V}(L', m, s') \to \mathbb{V}(f^*L, m, f^*s)$ over X'. Thus, we may assume that $X' = X$ and $f = \mathrm{id}_X$. Moreover, by Remark 4.12, we may assume that $L = kL'$, $\varepsilon = 1$, and $s = (s')^k$. By interchanging L and L', we are reduced to constructing a morphism $g_k : \mathbb{V}(L, m, s) \to \mathbb{V}(kL, m, s^k)$ over X such that

(a) it is equivariant with respect to the k-th power map $\boldsymbol{\mu}_m \to \boldsymbol{\mu}_m$, and
(b) it is an isomorphism when k is coprime to m.

For each $0 \le i < m$, by tensor product with $s^{\llcorner ik/m \lrcorner}$, we have an isomorphism

$$\varphi_i \colon \mathcal{O}_X(\llcorner ikL \lrcorner) \simeq \mathcal{O}_X(\llcorner m\langle ik/m\rangle L \lrcorner) \otimes \mathcal{O}_X(m \llcorner ik/m \lrcorner L) \to \mathcal{O}_X(\llcorner m\langle ik/m\rangle L \lrcorner),$$

since $ik = m \llcorner ik/m \lrcorner + m\langle ik/m\rangle$. For any $0 \le i, j < m$, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(\llcorner ikL \lrcorner) \otimes \mathcal{O}_X(\llcorner jkL \lrcorner) & \xrightarrow{\varphi_i \otimes \varphi_j} & \mathcal{O}_X(\llcorner m\langle ik/m\rangle L \lrcorner) \otimes \mathcal{O}_X(\llcorner m\langle jk/m\rangle L \lrcorner) \\ {\scriptstyle \tilde{\mu}_{i,j}} \big\downarrow & & \big\downarrow {\scriptstyle \tilde{\mu}_{m\langle ik/m\rangle, m\langle jk/m\rangle}} \\ \mathcal{O}_X(\llcorner (m\langle (i+j)/m\rangle kL \lrcorner) & \xrightarrow{\varphi_{m\langle (i+j)/m\rangle}} & \mathcal{O}_X(\llcorner m\langle (i+j)k/m\rangle L \lrcorner) \end{array}$$

is commutative, where $\tilde{\mu}_{\cdot,\cdot}$ are homomorphisms defining $\mathcal{O}_X$-algebra structures of $\mathcal{R}(kL, m, s^k)$ and $\mathcal{R}(L, m, s)$ (cf. Definition 4.11) and where we use

$$m\langle (m\langle ik/m\rangle + m\langle jk/m\rangle)/m\rangle = m\langle \langle ik/m\rangle + \langle jk/m\rangle\rangle = m\langle (i+j)k/m\rangle.$$

Thus, the sum of φ_i for all $0 \le i < m$ gives an $\mathcal{O}_X$-algebra homomorphism

$$\Phi_k \colon \mathcal{R}(kL, m, s^k) \to \mathcal{R}(L, m, s),$$

which corresponds to a finite morphism $g_k \colon \mathbb{V}(L, m, s) \to \mathbb{V}(kL, m, s^k)$ over X. It is equivariant with respect to the k-th power map $\boldsymbol{\mu}_m \to \boldsymbol{\mu}_m$, since each φ_i commutes with multiplication maps by

$$\zeta^{ik} = \zeta^{m\langle ik/m\rangle}$$

for any $\zeta \in \boldsymbol{\mu}_m$. This shows (a). If k is coprime to m, then the correspondence $i \mapsto m\langle ik/m\rangle$ gives a permutation of $\{0, 1, \ldots, m-1\}$, which is identified with the k-th power map of $\boldsymbol{\mu}_m$; hence, Φ_k and g_k are isomorphisms. This shows (b), and we are done. □

Definition 4.18. Let X be a normal variety and L a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.

(1) The *Cartier* (resp. *torsion*) *index* of L is either the smallest positive integer r such that rL is Cartier (resp. $rL \sim 0$), or ∞ if such r does not exist. For a point $P \in X$, the *local Cartier index* of L at P is the smallest positive integer r such that rL is Cartier at P.
(2) A finite morphism $Y \to X$ is called an *index* 1 *cover* (or a *global index* 1 *cover*) with respect to L if $Y \simeq \mathbb{V}(L, m, s)$ over X for the torsion index m of L and an isomorphism $s \colon \mathcal{O}_X(mL) \xrightarrow{\simeq} \mathcal{O}_X$. Note that the index 1 cover is normal and irreducible by Lemma 4.14 and Corollary 4.15.
(3) For a point $P \in X$, a *local index* 1 *cover* with respect to L and P is an index 1 cover with respect to $L|_U$ for an open neighborhood U of P such that the torsion index of $L|_U$ equals the local Carter index of L at P.
(4) For a point $P \in X$, an *index* 1 *cover* of the germ (X, P) with respect to L is a morphism $(\widetilde{X}, \widetilde{P}) \to (X, P)$ of germs (or the germ $(\widetilde{X}, \widetilde{P})$) induced by a local index 1 cover $\widetilde{X}$ with respect to L and P and for the point $\widetilde{P}$ lying over P.

Remark 4.19. Let $V = \mathbb{V}(L, m, s)$ and $V' = \mathbb{V}(L, m, s')$ be two index 1 covers with respect to L. Then $s = \alpha s'$ for a nowhere vanishing function α on X. We have a finite étale morphism $\tau\colon \widehat{X} \to X$ from a normal variety $\widehat{X}$ such that $\tau^*\alpha = \beta^m$ for a nowhere vanishing function β on $\widehat{X}$. In fact, $\widehat{X}$ is given as a connected component of $\mathbb{V}(0, m, \alpha)$ (cf. Lemma 4.14). Then $V \times_X \widehat{X} \simeq V' \times_X \widehat{X}$ over $\widehat{X}$ by Remark 4.12. If $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$, then α is constant, $\widehat{X} \to X$ is an isomorphism, and hence, $V \simeq V'$ over X. Similarly, every point $P \in X$ has an open neighborhood U such that $V \times_X U \simeq V' \times_X U$ over U. Consequently, the index 1 cover of the germ (X, P) with respect to L is unique up to isomorphism.

Remark. In [31], an index 1 cover is considered only for $K_X + D \sim_{\mathbb{Q}} 0$, where X is a normal surface and D is a reduced divisor.

Properties in Remark 4.19 are generalized to:

Lemma 4.20. *For* (X, L, m, s) *in* Definition 4.11 *with* $m > 1$, *let* $\tau\colon Y \to X$ *be a finite surjective morphism from a normal variety* Y *such that* $m = \deg \tau$ *and* $\tau^*L \sim 0$.

(1) *If* $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$ *and if* m *is the torsion index of* L, *then* τ *is an index* 1 *cover with respect to* L.

(2) *If* m *is the local Cartier index of* L *at a point* P, *then* $\tau^{-1}U \to U$ *is a local index* 1 *cover with respect to* L *and* P *for an open neighborhood* U *of* P.

Proof. Let $\pi\colon \mathbb{V} := \mathbb{V}(L, m, s) \to X$ be the associated cyclic cover over X. By assumption, there is a nowhere vanishing section t of $\mathcal{O}_Y(-\tau^*L)$. Then $\tau^*s = \alpha t^m$ in $H^0(Y, \mathcal{O}_Y(-m\tau^*L))$ for a nowhere vanishing function α on Y. Suppose that $\alpha = \beta^m$ for a nowhere vanishing function β on Y. Then $\tau^*s = (\beta t)^m$ and the normalization of $\mathbb{V} \times_X Y$ is isomorphic to

$$\mathbb{V}(\tau^*L, m, (\beta t)^m) \simeq \boldsymbol{\mu}_m \times \mathbb{V}(\tau^*L, 1, \beta t) \simeq \boldsymbol{\mu}_m \times Y$$

by Lemma 4.16 and Remark 4.12. Thus, there is a finite morphism $\theta\colon Y \to \mathbb{V}$ over X. If $\mathbb{V}$ is irreducible, then θ is an isomorphism, since $\mathbb{V}$ is normal (cf. Lemma 4.14) and since $\deg \tau = \deg \pi$. In the situation of (1), $H^0(Y, \mathcal{O}_Y) \simeq \mathbb{C}$, since it is integral over $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$ (cf. [7, §2.27, Integrity Lemma]); hence, such β exists and (1) holds, since $\mathbb{V}$ is irreducible (cf. Corollary 4.15).

In the situation of (2), by replacing X with an open neighborhood of P, we may assume that $mL \sim 0$. Then $\pi^{-1}U \to U$ is an index 1 cover with respect to $L|_U$ for any open neighborhood U of P; hence, $\pi^{-1}U$ are irreducible. It suffices to find an open neighborhood U and a function β_U on $\tau^{-1}U$ such that $\alpha|_{\tau^{-1}U} = (\beta_U)^m$. This is shown by the finiteness of τ as follows: Now, $\tau^{-1}(P)$ is a finite set $\{Q_1, Q_2, \ldots, Q_k\}$. For each $1 \leq i \leq k$, we have an open neighborhood $\mathcal{V}_i$ of Q_i and a nowhere vanishing function β_i on $\mathcal{V}_i$ such that $\bigcup_{i=1}^k \mathcal{V}_i$ is a disjoint union of $\mathcal{V}_i$ and that $\alpha|_{\mathcal{V}_i} = \beta_i^m$. Then $\tau^{-1}U \subset \bigcup_{i=1}^k \mathcal{V}_i$ for an open neighborhood U of P, and functions β_i define a nowhere vanishing function β_U on $\tau^{-1}U$ such that $\alpha|_{\tau^{-1}U} = (\beta_U)^m$. Thus, we are done. □

Lemma 4.21. *For a normal variety* X *with a connected open subset* X°, *let* $f\colon X^\circ \to X$ *be a non-degenerate morphism without exceptional divisor. Let* L *be a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X *such that* $L \sim_{\mathbb{Q}} 0$ *and that* $f^*L \sim kL|_{X^\circ}$ *for an integer* $k \in \mathbb{Z}$ *and let* $\pi\colon V \to X$ *be an index* 1 *cover with respect to* L.

(1) *If $H^0(X^\circ, \mathcal{O}_{X^\circ}) \simeq \mathbb{C}$, then there is a morphism $g\colon V^\circ \to V$ such that $\pi \circ g = f \circ \pi^\circ$, where $V^\circ = \pi^{-1}V$ and $\pi^\circ = \pi|_{V^\circ}\colon V^\circ \to X^\circ$.*

(2) *For any point $P \in X^\circ$, there exist an open neighborhood U of P in X° and a morphism $g_U\colon V^\circ_U \to V$ such that $\pi \circ g_U = f \circ \pi^\circ_U$, where $V^\circ_U := \pi^{-1}(U)$ and $\pi^\circ_U := \pi|_{V^\circ_U}\colon V^\circ_U \to U \hookrightarrow X^\circ$.*

(3) *Assume that k is coprime to the torsion index of L. Then the morphism g (resp. g_U) in* (1) *(resp.* (2)*) induces an isomorphism from V° (resp. V°_U) to the normalization of $V \times_{X,f} X^\circ$ (resp. $(V \times_{X,f} X^\circ) \times_{X^\circ} U$).*

Proof. Let m be the torsion index of L and we write $V = \mathbb{V}(L, m, s)$ for a nowhere vanishing section s of $\mathcal{O}_X(-mL)$. By $mf^*L \sim mkL|_{X^\circ}$, we have a nowhere vanishing section α of $\mathcal{O}_{X^\circ}(m(kL|_{X^\circ} - f^*L))$ such that $f^*s = \alpha s^k|_{X^\circ}$. For an open subset U of X°, assume that

($*$) $\alpha|_U = \beta_U^m$ for a nowhere vanishing section β of $\mathcal{O}_{X^\circ}(kL|_{X^\circ} - f^*L)|_U$.

Then there is a morphism $g_U\colon V^\circ_U = \pi^{-1}(U) \to V$ such that $\pi \circ g_U = f \circ \pi^\circ_U$ by Proposition 4.17(1), since $j^*(f^*s) = (\beta_U)^m s^k|_U$ for the open immersion $j\colon U \hookrightarrow X^\circ$. Moreover, if k is coprime to m, then V°_U is isomorphic to the normalization of $V \times_{X, f\circ j} U$ by Proposition 4.17(2). Thus, it is enough to verify ($*$) for $U = X^\circ$ in case (1) and for an open neighborhood U of P in case (2). This is trivial in case (2), and this is deduced from $\alpha \in \mathbb{C}$ in case (1). □

Remark. In (1), if $X^\circ = X$, then $g\colon V \to V$ is a lift of the endomorphism $f\colon X \to X$. In (2), if the torsion index of L equals the local Cartier index of L at P, then $V \to X$ and $V^\circ_U \to U$ are local index 1 covers with respect to L and P.

4.3. Essential blowings up of log-canonical pairs. We shall introduce the notion of an essential blowing up for a log-canonical pair (X, S) of a normal surface X and a reduced divisor S. This generalizes the notion of toroidal blowing up of a toroidal pair (cf. [41, §4.3]). We begin with some preliminary results on $\llcorner B \lrcorner$ for log-canonical pairs (X, B).

Lemma 4.22. *Let X be a normal surface with an effective $\mathbb{Q}$-divisor B such that (X, B) is log-canonical. Let $f\colon Y \to X$ be a bimeromorphic morphism from a normal surface Y and let B_f and T_f be the positive and negative parts, respectively, of the prime decomposition of $f^*B - R_f$, i.e., $K_Y + B_f = f^*(K_X + B) + T_f$. Then $\llcorner B_f \lrcorner = D + D'$ for two reduced divisors D and D', which might be zero, such that*

- *$D \cap D' = \emptyset$, $f(D) = \operatorname{Supp} \llcorner B \lrcorner$, $f(D') \cap \operatorname{Supp} \llcorner B \lrcorner = \emptyset$,*
- *$f(D')$ is at most 0-dimensional, and*
- *f induces an isomorphism $\mathcal{O}_{\llcorner B \lrcorner} \simeq f_*\mathcal{O}_D$ when $\llcorner B \lrcorner \neq 0$.*

Proof. Since $T_f - B_f - K_Y = -f^*(K_X + B)$ is f-nef, we have

$$R^1 f_* \mathcal{O}_Y(\ulcorner T_f \urcorner - \llcorner B_f \lrcorner) = 0 \tag{IV-6}$$

by Proposition 2.15. We set $T := \ulcorner T_f \urcorner$ and $C := \llcorner B_f \lrcorner$. Note that C is reduced, since $\ulcorner B_f \urcorner$ is so (cf. Lemma 2.10(1)). Let $\mathcal{F}$ be the cokernel of the canonical injection $\mathcal{O}_Y(T - C) \to \mathcal{O}_Y(T)$. Since $\mathcal{O}_Y(T - C) \cap \mathcal{O}_Y = \mathcal{O}_Y(-C)$ as a subsheaf of $\mathcal{O}_Y(T)$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Y(-C) & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathcal{O}_Y(T-C) & \longrightarrow & \mathcal{O}_Y(T) & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

of exact sequences of sheaves on Y in which α and β are also injective. By applying f_* to this diagram and by (IV-6), we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\mathcal{O}_Y(-C) & \longrightarrow & \mathcal{O}_X \simeq f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_C & & \\ & & \big\downarrow & & {\scriptstyle f_*\alpha}\big\downarrow & & \big\downarrow{\scriptstyle f_*\beta} & & \\ 0 & \longrightarrow & f_*\mathcal{O}_Y(T-C) & \longrightarrow & f_*\mathcal{O}_Y(T) & \longrightarrow & f_*\mathcal{F} & \longrightarrow & 0 \end{array}$$

of exact sequences in which $f_*\alpha$ is an isomorphism as T is f-exceptional. Hence, $f_*\beta$ is an isomorphism and $\mathcal{O}_X \to f_*\mathcal{O}_C$ is surjective. On the other hand, we have $f_*C = \llcorner B \lrcorner$ by $f_*B_f = B$. Hence, the ideal sheaf $\mathcal{O}_X(-\llcorner B \lrcorner)$ equals the double dual of $f_*\mathcal{O}_Y(-C)$, and there is a surjection $f_*\mathcal{O}_C \to \mathcal{O}_{\llcorner B \lrcorner}$ which is an isomorphism outside a discrete set Z. Since C is reduced, $\llcorner B \lrcorner \cap Z = \emptyset$. Thus, $C = D + D'$ for reduced divisors D and D' such that $D \cap D' = \emptyset$ and $f(D') \subset Z$ and $f(D) = \llcorner B \lrcorner$ with an isomorphism $f_*\mathcal{O}_D \simeq \mathcal{O}_{\llcorner B \lrcorner}$. □

Lemma 4.23. *In* Lemma 4.22*, the following hold for any $x \in \llcorner B \lrcorner$:*

(1) *If (X, B) is* 1*-log-terminal at x, then $f|_D : D \to \llcorner B \lrcorner$ is an isomorphism over an open neighborhood of x.*

(2) *If $x \in \operatorname{Sing} \llcorner B \lrcorner$ and if $f^{-1}(x)$ is contained in $\llcorner B_f \lrcorner$, then f is a toroidal blowing up with respect to $(X, \llcorner B \lrcorner)$ over an open neighborhood of x.*

Proof. (1): By shrinking X, we may assume that (X, B) is 1-log-terminal and that $D = \llcorner B_f \lrcorner$ by Lemma 4.22 and Definition 2.1. Then D is just the proper transform of $\llcorner B \lrcorner$ in Y, and the finite morphism $f|_D : D \to \llcorner B \lrcorner$ is an isomorphism by $\mathcal{O}_{\llcorner B \lrcorner} \simeq f_*\mathcal{O}_D$.

(2): By Lemma 2.6, $B = \llcorner B \lrcorner$ on an open neighborhood of x, since $x \in \operatorname{Sing} \llcorner B \lrcorner$. By shrinking X, we may assume that B is reduced, $\operatorname{Sing} X \subset \{x\}$, and f is an isomorphism outside $f^{-1}(x)$. Moreover, we may assume that $D = B_f$ and $\operatorname{Supp} D = (\operatorname{Supp} f^{[*]}B) \cup f^{-1}(x)$, since $\ulcorner B_f \urcorner$ is reduced and $f^{-1}(x) \subset \llcorner B_f \lrcorner$. In particular, $K_Y + D = f^*(K_X + B)$. Now, $K_X + B$ is Cartier (cf. Fact 2.5(1)). Thus, (Y, D) is log-canonical and $K_Y + D$ is Cartier, and it implies that (Y, D) is toroidal (cf. [41, Def. 3.12(2)], Fact 2.5). Therefore, f is a toroidal blowing up with respect to (X, B) (cf. [41, Def. 4.19]). □

Definition 4.24. Let (X, S) be a log-canonical pair of a normal surface X and a reduced divisor S. A bimeromorphic morphism $f : Y \to X$ from a normal surface Y is called an *essential blowing up* of (X, S) if $K_Y + S_Y = f^*(K_X + S)$ for a reduced divisor S_Y such that

- the f-exceptional locus is contained in S_Y, and
- (Y, S_Y) is 1-log-terminal on $Y \setminus \operatorname{Sing} S_Y$.

In this case, we say also that $f : (Y, S_Y) \to (X, S)$ is an essential blowing up. Furthermore, if $S = 0$, then X has only log-canonical singularities, and we call f an essential blowing up of X.

Remark. The pair (Y, S_Y) is log-canonical (cf. Lemma 2.10(1)), and S_Y is the union of $f^{-1}S$ and the f-exceptional locus, since $f_*S_Y = S$. If $(X, S = 0)$ is log-terminal, then any essential blowing up of X is an isomorphism.

Remark. The referee pointed out that the essential blowing up is very similar to the *dlt modification* (cf. [63, Def. 2.4]) for 2-dimensional log-canonical pairs. Since dlt is not analytically local (cf. Remark 2.3), the dlt modification does not cover the case of essential blowing up $(Y, S_Y) \to (X, S)$ in which Y is non-singular and S_Y contains a nodal rational curve (e.g. Example 4.29(3) below).

Lemma 4.25. *For a normal surface X with a reduced divisor S, assume that (X, S) is log-canonical and that (X, S) is 1-log-terminal outside* $\operatorname{Sing} S$. *Let $f\colon Y \to X$ be a bimeromorphic morphism from a normal surface Y. Then the following conditions are equivalent*:

(i) *f is an essential blowing up of (X, S);*
(ii) *f is a toroidal blowing up with respect to (X, S);*
(iii) *there is a reduced divisor S_Y on Y such that $K_Y + S_Y = f^*(K_X + S)$ and that S_Y contains the f-exceptional locus.*

Proof. We have (i) ⇒ (iii) by Definition 4.24. Assume (iii). Then any f-exceptional prime divisor is contracted to a point of $\operatorname{Sing} S$, since it is contained in S_Y and since (X, S) is 1-log-terminal outside $\operatorname{Sing} S$. Thus, f is an isomorphism over $X \setminus \operatorname{Sing} S$, and (ii) holds by Lemma 4.23(2).

Next assume (ii). Then $K_Y+S_Y = f^*(K_X+S)$, where $S_Y := f^{-1}S$ contains the f-exceptional locus. For a point $x \in X$, if $f^{-1}(x)$ is not a point, then (Y, S_Y) is toroidal along $f^{-1}(x)$, and (Y, S_Y) is 1-log-terminal along $f^{-1}(x) \setminus \operatorname{Sing} S_Y$. Hence, (Y, S_Y) is 1-log-terminal outside $\operatorname{Sing} S_Y$, since (X, S) is so outside $\operatorname{Sing} S$. This proves (ii) ⇒ (i). Thus, we are done. □

Lemma 4.26. *For a log-canonical pair (X, S) of a normal surface X and a reduced divisor S, let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface M such that the union of $\mu^{-1}S$ and the μ-exceptional locus is a normal crossing divisor. Let B_μ and T_μ be effective $\mathbb{Q}$-divisors on M without common prime components such that $K_M + B_\mu = \mu^*(K_X + S) + T_\mu$. Let $\sigma\colon M \to Y$ be the contraction morphism of all the μ-exceptional prime divisors not contained in $\llcorner B_\mu \lrcorner$. Let $f\colon Y \to X$ be the induced morphism such that $\mu = f \circ \sigma$, and set $S_Y := \sigma_* B_\mu$. Then $f\colon (Y, S_Y) \to (X, S)$ is an essential blowing up.*

Proof. The divisor $\ulcorner B_\mu \urcorner$ is reduced (cf. Lemma 2.10(1)). Since T_μ is σ-exceptional, by applying σ_* to $K_M + B_\mu = \mu^*(K_X + S) + T_\mu$, we have $K_Y + S_Y = f^*(K_X + S)$. Then (Y, S_Y) is also log-canonical (cf. Lemma 2.10(1)) and $S_Y = \sigma_* B_\mu$ is reduced. We set $D := \llcorner B_\mu \lrcorner$. Then $D = \sigma^{[*]} S_Y$ by construction, and $\sigma|_D\colon D \to S_Y$ is an isomorphism by Lemma 4.22 applied to σ and to the equality $K_M + B_\mu = \sigma^*(K_Y + S_Y) + T_\mu$ (cf. the proof of Lemma 4.23(1)). In particular, $\sigma(\operatorname{Sing} D) = \operatorname{Sing} S_Y$. Hence, (Y, S_Y) is 1-log-terminal on the open subset $U := Y \setminus \operatorname{Sing} Y$ by Proposition 2.12(2), since (M, B_μ) is 1-log-terminal on $\sigma^{-1}U$. Moreover, the f-exceptional locus is contained in $\sigma(D) = S_Y$, since the image of the μ-exceptional locus under σ is contained in the union of $\sigma(D)$ and a finite set. Therefore, f is an essential blowing up. □

Definition 4.27. The essential blowing up $(Y, S_Y) \to (X, S)$ in Lemma 4.26 is called the *standard partial resolution* if $\mu\colon M \to X$ is the minimal resolution of singularities.

Note that the union of $\mu^{-1}S$ and the μ-exceptional locus is normal crossing for the minimal resolution μ (cf. [30, Thm. 9.6]). We shall give local descriptions of standard partial resolutions in Examples 4.28 and 4.29 below:

Example 4.28. Let (X, S) be a log-canonical pair of a normal surface X and a reduced divisor S. Assume that $\operatorname{Sing} X = \{x\}$, $\operatorname{Sing} S \subset \{x\}$, and $x \in S$. Let $f\colon (Y, S_Y) \to (X, S)$ be the standard partial resolution, S' the proper transform $f^{[*]}S$ in Y, and C the exceptional divisor $f^{-1}(x)$. If $x \in \operatorname{Sing} S$, then (X, S) is toroidal at x by Fact 2.5(1), and hence:

- f is the minimal resolution of singularities;
- C is a *linear chain of rational curves* (cf. [41, Def. 4.1]);
- S' intersects C only at two points in C_{reg}, the intersection is transversal, and when C is reducible, each end component of C contains just one intersection point.

If $x \in S_{\mathrm{reg}}$ and (X, S) is 1-log-terminal at x, then, by Lemma 4.25, f is an isomorphism. Assume that $x \in S_{\mathrm{reg}}$ and (X, S) is not 1-log-terminal at x. Then the local description of (X, S) at x as in Fact 2.5(3). For the minimal resolution of singularities of X, the dual graph of the union of the exceptional locus and the inverse image of S is well known (cf. [30, Thm. 9.6(6)], [35, Ch. 3], [41, Thm 3.22(iii), Fig. 2]). As a consequence, the following hold:

- C is a linear chain $\sum_{i=1}^{k} C_i$ of rational curves;
- S' intersects C only at one point in $Y_{\mathrm{reg}} \cap C_1 \cap C_{\mathrm{reg}}$ for an end component C_1 of C, and the intersection is transversal;
- $\operatorname{Sing} Y$ consists of two A_1-singular points contained in C_{reg}, and when $k > 1$, these points are contained in the other end component C_k of C.

Example 4.29. Let X be a normal surface with a point $x \in X$ such that $(X, 0)$ is log-canonical and $\operatorname{Sing} X = \{x\}$. By the classification of 2-dimensional log-canonical singularities (cf. [55, App.], [30, Thm. 9.6], [35, Ch. 3]), the standard partial resolution $f\colon (Y, S_Y) \to (X, 0)$ is described as follows:

(1) If (X, x) is a quotient singularity, then f is an isomorphism.
(2) If (X, x) is a simple elliptic singularity, then f is the minimal resolution of singularities, and S_Y is an elliptic curve.
(3) If (X, x) is a cusp singularity, then f is the minimal resolution of singularities, and S_Y is a *cyclic chain of rational curves* (cf. [41, Def. 4.3]).
(4) If (X, x) is a rational singularity and its *index* 1 *cover* with respect to K_X (cf. Definition 4.18(4)) is a simple elliptic singularity, then S_Y is a non-singular rational curve, and $\operatorname{Sing} Y$ consists of three or four cyclic quotient singular points contained in S_Y.
(5) If (X, x) is a rational singularity and its index 1 cover with respect to K_X is a cusp singularity, then S_Y is a reducible linear chain of rational curves, $\operatorname{Sing} Y$ consists of four A_1-singular points contained in $(S_Y)_{\mathrm{reg}}$, and each end component of S_Y contains exactly two A_1-singular points.

Definition 4.30. Let Γ be a prime component of a reduced divisor S on a normal surface. We define $\boldsymbol{v}(\Gamma/S) := \#\Gamma \cap (S - \Gamma)$.

Lemma 4.31. *Let $f\colon (Y, S_Y) \to (X, S)$ be an essential blowing up of a log-canonical pair (X, S) of a normal surface X and a reduced divisor S. Let $\sigma\colon Z \to Y$ be a non-isomorphic bimeromorphic morphism from another normal surface Z with a reduced divisor S_Z such that S_Z contains the $f \circ \sigma$-exceptional locus and that $K_Z + S_Z = \sigma^*(K_Y + S_Y)$. Then*:

(1) *The composite $f \circ \sigma\colon (Z, S_Z) \to (X, S)$ is an essential blowing up, and $\sigma\colon (Z, S_Z) \to (Y, S_Y)$ is a toroidal blowing up with respect to (Y, S_Y).*
(2) *For any non-singular prime component Γ of S_Y and for the proper transform $\sigma^{[*]}\Gamma$ in Z, one has $\boldsymbol{v}(\Gamma/S_Y) = \boldsymbol{v}(\sigma^{[*]}\Gamma/S_Z)$.*
(3) *For any σ-exceptional prime divisor Θ, one has $\boldsymbol{v}(\Theta/S_Z) = 2$.*

Proof. By Lemma 4.25, σ is a toroidal blowing up with respect to (Y, S_Y) and is also an essential blowing up of (Y, S_Y). In particular, (Z, S_Z) is 1-log-terminal outside $\operatorname{Sing} S_Z$. This proves (1). Assertions (2) and (3) are deduced from properties of a toroidal blowing up. □

Lemma 4.32. *Let (X, S) be a log-canonical pair of a normal surface X and a reduced divisor S. For two essential blowings up $f_1\colon (Y_1, S_1) \to (X, S)$ and $f_2\colon (Y_2, S_2) \to (X, S)$, there exists an essential blowing up $f_3\colon (Y_3, S_3) \to (X, S)$ such that $f_i^{-1} \circ f_3\colon Y_3 \to Y_i$ is holomorphic and is a toroidal blowing up with respect to (Y_i, S_i) for any $i = 1, 2$.*

Proof. We can take a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface M such that the union of $\mu^{-1}S$ and the μ-exceptional locus is a normal crossing divisor and that $\nu_i := f_i^{-1} \circ \mu\colon M \to Y_i$ is holomorphic for any $i = 1, 2$. Let B_μ and T_μ be effective $\mathbb{Q}$-divisors on M without common prime components such that $K_M + B_\mu = \mu^*(K_X + S) + T_\mu$. For each $i = 1, 2$,

$$K_M + B_\mu = \nu_i^*(K_{Y_i} + S_i) + T_\mu,$$

and $\langle B_\mu \rangle + T_\mu$ is ν_i-exceptional, since f_i is an essential blowing up of (X, S). Let $\nu_3\colon M \to Y_3$ be the contraction morphism of all the prime divisors exceptional for both ν_1 and ν_2. Let $f_3\colon Y_3 \to X$ be the induced morphism such that $\mu = f_3 \circ \nu_3$. Then we have a commutative diagram

$$\begin{tikzcd}
& & Y_1 \arrow[dr, "f_1"] & \\
M \arrow[r, "\nu_3"] \arrow[urr, bend left, "\nu_1"] \arrow[drr, bend right, "\nu_2"'] & Y_3 \arrow[ur, "\sigma_1"] \arrow[rr, "f_3"] \arrow[dr, "\sigma_2"'] & & X \\
& & Y_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

of bimeromorphic morphisms. Now, $K_{Y_3} + S_3 = f_3^*(K_X + S)$ for the reduced divisor $S_3 := \nu_{3*}B_\mu = \nu_{3*}\llcorner B_\mu \lrcorner$, since $\langle B_\mu \rangle + T_\mu$ is ν_3-exceptional. Hence,

$$K_{Y_3} + S_3 = \sigma_i^*(K_{Y_i} + S_i) \tag{IV-7}$$

for any $i = 1, 2$. Here, $\sigma_i(S_3) \subset S_i$, since $Y_i \setminus S_i$ has only log-terminal singularities, and the induced morphism $\sigma_i|_{S_3}\colon S_3 \to S_i$ is an isomorphism over $S_i \setminus \operatorname{Sing} S_i$ by Lemma 4.23(1).

Hence, $S_i = \sigma_i(S_3)$ for any $i = 1, 2$.

Let Γ be an f_3-exceptional prime divisor on Y_3. Then $\sigma_i(\Gamma)$ is a prime divisor for $i = 1$ or 2, and in this case, $\sigma_i(\Gamma)$ is contained in the f_i-exceptional locus; thus, $\sigma_i(\Gamma) \subset S_i$. Here, the proper transform Γ of $\sigma_i(\Gamma)$ is contained in S_3 by $S_i = \sigma_i(S_3)$. Hence, S_3 contains the f_3-exceptional locus. Therefore, $\sigma_i \colon (Y_3, S_3) \to (Y_i, S_i)$ is a toroidal blowing up for any $i = 1$, 2, and $f_3 \colon (Y_3, S_3) \to (X, S)$ is an essential blowing up, by Lemmas 4.25 and 4.31. □

Corollary 4.33. *Let $f \colon (Y, S_Y) \to (X, S)$ be an essential blowing up of a log-canonical pair (X, S) of a normal surface X and a reduced divisor S.*

(1) *If an f-exceptional prime divisor Γ is non-singular, then $\boldsymbol{v}(\Gamma/S_Y) \leq 2$.*

(2) *Let Γ be a non-singular prime component of S_Y such that $\boldsymbol{v}(\Gamma/S_Y) \neq 2$. Then Γ is not contracted to a point by the meromorphic map $g^{-1} \circ f \colon Y \cdots\!\to Z$ for any essential blowing up $g \colon (Z, S_Z) \to (X, S)$, i.e., the proper transform of Γ in Z is a prime component of S_Z.*

(3) *If every f-exceptional prime divisor Γ is non-singular and satisfies $\boldsymbol{v}(\Gamma/S_Y) \leq 1$, then, for any essential blowing up $g \colon (Z, S_Z) \to (X, S)$, there is a toroidal blowing up $h \colon (Z, S_Z) \to (Y, S_Y)$ such that $g = f \circ h$.*

Proof. Let us take an arbitrary essential blowing up $g \colon (Z, S_Z) \to (X, S)$ and let $f_1 \colon (Y_1, S_1) \to (X, S)$ be the standard partial resolution. By Lemma 4.32, we have an essential blowing up $f_2 \colon (Y_2, S_2) \to (X, S)$ with a commutative diagram

$$\begin{tikzcd} & Y_1 \arrow[dr, "f_1"] & \\ Y_2 \arrow[ur, "\sigma_1"] \arrow[r, "\sigma"] \arrow[dr, "\tau"'] & Y \arrow[r, "f"] & X \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

of bimeromorphic morphisms such that $f_2 = f \circ \sigma$ and that $\sigma_1 \colon (Y_2, S_2) \to (Y_1, S_1)$, $\sigma \colon (Y_2, S_2) \to (Y, S_Y)$, and $\tau \colon (Y_2, S_2) \to (Z, S_Z)$ are toroidal blowings up.

Let Γ be a non-singular prime component of S_Y. Then the proper transform $\Gamma'' = \sigma^{[*]}\Gamma$ in Y_2 is also non-singular and $\boldsymbol{v}(\Gamma/S_Y) = \boldsymbol{v}(\Gamma''/S_2)$ by Lemma 4.31(2). If $\boldsymbol{v}(\Gamma''/S_2) \neq 2$, then Γ'' is not exceptional for both τ and σ_1 by Lemma 4.31(3). This shows (2). Assume that Γ is f-exceptional and that $\Gamma' = \sigma_1(\Gamma'')$ is a divisor, which is a prime component of S_1. If Γ' is non-singular, then $\boldsymbol{v}(\Gamma'/S_1) = \boldsymbol{v}(\Gamma''/S_2)$ by Lemma 4.31(2), and we have $\boldsymbol{v}(\Gamma'/S_1) \leq 2$ by Examples 4.28 and 4.29. If Γ' is singular, then $f(\Gamma) = f_1(\Gamma') \notin S$, X has a cusp singularity at $f(\Gamma)$, and Γ' is a nodal rational curve being a connected component of S_1, by Examples 4.28 and 4.29; in this case, $\boldsymbol{v}(\Gamma''/S_2) = 2$, since σ_1 is a toroidal blowing up with respect to (Y_1, S_1) and is not an isomorphism over the node of Γ' as Γ'' is non-singular. Therefore, $\boldsymbol{v}(\Gamma/S_Y) \leq 2$ for the both cases of Γ', and we have proved (1).

The remaining assertion (3) is deduced from (2). In fact, any f-exceptional prime divisor is not contracted to a point by the meromorphic map $\tau \circ \sigma^{-1} \colon Y \cdots\!\to Z$ by (2). Thus, every τ-exceptional divisor is σ-exceptional, and hence, $h := \sigma \circ \tau^{-1} \colon Z \to Y$ is holomorphic. This implies (3) by Lemma 4.25. □

Lemma 4.34. *Let (X, S) and (X', S') be log-canonical pairs of normal surfaces X and X' and reduced divisors S and S', respectively. Let $\tau\colon X' \to X$ be a morphism with only discrete fibers such that $S' = \tau^{-1}S$ and that $\tau|_{X'\setminus S'}\colon X' \setminus S' \to X \setminus S$ is étale in codimension 1. For an essential blowing up $f\colon (Y, D) \to (X, S)$, let Y' be the normalization of $Y \times_X X'$ with the induced commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ Y & \xrightarrow{f} & X. \end{array}$$

Then $f'\colon (Y', D') \to (X', S')$ is an essential blowing up for $D' := \sigma^{-1}D$, $\sigma\colon Y' \to Y$ is a morphism with only discrete fibers, and the induced morphism $Y' \setminus D' \to Y \setminus D$ is étale in codimension 1.

Proof. Note that $X' \times_X Y$ is irreducible and generically reduced by Lemma 1.13. Then σ has only discrete fibers, and it is étale in codimension 1 outside D, since D contains the f-exceptional locus and since τ is étale in codimension 1 outside S. The f'-exceptional locus is just the inverse image by σ of the f-exceptional locus, since σ and τ have only discrete fibers. Thus, $D' = \sigma^{-1}D$ contains the f'-exceptional locus. We have $K_{X'} + S' = \tau^*(K_X + S)$ and $K_{Y'} + D' = \sigma^*(K_Y + D)$ by Lemma 1.39, and moreover, $K_Y + D = f^*(K_X + S)$, since f is an essential blowing up. Hence, $K_{Y'} + D' = f'^*(K_{X'} + S')$. In particular, (Y', D') is log-canonical, and it is 1-log-terminal outside $\sigma^{-1}\operatorname{Sing} D$ by Lemma 2.10.

By Definition 4.24, it suffices to prove that $\sigma^{-1}\operatorname{Sing} D \subset \operatorname{Sing} D'$. For a point $y' \in \sigma^{-1}\operatorname{Sing} D$, by Corollary 1.8, we have an open neighborhood $\mathcal{U}'$ of y' in Y' such that $\mathcal{U} := \sigma(\mathcal{U}')$ is open and $\sigma_{\mathcal{U}} := \sigma|_{\mathcal{U}'}\colon \mathcal{U}' \to \mathcal{U}$ is finite and surjective. By shrinking $\mathcal{U}$, we may assume that $D|_{\mathcal{U}} = \Gamma_1 + \Gamma_2$ for two distinct prime divisors Γ_1 and Γ_2 and that $\sigma(y') \in \Gamma_1 \cap \Gamma_2$. Then $\sigma^*D|_{\mathcal{U}'} = \sigma_{\mathcal{U}}^*\Gamma_1 + \sigma_{\mathcal{U}}^*\Gamma_2$ and $y' \in \sigma_{\mathcal{U}}^{-1}\Gamma_1 \cap \sigma_{\mathcal{U}}^{-1}\Gamma_2$, where $\sigma_{\mathcal{U}}^*\Gamma_1$ and $\sigma_{\mathcal{U}}^*\Gamma_2$ have no common prime component, since $\sigma_{\mathcal{U}}$ is surjective. Hence, $y' \in \operatorname{Sing}\sigma^{-1}D$. This shows $\sigma^{-1}\operatorname{Sing} D \subset \operatorname{Sing} D'$, and we are done. □

4.4. Dual $\mathbb{R}$-divisors. We fix a normal surface X and a non-zero reduced connected compact divisor S on X such that the intersection matrix of prime components of S is negative definite; in other words, S is the inverse image of a point by a bimeromorphic morphism $X \to \overline{X}$ to a normal surface $\overline{X}$, by the contraction criterion (cf. [13, (e), page 366–367] and [52, Thm. (1.2)]). We shall introduce *primitive dual $\mathbb{Q}$-divisors* and *dual $\mathbb{R}$-divisors* for a prime component of S and study their basic properties.

Lemma-Definition 4.35. *Let Γ be a prime component of S.*

(1) *There is a unique $\mathbb{Q}$-divisor $\boldsymbol{D}(\Gamma/S)$ on X supported on S such that*

$$\operatorname{mult}_\Gamma A = \boldsymbol{D}(\Gamma/S)A$$

for any divisor A supported on S. We call $\boldsymbol{D}(\Gamma/S)$ the primitive dual $\mathbb{Q}$-divisor *of Γ with respect to S.*

(2) *For an effective $\mathbb{R}$-divisor H on X such that* $\operatorname{Supp} H = S$*, we define*

$$\boldsymbol{\Delta}(\Gamma, H) := -(\operatorname{mult}_\Gamma H)^{-1}\boldsymbol{D}(\Gamma/S)$$

and call it the dual $\mathbb{R}$-divisor *of* Γ *with respect to* H.

The following hold for $\boldsymbol{D}(\Gamma/S)$ *and* $\boldsymbol{\Delta}(\Gamma, H)$:

(3) *The* $\mathbb{Q}$*-divisor* $-\boldsymbol{D}(\Gamma/S)$ *is effective and* $\operatorname{Supp} \boldsymbol{D}(\Gamma/S) = S$.

(4) *If* Γ' *is a prime component of* $S - \Gamma$*, then* $\boldsymbol{D}(\Gamma/S)\Gamma' = 0$. *Moreover,*

$$A = \sum\nolimits_{\Gamma \subset S} (A\Gamma)\boldsymbol{D}(\Gamma/S).$$

for any $\mathbb{R}$*-divisor* A *supported on* S.

(5) *For any effective* $\mathbb{R}$*-divisor* H *on* X *such that* $\operatorname{Supp} H = S$*, the* $\mathbb{R}$*-divisor* $\boldsymbol{\Delta}(\Gamma, H)$ *is effective,* $\operatorname{Supp} \boldsymbol{\Delta}(\Gamma, H) = S$*,* $-\boldsymbol{\Delta}(\Gamma, H)$ *is nef on* S*, and* $\boldsymbol{\Delta}(\Gamma, H)H = -1$.

Proof. Since the intersection matrix of S is definite, the $\mathbb{Q}$-divisor $\boldsymbol{D}(\Gamma/S)$ satisfying (1) exists uniquely, and we have (4). Since $\boldsymbol{D}(\Gamma/S)$ is nef on S, we have (3) by Remark 1.25. Assertion (5) is deduced from (3) and (4). □

Lemma 4.36. *Let* $\pi \colon Y \to X$ *be a bimeromorphic morphism from a normal surface* Y*, and set* $S_Y := \pi^{-1}S$. *Let* H_Y *be an* $\mathbb{R}$*-divisor on* Y *such that* $\operatorname{Supp} H_Y = S_Y$*, and set* $H := \pi_* H_Y$. *Then, for any prime component* Γ *of* S *and its proper transform* $\pi^{[*]}\Gamma$ *in* Y*, one has*

$$\pi^* \boldsymbol{D}(\Gamma/S) = \boldsymbol{D}(\pi^{[*]}\Gamma/S_Y) \quad \textit{and} \quad \pi^* \boldsymbol{\Delta}(\Gamma, H) = \boldsymbol{\Delta}(\pi^{[*]}\Gamma, H_Y).$$

Proof. Note that S_Y is compact and connected, the intersection matrix of prime components of S_Y is also negative definite, and $\operatorname{Supp} H = S$. For any π-exceptional prime divisor E, we have $\boldsymbol{D}(\pi^{[*]}\Gamma/S_Y)E = 0$ by Lemma-Definition 4.35(4), since either $E \cap S_Y = \emptyset$ or $E \subset S_Y$. Thus, $\boldsymbol{D}(\pi^{[*]}\Gamma/S_Y) = \pi^* D$ for the pushforward $D := \pi_* \boldsymbol{D}(\pi^{[*]}\Gamma/S_Y)$. Then

$$D\Gamma^\dagger = (\pi^* D)\pi^{[*]}\Gamma^\dagger = \boldsymbol{D}(\pi^{[*]}\Gamma/S_Y)\pi^{[*]}\Gamma^\dagger = \begin{cases} 1, & \text{if} \quad \Gamma^\dagger = \Gamma, \\ 0, & \text{otherwise,} \end{cases}$$

for any prime component $\Gamma^\dagger$ of S, and $D = \boldsymbol{D}(\Gamma/S)$ by Lemma-Definition 4.35(1). Thus, we have the first equality. The second equality follows from the first one by Lemma-Definition 4.35(2), since $\operatorname{mult}_{\pi^{[*]}\Gamma} H_Y = \operatorname{mult}_\Gamma H$. □

We have the following generalization of the first equality in Lemma 4.36:

Lemma 4.37. *Let* $\pi \colon Y \to X$ *be a non-degenerate morphism from a normal surface* Y *such that* $S_Y := \pi^{-1}S$ *is compact. Let* Θ *be a prime component of* S_Y. *Then*

$$\pi_* \boldsymbol{D}(\Theta/S_Y) = \sum\nolimits_{\pi(\Theta) \subset \Gamma \subset S} (\operatorname{mult}_\Theta \pi^*\Gamma)\boldsymbol{D}(\Gamma/S).$$

In particular, if $\pi(\Theta)$ *is a prime divisor* Γ*, then*

$$\pi_* \boldsymbol{D}(\Theta/S_Y) = (\operatorname{mult}_\Theta \pi^*\Gamma)\boldsymbol{D}(\Gamma/S).$$

Conversely, for any prime component Γ *of* S*, one has*

$$\pi^* \boldsymbol{D}(\Gamma/S) = \sum\nolimits_{\Gamma \subset \pi(\Theta)} (\operatorname{mult}_\Gamma \pi_*\Theta)\boldsymbol{D}(\Theta/S_Y).$$

Proof. For any prime component Γ of S, we have

$$(\pi_* \boldsymbol{D}(\Theta/S_Y))\Gamma = \boldsymbol{D}(\Theta/S_Y)\pi^*\Gamma = \operatorname{mult}_\Theta \pi^*\Gamma$$

by Lemma-Definition 4.35(1). This implies the first equality, since $\operatorname{mult}_\Theta \pi^*\Gamma \neq 0$ if and only if $\pi(\Theta) \subset \Gamma$. The second equality is a special case of the first one. The third equality is deduced from equalities

$$(\pi^* \boldsymbol{D}(\Gamma/S))\Theta = \boldsymbol{D}(\Gamma/S)\pi_*\Theta = \operatorname{mult}_\Gamma \pi_*\Theta$$

and from Lemma-Definition 4.35(4). □

The following result almost corresponds to the last assertion of [6, Prop. 1.4].

Proposition 4.38. *Assume that (X, S) is log-canonical and let H be an effective $\mathbb{R}$-divisor on X such that* $\operatorname{Supp} H = S$. *Then there exist positive rational numbers $c_1 < c_2$ depending only on (X, S, H) such that*

$$c_1\pi^*H \leq \boldsymbol{\Delta}(\Theta, \pi^*H) \leq c_2\pi^*H \tag{IV-8}$$

for any non-degenerate morphism $\pi\colon Y \to X$ from a normal surface Y and any prime component Θ of $S_Y := \pi^{-1}S$ satisfying the following conditions:

(i) *$\pi(Y)$ is an open neighborhood of S, and $\pi\colon Y \to \pi(Y)$ is a bimeromorphic morphism inducing an isomorphism $Y \setminus S_Y \simeq \pi(Y) \setminus S$*;

(ii) $\operatorname{mult}_\Theta \Delta_\pi = 0$ *for the $\mathbb{Q}$-divisor Δ_π defined by $K_Y + S_Y = \pi^*(K_X + S) + \Delta_\pi$.*

Proof. We shall prove the assertion by three steps.

STEP 1. We shall reduce the assertion to the following two cases of (π, Θ):

(1) π is the identity morphism;

(2) $\pi(Y) = X$ and the exceptional locus of π equals the prime component Θ.

Note that in case (2), we have $\Delta_\pi = 0$ by $\operatorname{mult}_\Theta \Delta_\pi = 0$. Let c_1 and c_2 be positive rational numbers such that (IV-8) holds only in cases (1) and (2). Let $(\pi\colon Y \to X, \Theta)$ be an arbitrary pair satisfying (i) and (ii). First, assume that Θ is not π-exceptional. Then $\Theta = \pi^{[*]}\Gamma$ for a prime component Γ of S, and we have

$$\boldsymbol{\Delta}(\Theta, \pi^*H) = \pi^*\boldsymbol{\Delta}(\Gamma, H)$$

by Lemma 4.36 applied to the bimeromorphic morphism $Y \to \pi(Y)$. Hence, (IV-8) for this (π, Θ) is deduced from that for (id_X, Γ). Second, assume that Θ is π-exceptional and let $\varphi\colon Y \to \overline{Y}$ be the contraction morphism of the union of π-exceptional prime divisors except Θ. Then $\pi = \bar{\pi} \circ \varphi$ for a morphism $\bar{\pi}\colon \overline{Y} \to X$ satisfying (i), the $\bar{\pi}$-exceptional locus is $\overline{\Theta} := \varphi(\Theta)$, and

$$\boldsymbol{\Delta}(\Theta, \pi^*H) = \varphi^*\boldsymbol{\Delta}(\overline{\Theta}, \bar{\pi}^*H)$$

by Lemma 4.36. We can construct a bimeromorphic morphism $\hat{\pi}\colon \widehat{Y} \to X$ with an isomorphism $\hat{\pi}^{-1}(\pi(Y)) \simeq Y$ over X by gluing $Y \to \pi(Y)$ and the identity morphism of $X \setminus S$. Then $\widehat{\Theta} = \overline{\Theta}$ and $\hat{\pi}^*H = \bar{\pi}^*H$ are regarded as $\mathbb{Q}$-divisors on $\widehat{Y}$, and we have

$$\boldsymbol{\Delta}(\widehat{\Theta}, \hat{\pi}^*H) = \boldsymbol{\Delta}(\overline{\Theta}, \bar{\pi}^*H).$$

Thus, (IV-8) for (π, Θ) is deduced from that for $(\hat{\pi}, \widehat{\Theta})$. Therefore, it is enough to prove the assertion only in the cases (1) and (2).

STEP 2. We shall reduce the assertion to the case where X is non-singular and S is a simple normal crossing divisor. Since the assertion is on $\mathbb{R}$-divisors lying over S, we may replace X with an open neighborhood of S freely. Thus, we may assume that $X \setminus S$ is non-singular. There is a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface M such that $S_M := \mu^{-1}S$ is a simple normal crossing divisor and that μ is an isomorphism over $X \setminus S$. Then the $\mathbb{Q}$-divisor Δ_μ defined by $K_M + S_M = \mu^*(K_X + S) + \Delta_\mu$ is effective as (X, S) is log-canonical. Assume that the assertion holds for (M, S_M, μ^*H) instead of (X, S, H), i.e., the inequality corresponding to (IV-8) holds for (M, S_M, μ^*H) for some c_1 and c_2. By Step 1, it is enough to verify (IV-8) for (π, Θ) such that π is a bimeromorphic morphism, Θ is the exceptional locus of π, and $\Delta_\pi = 0$. Then (Y, S_Y) is log-canonical by $K_Y + S_Y = \pi^*(K_X + S)$ (cf. Lemma 2.10(1)). We can find a bimeromorphic morphism $\nu\colon N \to Y$ from a non-singular surface N and a bimeromorphic morphism $\phi\colon N \to M$ such that ν is an isomorphism over $Y \setminus S_Y$, ϕ is an isomorphism over $M \setminus S_M$, and the diagram

$$\begin{tikzcd} N \arrow[r, "\nu"] \arrow[d, "\phi"'] & Y \arrow[d, "\pi"] \\ M \arrow[r, "\mu"] & X \end{tikzcd}$$

is commutative. Then

$$\mathbf{\Delta}(\nu^{[*]}\Theta, \nu^*(\pi^*H)) = \nu^*\mathbf{\Delta}(\Theta, \pi^*H)$$

by Lemma 4.36. We set $S_N := \phi^{-1}S_M = \nu^{-1}S_Y$, and let Δ_ϕ and Δ_ν be $\mathbb{Q}$-divisors defined by

$$K_N + S_N = \phi^*(K_M + S_M) + \Delta_\phi \quad \text{and} \quad K_N + S_N = \nu^*(K_Y + S_Y) + \Delta_\nu.$$

Then Δ_ϕ is ϕ-exceptional and effective, and Δ_ν is ν-exceptional and effective, as (M, S_M) and (Y, S_Y) are log-canonical. Moreover, we have

$$\phi^*\Delta_\mu + \Delta_\phi = \Delta_\nu + \nu^*\Delta_\pi = \Delta_\nu.$$

Thus, $\nu^{[*]}\Theta \not\subset \operatorname{Supp}\Delta_\phi$ and $\phi(\nu^{[*]}\Theta) \not\subset \operatorname{Supp}\Delta_\mu$. As an inequality corresponding to (IV-8) for (M, S_M, π^*H), we have

$$c_1\phi^*(\mu^*H) \leq \mathbf{\Delta}(\nu^{[*]}\Theta, \phi^*(\mu^*H)) \leq c_2\phi^*(\mu^*H).$$

Applying ν_* to it, we have

$$c_1\pi^*H \leq \mathbf{\Delta}(\Theta, \pi^*H) \leq c_2\pi^*H$$

by Lemma 4.36, since $\phi^*(\mu^*H) = \nu^*(\pi^*H)$. Therefore, for the proof, we may replace (X, S, H) with (M, S_M, μ^*H).

STEP 3. The final step. We may assume that X is non-singular and S is a simple normal crossing divisor by Step 2. Since S has only finitely many prime components, we have positive rational numbers $c_1^0 < c_2^0$ satisfying

$$c_1^0 H \leq \mathbf{\Delta}(\Gamma, H) \leq c_2^0 H \tag{IV-9}$$

for any prime component Γ of S. We shall show that rational numbers $c_1 = c_1^0$ and $c_2 > c_2^0 + (2h^2)^{-1}$ satisfy the inequality (IV-8) for

$$h := \min\{\mathrm{mult}_\Gamma H \mid \Gamma \text{ is a prime component of } S\}.$$

By Step 1, it is enough to verify (IV-8) in the case where $\pi\colon Y \to X$ is a bimeromorphic morphism, Θ is the exceptional locus of π, and $\Delta_\pi = 0$. Since $K_Y + S_Y = \pi^*(K_X + S)$, the pair (Y, S_Y) is log-canonical and π is a toroidal blowing up at the node $x := \pi(\Theta)$ of S. Hence, $x \in \Gamma_1 \cap \Gamma_2$ for two prime components Γ_1, Γ_2 of S, and $\pi^{[*]}\Gamma_1 \cap \pi^{[*]}\Gamma_2 \cap \Theta = \emptyset$. Therefore, $x \notin \pi(\pi^{[*]}\Gamma_1 \cap \pi^{[*]}\Gamma_2)$, and

$$\Gamma_1\Gamma_2 = (\pi^{[*]}\Gamma_1)\pi^{[*]}\Gamma_2 + 1. \tag{IV-10}$$

For $i = 1, 2$, we set $a_i := \mathrm{mult}_\Theta \pi^*\Gamma_i \in \mathbb{Q}$, i.e., $\pi^*\Gamma_i = \pi^{[*]}\Gamma_i + a_i\Theta$. Then

$$(\pi^{[*]}\Gamma_1)\Theta = a_2^{-1}, \quad (\pi^{[*]}\Gamma_2)\Theta = a_1^{-1}, \quad \text{and} \quad \Theta^2 = -(a_1a_2)^{-1}. \tag{IV-11}$$

In fact, the second equality of (IV-11) is obtained by calculation

$$\Gamma_1\Gamma_2 = (\pi^*\Gamma_1)\pi^{[*]}\Gamma_2 = (\pi^{[*]}\Gamma_1)\pi^{[*]}\Gamma_2 + a_1\Theta\pi^{[*]}\Gamma_2 = \Gamma_1\Gamma_2 - 1 + a_1\Theta\pi^{[*]}\Gamma_2$$

using (IV-10): We have the first equality by interchanging (Γ_1, a_1) and (Γ_2, a_2), and the third one by calculation

$$0 = a_2(\pi^*\Gamma_1)\Theta = a_2(\pi^{[*]}\Gamma_1)\Theta + a_1a_2\Theta^2 = 1 + a_1a_2\Theta^2$$

using the first equality. We set $h_i := \mathrm{mult}_{\Gamma_i} H$ for $i = 1, 2$, and $h_3 := \mathrm{mult}_\Theta \pi^*H$. Then $h_3 = a_1h_1 + a_2h_2$ and we have

$$\begin{aligned} h_3\pi_*\mathbf{\Delta}(\Theta, \pi^*H) &= -\pi_*\boldsymbol{D}(\Theta/S_Y) = -a_1\boldsymbol{D}(\Gamma_1/S) - a_2\boldsymbol{D}(\Gamma_2/S) \\ &= a_1h_1\mathbf{\Delta}(\Gamma_1, H) + a_2h_2\mathbf{\Delta}(\Gamma_2, H) \end{aligned}$$

by Lemma 4.37 and Lemma-Definition 4.35(2). Therefore,

$$c_1^0 H \le \pi_*\mathbf{\Delta}(\Theta, \pi^*H) \le c_2^0 H \tag{IV-12}$$

by (IV-9). For the rational number e defined by

$$\mathbf{\Delta}(\Theta, \pi^*H) = \pi^*(\pi_*\mathbf{\Delta}(\Theta, \pi^*H)) + e\Theta,$$

we have $e = a_1a_2/h_3 > 0$ by calculation

$$-1/h_3 = \mathbf{\Delta}(\Theta, \pi^*H)\Theta = e\Theta^2 = -e/(a_1a_2)$$

using Lemma-Definition 4.35(2) and (IV-11). Therefore,

$$c_1^0\pi^*H \le \mathbf{\Delta}(\Theta, \pi^*H) \le c_2^0\pi^*H + a_1a_2h_3^{-1}\Theta \le (c_2^0 + a_1a_2h_3^{-2})\pi^*H$$

by (IV-12) and by $h_3\Theta \le \pi^*H$. Here, $a_1a_2h_3^{-2} \le (2h^2)^{-1}$ by

$$h_3^2 = (a_1h_1 + a_2h_2)^2 \ge 2a_1a_2h_1h_2 \ge 2a_1a_2h^2.$$

Thus, we have the expected inequality (IV-8) for $c_1 = c_1^0$ and $c_2 > c_2^0 + (2h^2)^{-1}$, and we are done. □

5. Endomorphisms of normal surface singularities

The purpose of this section is to prove Theorem 5.3 below from which Theorem 0.2 is deduced directly. This is stated for two cases (I) and (II), in Section 5.1. The proof in the case (I) (resp. (II)) is given in Section 5.4 (resp. 5.2). In Section 5.3, we shall prove Theorem 5.10 which is a key to the proof in the case (I).

5.1. Setting and statement. Let $\mathfrak{X} = (X, x)$ be a germ of a normal surface X at a point x. We consider a non-isomorphic finite surjective endomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ of the germ. Then $\mathfrak{X}$ is a log-canonical singularity by Corollary 3.7. Note that $\mathfrak{f}$ is induced by a morphism $f\colon X^{\circ} \to X$ of normal surfaces from an open neighborhood X° of x such that f has only discrete fibers, $f^{-1}(x) = \{x\}$, and $\deg_x f > 1$ (cf. Definition 1.9, Remark 3.2). Here, we may assume that $\operatorname{Sing} X \subset \{x\}$.

REMARK 5.1. By assumption and by Corollary 1.8, there is an open neighborhood $\mathcal{U}$ of x in X° such that $\mathcal{V} = f(\mathcal{U})$ is open and $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V}$ is a finite morphism of degree $= \deg_x f > 1$.

REMARK 5.2. If $\mathfrak{X} = (X, x)$ is a 2-dimensional quotient singularity, then any finite endomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ étale outside x is an isomorphism (cf. [6, §2.1]). This is shown as follows: For morphisms $f\colon X^{\circ} \to X$ and $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V} = f(\mathcal{U})$ above, we may assume that $\mathcal{U} \setminus \{x\}$ is étale over $\mathcal{V} \setminus \{x\}$. Since (X, x) is a quotient singularity, by shrinking $\mathcal{U}$ and $\mathcal{V}$, we may assume that the fundamental group $\pi_1(\mathcal{V} \setminus \{x\})$ of $\mathcal{V} \setminus \{x\}$ is finite. Then $\deg \mathfrak{f}$ is just the index of the subgroup $\pi_1(\mathcal{U} \setminus \{x\})$. As a consequence, $\deg \mathfrak{f}$ is bounded. If $\deg \mathfrak{f} > 1$, then $\deg \mathfrak{f}^k = (\deg \mathfrak{f})^k$ is sufficiently large for $k \gg 0$ for the k-th power $\mathfrak{f}^k = \mathfrak{f} \circ \mathfrak{f} \circ \cdots \circ \mathfrak{f}$. Thus, $\deg \mathfrak{f} = 1$ and $\mathfrak{f}$ is an isomorphism.

Theorem 0.2 is a direct consequence of:

Theorem 5.3. *Let X be a normal surface with a reduced divisor S such that* $\operatorname{Sing} X \cup \operatorname{Sing} S \subset \{x\}$ *for a point x. Let $f\colon X^{\circ} \to X$ be a morphism from an open neighborhood of x in X° such that f has only discrete fibers, $f^{-1}(x) = \{x\}$, $\deg_x f > 0$, $f^{-1}S = S|_{X^{\circ}}$, and f is étale over $X \setminus (\{x\} \cup \operatorname{Supp} S)$. Then (X, S) is log-canonical by* Theorem 3.5. *For any essential blowing up $\varphi\colon Y \to X$ of the log-canonical pair (X, S), the meromorphic map $f_Y^{(2)}\colon Y^{(2)} \cdots\!\to Y$ defined in* Definition 5.4 *below is holomorphic and has only discrete fibers in the following two cases*:

(I) *$S = 0$, and (X, x) is not a cusp singularity*;
(II) *$x \in S$, and $f^{*}S = dS|_{X^{\circ}}$ for a positive integer d.*

DEFINITION 5.4. For an integer $k \geq 1$ and for the morphism $f^{(k)}\colon X^{(k)} \to X$ in Definition 3.1, we set $Y^{(k)} := \varphi^{-1}(X^{(k)})$ and define

$$f_Y^{(k)}\colon Y^{(k)} \xrightarrow{\varphi|_{Y^{(k)}}} X^{(k)} \xrightarrow{f^{(k)}} X \overset{\varphi^{-1}}{\cdots\!\to} Y$$

as the composite of meromorphic maps. We write $Y^{\circ} := Y^{(1)}$ and $f_Y := f_Y^{(1)}$, since $X^{\circ} = X^{(1)}$ and $f = f^{(1)}$.

Remark 5.5. By the assumption of Theorem 5.3 and by Lemma 1.39, we have $K_{X^\circ}+S|_{X^\circ} = f^*(K_X + S)$.

5.2. Proof of Theorem 5.3 in the case (II). The case where $x \in \operatorname{Sing} S$ (resp. $x \in S_{\mathrm{reg}}$) is treated in Proposition 5.6 and Corollary 5.7 (resp. Proposition 5.9) below. Theorem 5.3 in the case (II) is just derived from Corollary 5.7 and Proposition 5.9. Proposition 5.8 below concerns the case where (X, S) is 1-log-terminal at x; it is not related to Theorem 5.3 directly, but where we consider a lifting problem of f by another kind of toroidal blowing up.

Proposition 5.6. *In the situation of* Theorem 5.3, *assume that* $\{x\} = S_1 \cap S_2$ *for two distinct prime components* S_1 *and* S_2 *of* S *and that*

$$f^*S_i = d_iS_i|_{X^\circ}$$

for some positive integer d_i *for* $i = 1$ *and* 2. *Then* $\deg_x f = d_1d_2$. *Moreover, the meromorphic map* $f_Y = f_Y^{(1)} : Y^\circ = Y^{(1)} \cdots\to Y$ *in* Definition 5.4 *is holomorphic if and only if* $d_1 = d_2$, *and in this case,* f_Y *has only discrete fibers.*

Proof. The pair (X, S) is toroidal at x by Fact 2.5. For the finite morphism $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V} = f(\mathcal{U})$ in Remark 5.1, by shrinking $\mathcal{V}$, we may assume that there is an open immersion $j: \mathcal{V} \hookrightarrow V$ to an affine toric surface $V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ (cf. Section 4.1), where $S|_{\mathcal{V}} = j^{-1}D$ for the boundary divisor D of V. We assume that $(\mathsf{N}, \boldsymbol{\sigma})$ is as in Fact 4.1 with primitive elements e_1 and e_2 of N and that $S_i|_{\mathcal{V}} = j^{-1}\Gamma_i$ for any $i = 1$ and 2, for the prime components $\Gamma_1 = \boldsymbol{\Gamma}(e_1)$ and $\Gamma_2 = \boldsymbol{\Gamma}(e_2)$ of D. Hence, $j(x)$ is the fixed point $*$ of the action of $\mathbb{T}_{\mathsf{N}}$. By shrinking $\mathcal{V}$ furthermore, we may assume that the open immersion $\mathcal{V} \setminus S \hookrightarrow V \setminus D \simeq \mathbb{T}_{\mathsf{N}}$ induces an isomorphism $\boldsymbol{\pi}_1(\mathcal{V} \setminus S) \simeq \boldsymbol{\pi}_1(V \setminus D) \simeq \mathsf{N}$ of fundamental groups (cf. [38, Cor. 3.1.2]). Let $\mathsf{N}^\dagger$ be a finite index subgroup of N isomorphic to the image of the homomorphism $\boldsymbol{\pi}_1(\mathcal{U} \setminus S) \to \boldsymbol{\pi}_1(\mathcal{V} \setminus S)$ associated with the finite étale morphism $f|_{\mathcal{U}\setminus S} : \mathcal{U} \setminus S \to \mathcal{V} \setminus S$. The inclusion $\mathsf{N}^\dagger \subset \mathsf{N}$ and the cone $\boldsymbol{\sigma} \subset \mathsf{N}^\dagger \otimes \mathbb{R} = \mathsf{N} \otimes \mathbb{R}$ define a *toric morphism*

$$\pi: V^\dagger := \mathbb{T}_{\mathsf{N}^\dagger}(\boldsymbol{\sigma}) \to V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$$

(cf. Definition 4.5), which is finite and surjective and is étale over $V \setminus D$. Moreover, $\mathcal{U} \setminus S \to \mathcal{V} \setminus S$ is isomorphic to the base change of π by the open immersion $\mathcal{V} \setminus S \hookrightarrow V$. Therefore, $\mathcal{U} \simeq V^\dagger \times_V \mathcal{V}$ over $\mathcal{V}$ by a theorem of Grauert–Remmert (cf. [14], [18, XII, Thm. 5.4]), since normal varieties $\mathcal{U}$ and $V^\dagger \times_V \mathcal{V}$ are finite over $\mathcal{V}$ and these are isomorphic to each other over the Zariski-open subset $\mathcal{V} \setminus S$. In particular, the singularity of $V^\dagger$ is the same as that of $\mathcal{U}$, and the *type* (n, q) of $(\mathsf{N}, \boldsymbol{\sigma})$ equals that of $(\mathsf{N}^\dagger, \boldsymbol{\sigma})$ (cf. Definition 4.2). Hence, we may assume that $\mathsf{N}^\dagger = \mathsf{N}$, $V^\dagger = V$, and π is a toric endomorphism $\mathbb{T}(\phi): V \to V$ associated with an injective endomorphism $\phi: \mathsf{N} \to \mathsf{N}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. The open immersion $j^\dagger : \mathcal{U} \hookrightarrow V^\dagger = V$ induced by $j: \mathcal{V} \hookrightarrow V$ is also a toroidal embedding such that $j^{\dagger-1}D = S|_{\mathcal{U}}$. Since $\pi^{-1}\Gamma_1$ is either Γ_1 or Γ_2, we have $\pi^*\Gamma_i = d_i\Gamma_i$ for $i = 1, 2$ by $f^*S_i = d_iS_i|_{X^\circ}$. Hence, $\deg_x f = \deg \pi = d_1d_2$ by Lemma 4.10. Note that $j^\dagger$ and j may not induce the same open immersion to V from a common open neighborhood of x.

By Lemma 4.25, the essential blowing up $\varphi: Y \to X$ is a toroidal blowing up and is an isomorphism over $X \setminus \{x\}$, since (X, S) is toroidal at x and $\operatorname{Sing} X \cup \operatorname{Sing} S \subset \{x\}$. Thus, φ is induced by a bimeromorphic toric morphism $\mu: W = \mathbb{T}_{\mathsf{N}}(\triangle) \to V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ associated with a fan $\triangle$ of N such that $|\triangle| = \boldsymbol{\sigma}$ (cf. Example 4.3). More precisely, φ is obtained by μ

as follows: Let $\theta\colon \mathcal{W} \to \mathcal{V}$ be the base change of μ by $j\colon \mathcal{V} \hookrightarrow V$. This is expressed as the blowing up of $\mathcal{V}$ along a closed subscheme Z of $\operatorname{Spec} \mathcal{O}_{V,x}/\mathfrak{m}_x^k$ for $k \gg 0$, where the defining ideal $\mathcal{J}$ of Z in $\mathcal{O}_V$ is written as in Remark 4.7. The morphism $\varphi\colon Y \to X$ is defined as the blowing up of X along the closed analytic subspace Z. In other words, Y is obtained by gluing X and $\mathcal{W}$ via the isomorphism $\mathcal{W} \setminus \theta^{-1}(x) \simeq \mathcal{V} \setminus \{x\}$. Here, $\triangle$ contains at least three 1-dimensional cones, since μ is not an isomorphism.

We can consider the following three commutative diagrams

$$\begin{array}{ccccccccccc} W^\dagger & \longrightarrow & W & & \mathcal{W}^\dagger & \longrightarrow & \mathcal{W} & & Y^\dagger & \longrightarrow & Y \\ {\scriptstyle\mu^\dagger}\downarrow & & \downarrow{\scriptstyle\mu} & & {\scriptstyle\theta^\dagger}\downarrow & & \downarrow{\scriptstyle\theta} & & {\scriptstyle\varphi^\dagger}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V & \xrightarrow{\pi} & V, & & \mathcal{U} & \xrightarrow{f|_{\mathcal{U}}} & \mathcal{V}, & & X^\circ & \xrightarrow{f} & X, \end{array}$$

where $W^\dagger$ (resp. $\mathcal{W}^\dagger$, resp. $Y^\dagger$) is the normalization of the fiber product $V \times_V W$ (resp. $\mathcal{U} \times_{\mathcal{V}} \mathcal{W}$, resp. $X^\circ \times_X Y$) of $\pi\colon V = V^\dagger \to V$ and μ (resp. $f|_{\mathcal{U}}$ and θ, resp. f and φ), and where $\mu^\dagger$ (resp. $\theta^\dagger$, resp. $\varphi^\dagger$) is induced by the first projection. In the first diagram, $W^\dagger$ is a toric variety expressed as $\mathbb{T}_{\mathsf{N}}(\triangle^\dagger)$ for the fan $\triangle^\dagger$ consisting of cones $\phi_{\mathbb{R}}^{-1}\boldsymbol{\tau}$ for all $\boldsymbol{\tau} \in \triangle$, and $\mu^\dagger$ is a bimeromorphic toric morphism defined by $|\triangle^\dagger| = \boldsymbol{\sigma}$. In particular, $\triangle$ and $\triangle^\dagger$ give subdivisions of $\boldsymbol{\sigma}$ and $\#\triangle = \#\triangle^\dagger$. The second diagram is obtained from the first one by base change by $j\colon \mathcal{V} \hookrightarrow V$, since $f|_{\mathcal{U}} = \pi \circ j^\dagger$. It is also obtained from the third diagram by base change by open immersions $\mathcal{V} \hookrightarrow X$ and $\mathcal{U} \hookrightarrow X^\circ$. Thus, $\varphi^\dagger\colon Y^\dagger \to X^\circ$ is a toroidal blowing up induced by the bimeromorphic toric morphism $\mu^\dagger$ via the open immersion $j^\dagger\colon \mathcal{U} \hookrightarrow V^\dagger$.

On the other hand, $\varphi^\circ := \varphi|_{Y^\circ}\colon Y^\circ = \varphi^{-1}(X^\circ) \to X^\circ$ is also a toroidal blowing up and it is induced by $\mu\colon W \to V$ via $j\colon \mathcal{V} \hookrightarrow V$. Note that $f_Y\colon Y^\circ \cdots\to Y$ is holomorphic if and only if $(\varphi^\dagger)^{-1} \circ \varphi^\circ\colon Y^\circ \cdots\to Y^\dagger$ is so. Since φ (resp. μ) is an isomorphism over $X \setminus \{x\}$ (resp. $V \setminus \{j(x)\}$), by the relation of three diagrams, we see that f_Y is holomorphic if and only if $(\mu^\dagger)^{-1} \circ \mu\colon W \cdots\to W^\dagger$ is so: This is equivalent to $\triangle = \triangle^\dagger$ by Lemma 4.8, since $|\triangle| = |\triangle^\dagger| = \boldsymbol{\sigma}$ and $\#\triangle = \#\triangle^\dagger$. Moreover, if f_Y is holomorphic, then it has only discrete fibers, since the morphism $W^\dagger \to W$ induced by the second projection is finite and surjective.

Assume that $d_1 = d_2$. Then $\phi\colon \mathsf{N} \to \mathsf{N}$ is the multiplication map by d_1, by Lemma 4.10. It implies that $\triangle = \triangle^\dagger$, and hence, f_Y is holomorphic. Conversely, assume that f_Y is holomorphic. Then $\phi\colon \mathsf{N}^\dagger = \mathsf{N} \to \mathsf{N}$ is *compatible* with $\triangle^\dagger (= \triangle)$ and $\triangle$ (cf. Definition 4.5). In particular, $\phi_{\mathbb{R}}$ has at least three eigenvectors, since $\triangle$ contains at least three 1-dimensional cones. This implies that $\phi_{\mathbb{R}}$ is a scalar map, and hence, $d_1 = d_2$ by Lemma 4.9. Thus, we are done. □

Corollary 5.7. *In the situation of* Theorem 5.3*, assume that $x \in \operatorname{Sing} S$ and $f^*S = dS|_{X^\circ}$ for a positive integer d. Then $\deg_x f = d^2$, and $f_Y^{(2)}\colon Y^{(2)} \to Y$ is holomorphic with only discrete fibers.*

Proof. By replacing X with an open neighborhood of x, we may assume that $\{x\} = S_1 \cap S_2$ for two distinct prime components S_1 and S_2 of S. Thus, the assertion follows from Proposition 5.6 applied to $f^{(2)}\colon X^{(2)} \to X$ instead of $f\colon X^\circ \to X$. □

Proposition 5.8. *In the situation of* Theorem 5.3*, assume that $x \in S$ and that (X, S) is 1-log-terminal at x. Then $f|_{S \cap X^\circ}\colon S \cap X^\circ \to S$ is an isomorphism at x. Moreover, for any integer $k > 0$ and for any non-isomorphic toroidal blowing up $\varphi\colon Y \to X$ at x in the sense*

($\diamond$) *below, the meromorphic map* $f_Y^{(k)}\colon Y^{(k)} \dashrightarrow Y$ *in* Definition 5.4 *is not holomorphic*:

($\diamond$) *By* Fact 2.5*, x has an open neighborhood U with a prime divisor S' on U such that $x \in S|_U \cap S'$ and that $(U, S|_U + S')$ is toroidal at x. The bimeromorphic morphism $\varphi\colon Y \to X$ is a toroidal blowing up with respect to $(U, S|_U + S')$ for such U and S'.*

Proof. For the finite morphism $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V} = f(\mathcal{U})$ in Remark 5.1, we may assume the existence of an open immersion $j\colon \mathcal{V} \hookrightarrow V$ to a toric surface $V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ satisfying the following conditions by Fact 2.5 and by an argument in the proof of Proposition 5.6:

- $j(x)$ is the fixed point $*$ by an action of $\mathbb{T}_{\mathsf{N}}$;
- $j^{-1}\Gamma_2 = S|_{\mathcal{V}}$ for a prime component Γ_2 of the boundary divisor $D = \Gamma_1 + \Gamma_2$ of V;
- φ is a toroidal blowing up with respect to $(\mathcal{V}, j^{-1}D)$;
- the homomorphism $\boldsymbol{\pi}_1(\mathcal{V} \setminus j^{-1}D) \to \boldsymbol{\pi}_1(V \setminus D) = \mathsf{N}$ of fundamental groups is an isomorphism.

Let $\mathsf{N}^{\ddagger}$ be the subgroup of N isomorphic to the image of the homomorphism

$$\boldsymbol{\pi}_1(\mathcal{U} \setminus f^{-1}(j^{-1}D)) \to \boldsymbol{\pi}_1(\mathcal{V} \setminus j^{-1}D)$$

associated with the finite étale morphism $\mathcal{U} \setminus f^{-1}(j^{-1}D) \to \mathcal{V} \setminus j^{-1}D$. Let $\pi\colon V^{\ddagger} = \mathbb{T}_{\mathsf{N}^{\ddagger}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ be the toric morphism associated with the inclusion $\mathsf{N}^{\ddagger} \subset \mathsf{N}$ and $\boldsymbol{\sigma} \subset \mathsf{N}^{\ddagger} \otimes \mathbb{R} = \mathsf{N} \otimes \mathbb{R}$. Then $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V}$ is isomorphic to the base change of π by j by the same argument as in the proof of Proposition 5.6. In particular, the type (n, q) of $(\mathsf{N}, \boldsymbol{\sigma})$ equals that of $(\mathsf{N}^{\ddagger}, \boldsymbol{\sigma})$. Hence, π is isomorphic to a toric morphism $\mathbb{T}(\phi)\colon \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ associated with an injective homomorphism $\phi\colon \mathsf{N} \to \mathsf{N}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. Since $f|_{\mathcal{U}}$ is étale over $\mathcal{V} \setminus j^{-1}\Gamma_2$, we have $\pi^*\Gamma_1 = \Gamma_1$ and $\pi^*\Gamma_2 = d\Gamma_2$ for a positive integer $d > 0$. Hence, $\deg_x f = \deg \pi = d > 1$ by Lemma 4.9. In particular, $\pi|_{\Gamma_2}\colon \Gamma_2 \to \Gamma_2$ is an isomorphism, and hence, $f|_{S \cap X^{\circ}}\colon S \cap X^{\circ} \to S$ is an isomorphism at x.

Let $\mu\colon W = \mathbb{T}_{\mathsf{N}}(\triangle) \to V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ be a toric morphism defined by a fan $\triangle$ such that $|\triangle| = \boldsymbol{\sigma}$ and assume that the toroidal blowing up $\varphi\colon Y \to X$ in the sense of ($\diamond$) is induced by μ in the same way as in the proof of Proposition 5.6. For an integer $k > 0$, let $W^{(k)}$ be the normalization of the fiber product $V \times_V W$ of μ and the k-th power $\pi^k\colon V \to V$. Then $W^{(k)} \simeq \mathbb{T}_{\mathsf{N}}(\triangle^{(k)})$ for the fan $\triangle^{(k)}$ consisting of cones $(\phi_{\mathbb{R}}^k)^{-1}\boldsymbol{\tau}$ for all $\boldsymbol{\tau} \in \triangle$, and the morphism $W^{(k)} \to V$ induced by the first projection is a toric morphism defined by $|\triangle^{(k)}| = \boldsymbol{\sigma}$. As in the proof of Proposition 5.6, if $f_Y^{(k)}$ is holomorphic, then $\triangle^{(k)} = \triangle$, and $\phi_{\mathbb{R}}^k$ is a scalar map. However, $\phi_{\mathbb{R}}^k$ has two eigenvalues 1 and $d > 1$; thus, it is not a scalar map. Therefore, $f_Y^{(k)}$ is not holomorphic for any $k > 0$. □

Proposition 5.9. *In the situation of* Theorem 5.3*, assume that $x \in S_{\mathrm{reg}}$ and (X, S) is not* 1*-log-terminal at x. Then there is a positive integer d such that $f^*S = dS|_{X^{\circ}}$ and* $\deg_x f = d^2$*. Moreover, the meromorphic map $f_Y^{(2)}$ in* Definition 5.4 *is holomorphic and has only discrete fibers for any essential blowing up $\varphi\colon Y \to X$ of the log-canonical pair (X, S).*

Proof. For the proof, we may replace X with an open neighborhood of x freely. Hence, we may assume that $\operatorname{Sing} X = \{x\}$, S is a non-singular prime divisor, and $2(K_X + S) \sim 0$ (cf. Fact 2.5(3)). In particular, $f^*S = dS|_{X^{\circ}}$ for a positive integer d. Let $\lambda\colon \widetilde{X} \to X$ be an index 1 cover with respect to $K_X + S$. Then

- λ is a double cover étale over $X \setminus \{x\}$,

- $\lambda^{-1}(x) = \{\tilde{x}\}$ for a point $\tilde{x}$, and
- $(\widetilde{X}, \widetilde{S})$ is toroidal and $\tilde{x} \in \operatorname{Sing} \widetilde{S}$ for the divisor $\widetilde{S} := \lambda^* S$,

by Fact 2.5(3). Since $K_{X^\circ} + S|_{X^\circ} = f^*(K_X + S)$ (cf. Remark 5.5), by Lemma 4.21(2), after replacing X° with an open neighborhood of x, we have a morphism $\tilde{f} \colon \widetilde{X}^\circ = \lambda^{-1}(X^\circ) \to \widetilde{X}$ such that $\lambda \circ \tilde{f} = f \circ (\lambda|_{\widetilde{X}^\circ})$. Here, $\tilde{f}$ has only discrete fibers, $\tilde{f}^{-1}(\tilde{x}) = \{\tilde{x}\}$, and $\tilde{f}^*\widetilde{S} = d\widetilde{S}|_{\widetilde{X}^\circ}$. Then $\deg_x f = \deg_{\tilde{x}} \tilde{f} = d^2$ by Corollary 5.7. By iterating f, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{X}^{(2)} & \xrightarrow{\tilde{f}^{(2)}} & \widetilde{X} \\ {\scriptstyle \lambda|_{\widetilde{X}^{(2)}}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ X^{(2)} & \xrightarrow{f^{(2)}} & X, \end{array}$$

where $\widetilde{X}^{(2)} := \lambda^{-1}(X^{(2)})$ and $\tilde{f}^{(2)} := \tilde{f} \circ (\tilde{f}|_{\widetilde{X}^{(2)}})$.

We set $T := \varphi^{-1}S$ and apply Lemma 4.34 to the essential blowing up $\varphi \colon (Y, T) \to (X, S)$ and the index 1 cover $\lambda \colon \widetilde{X} \to X$. Then we have a commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\tilde{\varphi}} & \widetilde{X} \\ {\scriptstyle \sigma}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ Y & \xrightarrow{\varphi} & X \end{array}$$

in which $\widetilde{Y}$ is the normalization of the fiber product $Y \times_X \widetilde{X}$, $\tilde{\varphi} \colon (\widetilde{Y}, \widetilde{T}) \to (\widetilde{X}, \widetilde{S})$ is an essential blowing up for the reduced divisor $\widetilde{T} = \sigma^{-1}T$, and σ is étale in codimension 1 over $Y \setminus T$. Moreover, σ is an index 1 cover with respect to $K_Y + T = \varphi^*(K_X + S)$ by Lemma 4.21(3), since $K_{X^\circ} + S|_{X^\circ} = f^*(K_X + S)$. Then $\sigma \circ \tilde{f}^{(2)}_{\widetilde{Y}} = f^{(2)}_Y \circ (\sigma|_{\widetilde{Y}^{(2)}})$ for the meromorphic map

$$\tilde{f}^{(2)}_{\widetilde{Y}} \colon \widetilde{Y}^{(2)} := \sigma^{-1}(Y^{(2)}) = \tilde{\varphi}^{-1}(\widetilde{X}^{(2)}) \xrightarrow{\tilde{\varphi}} \widetilde{X}^{(2)} \xrightarrow{\tilde{f}^{(2)}} \widetilde{X} \overset{\tilde{\varphi}^{-1}}{\dashrightarrow} \widetilde{Y}.$$

By Lemma 4.25, $\tilde{\varphi}$ is a toroidal blowing up at $\tilde{x}$. Hence, $\tilde{f}^{(2)}_{\widetilde{Y}}$ is a holomorphic map with only discrete fibers by Corollary 5.7. Thus, $f^{(2)}_Y$ is so. □

5.3. A key theorem. We shall prove the following theorem, which is a key to the proof of Theorem 5.3 in the case (I).

Theorem 5.10. *Let X be a normal surface with a point x and let $f \colon X^\circ \to X$ be a morphism from an open neighborhood X° of x such that $f^{-1}(x) = \{x\}$, $\deg_x f > 1$, and f is étale over $X \setminus \{x\}$. Let $\varphi \colon Y \to X$ be a bimeromorphic morphism from a normal surface Y such that $B := \varphi^{-1}(x)$ is a divisor, φ is an isomorphism over $X \setminus \{x\}$, and $K_Y + B = \varphi^* K_X$. We define $g \colon Y^\circ \dashrightarrow Y$ to be the meromorphic map f_Y in* Definition 5.4 *and assume that*

($\sharp$) *any prime component of B is not contracted to a point by g.*

Then g is holomorphic and induces an automorphism of the set of prime components of B by $\Gamma \mapsto \operatorname{Supp} g_{[*]}\Gamma$ (cf. Definition 1.30(3)). *Moreover, the following hold for $b := (\deg_x f)^{1/2} > 0$:*

1. *If* $\operatorname{Supp} g_{[*]}\Gamma = \Gamma$ *for a prime component Γ of B, then $b \in \mathbb{Z}$ and $g^*\Gamma = b\Gamma$.*
2. *There exists an effective $\mathbb{R}$-divisor H on Y such that* $\operatorname{Supp} H = B$, *$g^*H = bH|_{Y^\circ}$, and*

$H\Gamma < 0$ for any prime component Γ of B.

We shall prove Theorem 5.10 by applying results in Sections 1.4 and 4.4. The final part of the proof is given at the end of Section 5.3 after showing necessary results under the condition of Theorem 5.10. We begin with the following lemma on the graph of the meromorphic map g:

Lemma 5.11. *Let V be the normalization of the fiber product $Y \times_X X^\circ$ of φ and f over X. Let $\phi\colon V \to Y$ and $\varphi_V\colon V \to X^\circ$ be morphisms induced by projections from the fiber product. Then there is a bimeromorphic morphism $\mu\colon V \to Y^\circ$ such that $\phi = g \circ \mu$ and $\varphi_V = \varphi^\circ \circ \mu$ for $\varphi^\circ := \varphi|_{Y^\circ}\colon Y^\circ \to X^\circ$. In particular, there is a commutative diagram*

(V-1)
$$\begin{tikzcd}
& V \arrow[dl, "\mu"'] \arrow[dr, "\phi"] & \\
Y^\circ \arrow[rr, dashed, "g"] \arrow[d, "\varphi^\circ"'] & & Y \arrow[d, "\varphi"] \\
X^\circ \arrow[rr, "f"] & & X
\end{tikzcd}$$

and V is isomorphic to the normalization of the graph of g.

Proof. Let W be the normalization of the graph of the bimeromorphic map $\varphi_V^{-1} \circ \varphi^\circ\colon Y^\circ \dashrightarrow V$. Let $\nu\colon W \to Y^\circ$ and $\psi\colon W \to V$ be induced morphisms such that $\varphi^\circ \circ \nu = \varphi_V \circ \psi$. Then we have a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\psi"] \arrow[d, "\nu"'] & V \arrow[r, "\phi"] \arrow[d, "\varphi_V"'] & Y \arrow[d, "\varphi"] \\
Y^\circ \arrow[r, "\varphi^\circ"] & X^\circ \arrow[r, "f"] & X
\end{tikzcd}$$

and the meromorphic map $g = f_Y$ is expressed as the composite $\phi \circ \psi \circ \nu^{-1}$. If a prime divisor Ξ on W is ψ-exceptional, then $\Xi \subset \psi^{-1}(\phi^{-1}B) = \nu^{-1}B$, and Ξ is not expressed as $\nu^{[*]}\Gamma$ for any prime component Γ of B by ($\sharp$) in Theorem 5.10; hence, Ξ is ν-exceptional. Therefore, the meromorphic map $\mu := \nu \circ \psi^{-1}\colon V \dashrightarrow Y^\circ$ is holomorphic, and $\varphi_V = \varphi^\circ \circ \mu$. Hence, $\psi\colon W \to V$ is an isomorphism, since W is the normalization of the graph of $\mu^{-1} = \varphi_V^{-1} \circ \varphi^\circ$. Thus, $g \circ \mu = \phi$, and V is isomorphic to the normalization of the graph of g. □

Remark. The following hold for the diagram (V-1):

- φ, φ°, and $\mu = \nu \circ \psi^{-1}$ are bimeromorphic morphisms;
- ϕ has only discrete fibers and is étale over $Y \setminus B$;
- the restriction $\mu^{-1}((\varphi^\circ)^{-1}\mathcal{U}) \to \varphi^{-1}\mathcal{V}$ of ϕ is a finite and surjective morphism of degree $\deg_x f$ for some open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of x (cf. Remark 5.1).

Definition 5.12. As reduced divisors on Y° and V, we define

$$B^\circ := B|_{Y^\circ} \quad \text{and} \quad B_V := \phi^{-1}B = \mu^{-1}(B^\circ),$$

respectively. For an $\mathbb{R}$-divisor D on Y such that $\operatorname{Supp} D \subset B$, we write $D^\circ = D|_{Y^\circ}$ as an $\mathbb{R}$-divisor on Y°, and set

$$D^V := \mu^*(D^\circ) \quad \text{and} \quad D_{(V)} := \mu^{[*]}(D^\circ)$$

as $\mathbb{R}$-divisors on V (cf. Definition 1.22). However, sometimes, we write $B = B^\circ$ and $D = D^\circ$ for simplicity. Note that $B_V = (B^V)_{\mathrm{red}}$.

Remark 5.13. For the $\mathbb{R}$-divisor D above, the pullbacks $g^{[*]}D$ and g^*D and the pushforwards $g_{[*]}D^\circ = g_{[*]}D$ and $g_*D^\circ = g_*D$ by the meromorphic map g are defined in Definition 1.30. Here, $g_*D = \phi_*D^V$ and $g_{[*]}D = \phi_*D_{(V)}$ by definition, and $g^{[*]}D = g^*D = \mu_*(\phi^*D)$, since ϕ has no exceptional divisor. If g is holomorphic, then $g_*D = g_{[*]}D$.

Definition 5.14. For an integer $k \geq 0$, we define $g^{(k)} : Y^{(k)} \cdots\to Y$ to be the meromorphic map $f_Y^{(k)}$ in Definition 5.4.

Remark 5.15. For an $\mathbb{R}$-divisor D on Y such that $\operatorname{Supp} D \subset B$, we can consider $g_*^{(k)}D$, $g_{[*]}^{(k)}D$, and $g^{(k)*}D$ as in Remark 5.13. Then

$$g_{[*]}^{(k+l)}D = g_{[*]}^{(k)}(g_{[*]}^{(l)}D) \quad \text{and} \quad g^{(k+l)*}D = g^{(l)*}(g^{(k)*}D)$$

for any $k, l \geq 0$ by Lemma 1.32, since ϕ has no exceptional divisor. However, we can not expect the equality $g_*^{(k+l)}D = g_*^{(k)}(g_*^{(l)}D)$ in general.

Definition 5.16. Let $\mathbb{I}$ be the set of prime components of B. We define a map $f_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ and a function $\boldsymbol{a} : \mathbb{I} \to \mathbb{Q}$ by

$$f_{\mathbb{I}}(\Gamma) := \operatorname{Supp} g_{[*]}\Gamma \quad \text{and} \quad \boldsymbol{a}(\Gamma) := \operatorname{mult}_\Gamma g^*B$$

(cf. (♯) in Theorem 5.10). Let $\mathbb{J}$ be the set of prime components of $B_V = \phi^{-1}B = \mu^{-1}B^\circ$, and for each $\Gamma \in \mathbb{I}$, let $\mathbb{J}_\Gamma$ be the set of prime components Θ of B_V such that $\phi(\Theta) = \Gamma$. Then $\mathbb{J} = \bigsqcup_{\Gamma\in\mathbb{I}} \mathbb{J}_\Gamma$. For $\Theta \in \mathbb{J}_\Gamma$, we define

$$\boldsymbol{a}_\Theta := \operatorname{mult}_\Theta \phi^*B = \operatorname{mult}_\Theta \phi^*\Gamma \quad \text{and} \quad \boldsymbol{m}_\Theta := \operatorname{mult}_\Gamma \phi_*\Theta = \deg(\phi|_\Theta : \Theta \to \Gamma).$$

Remark 5.17. For any $\Gamma \in \mathbb{I}$ and for the proper transform $\Gamma_{(V)} = \mu^{[*]}\Gamma^\circ$, we have

$$f_{\mathbb{I}}(\Gamma) = \phi(\Gamma_{(V)}) \quad \text{and} \quad \boldsymbol{a}(\Gamma) = \operatorname{mult}_{\Gamma_{(V)}} \phi^*B = \operatorname{mult}_{\Gamma_{(V)}} \phi^*(f_{\mathbb{I}}(\Gamma)).$$

In particular, $\boldsymbol{a}(\Gamma)$ is a positive integer, since ϕ has only discrete fibers and since ϕ^*B is a divisor (cf. Lemma 1.19 and Remarks 1.20 and 1.24(5)). Moreover,

$$\Gamma_{(V)} \in \mathbb{J}_{f_{\mathbb{I}}(\Gamma)}, \quad \boldsymbol{a}(\Gamma) = \boldsymbol{a}_{\Gamma_{(V)}}, \quad \text{and} \quad g_{[*]}\Gamma = \phi_*\Gamma_{(V)} = \boldsymbol{m}_{\Gamma_{(V)}} f_{\mathbb{I}}(\Gamma)$$

for any $\Gamma \in \mathbb{I}$. If $f_{\mathbb{I}}^{-1}(f_{\mathbb{I}}(\Gamma)) = \{\Gamma\}$, then

$$g^*(f_{\mathbb{I}}(\Gamma)) = \mu_*\phi^*(f_{\mathbb{I}}(\Gamma)) = \boldsymbol{a}(\Gamma)\mu_*\Gamma_{(V)} = \boldsymbol{a}(\Gamma)\Gamma, \tag{V-2}$$

since $\mu_*\Theta = 0$ for any $\Theta \in \mathbb{J}_{f_{\mathbb{I}}}(\Gamma) \setminus \{\Gamma_{(V)}\}$. For an integer $k \geq 1$, we can consider the map $(f^{(k)})_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ associated with $f^{(k)} : X^{(k)} \to X$ similarly to $f_{\mathbb{I}}$, where $(f^{(k)})_{\mathbb{I}}(\Gamma) = \operatorname{Supp} g_{[*]}^{(k)}\Gamma$ for any $\Gamma \in \mathbb{I}$. Then

(1) $(f^{(k)})_{\mathbb{I}}$ equals the k-th power $(f_{\mathbb{I}})^k = f_{\mathbb{I}} \circ \cdots \circ f_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ for any $k \geq 1$, and
(2) the equality

$$\operatorname{mult}_\Gamma (g^{(k)})^*B = \prod\nolimits_{i=0}^{k-1} \boldsymbol{a}((f_{\mathbb{I}})^i(\Gamma))$$

holds for any $\Gamma \in \mathbb{I}$ and $k \geq 1$.

These are shown by equalities in Remark 5.15.

Remark 5.18. For $\Gamma \in \mathbb{I}$ and $\Theta \in \mathbb{J}_\Gamma$, we have

$$\phi^*\Gamma = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} \boldsymbol{a}_\Theta \Theta \quad \text{and} \quad \phi_*\Theta = \boldsymbol{m}_\Theta \Gamma$$

by Definition 5.16, and moreover, by Lemma 4.37,

$$\phi_*\boldsymbol{D}(\Theta/B_V) = \boldsymbol{a}_\Theta \boldsymbol{D}(\Gamma/B) \quad \text{and} \quad \phi^*\boldsymbol{D}(\Gamma/B) = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} \boldsymbol{m}_\Theta \boldsymbol{D}(\Theta/B_V).$$

Lemma 5.19. *Let D be a non-zero effective $\mathbb{R}$-divisor on Y such that* $\operatorname{Supp} D \subset B$. *We set* $H := H_D := \sum_{\Gamma \in \mathbb{I}} h_\Gamma \boldsymbol{D}(\Gamma/B)$, *where*

$$h_\Gamma = \begin{cases} 0, & \textit{if } \operatorname{mult}_\Gamma D = 0, \\ -(\operatorname{mult}_\Gamma D)^{-1}, & \textit{otherwise.} \end{cases}$$

Then H is effective, $\operatorname{Supp} H = B$, *and $-H$ is* nef on B (*cf.* Remark 1.25). *If $f_\mathbb{I} \colon \mathbb{I} \to \mathbb{I}$ is bijective and if $g^*D = bD$ for a real number $b > 0$, then $g_*^{(k)} H = b^k H$ for any $k \geq 1$.*

Proof. By Lemma-Definition 4.35(3), H is effective and $\operatorname{Supp} H = B$. Moreover, $H\Gamma = h_\Gamma \leq 0$ for any $\Gamma \in \mathbb{I}$ by Lemma-Definition 4.35(1). Thus, $-H$ is nef on B, and we have proved the first assertion. Assume that $g^*D = bD$. Then

$$\boldsymbol{a}(\Gamma) \operatorname{mult}_{f_\mathbb{I}(\Gamma)} D = \operatorname{mult}_\Gamma g^*D = b \operatorname{mult}_\Gamma D$$

for any $\Gamma \in \mathbb{I}$ by the definition of $\boldsymbol{a}(\Gamma)$. In particular, $\Gamma \subset \operatorname{Supp} D$ if and only if $f_\mathbb{I}(\Gamma) \subset \operatorname{Supp} D$, and we have

$$\boldsymbol{a}(\Gamma) h_\Gamma = b h_{f_\mathbb{I}(\Gamma)}$$

for any $\Gamma \subset \operatorname{Supp} D$. On the other hand,

$$\mu^*\boldsymbol{D}(\Gamma/B) = \boldsymbol{D}(\Gamma_{(V)}/B_V) \quad \text{and} \quad g_*\boldsymbol{D}(\Gamma/B) = \phi_*\boldsymbol{D}(\Gamma_{(V)}/B_V) = \boldsymbol{a}(\Gamma)\boldsymbol{D}(f_\mathbb{I}(\Gamma)/B)$$

for any $\Gamma \in \mathbb{I}$ by Lemma 4.36 and Remarks 5.17 and 5.18. Therefore,

$$\begin{aligned} g_*H = \sum\nolimits_{\Gamma \subset \operatorname{Supp} D} h_\Gamma g_*\boldsymbol{D}(\Gamma/B) &= \sum\nolimits_{\Gamma \subset \operatorname{Supp} D} h_\Gamma \boldsymbol{a}(\Gamma)\boldsymbol{D}(f_\mathbb{I}(\Gamma)/B) \\ &= b \sum\nolimits_{\Gamma \subset \operatorname{Supp} D} h_{f_\mathbb{I}(\Gamma)} \boldsymbol{D}(f_\mathbb{I}(\Gamma)/B), \end{aligned}$$

and we have $g_*H = bH$ when $f_\mathbb{I}$ is bijective. For any $k \geq 1$, we have $g^{(k)*}D = b^k D$ by Remark 5.15, and if $f_\mathbb{I}$ is bijective, then $(f^k)_\mathbb{I} = (f_\mathbb{I})^k$ is bijective by Remark 5.17(1). Hence, if $f_\mathbb{I}$ is bijective, then $g_*^{(k)} H = b^k H$ by the argument above applied to $f^{(k)}$ instead of f. □

Lemma 5.20. *Assume that $X \setminus \{x\}$ is non-singular. Then (Y, B) and (V, B_V) are log-canonical, and $K_V + B_V = \mu^*(K_{Y^\circ} + B^\circ)$.*

Proof. The pair (Y, B) is log-canonical by $K_Y + B = \varphi^*(K_X)$ and by Lemma 2.10(1). Since ϕ is étale over $Y \setminus B$ and since f is étale over $X \setminus \{x\}$, we have

$$K_V + B_V = \phi^*(K_Y + B) = \phi^*(\varphi^*K_X) = \varphi_V^*(f^*K_X) = \varphi_V^*(K_{X^\circ})$$

by Lemma 1.39 for the morphism $\varphi_V\colon V \to X^\circ$ in Lemma 5.11. Thus, (V, B_V) is also log-canonical by Lemma 2.10(1), and we have

$$\mu^*(K_{Y^\circ} + B^\circ) = \mu^*(\varphi^{\circ *} K_{X^\circ}) = \varphi_V^*(K_{X^\circ}) = K_V + B_V$$

by $\varphi_V = \varphi^\circ \circ \mu$. □

Proposition 5.21. *Let H be a non-zero $\mathbb{R}$-divisor on Y and let b be a positive real number such that* $\operatorname{Supp} H \subset B$, *$-H$ is nef on B, and $g_*^{(k)} H = b^k H$ for any $k \geq 1$. Then $\phi^* H = bH^V$ and* $\deg_x f = b^2$, *where $H^V = \mu^* H^\circ$ (cf.* Definition 5.12).

Proof. By Remark 1.25, H is effective and $\operatorname{Supp} H = B$. Moreover, we can write

$$H = \sum\nolimits_{\Gamma \in \mathbb{I}} \beta_\Gamma \mathbf{\Delta}(\Gamma, H) \tag{V-3}$$

for non-negative real numbers $\beta_\Gamma = -(H\Gamma) \operatorname{mult}_\Gamma H$ by (2) and (4) of Lemma-Definition 4.35. Note that $\beta := \sum_{\Gamma \in \mathbb{I}} \beta_\Gamma > 0$ as $H \neq 0$. For the assertion, we may replace X with an open neighborhood of x. Thus, we may assume that $X \setminus \{x\}$ is non-singular. Then there exist positive integers $c_1 < c_2$ depending on (Y, B, H) such that

$$c_1 H^V \leq \mathbf{\Delta}(\Theta, H^V) \leq c_2 H^V \tag{V-4}$$

for any $\Theta \in \mathbb{J}$, by Lemma 5.20 and by Proposition 4.38 applied to $(Y^\circ, B^\circ, H^\circ)$, $\mu\colon V \to Y^\circ$, and Θ.

For a prime component Θ of B_V, we define

$$t_\Theta := \frac{\operatorname{mult}_\Theta H^V}{\operatorname{mult}_\Gamma H},$$

where $\Gamma = \phi(\Theta)$, i.e., $\Theta \in \mathbb{J}_\Gamma$. Then

$$\phi^* \mathbf{\Delta}(\Gamma, H) = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} \boldsymbol{m}_\Theta t_\Theta \mathbf{\Delta}(\Theta, H^V)$$

by Lemma 4.37 and Lemma-Definition 4.35(2). It implies that

$$b = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} \boldsymbol{m}_\Theta t_\Theta. \tag{V-5}$$

In fact, by $\phi_* H^V = g_* H = bH$ and by Lemma-Definition 4.35(5), we have

$$(\phi^* \mathbf{\Delta}(\Gamma, H)) H^V = \mathbf{\Delta}(\Gamma, H) \phi_* H^V = b \mathbf{\Delta}(\Gamma, H) H = -b,$$
$$(\phi^* \mathbf{\Delta}(\Gamma, H)) H^V = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} \boldsymbol{m}_\Theta t_\Theta \mathbf{\Delta}(\Theta, H^V) H^V = -\sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} \boldsymbol{m}_\Theta t_\Theta.$$

Then, for any $\Gamma \in \mathbb{I}$,

$$c_1 b H^V \leq \phi^* \mathbf{\Delta}(\Gamma, H) \leq c_2 b H^V$$

by (V-4) and (V-5), and moreover, by applying ϕ_*, we have

$$c_1 b^2 H \leq (\deg_x f) \mathbf{\Delta}(\Gamma, H) \leq c_2 b^2 H.$$

Therefore,

$$c_1 \beta b^2 \leq \deg_x f \leq c_2 \beta b^2$$

for $\beta = \sum_{\Gamma \in \mathbb{I}} \beta_\Gamma > 0$ by (V-3). We can apply the argument above to $f^{(k)}$ for any $k \geq$

1 instead of f, since $g_*^{(k)}H = b^k H$ and since c_1, c_2, and β depend only on (Y, B, H) (cf. Proposition 4.38). Hence,

$$c_1\beta b^{2k} \le \deg_x f^{(k)} = (\deg_x f)^k \le c_2\beta b^{2k}$$

for any $k \ge 1$. Taking limits for $k \to \infty$, we have $\deg_x f = b^2$. Then

$$(\phi^* H - bH^V)^2 = (\phi^* H)^2 - 2b(\phi^* H)H + b^2(\mu^* H^\circ)^2 = (\deg_x f)H^2 - 2b^2 H^2 + b^2 H^2 = 0,$$

by $H^V = \mu^* H^\circ$. This implies that $\phi^* H = bH^V$, since the intersection matrix of prime components of B is negative definite. □

REMARK. The method in the proof above is borrowed from the proof of [6, Prop. 2.1].

Lemma 5.22. Theorem 5.10 *holds true if* $f_{\mathbb{I}}\colon \mathbb{I} \to \mathbb{I}$ *is bijective.*

Proof. We shall prove by three steps:

STEP 1. Let D and $H = H_D$ be $\mathbb{R}$-divisors in Lemma 5.19, and assume that $g^* D = bD$ for a real number $b > 0$. Then $\phi^* H = bH^V = b\mu^* H$ and $\deg_x f = b^2$ by Lemma 5.19 and Proposition 5.21. Assuming that $\operatorname{Supp} D = B$, we shall show that g is holomorphic and that H satisfies the condition of Theorem 5.10(2). By assumption, $H\Gamma = h_\Gamma < 0$ for any $\Gamma \in \mathbb{I}$, and H satisfies the condition of Theorem 5.10(2) by Lemma 5.19. On the other hand, $\phi^* H = bH^V$ implies that

$$H(\phi_*\Theta) = (\phi^* H)\Theta = b(\mu^* H)\Theta = 0$$

for any μ-exceptional prime divisor Θ. Hence, $\phi_*\Theta = 0$ for any μ-exceptional prime divisor Θ, and consequently, μ is an isomorphism and g is holomorphic.

STEP 2. We shall show that $\boldsymbol{a}(\Gamma)^2 = \deg_x f$ for any $\Gamma \in \mathbb{I}$ satisfying $f_{\mathbb{I}}(\Gamma) = \Gamma$. Now $g^*\Gamma = \boldsymbol{a}(\Gamma)\Gamma$ by (V-2) in Remark 5.17. By applying an argument in Step 1 to $D = \Gamma$, we have $\boldsymbol{a}(\Gamma)^2 = \deg_x f$. As a consequence, Theorem 5.10(1) holds. Moreover, $g^* B = bB$ for $b := (\deg_x f)^{1/2} > 0$ provided that $f_{\mathbb{I}}$ is the identity map.

STEP 3. Final step. By Step 1, it is enough to construct an effective $\mathbb{R}$-divisor D on Y such that $\operatorname{Supp} D = B$ and $g^* D = bD$ for $b := (\deg_x f)^{1/2}$. Let n be the order of the bijection $f_{\mathbb{I}}\colon \mathbb{I} \to \mathbb{I}$. Then $(\deg_x f)^n = b^{2n} = \deg_x f^{(n)}$ and $(f^{(n)})_{\mathbb{I}} = (f_{\mathbb{I}})^n = \mathrm{id}_{\mathbb{I}}$ by Remark 5.17(1), and $g^{(n)*}B = b^n B$ by Step 2 applied to $f^{(n)}\colon X^{(n)} \to X$ instead of f. By Remark 5.17(2), we have

$$b^n = \operatorname{mult}_\Gamma g^{(n)*}B = \prod\nolimits_{k=0}^{n-1} \boldsymbol{a}((f_{\mathbb{I}})^k\Gamma) \tag{V-6}$$

for any $\Gamma \in \mathbb{I}$. Let $\mathbb{M}$ be the multiplicative abelian group defined as the set of maps $\mathbb{I} \to \mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$. The bijection $f_{\mathbb{I}}$ defines an action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{M}$ in which the transform γ^{T} of $\gamma \in \mathbb{M}$ by the action of $1 \in \mathbb{Z}/n\mathbb{Z}$ is given by $\gamma^{\mathsf{T}}(\Gamma) = \gamma(f_{\mathbb{I}}(\Gamma))$. We define a map $\varepsilon\colon \mathbb{I} \to \mathbb{R}_+$ by $\varepsilon(\Gamma) = b^{-1}\boldsymbol{a}(\Gamma)$. Then

$$\prod\nolimits_{k=0}^{n-1} \varepsilon^{\mathsf{T}^k} = 1$$

by (V-6), and hence, ε defines a 1-cocycle of the $\mathbb{Z}/n\mathbb{Z}$-module $\mathbb{M}$. The group cohomology $H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{M})$ is trivial, since the n-th power map is bijective for $\mathbb{R}_+$ and for $\mathbb{M}$. Thus, we have a map $\delta\colon \mathbb{I} \to \mathbb{R}_+$ such that $\varepsilon = \delta \cdot (\delta^{\mathsf{T}})^{-1}$, i.e.,

$$\varepsilon(\Gamma) = \delta(\Gamma)\delta(f_{\mathbb{I}}(\Gamma))^{-1}$$

for any $\Gamma \in \mathbb{I}$. Then $D = \sum_{\Gamma\in\mathbb{I}} \delta(\Gamma)\Gamma$ satisfies $\operatorname{Supp} D = B$ and

$$\begin{aligned} g^*D = \sum\nolimits_{\Gamma\in\mathbb{I}} \delta(f_{\mathbb{I}}(\Gamma))g^*(f_{\mathbb{I}}(\Gamma)) &= \sum\nolimits_{\Gamma\in\mathbb{I}} \delta(f_{\mathbb{I}}(\Gamma))\boldsymbol{a}(\Gamma)\Gamma \\ &= \sum\nolimits_{\Gamma\in\mathbb{I}} \varepsilon(\Gamma)^{-1}\boldsymbol{a}(\Gamma)\delta(\Gamma)\Gamma = bD \end{aligned}$$

by (V-2) in Remark 5.17. Thus, we are done. □

Now, we shall finish the proof of Theorem 5.10:

Proof of Theorem 5.10. We set $\mathbb{I}_\infty := \bigcap_{k\geq 1}(f_{\mathbb{I}})^k(\mathbb{I})$. Then $\mathbb{I}_\infty = (f_{\mathbb{I}})^m(\mathbb{I})$ for some $m > 0$, and $f_{\mathbb{I}}$ induces a bijection $\mathbb{I}_\infty \to \mathbb{I}_\infty$. By Lemma 5.22, it is enough to derive a contradiction assuming that $\mathbb{I}_\infty \neq \mathbb{I}$. Let $\pi\colon Y \to \overline{Y}$ be the contraction morphism of all the prime components of B not belonging to $\mathbb{I}_\infty$. Let $\bar\varphi\colon \overline{Y} \to X$ be the induced bimeromorphic morphism satisfying $\varphi = \bar\varphi \circ \pi$ and let

$$\bar g\colon \overline{Y}^\circ := \bar\varphi^{-1}(X^\circ) \xrightarrow{\bar\varphi^\circ} X^\circ \xrightarrow{f} X \overset{\bar\varphi^{-1}}{\dashrightarrow} \overline{Y}$$

be the composite of meromorphic maps. Then we have a commutative diagram

(V-7)
$$\begin{tikzcd} & V \arrow[dl, "\mu"'] \arrow[dr, "\phi"] & \\ Y^\circ \arrow[rr, dashed, "g"] \arrow[d, "\pi^\circ"'] \arrow[dd, bend right=60, "\varphi^\circ"'] & & Y \arrow[d, "\pi"] \arrow[dd, bend left=60, "\varphi"] \\ \overline{Y}^\circ \arrow[rr, dashed, "\bar g"] \arrow[d, "\bar\varphi^\circ"'] & & \overline{Y} \arrow[d, "\bar\varphi"] \\ X^\circ \arrow[rr, "f"] & & X \end{tikzcd}$$

extending (V-1) in Lemma 5.11, where $\pi^\circ = \pi|_{Y^\circ}$. The set $\overline{\mathbb{I}}$ of prime components of $\overline{B} = \pi(B) = \bar\varphi^{-1}(x)$ is identified with $\mathbb{I}_\infty$, and the map $f_{\overline{\mathbb{I}}}\colon \overline{\mathbb{I}} \to \overline{\mathbb{I}}$ defined by $\overline{\Gamma} \mapsto \operatorname{Supp} \bar g_{[*]}\overline{\Gamma}$ is identical to the bijection $\mathbb{I}_\infty \to \mathbb{I}_\infty$ induced by $f_{\mathbb{I}}$. Hence, by Lemma 5.22, $\bar g$ is holomorphic, and $\bar g^*\overline{H} = b\overline{H}$ for an $\mathbb{R}$-divisor $\overline{H}$ on $\overline{Y}$ such that $\overline{H}\overline{\Gamma} < 0$ for any $\overline{\Gamma} \in \overline{\mathbb{I}}$, where $b = (\deg_x f)^{1/2} > 0$. Then

$$b\mu^*(\pi^{\circ *}\overline{H}) = \mu^*(\pi^{\circ *}(\bar g^*\overline{H})) = \phi^*(\pi^*\overline{H})$$

by $\bar g \circ \pi^\circ \circ \mu = \pi \circ \phi$ (cf. (V-7)). For $\Gamma \in \mathbb{I}$, if $f_{\mathbb{I}}(\Gamma) \in \mathbb{I}_\infty$, then $\Gamma \in \mathbb{I}_\infty$, by

$$\begin{aligned} b\overline{H}(\pi_*\Gamma) = b(\pi^*\overline{H})\Gamma = b(\pi^{\circ *}\overline{H})\Gamma^\circ = b(\pi^{\circ *}\overline{H})\mu_*\Gamma_{(V)} = b\mu^*(\pi^{\circ *}\overline{H})\Gamma_{(V)} \\ = \phi^*(\pi^*\overline{H})\Gamma_{(V)} = (\pi^*\overline{H})\phi_*\Gamma_{(V)} = \boldsymbol{m}_{\Gamma_{(V)}}(\pi^*\overline{H})f_{\mathbb{I}}(\Gamma) = \boldsymbol{m}_{\Gamma_{(V)}}\overline{H}\pi_*(f_{\mathbb{I}}(\Gamma)) < 0 \end{aligned}$$

(cf. Remark 5.17). Therefore, $\mathbb{I} = \mathbb{I}_\infty$, a contradiction. Thus, we are done. □

5.4. Proof of Theorem 5.3 in the case (I). We shall complete the proof of Theorem 5.3.

Lemma 5.23. *In the situation of the case* (I) *of* Theorem 5.3*, assume that the index* 1 *cover of* (X, x) *with respect to* K_X *is a simple elliptic singularity. Then the exceptional locus*

$C = \varphi^{-1}(x)$ is irreducible, and the meromorphic map $f_Y \colon Y^\circ \dashrightarrow Y$ is holomorphic and has only discrete fibers. Moreover, $\deg_x f = b^2$ *and* $f_Y^* C = bC|_{Y^\circ}$ *for a positive integer b.*

Proof. Every essential blowing up $\varphi\colon Y \to X$ is isomorphic to the *standard partial resolution* (cf. Definition 4.27) and $C = \varphi^{-1}(x)$ is irreducible by Example 4.29. Let V be the normalization of the fiber product $Y \times_X X^\circ$ of φ and f over X. Then the induced morphism $\varphi_V\colon V \to X^\circ$ is also an essential blowing up by Lemma 4.34. Thus, the bimeromorphic map $\varphi_V^{-1} \circ (\varphi|_{Y^\circ})\colon Y^\circ \dashrightarrow V$ is an isomorphism by Corollary 4.33(3), and f_Y is holomorphic with only discrete fibers. We have $f_Y^* C = bC$ for a positive integer b by construction, where $b^2 = \deg_x f$ by $C^2 < 0$. □

Remark. We can prove Lemma 5.23 by another method as follows. When (X, x) is a simple elliptic singularity, φ is the minimal resolution of singularities and C is an elliptic curve (cf. Example 4.29(2)); in this case, it is easy to prove the assertion. Next, we consider the case where (X, x) is a rational singularity. By localizing X, we may have an index 1 cover $\lambda\colon \widetilde{X} \to X$ with respect to K_X such that $(\widetilde{X}, \tilde{x})$ is a simple elliptic singularity for the point $\tilde{x}$ lying over x. Moreover, we may assume that $f\colon X^\circ \to X$ lifts to a morphism $\tilde{f}\colon \widetilde{X}^\circ = \lambda^{-1}(X^\circ) \to \widetilde{X}$ by Lemma 4.21(2). Thus, in this case, we can prove that f_Y is holomorphic and has only discrete fibers, by the same method as in the proof of Proposition 5.9 using Lemma 4.34.

Lemma 5.24. *In the situation of the case* (I) *of* Theorem 5.3*, assume that* (X, x) *is a rational singularity whose index* 1 *cover with respect to K_X is a cusp singularity. Assume also that the essential blowing up $\varphi\colon Y \to X$ is obtained from the standard partial resolution of X by contracting all the non-end components of the exceptional divisor* (*cf.* Example 4.29(5))*. Then $f_Y\colon Y^\circ \dashrightarrow Y$ is holomorphic and has only discrete fibers. Moreover,* $(f_Y^{(2)})^*\Gamma = (\deg_x f)\Gamma|_{Y^{(2)}}$ *for any φ-exceptional prime divisor Γ.*

Proof. The exceptional locus $\varphi^{-1}(x)$ is a linear chain $\Gamma_1 + \Gamma_2$ of two rational curves by construction and by Example 4.29(5). In particular, $\#\Gamma_1 \cap \Gamma_2 = 1$. For the normalization V of the fiber product $Y \times_X X^\circ$ of φ and f over X, the induced morphism $\varphi_V\colon V \to X^\circ$ is also an essential blowing up by Lemma 4.34. Thus, the bimeromorphic map $\varphi_V^{-1} \circ (\varphi|_{Y^\circ})\colon Y^\circ \dashrightarrow V$ does not contract Γ_1 and Γ_2 to points by Corollary 4.33(2). Hence, f_Y does not contract Γ_1 and Γ_2 to points and the image of Γ_1 under f_Y is either Γ_1 or Γ_2, and vice versa. Therefore, the assertion is a consequence of Theorem 5.10. □

Theorem 5.3 has been proved in the case (II) by Corollary 5.7 and Proposition 5.9 in Section 5.2. Finally, we shall prove Theorem 5.3 in the case (I):

Proof of Theorem 5.3 in the case (I). If (X, x) is a quotient singularity, then the essential blowing up $\varphi\colon Y \to X$ is an isomorphism (cf. Definition 4.24), and we have nothing to do. Since (X, x) is not a cusp singularity, we may assume one of (a) and (b) below by the classification of 2-dimensional log-canonical singularities (cf. [30, Thm. 9.6]):

(a) the index 1 cover of (X, x) with respect to K_X is a simple elliptic singularity;
(b) (X, x) is a rational singularity whose index 1 cover with respect to K_X is a cusp singularity.

In case (a), Theorem 5.3 is a consequence of Lemma 5.23. Thus, we may assume (b). Let $\widehat{\varphi}\colon \widehat{Y} \to X$ be the essential blowing up φ in Lemma 5.24. Then any essential blowing up $\varphi\colon Y \to X$ factors through $\widehat{Y}$ by a toroidal blowing up $Y \to \widehat{Y}$, by Lemma 4.32 and Corollary 4.33(3). By Lemma 5.24, $f^{(2)}\colon X^{(2)} \to X$ lifts to a morphism

$$\hat{f}^{(2)}\colon \widehat{Y}^{(2)} := \hat{\varphi}^{-1}(X^{(2)}) \to \widehat{Y}$$

with only discrete fibers such that

$$(\hat{f}^{(2)})^{*}\Gamma_i = (\deg_x f)\Gamma_i|_{\widehat{Y}^{(2)}}$$

for any $i = 1, 2$ for the exceptional locus $\hat{\varphi}^{-1}(x) = \Gamma_1 \cup \Gamma_2$. Hence, the lift $f_Y^{(2)}\colon Y^{(2)} \to Y$ of $\hat{f}^{(2)}$ is also holomorphic and has only discrete fibers by Proposition 5.6. Thus, we have completed the proof of Theorem 5.3. □

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