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CONSTRUCTION OF ONE-FIXED-POINT ACTIONS ON SPHERES OF NONSOLVABLE GROUPS I

Dedicated to Japan Securities Scholarship Foundation for its 50th anniversary

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Abstract

Let G be a finite group. It is known that if a homotopy sphere X has a one-fixed-point smooth G -action then the dimension of X is greater than or equal to 6. It is also known that there is an effective 2-pseudofree one-fixed-point smooth G -action on the sphere S^n of dimension n if and only if n is equal to 6 and G is isomorphic to the alternating group A_5 on five letters. E. Stein proved that for the group $G = \mathrm{SL}(2, 5) \times Z_m$ such that m is prime to 30, there is a 3-pseudofree one-fixed-point smooth G -action on S^7 , where Z_m is a cyclic group of order m . In this article, we determine the finite groups G possessing 3-pseudofree one-fixed-point smooth G -actions on S^6 . In addition, for an arbitrary finite group G isomorphic to A_5 , $A_5 \times Z_2$, or $\mathrm{SL}(2, 5) \times Z_m$ such that m is prime to 30, we prove that there is a 3-pseudofree one-fixed-point smooth G -action on S^7 .

1. Introduction

In this paper, G is a finite group and we read a G -manifold as a smooth manifold with a smooth G -action. Let $S(G)$ denote the set of all subgroups of G and E the trivial group. The set $S(G)$ is an ordered set (possibly not a totally ordered set), i.e. for $H, K \in S(G)$, we say $H < K$ if H is a proper subgroup of K . For a subset A of $S(G)$, let $\max(A)$ denote the set of maximal elements of A with respect to the order on A inherited from $S(G)$. A real G -representation V is called *free* if $\dim V^H = 0$ for all $H \in S(G) \setminus \{E\}$. Let m be a non-negative integer. We call a G -action on a manifold X *m-pseudofree* if $\dim X^H \leq m$ for all $H \in S(G) \setminus \{E\}$. We call an m -pseudofree G -action on X *properly m-pseudofree* if there is a subgroup $H \in S(G) \setminus \{E\}$ such that $\dim X^H = m$. We call a G -action on X a *one-fixed-point action* if X^G consists of exactly one point. It is known that the Poincaré sphere (a homology sphere of dimension 3) admits a one-fixed-point action of the alternating group A_5 on five letters. However the works M. Furuta [12], S. Demichelis [9] and N. Buchdahl–S. Kwasik–R. Schultz [7] together show that any homotopy sphere of dimension ≤ 5 does not admit a one-fixed-point action of finite group. Therefore a homotopy sphere Σ possessing a one-fixed-point action of finite group satisfies $\dim \Sigma \geq 6$. The existence of one-fixed-point G -action on a homotopy sphere makes it look like there exists a one-fixed-point G -action on the same dimensional sphere.

Our present study was motivated by the following results of E. Laitinen–P. Traczyk [17]. Unless otherwise stated, let Σ be a homotopy sphere of dimension ≥ 5 equipped with a G -action and let x_0 be a G -fixed point of Σ . For a G -fixed point x of Σ , let $T_x(\Sigma)$ denote the tangential G -representation of Σ at x . The trivial real G -representation of dimension 1 will be denoted by $\mathbb{R}$.

Laitinen–Traczyk Theorem 1. *Suppose the G -action on $T_{x_0}(\Sigma)$ is 2-pseudofree. If the G -fixed-point set Σ^G contains at least 2 points then Σ^H is diffeomorphic to the k -dimensional sphere, where k is 0, 1, or 2, for any $H \in S(G) \setminus \{E\}$.*

They obtain the next result as a corollary to the theorem above from S. Illman [13, Theorem 5].

Laitinen–Traczyk Theorem 2. *Suppose the G -action on $T_{x_0}(\Sigma)$ is 2-pseudofree. Then for any $x \in \Sigma^G$, the tangential G -representation $T_x(\Sigma)$ is G -homeomorphic to $T_{x_0}(\Sigma)$. In addition, Σ is G -homeomorphic to the unit sphere of $\mathbb{R} \oplus V$, where $V = T_{x_0}(\Sigma)$.*

They obtained a necessary condition on 2-pseudofree one-fixed-point G -actions on homotopy spheres.

Laitinen–Traczyk Theorem 3. *If $\Sigma^G = \{x_0\}$ and $T_{x_0}(\Sigma)$ is a 2-pseudofree G -representation then $\dim \Sigma = 6$, the group G is isomorphic to A_5 , and $T_{x_0}(\Sigma)$ is the direct sum of two irreducible real G -representations of dimension 3.*

We recall the following facts concerning the existence of one-fixed-point actions of finite group on spheres. Let S^n denote the sphere of dimension n .

- (F1) In [32, Proposition 4.3], E. Stein showed the existence of 3-pseudofree one-fixed-point actions on S^7 of the group $\mathrm{SL}(2, 5) \times Z_m$ satisfying $(m, 30) = 1$.
- (F2) In [19, Theorem A], [22, Theorem A], we showed the existence of 2-pseudofree one-fixed-point actions on S^6 of A_5 .
- (F3) In [3, Theorem 7], A. Bak and the author showed the existence of 3-pseudofree one-fixed-point actions on S^7 of A_5 .

Therefore, putting Laitinen–Traczyk Theorem 3 and (F2) together, we see that a homotopy sphere Σ admits a 2-pseudofree one-fixed-point action of finite group if and only if $\dim \Sigma = 6$.

In the present paper, we will obtain the following two theorems from Laitinen–Traczyk Theorems 1–3.

Theorem 1.1. *Suppose that Σ is of even dimension ≥ 6 and the G -action on $T_{x_0}(\Sigma)$ is 3-pseudofree. If the G -fixed-point set contains at least 2 points then Σ^G is a $\mathbb{Z}_2$ -homology sphere of dimension ≤ 3 , and for any $x \in \Sigma^G$, $T_x(\Sigma)$ is $\langle g \rangle$ -homeomorphic to $T_{x_0}(\Sigma)$ for any $g \in G$.*

Theorem 1.2. *If Σ is of even dimension ≥ 6 , $T_{x_0}(\Sigma)$ is a properly 3-pseudofree G -representation, and $\Sigma^G = \{x_0\}$, then $\dim \Sigma = 6$ and either (1) or (2) below holds.*

- (1) G is isomorphic to the symmetric group S_5 on five letters, and $T_{x_0}(\Sigma)$ is an irreducible real G -representation.
- (2) G is isomorphic to $A_5 \times Z$ such that Z is a group of order 2, and the real G -representations V^Z and V_Z are irreducible and 3-dimensional, where $V = T_{x_0}(\Sigma)$, V^Z is the Z -fixed-point set of V , and V_Z is the orthogonal complement of V^Z in V .

In addition, S. Tamura and the author [27] showed that S_5 does not admit a one-fixed-point action on S^7 , and P. Mizerka [18] showed that $\text{TL}(2, 5)$ (the GAP ID is 240(89)) does not admit an effective one-fixed-point action on S^n for any $n \leq 13$.

In the present paper, we will also prove the next existence result of one-fixed-point actions of finite group on spheres.

Theorem 1.3. *For the integer n , the finite group G , and the real G -representation V described below, there is an effective one-fixed-point G -action on the sphere S of dimension n such that $T_{x_0}(S)$ is isomorphic to V as real G -representations, where x_0 is the G -fixed point of S .*

- (1) $n = 6$:
 - (i) $G = A_5$ and V is a direct sum of two irreducible real G -representations of dimension 3. In this case, the G -action on V is properly 2-pseudofree.
 - (ii) $G = S_5$ and V is an irreducible real G -representation of dimension 6. In this case, the G -action on V is properly 3-pseudofree.
 - (iii) $G = A_5 \times Z$, where Z is a group of order 2, and V has the form $V = V^Z \oplus V_Z$ such that V^Z and V_Z are irreducible real G -representations of dimension 3. In this case, the G -action on V is properly 3-pseudofree.
- (2) $n = 7$:
 - (iv) $G = A_5$ and V is a direct sum of irreducible real G -representations of dimension 3 and 4. In this case, the G -action on V is properly 3-pseudofree.
 - (v) $G = A_5 \times Z$, where Z is a group of order 2, and V has the form $V = V^Z \oplus V_Z$ such that V^Z is an irreducible real G -representation of dimension 3 and V_Z is an irreducible real G -representation of dimension 4. In this case, the G -action on V is properly 3-pseudofree.
- (3) $n = 3 + 4k$ with $k \in \mathbb{N}$:
 - (vi) $G = \text{SL}(2, 5) \times Z_m$, where Z_m is a cyclic group of order m satisfying $(m, 30) = 1$, and V has the form $V = V^{Z \times Z_m} \oplus W$, where $Z = \text{Center}(\text{SL}(2, 5))$, such that $V^{Z \times Z_m}$ is an irreducible real G -representation of dimension 3 and W is a free real G -representation of dimension $4k$. In this case, the G -action on V is properly 3-pseudofree.
- (4) $n = 6 + 8k$ with $k \in \mathbb{N}$:
 - (vii) $G = \text{TL}(2, 5) \times Z_m$, where $\text{TL}(2, 5)$ is the double cover of S_5 of minus type (the GAP ID is 240(89)) with $Z = \text{Center}(\text{TL}(2, 5))$, Z_m is a cyclic group of order m satisfying $(m, 30) = 1$, and V has the form $V = V^{Z \times Z_m} \oplus W$ such that $V^{Z \times Z_m}$ is an irreducible real G -representation of dimension 6 and W is a free real G -representation of dimension $8k$. In this case, the G -action on V is properly 6-pseudofree.

Concerning this result, we note that there exist a free real G -representation of dimension 4 for $G = \text{SL}(2, 5) \times Z_m$ and a free real G -representation of dimension 8 for $G = \text{TL}(2, 5) \times Z_m$ whenever $(m, 30) = 1$. We remark that Theorem 1.3 implies the facts (F1)–(F3) mentioned above.

Next note that the sphere S^n of dimension n admits a properly 3-pseudofree one-fixed-point action of finite group if $n = 6$ or $3 + 4k$ with $k \in \mathbb{N}$. There arises a question: we wonder whether the sphere S^n of dimension $n = 5 + 4k$ with $k \in \mathbb{N}$ admits a properly 3-pseudofree

one-fixed-point action of finite group.

2. Proof of Theorems 1.1 and 1.2

Let Σ be a $\mathbb{Z}$ -homology sphere of even dimension ≥ 6 equipped with a 3-pseudofree G -action and with a G -fixed point x_0 , let V denote the tangential G -representation $T_{x_0}(\Sigma)$ of Σ at x_0 , and let G_0 denote the subgroup

$$\{g \in G \mid \text{the transformation } g : V \rightarrow V \text{ preserves an orientation of } V\}$$

of G . Therefore $|G/G_0| = 1$ or 2 .

Proposition 2.1. *Let H be a subgroup of G . If $\dim V^H = 3$ then the order of H is 2, the generator σ of H acts on V as the scalar -1 , and $\sigma \notin G_0$.*

Proof. We have the decomposition $V = V^H \oplus V_H$ as real H -representations. Since the G -action on V is 3-pseudofree and $\dim V^H = 3$, V_H is a free H -representation. Since $\dim V$ is even and $\dim V^H = 3$, $\dim V_H$ is odd. Therefore $|H| = 2$ and the generator of H acts on V as the scalar -1 . Since $\dim V_H$ is odd, the action of the generator reverses orientations of V_H and V . $\square$

Proposition 2.2. *The G_0 -action on V is 2-pseudofree.*

Proof. Let $H \in \mathcal{S}(G_0) \setminus \{E\}$. Proposition 2.1 says $\dim V^H \leq 2$. $\square$

Proposition 2.3. *Suppose Σ is a homotopy sphere. Then the following holds.*

- (1) *If $|\Sigma^{G_0}| \geq 2$ then for any $H \in \mathcal{S}(G_0) \setminus \{E\}$, Σ^H is diffeomorphic to S^k , where $0 \leq k \leq 2$.*
- (2) *If $|\Sigma^{G_0}| = 1$ then G_0 is isomorphic to A_5 , Σ is diffeomorphic to S^6 , and $\text{res}_{G_0}^G V$ is a direct sum of two irreducible real G_0 -representations of dimension 3.*

Proof. This follows from Proposition 2.2 and Laitinen–Traczyk Theorems 1–3. $\square$

Proof of Theorem 1.1. If $G = G_0$ then the G -action on V is 2-pseudofree and the theorem is clear from Laitinen–Traczyk Theorems 1 and 2. Thus it suffices to prove the theorem in the case $|G/G_0| = 2$. We suppose $|G/G_0| = 2$.

If $G_0 \neq E$ then $\Sigma^{G_0} \cong S^k$, where $0 \leq k \leq 2$, we obtain $\Sigma^G = (\Sigma^{G_0})^{G/G_0} \cong (S^k)^{G/G_0} \cong S^h$ for $h = 0, 1$, or 2 . If $G_0 = E$ then G is a group of order 2, and hence Σ^G is a $\mathbb{Z}_2$ -homology sphere.

Let $x \in \Sigma^G \setminus \{x_0\}$ and set $W = T_x(\Sigma)$. If $\dim \Sigma^G \geq 1$ then Σ^G is connected and hence W is isomorphic to V as real G -representations.

Suppose $\dim \Sigma^G = 0$. Since Σ^G is a $\mathbb{Z}_2$ -homology sphere, we have $\Sigma^G = \{x_0, x\}$. Let $g \in G$. If g belongs to G_0 or $\dim V^g \leq 2$ then Laitinen–Traczyk Theorem 2 implies that W is $\langle g \rangle$ -homeomorphic to V . Finally we suppose $\dim V^g = 3$. Proposition 2.1 says that g is of order 2. Since Σ^g is a $\mathbb{Z}_2$ -homology sphere, we get $\dim W^g = \dim V^g = 3$. Clearly we have $\dim W = \dim V = \dim \Sigma$. Therefore W is isomorphic to V as real $\langle g \rangle$ -representations. $\square$

Proof of Theorem 1.2. Recall that if $|\Sigma^{G_0}| \geq 2$ then Laitinen–Traczyk Theorem 1 implies that Σ^{G_0} is diffeomorphic to S^k for $k = 0, 1$, or 2 . In this case Σ^G is also diffeomorphic to S^h for $h = 0, 1$, or 2 , which is a contradiction. Therefore we get $\Sigma^{G_0} = \Sigma^G = \{x_0\}$, which

implies $\dim \Sigma = 6$, G_0 is isomorphic to A_5 , and $\text{res}_{G_0}^G V$ is a direct sum of two irreducible real G_0 -representations V_1 and V_2 of dimension 3. Since V is properly 3-pseudofree, there is an element $g \in G$ such that $\dim V^g = 3$. Then the order of g is 2 and $g \notin G_0$. Therefore G is isomorphic to S_5 or $A_5 \times Z$ with $Z = \langle g \rangle$. In the case $G \cong S_5$, the irreducibility of V follows from the property $\text{res}_{G_0}^G V = V_1 \oplus V_2$. In the case $G \cong A_5 \times Z$, V is isomorphic to $(V_1 \otimes W_1) \oplus (V_2 \otimes W_2)$, where W_1 and W_2 are real Z -representations of dimension 1. Since V is 3-pseudofree, one of W_1 or W_2 has a nontrivial Z -action and the other has the trivial Z -action. $\square$

3. The element β_G of the Burnside ring of G

Let G be a finite group and let $\Omega(G)$ denote the Burnside ring of G . Each element of $\Omega(G)$ is an equivalence class $[F_1] - [F_2]$ of a pair (F_1, F_2) consisting of finite G -sets F_1 and F_2 . A subgroup H gives the homomorphism $\chi_H : \Omega(G) \rightarrow \mathbb{Z}$ defined by $\chi_H([F_1] - [F_2]) = |F_1^H| - |F_2^H|$.

Suppose that G is nonsolvable. Let $S(G)_{\text{sol}}$ be the set of all solvable subgroups of G and set $S(G)_{\text{nonsol}} = S(G) \setminus S(G)_{\text{sol}}$. Then by [8, (1.3.2), (1.3.3), Proposition 1.3.5], there is a unique element $\beta (= \beta_G)$ of $\Omega(G)$ such that

$$(3.1) \quad \chi_H(\beta) = \begin{cases} 0 & \text{for } H \in S(G)_{\text{nonsol}} \\ 1 & \text{for } H \in S(G)_{\text{sol}}. \end{cases}$$

For a subgroup H of G , we denote by $(H)_G$ the G -conjugacy class of H , i.e.

$$(H)_G = \{gHg^{-1} \mid g \in G\} \subset S(G).$$

For $H, K \in S(G)$, we say that H is *subconjugate* (or *G -subconjugate*) to K and write $(H)_G \leq (K)_G$ if gHg^{-1} is a subgroup of K for some element $g \in G$. There is a unique subset $\text{Iso}(G, \beta)$ of $S(G)$ which is closed under conjugations of elements in G and satisfies

$$(3.2) \quad \beta = \sum_{(H)_G \in \text{Iso}(G, \beta)} a_{(H)_G} [G/H] \quad \text{for some integers } a_{(H)_G} \neq 0.$$

It immediately follows that $\text{Iso}(G, \beta) \subset S(G)_{\text{sol}}$, $\max(S(G)_{\text{sol}}) \subset \text{Iso}(G, \beta)$, and $a_{(H)_G} = 1$ holds for each $H \in \max(S(G)_{\text{sol}})$. By (3.1), β is an idempotent of $\Omega(G)$.

The subgroup lattice of A_5 up to conjugations is as in Figure 1.

In Figure 1, C_m and D_n denote a cyclic group of order m and a dihedral group of order n , respectively, and A_4 denote the alternating group on four letters. There a real line between two subgroups H and K indicates that $gHg^{-1} \triangleleft K$ holds for some $g \in G$, and a dotted line indicates that $gHg^{-1} < K$ holds for some $g \in G$ and $gHg^{-1} \triangleleft K$ does not hold for any $g \in G$.

Proposition 3.1. *Let G be A_5 . Then the idempotent β_G in $\Omega(G)$ has the form*

$$(3.3) \quad \beta_G = [G/A_4] + [G/D_{10}] + [G/D_6] - [G/C_3] - 2[G/C_2] + [G/E],$$

and therefore

$$(3.4) \quad \text{Iso}(G, \beta_G) = (A_4)_G \cup (D_{10})_G \cup (D_6)_G \cup (C_3)_G \cup (C_2)_G \cup (E)_G.$$

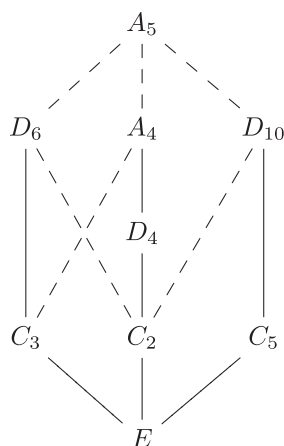


Fig. 1.

Proof. We tabulate the data $|(G/H)^K|$ necessary to determine β_G in Table 1. The proposition readily follows from Table 1. $\square$

Table 1.

K	G	A_4	D_{10}	D_6	C_5	D_4	C_3	C_2	E
G/G	1	1	1	1	1	1	1	1	1
G/A_4	0	1	0	0	0	1	2	1	5
G/D_{10}	0	0	1	0	1	0	0	2	6
G/D_6	0	0	0	1	0	0	1	2	10
G/C_5	0	0	0	0	2	0	0	0	12
G/D_4	0	0	0	0	0	3	0	3	15
G/C_3	0	0	0	0	0	0	2	0	20
G/C_2	0	0	0	0	0	0	0	2	30
G/E	0	0	0	0	0	0	0	0	60

The subgroup lattice of S_5 up to conjugations is as in Figure 2. There $\mathfrak{C}_m$ and $\mathfrak{D}_n$ are a cyclic subgroup and a dihedral subgroup (not of A_5 but) of S_5 of order m and n , respectively, $\mathfrak{F}_{20}$ is a subgroup of order 20, S_3 is a subgroup isomorphic to the symmetric group on 3 letters, $\mathfrak{S}_3\mathfrak{C}_2$ is a subgroup of order 12 isomorphic to $\mathfrak{S}_3 \times \mathfrak{C}_2$, where $\mathfrak{S}_3$ is a subgroup conjugate to S_3 in S_5 .

Proposition 3.2. *Let G be S_5 . Then the idempotent β_G in $\Omega(G)$ has the form*

$$(3.5) \quad \beta_G = [G/S_4] + [G/\mathfrak{F}_{20}] + [G/\mathfrak{S}_3\mathfrak{C}_2] - [G/S_3] - [G/\mathfrak{D}_4] - [G/\mathfrak{C}_4] + [G/\mathfrak{C}_2]$$

and hence

$$(3.6) \quad \text{Iso}(G, \beta_G) = (S_4)_G \cup (\mathfrak{F}_{20})_G \cup (G/\mathfrak{S}_3\mathfrak{C}_2)_G \cup (S_3)_G \cup (\mathfrak{D}_4)_G \cup (\mathfrak{C}_4)_G \cup (\mathfrak{C}_2)_G.$$

Proof. The proposition is easily obtained from Table 2 of the numbers $|(G/H)^K|$. $\square$

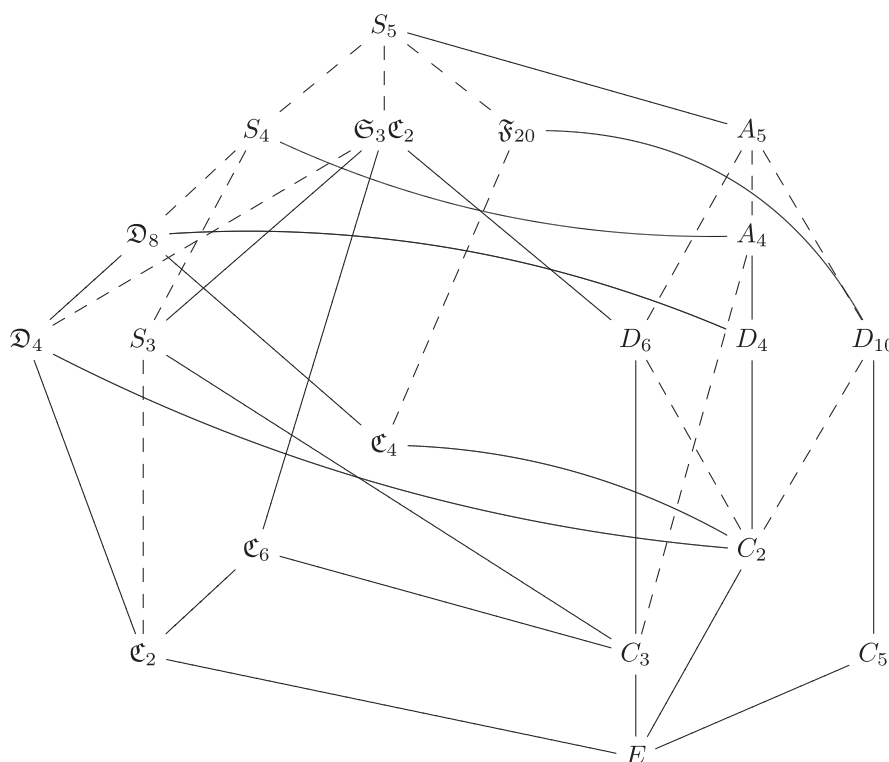


Fig.2.

- (1) For the case $G = A_5 \times Z$ with $|Z| = 2$, β_G is obtained as $f^*\beta_{A_5}$, where $f : G \rightarrow A_5$ is the canonical projection.
- (2) For the case $G = \text{SL}(2, 5) \times Z_m$, β_G is obtained as $g^*\beta_{A_5}$, where $g : \text{SL}(2, 5) \times Z_m \rightarrow A_5$ is an epimorphism.
- (3) For the case $G = \text{TL}(2, 5) \times Z_m$, β_G is obtained as $h^*\beta_{S_5}$, where $h : \text{TL}(2, 5) \times Z_m \rightarrow S_5$ is an epimorphism.

DEFINITION 3.1. We say that V is *ample* for β_G if $\text{Iso}(G, \beta_G) \setminus \max(\mathcal{S}_{\text{sol}}(G))$ is contained in $\text{Iso}(G, V \setminus \{0\})$.

- (1) Case $G = A_5$ and V containing an irreducible real G -representation of dimension 3.
- (2) Case $G = S_5$ and V containing an irreducible real G -representation of dimension 6.
- (3) Case $G = A_5 \times Z$, where $|Z| = 2$, and V such that V^Z contains an irreducible real G -representation of dimension 3.
- (4) Case $G = \mathrm{SL}(2, 5) \times Z_m$, where $(m, 30) = 1$, and V such that $V^{Z \times Z_m}$ contains an irreducible real G -representation of dimension 3, where Z is the center of $\mathrm{SL}(2, 5)$.
- (5) $G = \mathrm{TL}(2, 5) \times Z_m$, where $(m, 30) = 1$, and V such that $V^{Z \times Z_m}$ contains an irreducible

Table 2.

[illegible]

real G -representation of dimension 6, where Z is the center of $\mathrm{TL}(2, 5)$.

Proof. First we consider the case $G = A_5$. Clearly we have

$$\max(S(G)_{\mathrm{sol}}) = (A_4)_G \cup (D_{10})_G \cup (D_6)_G.$$

Let W be an irreducible real G -representation of dimension 3. Then the H -fixed-point set W^H , $H \in S(G)$, has the dimension as in Table 3.

Table 3.

H	C_5	C_3	C_2	E	Non Cyclic
$\dim W^H$	1	1	1	3	0

Table 3 shows

$$(3.7) \quad \mathrm{Iso}(G, W \setminus \{0\}) = (E)_G \cup (C_2)_G \cup (C_3)_G \cup (C_5)_G.$$

It follows from (3.4) and (3.7) that W is ample for β_G .

Second we consider the case $G = S_5$. It follows readily that

$$\max(S(G)_{\mathrm{sol}}) = (S_4)_G \cup (\mathfrak{F}_{20})_G \cup (\mathfrak{S}_3\mathfrak{C}_2)_G.$$

Let W be an irreducible real G -representation of dimension 6. Then the H -fixed-point set W^H , $H \in S(G)$, has the dimension as in Table 4.

Table 4.

H	S_3	$\mathfrak{C}_6$	C_5	$\mathfrak{D}_4$	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E	$H \in \mathcal{K}$
$\dim W^H$	1	1	2	1	1	2	3	2	6	0

where $\mathcal{K} = \{G, A_5, S_4, \mathfrak{S}_3\mathfrak{C}_2, \mathfrak{F}_{20}, A_4, D_{10}, \mathfrak{D}_8, D_6, D_4\}$. Table 4 shows

$$(3.8) \quad \begin{aligned} \mathrm{Iso}(G, W \setminus \{0\}) = & (E)_G \cup (C_2)_G \cup (\mathfrak{C}_2)_G \cup (C_3)_G \\ & \cup (\mathfrak{C}_4)_G \cup (\mathfrak{D}_4)_G \cup (C_5)_G \cup (\mathfrak{C}_6)_G \cup (S_3)_G. \end{aligned}$$

It follows from (3.6) and (3.8) that W is ample for β_G .

The ampleness of V for β_G in the cases (3), (4) and (5) follows from that in the cases (1) and (2). $\square$

4. Definition of G -framed maps

Let G be a finite nonsolvable group and let I denote the closed unit interval $[0, 1]$. For a space A and a map $g : P \rightarrow Q$, we denote by $A \times g$ the map $id_A \times g : A \times P \rightarrow A \times Q$. For a space A and a pair $\mathbf{g} = (g, c)$ of maps $g : P \rightarrow Q$ and $c : S \rightarrow T$, we denote by $A \times \mathbf{g}$ the pair $(A \times g, A \times c)$. In this paper, we mean by a G -framed map $\mathbf{f}$ a pair (f, b) consisting of a G -map $f : (X, \partial X) \rightarrow (Y, \partial Y)$ between G -manifolds X and Y with boundaries ∂X and ∂Y , respectively, where the case $\partial X = \partial Y = \emptyset$ is possible, and a G -bundle isomorphism $b : \tau_X \rightarrow f^*\tau_Y$, where $\tau_X = \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^\ell)$ and $\tau_Y = \varepsilon_Y(\mathbb{R}) \oplus T(Y) \oplus \varepsilon_Y(\mathbb{R}^\ell)$ and we suppose $\ell \geq \dim X + 2$. In this situation, the equality $\dim X^H = \dim Y^H$ holds for all $H \in S(G)$ such that $X^H \neq \emptyset$, because $\dim X^H$ is equal to the fiber dimension of the real vector bundle $T(X)^H$ and it is true for X replaced by Y .

In this paper, unless otherwise stated, for G -framed maps $\mathbf{f} = (f, b)$, $\mathbf{f}' = (f', b')$, $\mathbf{f}'' = (f'', b'')$, $\dots$, the source manifolds of $f, f', f'', \dots$, are $(X, \partial X), (X', \partial X'), (X'', \partial X''), \dots$ and the target manifolds of them are same $(Y, \partial Y)$, and we suppose that $\partial X = \partial X' = \partial X'' = \dots = \partial Y$. A homotopy $\mathbf{F}$ from $\mathbf{f}$ to $\mathbf{f}'$ means a pair (F, B) consisting of a G -map $F : I \times X \rightarrow I \times Y$ and a G -bundle isomorphism

$$B : T(I \times X) \oplus \varepsilon_{I \times X}(\mathbb{R}^\ell) \rightarrow F^*T(I \times Y) \oplus \varepsilon_{I \times Y}(\mathbb{R}^\ell)$$

satisfying the following conditions.

- (1) $p_I(F(t, x)) = t$ for all $t \in I$ and $x \in X$, where p_I is the projection $I \times Y \rightarrow I$.
- (2) The restriction of $\mathbf{F}$ to $\{0\} \times X$ coincides with $\{0\} \times \mathbf{f}$.
- (3) The restriction of $\mathbf{F}$ to $\{1\} \times X$ coincides with $\{1\} \times \mathbf{f}'$.
- (4) The restriction of $\mathbf{F}$ to $I \times \partial X$ coincides with $I \times \mathbf{f}|_{\partial X}$, where $\mathbf{f}|_{\partial X}$ is the restriction of $\mathbf{f}$ to ∂X .

A G -framed cobordism $\mathbf{F}$ from $\mathbf{f}$ to $\mathbf{f}'$ rel. boundary (or rel. ∂) means a pair (F, B) consisting of a G -map

$$(4.1) \quad F : (W, \partial_0 W, \partial_1 W, \partial_{01} W) \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$

and a G -bundle isomorphism

$$B : T(W) \oplus \varepsilon_W(\mathbb{R}^\ell) \rightarrow F^*T(Z) \oplus \varepsilon_Z(\mathbb{R}^\ell),$$

where $\partial_0 W, \partial_1 W$, and $\partial_{01} W$ are G -manifolds canonically identified with X, X' , and $I \times \partial Y$, respectively, and $Z = I \times Y$, $\partial_0 Z = \{0\} \times Y$, $\partial_1 Z = \{1\} \times Y$ and $\partial_{01} Z = I \times \partial Y$, satisfying the following conditions.

- (1) $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_{01} W$, $\partial_0 W \cap \partial_1 W = \emptyset$, $\partial_0 W \cap \partial_{01} W = \partial(\partial_0 W)$, $\partial_1 W \cap \partial_{01} W = \partial(\partial_1 W)$, and $\partial(\partial_{01} W) = \partial(\partial_0 W) \amalg \partial(\partial_1 W)$.
- (2) The restriction of $\mathbf{F}$ to $\partial_0 W$ coincides with $\mathbf{f}$ up to homotopies of G -framed maps rel. ∂ .
- (3) The restriction of $\mathbf{F}$ to $\partial_1 W$ coincides with $\mathbf{f}'$ up to homotopies of G -framed maps rel. ∂ .
- (4) The restriction of $\mathbf{F}$ to $\partial_{01} W$ coincides with $I \times \mathbf{id}_Y|_{\partial Y} (= I \times \mathbf{id}_X|_{\partial X})$, where $\mathbf{id}_Y|_{\partial Y}$ is the restriction of $\mathbf{id}_Y$ to ∂Y .

Here the G -cobordism W from X and X' is not necessarily diffeomorphic to $I \times X$. For a subset A of X , if $I \times A \subset W$ and the restriction of $\mathbf{F}$ to $I \times A$ coincides with $I \times \mathbf{f}|_A$ up to G -homotopies of G -framed maps then we call $\mathbf{F}$ a G -framed cobordism rel. A . Let $\mathcal{F}$ be a G -conjugation-invariant set of subgroups of G , i.e. if $H \in \mathcal{F}$ then $(H)_G \subset \mathcal{F}$. If $\mathbf{F}$ is a G -framed map rel. a G -regular neighborhood of $\bigcup_{K \in \mathcal{F}} X^K$, we say that $\mathbf{F}$ is a G -framed map rel. $\mathcal{F}$. For a G -framed map $\mathbf{F} = (F, B)$, the map F in (4.1) will be written as $F : W \rightarrow I \times Y$ for simplicity of notation when the context is clear.

Let M be a subgroup of G . Hereafter $\mathbf{F}_M = (F_M, B_M)$, $\mathbf{F}'_M = (F'_M, B'_M)$, $\mathbf{F}''_M = (F''_M, B''_M)$, $\dots$, are M -framed cobordisms with M -maps $F_M : W_M \rightarrow I \times Y$, $F'_M : W'_M \rightarrow I \times Y$, $F''_M : W''_M \rightarrow I \times Y$, $\dots$, respectively.

Let V be a real G -representation being $S(G)_{\text{nonsol}}$ -free, i.e.

$$\dim V^H = 0 \text{ for all } H \in S(G)_{\text{nonsol}}.$$

Hereafter, unless otherwise stated, Y will be the unit disk $D(V)$ of V with respect to a G -invariant inner product. Clearly, the boundary of Y is obviously the unit sphere $S(V)$. Remark that if V is faithful then the G -action Y is effective and therefore the G -action on X is also effective. We assume that a G -framed map $f = (f, b)$, where $f : (X, \partial X) \rightarrow (Y, \partial Y)$, satisfies the boundary condition that $\partial X = \partial Y$ and there is a G -collar neighborhood C of ∂X in X such that the restriction $f|_C = (f|_C, b|_C)$ of f to C is the identity G -framed map on C . This clearly requires that C is also a G -collar neighborhood of ∂Y in Y .

5. G -connected sums of G -framed maps

Let $f = (f, b)$ be a G -framed map with target $Y = D(V)$. We have the canonical G -bundle isomorphisms $f^* \varepsilon_Y(\mathbb{R}) \rightarrow \varepsilon_X(\mathbb{R})$, $f^* \varepsilon_Y(\mathbb{R}^\ell) \rightarrow \varepsilon_X(\mathbb{R}^\ell)$, $T(Y) \rightarrow \varepsilon_Y(V)$, and $f^* T(Y) \rightarrow \varepsilon_X(V)$. Let $\mathfrak{o}^1$ and $\mathfrak{o}^\ell$ be the canonical orientations of $\mathbb{R}$ and $\mathbb{R}^\ell$, respectively. For a subgroup H of G , we get the induced orientations $\mathfrak{o}_{Y^H}^1$, $\mathfrak{o}_{Y^H}^\ell$, $\mathfrak{o}_{X^H}^1$, $\mathfrak{o}_{X^H}^\ell$, of $\varepsilon_{Y^H}(\mathbb{R})$, $\varepsilon_{Y^H}(\mathbb{R}^\ell)$, $\varepsilon_{X^H}(\mathbb{R})$, $\varepsilon_{X^H}(\mathbb{R}^\ell)$, respectively. Note that $T(Y^H) = T(Y)^H = \varepsilon_{Y^H}(V^H)$, $(f^* T(Y))^H = f^{H*} T(Y^H)$. Let $\tau_X^H = \varepsilon_{X^H}(\mathbb{R}) \oplus T(X^H) \oplus \varepsilon_{X^H}(\mathbb{R}^\ell)$. There are two possibilities in choice of an orientation of $sV^H = \mathbb{R} \oplus V^H \oplus \mathbb{R}^\ell$ even if $\dim V^H = 0$. Fix an orientation $\mathfrak{o}_{sV^H}$ of sV^H . This induces the orientation $\mathfrak{o}_{\tau_Y^H}$ of $\tau_Y^H = \varepsilon_{Y^H}(\mathbb{R}) \oplus T(Y^H) \oplus \varepsilon_{Y^H}(\mathbb{R}^\ell)$, and $\mathfrak{o}_{\tau_X^H}$ of τ_X^H via b^H . In this paper we refer to $\mathfrak{o}_{\tau_Y^H}$ and $\mathfrak{o}_{\tau_X^H}$ as orientations of Y^H and X^H , respectively. Without loss of any generality, we can assume that the restriction $\mathfrak{o}_{\tau_Y^H}|_{y_0}$ of $\mathfrak{o}_{\tau_Y^H}$ to $y_0 = 0 \in Y$ coincides with $\mathfrak{o}^1 \cup \mathfrak{o}'$ for some orientation $\mathfrak{o}'$ of $V^H \oplus \mathbb{R}^\ell$.

Let $\Sigma(X, Y)$ denote the union $X \cup_{\partial} Y$ of X and Y glued along the boundary $\partial Y = \partial X$. Here $\Sigma(X, Y)$ is a G -manifold. We have the G -map $\Sigma(f, id_Y) : \Sigma(X, Y) \rightarrow Y$ such that the restrictions $\Sigma(f, id_Y)|_X$ and $\Sigma(f, id_Y)|_Y$ are f and id_Y , respectively. We call $\Sigma(X, Y)$ and $\Sigma(f, id_Y)$ the *quasisphericalizations* of X and f , respectively. For a while let Z be the quasisphericalization of X . The stable tangent bundle $\tau_Z^H = \varepsilon_{Z^H}(\mathbb{R}) \oplus T(Z^H) \oplus \varepsilon_{Z^H}(\mathbb{R}^\ell)$ of Z^H has the orientation $\mathfrak{o}_{\tau_Z^H}$ extending $\mathfrak{o}_{\tau_X^H}$ such that the restriction $\mathfrak{o}_{\tau_Z^H}|_{y_0}$ of $\mathfrak{o}_{\tau_Z^H}$ to y_0 coincides with $(-\mathfrak{o}^1) \cup \mathfrak{o}'$. The restriction of $\Sigma(f, id_Y)^H$ to a small disk-neighborhood of y_0 in $\Sigma(X, Y)^H$ is orientation reversing.

Let $D_X(x)$ and $D_Y(y_0)$ be small H - and G -disk-neighborhoods of x and y_0 in X and Y , respectively. For a subset A of X or Y , the interior of A in X or Y is denoted by $\overset{\circ}{A}$. Suppose that the isotropy subgroup of G at x is H and the restriction $f|_{D_X(x)} : D_X(x) \rightarrow D_Y(y_0)$ of f is an H -diffeomorphism such that $f^K : X^K \rightarrow Y^K$ is locally orientation preserving at x for any $K \leq H$. Then $\psi = f|_{G \cdot D_X(x)} : G \cdot D_X(x) \rightarrow G \times_H D_Y(y_0)$ is a G -diffeomorphism. The G -manifold

$$X \#_{G, H, x, y_0} (G \times_H \Sigma(X, Y)) = (X \setminus G \cdot \overset{\circ}{D}_X(x)) \cup_{\varphi} G \times_H (\Sigma(X, Y) \setminus \overset{\circ}{D}_Y(y_0)),$$

where

$$\varphi : G \times_H \partial D_Y(y_0) \rightarrow G \cdot \partial D_X(x)$$

is the restriction of ψ^{-1} , is called the G -connected sum of X and $\Sigma(X, Y)$ of isotropy type $(H)_G$ with respect to points x and y_0 . For any subgroup K of G , the manifold $(X \#_{G, H, x, y_0} (G \times_H \Sigma(X, Y)))^K$ has the orientation of which the restriction to $(X \setminus G \cdot \overset{\circ}{D}_X(x))^K$ coincides with the restriction of $\mathfrak{o}_{\tau_X^K}$ to $(X \setminus G \cdot \overset{\circ}{D}_X(x))^K$. We get the G -map $f \#_{G, H, x, y_0} G \times_H \Sigma(f, id_Y)$ glu-

ing the restriction of f to $X \setminus G \cdot \overset{\circ}{D}_X(x)$ and the restriction of $G \times_H \Sigma(f, id_Y)$ to $G \times_H (\Sigma(X, Y) \setminus \overset{\circ}{D}_Y(y_0))$. We mean by $([G/G] + [G/H])X$ and $([G/G] + [G/H])f$ the G -manifold $X \#_{G, H, x, y_0} (G \times_H \Sigma(X, Y))$ and the G -map $f \#_{G, H, x, y_0} (G \times_H \Sigma(f, id_Y))$, respectively.

On the other hand, the G -manifold

$$X \#_{G, H, x, x} (G \times_H -\Sigma(X, Y)) = (X \setminus G \cdot \overset{\circ}{D}_X(x)) \cup_{\iota} G \times_H (\Sigma(X, Y) \setminus \overset{\circ}{D}_X(x)),$$

where

$$\iota : G \times_H \partial D_X(x) \rightarrow G \cdot \partial D_X(x)$$

is the canonical map, is called the G -connected sum of X and $-\Sigma(X, Y)$ of isotropy type $(H)_G$ with respect to points $x \in X$ and $x \in -\Sigma(X, Y)$. For any subgroup K of G , the manifold $(X \#_{G, H, x, x} (G \times_H -\Sigma(X, Y)))^K$ has the orientation of which the restriction to $(X \setminus G \cdot \overset{\circ}{D}_X(x))^K$ coincides with the restriction of $\nu_{\tau_X^K}$ to $(X \setminus G \cdot \overset{\circ}{D}_X(x))^K$. We get the G -map $f \#_{G, H, x, x} G \times_H -\Sigma(f, id_Y)$ gluing the restriction of f to $X \setminus G \cdot \overset{\circ}{D}_X(x)$ and the restriction of $G \times_H \Sigma(f, id_Y)$ to $\Sigma(X, Y) \setminus \overset{\circ}{D}_X(x)$. We mean by $([G/G] - [G/H])X$ and $([G/G] - [G/H])f$ the G -manifold $X \#_{G, H, x, x} (G \times_H -\Sigma(X, Y))$ and the G -map $f \#_{G, H, x, x} (G \times_H -\Sigma(f, id_Y))$, respectively.

Let γ_0 and $\gamma = \gamma_0 + [G/H]$ (resp. $\gamma = \gamma_0 - [G/H]$) be elements of the Burnside ring $\Omega(G)$. Suppose that $\text{Iso}(G, \gamma_0) \cup \text{Iso}(G, \gamma) \subset \text{Iso}(G, V \setminus \{0\})$. As an inductive step, we assume that we have obtained $\gamma_0 X$ and $\gamma_0 f$. Suppose there is $x \in (\gamma_0 f)^{-1}(y_0)$ with $G_x = H$ such that $\gamma_0 f$ is transverse regular to $\{y_0\}$ in Y and $(\gamma_0 f)^K$ is locally orientation preserving at x for every $K \leq H$. Then similarly to the construction above of $([G/G] \pm [G/H])X$ and $([G/G] \pm [G/H])f$, we can obtain the equivariant connected sums

$$\begin{aligned} (5.1) \quad \gamma X &= \gamma_0 X \#_{G, H, x, y_0} (G \times_H \Sigma(X, Y)), \\ \gamma f &= \gamma_0 f \#_{G, H, x, y_0} (G \times_H \Sigma(f, id_Y)), \\ (\text{resp. } \gamma X &= \gamma_0 X \#_{G, H, x, x} (G \times_H -\Sigma(X, Y)), \\ \gamma f &= \gamma_0 f \#_{G, H, x, x} (G \times_H -\Sigma(f, id_Y))). \end{aligned}$$

6. Basic lemmas on the reflection method

Let $M \in \mathcal{S}(G)_{\text{sol}}$, $\mathbf{f} = (f, b)$ a G -framed map and $\mathbf{F}_M = (F_M, B)$ a G -framed cobordism from $\text{res}_M^G \mathbf{f}$ to $\text{res}_M^G \mathbf{id}_Y$ rel. ∂ . Here we recall that $Y = D(V)$, $f : (X, \partial X) \rightarrow (Y, \partial Y)$, and $F_M : W_M \rightarrow I \times Y$. For a submanifold Z of X and an embedding $\Psi : I \times Z \rightarrow W_M$, we call Ψ a *product embedding* if

- (1) $\Psi(t, x) = (t, x)$ in $\partial_{01} W_M$ for all $x \in Z \cap \partial X$ and $t \in I$,
- (2) $\Psi(t, x) = (t, x)$ in a collar neighborhood $C_X = [0, \delta] \times X$ of $\{0\} \times X$ in W_M for all $t \in [0, \delta]$ and $x \in Z$, and
- (3) $\Psi(1 - t, x) = (1 - t, \psi(x))$ in a collar neighborhood $C_Y = [1 - \delta, 1] \times Y$ of $\{1\} \times Y$ in W_M for all $t \in [0, \delta]$ and $x \in Z$, for some embedding $\psi : Z \rightarrow Y$.

Here δ is a small positive real number and $[0, \delta]$ and $[1 - \delta, 1]$ are the closed intervals $\subset \mathbb{R}$. For $K \in \mathcal{S}(G)$ and a K -subcomplex Z of X with respect to a smooth G -triangulation of X , let $N_K(Z, X)$ denote a K -regular neighborhood of Z in X . Therefore for $H \in \mathcal{S}(G)$, $N_K(X^H, X)$

is a K -tubular neighborhood of X^H , where $K = N_G(H)$. By virtue of the G -isomorphism b , the restriction $f^H : X^H \rightarrow Y^H$ of f is K -homotopic to a diffeomorphism if and only if the restriction $f|_{N_K(X^H, X)} : N_K(X^H, X) \rightarrow N_K(Y^H, Y)$ of f is K -homotopic to a diffeomorphism, where $K = N_G(H)$. For a subgroup H of G , we denote by $\mathcal{U}_G(H)$ the set of subgroups K of G satisfying $H < K$. For $H \in \mathcal{S}(M)$, we call the set

$$X^{>H} = \bigcup_{K \in \mathcal{U}_G(H)} X^K$$

the G -singular set of X at H .

DEFINITION 6.1. Let H be a subgroup of G satisfying $N_G(H) \subset M$. We say that (X, Y, W_M) has the (G, M) -tame singular set at H (or $X^{>H}$ is (G, M) -tame in (X, W_M)) if there is a product M -embedding $\Psi_M : I \times N_M(M \cdot X^{>H}, X) \rightarrow W_M$ such that $\text{Im}(\Psi_M)^{>H} = W_M^{>H}$.

For a subgroup $K \in \mathcal{S}(G)$, let

$$(6.1) \quad \begin{aligned} \mathcal{V}_G(K) &= \mathcal{S}(G) \setminus \bigcup_{L \in (K)_G} \mathcal{S}(L), \text{ and} \\ \mathcal{V}_{M,G}(K) &= \mathcal{S}(M) \setminus \bigcup_{L \in (K)_G} \mathcal{S}(M \cap L). \end{aligned}$$

We remark that if $H \in \mathcal{S}(M)$, $N_G(H) \subset M$, and $(H)_G \cap \mathcal{S}(M) = (H)_M$ then

$$(6.2) \quad \{g \in G \mid gHg^{-1} \subset M\} \subset M.$$

The modification of G -framed maps by following Lemmas 6.1, 6.2, and 6.4 is called the *reflection method* in G -surgery theory.

Lemma 6.1. Let $M \in \mathcal{S}(G)_{\text{sol}}^*$ and $H \in \mathcal{S}(M)$ satisfying $N_G(H) \subset M$. Suppose the following.

- (i) (X, Y, W_M) has the (G, M) -tame singular set at H with respect to a product M -embedding $\Psi_M : I \times N_M(M \cdot X^{>H}, X) \rightarrow W_M$.
- (ii) There is an M -homotopy

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ such that $\mathbb{H}_M|_{W_M \times \{0\}}$ coincides with F_M and $\mathbb{H}_M|_{\text{Im}(\Psi_M) \times \{1\}}$ is a diffeomorphism.

Then there are

- a G -framed map $\mathbf{f}'$ rel. ∂ , where $\mathbf{f}' = (f', b')$ and $f' : (X', \partial X') \rightarrow (Y, \partial Y)$,
- a G -framed cobordism $\mathbf{F}_G$ from $\mathbf{f}$ to $\mathbf{f}'$ rel. ∂ and $\mathcal{V}_G(H)$,
- an M -framed cobordism $\mathbb{F}_M$ from $\text{res}_M^G \mathbf{F}_G \cup_{\text{res}_M^G \mathbf{f}} \mathbf{F}_M$ to $\mathbf{F}'_M$ rel. ∂ and $\mathcal{V}_M(H)$, where $\mathbf{F}'_M = (F'_M, B'_M)$ is an M -framed cobordism from $\text{res}_M^G \mathbf{f}'$ to $\text{res}_M^G \text{id}_Y$ rel. ∂ and $\mathcal{V}_{M,G}(H)$, and

$$F'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$

- a product M -embedding $\Phi'_M : I \times N_M(M \cdot X'^H, X') \rightarrow W'_M$ with $\text{Im}(\Phi'_M) = N_M(M \cdot W_M'^H, W'_M)$, and

- an M -homotopy

$$\mathbb{H}'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

$$\text{rel. } \partial_1 W'_M \cup \partial_{01} W'_M$$

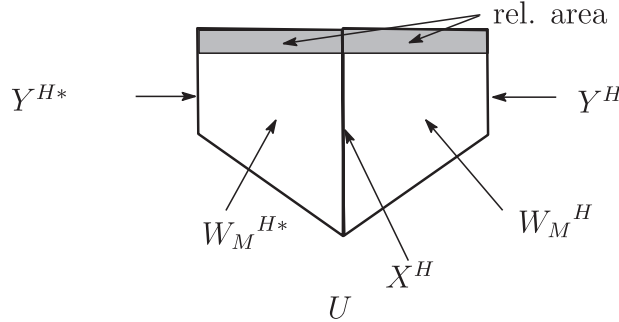
possessing the following properties.

- (1) $N_M(M \cdot X'^{>H}, X') = N_M(M \cdot X^{>H}, X)$, $N_M(M \cdot W_M'^{>H}, W'_M) = N_M(M \cdot W_M^{>H}, W_M)$, and $\Phi'_M|_{I \times N} = \Psi_M|_{I \times N}$ for $N = N_M(M \cdot X'^H, X') \cap N_M(M \cdot X^{>H}, X)$.
- (2) $\mathbb{H}'_M|_{W'_M \times \{0\}}$ coincides with F'_M , $\mathbb{H}'_M|_{N_M(M \cdot W_M'^H, W'_M) \times \{1\}}$ is a diffeomorphism, and $\mathbb{H}'_M|_{\Phi'_M(I \times N) \times I}$ coincides with $\mathbb{H}_M|_{\Psi_M(I \times N) \times I}$ for N above.

In particular, X'^H is $N_G(H)$ -diffeomorphic rel. ∂ to Y^H and $f'^H : X'^H \rightarrow Y^H$ is $N_G(H)$ -homotopic rel. ∂ to a diffeomorphism.

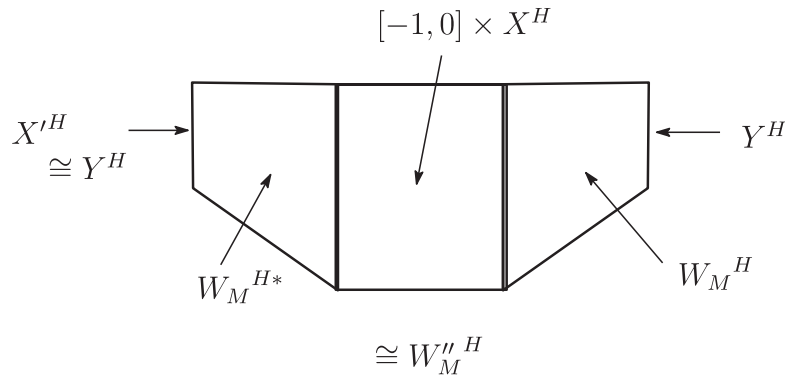
REMARK 6.1. If $(H)_G|_M = (H)_M$, where $(H)_G|_M = (H)_G \cap S(M)$, then the properties (1) and (2) in Lemma 6.1 are true for H replaced by arbitrary $H' \in (H)_G|_M$.

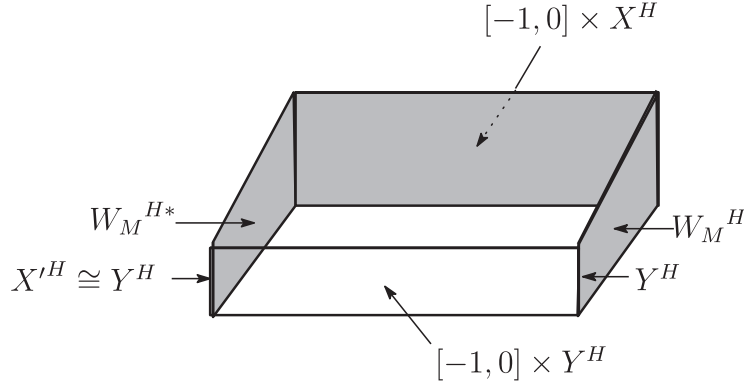
Proof. By virtue of Ψ_M , we can regard W_M^H is an $N_G(H)$ -cobordism from X^H to Y^H rel. $X^{>H} \cup \partial X^H$. Let W_M^{H*} be a copy of W_M^H and let Y^{H*} and $\Psi_M(\{1\} \times N_M(M \cdot X^{>H}, X))^{H*}$ be the copies of Y^H and $\Psi_M(\{1\} \times N_M(M \cdot X^{>H}, X))^H$, respectively, in W_M^{H*} . Then the union $U = W_M^{H*} \cup_{X^H} W_M^H$ of W_M^{H*} and W_M^H attached along X^H can be regarded as an $N_G(H)$ -cobordism rel. ∂ and $\Psi_M(\{1\} \times N_M(M \cdot X^{>H}, X))^{H*}$ from Y^{H*} to Y^H .



In addition, the associated map $f^{H*} : Y^{H*} \rightarrow Y^H$ is a copy of the identity map on Y^H . Let $F_G = (F_G, B_G)$, where $F_G : W_G \rightarrow I \times Y$, be the G -framed cobordism from $\mathbf{f}$ to $\mathbf{f}'$ rel. ∂ obtained by G -surgeries on X of isotropy type $(H)_G$ such that $W_G^H = W_M^{H*}$. Then $\mathbf{f}'$ is a desired G -framed map.

Let us observe F_G above. Set $W''_M = W_G \cup_X W_M$ and $\mathbf{F}''_M = (F''_M, B''_M)$, where $F''_M = F_G \cup_f F_M$ and $B''_M = B_G \cup_b B_M$. The following two pictures





show that $W_M^{\prime\prime\prime H} = W_G^H \cup_{X^H} W_M^H$ is $N_M(H)$ -cobordant rel. ∂ to the product cobordism $I \times Y^H$. Therefore $F_M^{\prime\prime\prime}$ is M -framed cobordant rel. ∂ to an M -framed cobordism $F'_M = (F'_M, B'_M)$, where $F'_M : W'_M \rightarrow I \times Y$, such that $W_M^{\prime H}$ is $N_G(H)$ -diffeomorphic rel. ∂ to $I \times Y^H$ and $F_M^{\prime H} : W_M^{\prime H} \rightarrow I \times Y^H$ is $N_G(H)$ -homotopic to a diffeomorphism. We can formalize the above observation to Lemma 6.1. $\square$

Lemma 6.2. *Let M , H and Z be as in Lemma 6.1. Invoke the following hypotheses (i)–(iii).*

- (i) $(\text{res}_M^G X, \text{res}_M^G Y, W_M)$ has the (M, M) -tame singular set at H with respect to a product M -embedding $\Psi_M : I \times N_M(M \cdot (\text{res}_M^G X)^{>H}, \text{res}_M^G X) \rightarrow W_M$.
- (ii) *There is an M -homotopy*

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ such that $\mathbb{H}_M|_{W_M \times \{0\}}$ coincides with F_M and $\mathbb{H}_M|_{\text{Im}(\Psi_M) \times \{1\}}$ is a diffeomorphism.

- (iii) *There is $K \in \mathcal{U}_M(H)$ such that $\dim Y^H = \dim Y^K > 0$ and $X^{>H} = X^K$.*

Then the conclusion same as Lemma 6.1 holds. In particular, $X'^H = X'^K = X^K$, $f'^H = f'^K = f^K$, and $W_M^{\prime H} = W_M^{\prime K} = W_M^K$ for some $K \in \mathcal{U}_M(H)$.

Proof. If K_1 and K_2 both satisfy the conditions required for K in (iii) then so does $K_1 \cap K_2$. Let K be the smallest subgroup satisfying the conditions in (iii). Then we have $X^{>H} = X^K$ and $X^H = X^K \amalg X^{=H}$. In addition $W_M^H = W_M^{>H} \amalg W_M^{=H} = W_M^K \amalg W_M^{=H}$ follows from (i) and (iii). Let W_M^* be a copy of W_M . Then $W_M^{*H} \cup_{X^H} W_M^H$ is $N_M(H)$ -cobordant rel. ∂ to $W_M^{*K} \cup_{X^H} W_M^K$ by M -surgeries of isotropy type $(H)_M$. Therefore, we can remove $X^{=H}$ and $W_M^{=H}$ by G -surgeries on f and M -surgeries on F_M of isotropy types $(H)_G$ and $(H)_M$, respectively. $\square$

Define $\mathcal{Y}(G, M, H)$ by

$$(6.3) \quad \mathcal{Y}(G, M, H) = \{K \in \mathcal{U}_G(H) \mid K \cap M = H\}.$$

Let Z be a G -manifold. We say that Z satisfies the *primitive gap condition* for (G, M, H) if the following conditions are satisfied.

- (1) $\dim Z_\alpha^H > \dim Z_\beta^K$ for all $K \in \mathcal{U}_M(H)$, $\alpha \in \pi_0(Z^H)$ and $\beta \in \pi_0(Z^K)$ with $Z_\beta \subset Z_\alpha$.
- (2) $\dim Z^K = 0$ for all $K \in \mathcal{Y}(G, M, H)$.

Lemma 6.3. *Let $M \in \mathcal{S}(G)_{\text{sol}}$ and $H \in \mathcal{S}(M)$ such that $N_G(H) \subset M$. Suppose the following conditions are fulfilled.*

- (1) $(\text{res}_M^G X, \text{res}_M^G Y, W_M)$ has the (M, M) -tame singular set at H .
- (2) X satisfies the primitive gap condition for (G, M, H) .
- (3) W_M^H is connected.

Then (X, Y, W_M) has the (G, M) -tame singular set at H .

Proof. The set $X(\mathcal{Y}) = \bigcup_{K \in \mathcal{Y}(G, M, H)} X^K$ is a finite set. Therefore it is easy to obtain a product $N_M(H)$ -embedding $I \times X(\mathcal{Y}) \rightarrow W_M^H \setminus W_M^{>H}$ and to obtain a product M -embedding $\Psi_M : I \times N_M(M \cdot X^{>H}, X) \rightarrow W_M$ such that $\text{Im}(\Psi_M)^{>H} = W_M^{>H}$. $\square$

The next lemma follows from Lemmas 6.1 and 6.3.

Lemma 6.4. *Let M, H, Z be as in Lemma 6.1. Suppose the following (i)–(iv).*

- (i) $(\text{res}_M^G X, \text{res}_M^G Y, W_M)$ has the (M, M) -tame singular set with respect to a product M -embedding $\Psi_M : I \times N_M(M \cdot (\text{res}_M^G X)^{>H}, X) \rightarrow W_M$.
- (ii) *There is an M -homotopy*

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ such that $\mathbb{H}_M|_{W_M \times \{0\}}$ coincides with F_M and $\mathbb{H}_M|_{\text{Im}(\Psi_M) \times \{1\}}$ is a diffeomorphism.

- (iii) X satisfies the primitive gap condition for (G, M, H) .
- (iv) W_M^H is connected.

Then the conclusion same as Lemma 6.1 holds.

In the rest of this section we give a remark on the (G, M) -tame singularity. Let Z be a G -manifold. We say that Z satisfies the *gap condition at H* if

$$(6.4) \quad 2 \dim Z_\beta^K < \dim Z_\alpha^H$$

holds for all $K \in \mathcal{U}_G(H)$, $\alpha \in \pi_0(Z^H)$, $\beta \in \pi_0(Z^K)$ with $Z_\beta^K \subset Z_\alpha^H$, where Z_α^H and Z_β^K stand for the underlying spaces of α and β . We say that Z satisfies the *cobordism gap condition at H* if

- (1) $\dim Z_\beta^K + \dim Z_\gamma^L + 1 < \dim Z_\alpha^H$ holds for all $K \in \mathcal{U}_G(H) \setminus \mathcal{U}_M(H)$, $L \in \mathcal{U}_M(H)$, $\alpha \in \pi_0(Z^H)$, $\beta \in \pi_0(Z^K)$ with $Z_\beta^K \subset Z_\alpha^H$, $\gamma \in \pi_0(Z^L)$ with $Z_\gamma^L \subset Z_\alpha^H$, and
- (2) $2 \dim Z_\beta^K + 1 < \dim Z_\alpha^H$ holds for all $K \in \mathcal{U}_G(H) \setminus \mathcal{U}_M(H)$, $\alpha \in \pi_0(Z^H)$, $\beta \in \pi_0(Z^K)$ with $Z_\beta^K \subset Z_\alpha^H$.

REMARK 6.2. Suppose the following conditions are fulfilled.

- (1) $(\text{res}_M^G X, \text{res}_M^G Y, W_M)$ has the (M, M) -tame singular set at H .
- (2) Y satisfies the cobordism gap condition at H .
- (3) $f^H : X^H \rightarrow Y^H$ and $F_M^H : W_M^H \rightarrow I \times Y^H$ are connected up to the middle dimensions, respectively.

Then (X, Y, W_M) has the (G, M) -tame singular set at H .

7. Remarks on specific representations

Let $\mathcal{F}$ and $\mathcal{H}$ be sets of subgroups of G such that $\mathcal{F} \subset \mathcal{H}$. We call $\mathcal{F}$ *upper closed* in $\mathcal{H}$ if K belongs to $\mathcal{F}$ whenever $H \in \mathcal{F}$, $K \in \mathcal{H}$, and $H \subset K$.

DEFINITION 7.1. Let $\mathcal{F}$ be a subset of $S(G)_{\text{sol}}$ which is G -conjugation invariant and upper closed in $S(G)_{\text{sol}}$. We say that $\mathcal{F}$ is *G -simply organized* (for equivariant surgeries) if there are a complete set $\mathcal{F}^*$ of representatives of $\mathcal{F}$ and a map $\rho_{\max} : \mathcal{F}^* \rightarrow \max(\mathcal{F})^*$, where $\max(\mathcal{F})^* = \mathcal{F}^* \cap \max(\mathcal{F})$, satisfying the following conditions.

- (1) $H \subset N_G(H) \subset \rho_{\max}(H)$ for any $H \in \mathcal{F}^*$.
- (2) $\rho_{\max}(K^*) = \rho_{\max}(H)$ for any $H \in \mathcal{F}^*$ and $K \in \mathcal{U}_{\rho_{\max}(H)}(H)$, where K^* is the representative of $(K)_G$ in $\mathcal{F}^*$.
- (3) $(H)_G \cap S(\rho_{\max}(H)) = (H)_{\rho_{\max}(H)}$ for any $H \in \mathcal{F}^*$.

We remark that if $\mathcal{F}$ is G -simply organized as above then by (6.2) we have

$$\{g \in G \mid gHg^{-1} \subset \rho_{\max}(H)\} \subset \rho_{\max}(H).$$

for all $H \in \mathcal{F}^*$, and furthermore if $\mathcal{F}'$ is a subset of $\mathcal{F}$ such that $\mathcal{F}'$ is G -invariant and upper closed in $S(G)_{\text{sol}}$ then $\mathcal{F}'$ is G -simply organized.

Let H be a subgroup of G and Z a G -manifold. We say that Z satisfies the *weak gap condition* at H if

$$(7.1) \quad 2 \dim Z_{\delta}^K \leq \dim Z_{\gamma}^H$$

holds for all $\gamma \in \pi_0(Z^H)$, $K \in \mathcal{U}_G(H)$, and $\delta \in \pi_0(Z^K)$ with $Z_{\delta}^K \subset Z_{\gamma}^H$. For $\gamma \in \pi_0(Z^H)$, let $\overline{H}_{\gamma}$ denote the set of elements g of $\overline{H} = N_G(H)/H$ such that $g\gamma = \gamma$, and let $\Pi(H, \gamma)_{1/2}$ denote the set of pairs (K, δ) of $K \in \mathcal{U}_G(H)$ and $\delta \in \pi_0(Z^K)$ such that $Z_{\delta}^K \subset Z_{\gamma}^H$ and $2 \dim Z_{\delta}^K = \dim Z_{\gamma}^H$. We say that Z satisfies the *modified weak gap condition* at H if the following conditions are fulfilled.

- (1) Z satisfies the weak gap condition at H .
- (2) For all $\gamma \in \pi_0(Z^H)$ with $\dim Z_{\gamma}^H > 0$ and $(K, \delta) \in \Pi(H, \gamma)_{1/2}$,
 - (a) $K \subset N_G(H)$ and $K/H \subset \overline{H}_{\gamma}$,
 - (b) $|(K/H) \cap \overline{H}_{\gamma}(2)| \leq 1$, where $\overline{H}_{\gamma}(2)$ is the set of elements in $\overline{H}_{\gamma}$ of order 2, and
 - (c) $\dim Z_{\omega}^L + 1 < \dim Z_{\delta}^K$ for all $L \in \mathcal{U}_G(K)$ and $\omega \in \pi_0(Z^L)$ with $Z_{\omega}^L \subset Z_{\delta}^K$.
- (3) For all $\gamma \in \pi_0(Z^H)$ with $\dim Z_{\gamma}^H > 0$ and $(K_1, \delta_1), (K_2, \delta_2) \in \Pi(H, \gamma)_{1/2}$, the smallest subgroup $\langle K_1, K_2 \rangle$ of G containing $K_1 \cup K_2$ is solvable.

Let S_5 (resp. A_5) denote the symmetric group (resp. the alternating group) on the five letters $1, 2, \dots, 5$. We fix subgroups of S_5 as follows.

S_4 (resp. A_4) the symmetric group (resp. the alternating group) on the letters $2, 3, 4, 5$.

S_3 the symmetric group on the letters $1, 2, 3$.

$\mathfrak{C}_2 = \langle (4, 5) \rangle$, $\mathfrak{C}_4 = \langle (2, 4, 3, 5) \rangle$, and $\mathfrak{C}_6 = \langle (1, 2, 3)(4, 5) \rangle$ (cyclic groups).

$\mathfrak{S}_3 \mathfrak{C}_2 = \langle (1, 2), (1, 2, 3), (4, 5) \rangle (\cong S_3 \times \mathfrak{C}_2)$.

$C_2 = \langle (2, 3)(4, 5) \rangle$, $C_3 = \langle (1, 2, 3) \rangle$, and $C_5 = \langle (1, 2, 3, 4, 5) \rangle$ (cyclic groups).

$D_4 = \langle (2, 3)(4, 5), (2, 4)(3, 5) \rangle$, $D_6 = \langle (1, 2, 3), (2, 3)(4, 5) \rangle$, and

$D_{10} = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4) \rangle$ (dihedral groups).

$\mathfrak{D}_4 = \langle (2, 3), (2, 3)(4, 5) \rangle$, and $\mathfrak{D}_8 = \langle (2, 4, 3, 5), (2, 3) \rangle$ (dihedral groups).

The normalizers of subgroups H of $G = A_5$ are as in Table 5.

Table 5.

H	A_4	D_{10}	D_6	D_4	C_5	C_3	C_2	E
$N_G(H)$	A_4	D_{10}	D_6	A_4	D_{10}	D_6	D_4	G

We assign $\rho_{\max}(H)$ to H as in Table 6.

Table 6.

H	A_4	D_{10}	D_6	D_4	C_5	C_3	C_2
$\rho_{\max}(H)$	A_4	D_{10}	D_6	A_4	D_{10}	D_6	A_4

We immediately obtain the proposition:

Proposition 7.1. *Let $G = A_5$, $\mathcal{F} = \mathcal{S}(A_5)_{\text{sol}} \setminus \{E\}$, and $\mathcal{F}^* = \{A_4, D_{10}, D_6, D_4, C_5, C_3, C_2\}$. Then $\mathcal{F}$ is G -simply organized with respect to $\rho_{\max} : \mathcal{F}^* \rightarrow \max(\mathcal{F})^*$ given by Table 6.*

The next result follows from Table 3.

Proposition 7.2. *Let $G = A_5$. Let W_3 and W'_3 be irreducible real G -representations of dimension 3 and let $W = W_3 \oplus W'_3$. Then*

- (1) $\dim W_3^H = 0$ for $H = A_4, D_{10}, D_6, D_4$,
- (2) $\dim W_3^H = 1$, W_3 satisfies the gap condition at H for $H = C_5, C_3, C_2$,
- (3) $\dim W^H = 2$ and W satisfies the primitive gap condition for $(G, \rho_{\max}(H), H)$ (as well as the gap condition at H) for $H = C_5, C_3, C_2$, and
- (4) W satisfies the gap condition at $H = E$.

Let $G = A_5$. Let W_3 and W_4 be irreducible real G -representations of dimensions 3 and 4, respectively, and let $W = W_3 \oplus W_4$. Then the dimensions of the H -fixed-point sets W^H are as in the next table.

Table 7.

H	G	A_4	D_{10}	D_6	C_5	D_4	C_3	C_2	E
$\dim W^H$	0	1	0	1	1	1	3	3	7

We immediately obtain the proposition:

Proposition 7.3. *Let $G = A_5$. Let W_3 and W_4 be irreducible real G -representations of dimensions 3 and 4, respectively, and let $W = W_3 \oplus W_4$. Then*

- (1) $\dim W^H = 0$ for $H = D_{10}$,
- (2) $\dim W^H = 1$ for $H = A_4, D_6, C_5, D_4$, and W satisfies the gap condition at H for $H = A_4, D_6, C_5$,
- (3) $\dim W^H = 3$ and W satisfies the gap condition at H for $H = C_3, C_2$, and
- (4) W satisfies the gap condition at $H = E$.

Next we consider the case $G = S_5$. The normalizers of subgroups H of S_5 are as in the Table 8.

Table 8.

H	A_5	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
$N_G(H)$	G	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4	$\mathfrak{F}_{20}$	$\mathfrak{D}_8$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E
$N_G(H)$	$\mathfrak{F}_{20}$	$\mathfrak{D}_8$	S_4	$\mathfrak{D}_8$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{D}_8$	G

We assign $\rho_{\max}(H)$ to H as Table 9.

Table 9.

H	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
$\rho_{\max}(H)$	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4	$\mathfrak{F}_{20}$	S_4	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	C_2
$\rho_{\max}(H)$	$\mathfrak{F}_{20}$	S_4	S_4	S_4	$\mathfrak{S}_3\mathfrak{C}_2$	S_4

We immediately obtain the proposition.

Proposition 7.4. *Let $G = S_5$, $\mathcal{F} = S(G)_{\text{sol}} \setminus (\{E\} \cup (\mathfrak{C}_2)_G)$, and $\mathcal{F}^*$ the set of subgroups H in Table 9. Then $\mathcal{F}$ is G -simply organized with respect to $\rho_{\max} : \mathcal{F}^* \rightarrow \max(S(G)_{\text{sol}})^*$ given by Table 9.*

The next result follows from Table 4.

Proposition 7.5. *Let $G = S_5$ and let W be an irreducible real G -representation of dimension 6. Then*

- (1) $\dim W^H = 0$ for $H = S_4, \mathfrak{F}_{20}, \mathfrak{S}_3\mathfrak{C}_2, A_4, D_{10}, \mathfrak{D}_8, D_6, D_4$,
- (2) $\dim W^H = 1$ and W satisfies the gap condition at H for $H = S_3, \mathfrak{C}_6, \mathfrak{D}_4, \mathfrak{C}_4$,
- (3) $\dim W^H = 2$ and W satisfies the primitive gap condition for $(G, \rho_{\max}(H), H)$ for $H = C_5, C_3, C_2$,
- (4) $\dim W^H = 3$ and W satisfies the gap condition at H for $H = \mathfrak{C}_2$, and
- (5) W satisfies the modified weak gap condition at $H = E$.

Now let $G = A_5 \times Z$, where Z is a group of order 2. We identify subgroups $H \in S(A_5)$ with $H \times \{e\} \in S(G)$, respectively, and Z with $\{e\} \times Z \in S(G)$. Let C_2 be the subgroup of order 2 belonging to $S(C_2Z) \setminus \{C_2, Z\}$. Let D_{2n} be the dihedral subgroup of order $2n$ generated by C_n and C_2 . We tabulate subgroups H giving a complete set of representatives of conjugacy classes of subgroups of $G = A_5 \times Z$ and the normalizers of subgroups H in Table 10.

Table 10.

H	G	A_5	A_4Z	$D_{10}Z$	D_6Z	A_4	D_{10}	D_{10}	C_5Z	D_4Z	C_3Z
$N_G(H)$	G	G	A_4Z	$D_{10}Z$	D_6Z	A_4Z	$D_{10}Z$	$D_{10}Z$	$D_{10}Z$	A_4Z	D_6Z

H	D_6	D_6	C_5	D_4	C_2Z	D_4	C_3	C_2	C_2	Z	E
$N_G(H)$	D_6Z	D_6Z	$D_{10}Z$	A_4Z	D_4Z	A_4Z	D_6Z	D_4Z	D_4Z	G	G

Table 11.

H	A_4Z	$D_{10}Z$	D_6Z	A_4	D_{10}	D_{10}	C_5Z	D_4Z	C_3Z
$\rho_{\max}(H)$	A_4Z	$D_{10}Z$	D_6Z	A_4Z	$D_{10}Z$	$D_{10}Z$	$D_{10}Z$	A_4Z	D_6Z

H	D_6	D_6	C_5	D_4	C_2Z	D_4	C_3	C_2
$\rho_{\max}(H)$	D_6Z	D_6Z	$D_{10}Z$	A_4Z	A_4Z	A_4Z	D_6Z	A_4Z

In the case $G = A_5 \times Z$ above, we assign $\rho_{\max}(H)$ to H as in Table 11.

We immediately obtain the next proposition.

Proposition 7.6. *Let $G = A_5 \times Z$, where Z is a group of order 2, $\mathcal{F} = S(G)_{\text{sol}} \setminus (\{E, Z\} \cup (C_2)_G)$, and $\mathcal{F}^*$ the set of subgroups H in Table 11. Then $\mathcal{F}$ is G -simply organized with respect to $\rho_{\max} : \mathcal{F}^* \rightarrow \max(S(G)_{\text{sol}})^*$ given by Table 11.*

Let W_3 and W'_3 be irreducible real A_5 -representations of dimension 3 and let $\mathbb{R}$ and $\mathbb{R}_{\pm}$ be 1-dimensional real Z -representations with trivial and nontrivial Z -actions, respectively. The dimensions of the H -fixed-point sets W^H of $W = (W_3 \otimes \mathbb{R}) \oplus (W'_3 \otimes \mathbb{R}_{\pm})$ are as in Table 12.

Table 12.

H	D_{10}	C_5Z	C_3Z	D_6	C_5	D_4	C_2Z	C_3	C_2	C_2	Z	E	$H \in \mathcal{K}$
$\dim W^H$	1	1	1	1	2	1	1	2	3	2	3	6	0

where $\mathcal{K} = \{G, A_5, A_4Z, D_{10}Z, D_6Z, A_4, D_{10}, D_4Z, D_6, D_4\}$. The next proposition follows.

Proposition 7.7. *Let $G = A_5 \times Z$, where Z is a group of order 2, and let $W = (W_3 \otimes \mathbb{R}) \oplus (W'_3 \otimes \mathbb{R}_{\pm})$ be a real G -representation of dimension 6 described above. Then*

- (1) $\dim W^H = 0$ for $H = A_4Z, D_{10}Z, D_6Z, A_4, D_{10}, D_4Z, D_6, D_4$,
- (2) $\dim W^H = 1$ and W satisfies the gap condition at H for $H = D_{10}, C_5Z, C_3Z, D_6, D_4, C_2Z$,
- (3) $\dim W^H = 2$ and W satisfies the primitive gap condition for $(G, \rho_{\max}(H), H)$ for $H = C_5, C_3, C_2$,
- (4) $\dim W^H = 3$ and W satisfies the gap condition at H for $H = C_2, Z$, and
- (5) W satisfies the modified weak gap condition at $H = E$.

Next we consider the case where $W = (W_3 \otimes \mathbb{R}) \oplus (W_4 \otimes \mathbb{R}_{\pm})$, where W_4 is an irreducible real A_5 -representation of dimension 4. Then the dimensions of the H -fixed-point sets W^H of W are as in the Table 13.

Table 13.

H	A_4	C_5Z	C_3Z	D_6	D_6	C_5	D_4	D_4	C_2Z	C_3	C_2	C_2	Z	E	$H \in \mathcal{K}$
$\dim W^H$	1	1	1	1	1	1	1	1	1	3	3	3	3	7	0

where $\mathcal{K} = \{G, A_5, A_4Z, D_{10}Z, D_6Z, D_{10}, D_{10}, D_4Z\}$. The next proposition follows.

Proposition 7.8. *Let $G = A_5 \times Z$, where Z is a group of order 2, and let $W = (W_3 \otimes \mathbb{R}) \oplus (W_4 \otimes \mathbb{R}_{\pm})$ be a real G -representation of dimension 7 described above. Then*

- (1) $\dim W^H = 0$ for $H = A_4Z, D_{10}Z, D_6Z, D_{10}, D_{10}, D_4Z$,

- (2) $\dim W^H = 1$ and W satisfies the gap condition at H for $H = A_4, C_5Z, D_6, D_6, C_3Z, C_5, D_4, C_2Z$,
 (3) $\dim W^H = 3$ and W satisfies the gap condition at H for $H = C_3, C_2, C_2, Z$,
 (4) W satisfies the gap condition at $H = E$.

8. G -surgery obstructions of isotropy type $(H)_G$

Let $f = (f, b)$ be a G -framed map as in Section 6. Recall that $f : (X, \partial X) \rightarrow (Y, \partial Y)$, $Y = D(V)$, $b : \tau_X \rightarrow f^* \tau_Y$, and $\partial f : \partial X \rightarrow \partial Y$ is the identity map on $\partial X = \partial Y$. Hence the mapping degree of $f^H : (X^H, \partial X^H) \rightarrow (Y^H, \partial Y^H)$ is 1 whenever $H \in S(G)$ and $\dim V^H > 0$.

Let H be a subgroup and set $\overline{H} = N_G(H)/H$. Let $G(2)$ denote the set of elements of order 2 in G . Thus $\overline{H}(2)$ is the set of elements of order 2 in $\overline{H}$. For a principal ideal domain R satisfying $a^2 = a$ in $R/2R$ for all $a \in R$, let $A_{\overline{H}} = R[\overline{H}]$ denote the group algebra of $\overline{H}$ over R . Therefore $A_{\overline{H}} = \text{Map}(\overline{H}, R)$. Let $w_{\overline{H}} : \overline{H} \rightarrow \{1, -1\}$ denote the orientation homomorphism of V^H with $\overline{H}$ -action. Set $n_H = \dim V^H$, let k_H be the integer satisfying $n_H = 2k_H$ or $2k_H + 1$, and set $\lambda_H = (-1)^{k_H}$. $A_{\overline{H}}$ has the involution $- : A_{\overline{H}} \rightarrow A_{\overline{H}}; x \mapsto \bar{x}$, defined by

$$(8.1) \quad \overline{\sum_{g \in \overline{H}} r_g g} = \sum_{g \in \overline{H}} r_g w_{\overline{H}}(g) g^{-1},$$

where $r_g \in R$. Depending on $\varepsilon \in \{1, -1\}$, we define the submodule $\min_{\varepsilon}(A_{\overline{H}})$ of $A_{\overline{H}}$ by

$$\min_{\varepsilon}(A_{\overline{H}}) = \{x - \varepsilon \bar{x} \mid x \in A_{\overline{H}}\} \quad (\text{see (8.1)}).$$

Case $n_H = 2k_H \geq 6$. Let $Q_{\overline{H}}$ (resp. $S_{\overline{H}}$) denote the set of elements $g \in \overline{H}(2)$ satisfying $\dim(V^H)^g = k_H - 1$ (resp. $\dim(V^H)^g = k_H$). Let

$$\begin{aligned} A_{\overline{H},s} &= R[S_{\overline{H}}], \\ \Gamma_{\overline{H}} &= \min_{-\lambda_H}(A_{\overline{H}}) + R[S_{\overline{H}}], \\ \Lambda_{\overline{H}} &= \min_{\lambda_H}(A_{\overline{H}}) + R[Q_{\overline{H}}], \end{aligned}$$

where $R[S_{\overline{H}}] = \text{Map}(S_{\overline{H}}, R)$ and $R[Q_{\overline{H}}] = \text{Map}(Q_{\overline{H}}, R)$. We call

$$A_{\overline{H}} = (A_{\overline{H}}, (-, \lambda_H), \Gamma_{\overline{H}}, \overline{H}, A_{\overline{H},s}, A_{\overline{H},s} + \Lambda_{\overline{H}})$$

the *double parameter algebra* of the $\overline{H}$ -manifold Y^H , see [6, Definition 2.5] and [6, p. 538].

Let $\Theta_{\overline{H},2}$ be the set of all generators of $H_{k_H}((Y^H)^K, \partial(Y^H)^K; \mathbb{Z}_2) \cong \mathbb{Z}_2$, where K runs over $S(\overline{H})$ such that $\dim(Y^H)^K = k_H$, and $\widetilde{\Theta}_{\overline{H}}$ the set of all generators of $H_{k_H}((Y^H)^K, \partial(Y^H)^K; \mathbb{Z}) \cong \mathbb{Z}$, where K runs over $S(\overline{H})$ such that $\dim(Y^H)^K = k_H$. The canonical map $pr_{\overline{H}} : \widetilde{\Theta}_{\overline{H}} \rightarrow \Theta_{\overline{H},2}$ is a double covering. We have the map $\rho_{\overline{H}} : \Theta_{\overline{H},2} \rightarrow \mathfrak{P}(S_{\overline{H}})$, where $\mathfrak{P}(S_{\overline{H}})$ is the set of subsets of $S_{\overline{H}}$, defined by $\rho_{\overline{H}}(t) = K \cap S_{\overline{H}}$ for a generator t of $H_{k_H}((Y^H)^K, \partial(Y^H)^K; \mathbb{Z}_2)$ with $\dim(Y^H)^K = k_H$. We call

$$\Theta_{\overline{H}} = (pr_{\overline{H}} : \widetilde{\Theta}_{\overline{H}} \rightarrow \Theta_{\overline{H},2}, \rho_{\overline{H}} : \Theta_{\overline{H},2} \rightarrow \mathfrak{P}(S_{\overline{H}}))$$

the *positioning data* of the $\overline{H}$ -manifold Y^H , see [6, pp. 533, 538]. By the definition [6, p. 545], we obtain the abelian group

$$\mathcal{L}_{V,H}(R[\overline{H}]) = W_{n_H}(R, \overline{H}, Q_{\overline{H}}, S_{\overline{H}}, \Theta_{\overline{H}})_{\text{free}}.$$

Case $n_H = 2k_H + 1 \geq 3$. Let $Q_{\overline{H}}$ denote the set of elements g with order 2 of $\overline{H}$ satisfying $\dim(V^H)^g = k_H$ and

$$\Lambda_{\overline{H}} = \min_{\lambda_H}(A_{\overline{H}}) + R[Q_{\overline{H}}].$$

We call

$$A_{\overline{H}} = (A_{\overline{H}}, (-, \lambda_H), \Lambda_{\overline{H}})$$

the *form algebra* of the $\overline{H}$ -manifold Y^H . By [20, Definition 1.5], we obtain the abelian group

$$\mathcal{L}_{V,H}(R[\overline{H}]) = W_1^{\lambda_H}(A_{\overline{H}}, \Lambda_{\overline{H}}).$$

Suppose V is $S(G)_{\text{non-sol}}$ -free, i.e. $V^L = \{0\}$ for all $L \in S(G)_{\text{non-sol}}$. Let $H \in S(G)_{\text{sol}}$. We obtain the $\overline{H}$ -framed map $\mathbf{f}^H = (f^H, b^H)$ from the G -framed map $\mathbf{f}$, where $f^H : (X^H, \partial X^H) \rightarrow (Y^H, \partial Y^H)$ and $b^H : \tau_{X^H} \rightarrow f^{H*} \tau_{Y^H}$.

We say that $\mathbf{f}$ is $\mathcal{P}$ -adjusted at H if $f^K : X^K \rightarrow Y^K$ is a $\mathbb{Z}_p$ -homology equivalence for every prime p and every $K \in \mathcal{U}_{N_G(H)}(H)$ such that $|K/H|$ is a power of p . We suppose that Y satisfies the modified weak gap condition at H and $\mathbf{f}$ is $\mathcal{P}$ -adjusted at H . The G -framed map $\mathbf{f}$ is G -framed cobordant rel. ∂ by G -surgeries of isotropy type $(H)_G$ to $\mathbf{f}' = (f', b')$, where $f' : (X', \partial X') \rightarrow (Y, \partial Y)$, such that $f : X' \rightarrow Y$ is k_H -connected, where $\dim Y^H = 2k_H$ or $2k_H + 1$. Suppose $f^H : X^H \rightarrow Y^H$ is k_H -connected. We define the *surgery kernel* $L(f^H; R)$ to be the $\overline{H}$ -module

$$(8.2) \quad \begin{aligned} \text{Ker}[f^H_* : H_{k_H}(X^H; R) \rightarrow H_{k_H}(Y^H; R)] &= H_{k_H}(X^H; R) \quad \text{if } \dim Y^H = 2k_H \geq 6, \\ K_{k_H+1}(X^H_0, \partial \overline{H} U) \otimes_{\mathbb{Z}} R &\quad \text{if } \dim Y^H = 2k_H + 1 \geq 5, \text{ see [20, Diagram 4.2], and} \\ K_2(X^H_0, \partial \overline{H} U; R) &\quad \text{if } \dim Y^H = 3, \text{ see [24, Diagram 3.1],} \end{aligned}$$

where U is a submanifold of $\overline{H}$ -manifold X^H and $X^H_0 = X^H \setminus \overset{\circ}{\overline{H} U}$.

Lemma 8.1. *Let $R = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$ for a prime p . Suppose the following (i)–(iii).*

- (i) V satisfies the modified weak gap condition at H .
- (ii) $\mathbf{f}$ is $\mathcal{P}$ -adjusted at H .
- (iii) $f^H : X^H \rightarrow Y^H$ is k_H -connected.

If the surgery kernel $L(f^H; R)$ is stably free over $R[\overline{H}]$ then there is an element

$$\sigma_{G,H}(\mathbf{f}) (= \sigma_{\overline{H}}(\mathbf{f}^H)) \text{ of } \mathcal{L}_{V,H}(R[\overline{H}])$$

having the property: if $\sigma_{G,H}(\mathbf{f}) = 0$ then $\mathbf{f}$ is G -framed cobordant rel. ∂ by G -surgeries of isotropy type $(H)_G$ to $\mathbf{f}' = (f', b')$, where $f' : (X', \partial X') \rightarrow (Y, \partial Y)$, such that

- (1) X'^H is 1-connected and R -acyclic if $\dim V^H \geq 5$, and
- (2) X'^H is (connected and) R -acyclic if $\dim V^H = 3$.

Proof. The lemma follows from the proofs of [6, Theorems 1.1 and 1.2], [20, Theorem A], and [24, Theorem 1.1]. $\square$

9. Construction of G -framed maps

Let G be a nonsolvable group, $\beta = \beta_G$ the idempotent of $\Omega(G)$ defined by (3.1), and V a real G -representation of positive dimension being $S(G)_{\text{nonsol}}$ -free and ample for β_G . Recall that $V^L = \{0\}$ for all $L \in S(G)_{\text{nonsol}}$. Let $Z = S(\mathbb{R} \oplus V)$ and $Z^+ = Z \amalg \{pt\}$. The sphere Z is the union of the hemispheres $S_+ = \{(u, v) \in S(\mathbb{R} \oplus V) \mid u \geq 0\}$ and $S_- = \{(u, v) \in S(\mathbb{R} \oplus V) \mid u \leq 0\}$, where $u \in \mathbb{R}$ and $v \in V$. Let $y_+ = (1, 0) \in S(\mathbb{R} \oplus V)$ and $y_- = (-1, 0) \in S(\mathbb{R} \oplus V)$, where $\pm 1 \in \mathbb{R}$ and $0 \in V$. We have the canonical G -diffeomorphism $S_+ \rightarrow D(V)$, which carries y_+ to $y_0 = 0$, and identify S_+ with $D(V)$ via the diffeomorphism. Recall the generalized cohomology

$$\omega_G^0(Z) = \lim_{m \rightarrow \infty} [Z^+ \wedge M^\bullet, M^\bullet]_0^G,$$

where $M^\bullet$ is the one-point compactification of $M = \mathbb{R}[G]^m$. For $\alpha = 1 - \beta$, the set $S = \{\alpha\}$ is a multiplicatively closed subset of $\Omega(G)$ and the restriction map

$$(9.1) \quad \begin{array}{ccc} S^{-1}\omega_G^0(Z) & \longrightarrow & S^{-1}\omega_G^0(Z^G) \\ & \downarrow = & \\ S^{-1}\omega_G^0(\{y_+\}) \oplus S^{-1}\omega_G^0(\{y_-\}) & \xrightarrow{\cong} & S^{-1}\Omega(G) \oplus S^{-1}\Omega(G) \end{array}$$

is an isomorphism. The module $\Omega(G) \oplus \Omega(G)$ contains the element $(\alpha, 0)$. By the arguments in [23] and [26, Section 4], originally due to T. Petrie [30, Sections 1 and 2], we obtain the next lemma.

Lemma 9.1. *There are a G -framed map $f = (f, b)$, where $Y = D(V)$, $f : (X, \partial X) \rightarrow (Y, \partial Y)$ with $\partial f = id_{\partial Y}$ and $b : \tau_X \rightarrow f^* \tau_Y$, and M -framed cobordisms $F_M = (F_M, B_M)$ from $\text{res}_M^G f$ to $\text{res}_M^G id_Y \text{ rel. } \partial$, where $F_M : W_M \rightarrow I \times Y$ is an M -map and $B_M : T(W_M) \oplus \varepsilon_{W_M}(\mathbb{R}^\ell) \rightarrow F_M^* T(I \times Y) \oplus \varepsilon_{W_M}(\mathbb{R}^\ell)$ is an M -bundle isomorphism, for all $M \in \max(S(G)_{\text{sol}})$, satisfying the following conditions (C1)–(C3).*

- (C1) $X^L = \emptyset$ for any $L \in S(G)_{\text{nonsol}}$.
- (C2) $f^{-1}(y_0)^H$ consists of one point, say x_H , $f : X \rightarrow Y$ is transverse regular at x_H to y_0 in Y , and $f^K : X^K \rightarrow Y^K$ is locally an orientation-preserving diffeomorphism from a neighborhood of x_H in X^K to a neighborhood of y_0 in Y^K , for any $H \in \max(S(G)_{\text{sol}})$ with $\dim V^H = 0$ and $K \in S(H)$.
- (C3) $f^{-1}(y_0)^{=H} = \emptyset$ for each $H \in S(G)_{\text{sol}} \setminus \max(S(G)_{\text{sol}})$ with $\dim V^H = 0$, where $f^{-1}(y_0)^{=H}$ is the subset of $f^{-1}(y_0)^H$ consisting of points with isotropy subgroup H .

In the lemma above, it holds that

- (C4) $\text{Iso}(G, X) \supset \text{Iso}(G, Y \setminus \{y_0\}) \cup \max(S(G)_{\text{sol}}) \supset \text{Iso}(G, \beta)$, and
- (C5) $\deg[f^H : (X^H, \partial X^H) \rightarrow (Y^H, \partial Y^H)] = 1$ for any $H \in S(G)$ with $\dim V^H > 0$.

Lemma 9.2. *For the M -framed cobordism F_M , where $M \in \max(S(G)_{\text{sol}})$, in Lemma 9.1, we can adjust it so as to satisfy the following conditions.*

- (C6) X^M is diffeomorphic to Y^M and W_M^M is a product cobordism, i.e. diffeomorphic to $I \times Y^M$, and furthermore

$$F_M^M : (W_M^M, \partial_0 W_M^M, \partial_1 W_M^M, \partial_{01} W_M^M) \rightarrow (Z^M, \partial_0 Z^M, \partial_1 Z^M, \partial_{01} Z^M),$$

where $Z = I \times Y$, is homotopic rel. $\partial_1 W_M^M \cup \partial_{01} W_M^M$ to a diffeomorphism. Therefore $f^M : X^M \rightarrow Y^M$ is homotopic rel. ∂ to a diffeomorphism.

(C7) If $H \in S(M)$ and $\dim V^H = 0$ then $W_M^H = W_M^M$ (and W_M^H is diffeomorphic to the closed interval $[0, 1]$).

Proof. The properties in (C6) is readily achieved by the reflection method.

To show (C7), let $H \in S(M)$ with $\dim V^H = 0$. If $X^H \neq \emptyset$ then we get $H = M$ by (C2) and (C3). In the case $H = M$, we get $X^H = \{x_M\}$ and $\dim W_M^H = 1$ and it holds that one of the connected components of $\dim W_M^H$ is diffeomorphic to $[0, 1]$ and the others are diffeomorphic to the circle. It is easy to convert W_M by M -surgeries of isotropy type $(H)_M$ ($H = M$) so that W_M^H is diffeomorphic to $[0, 1]$. Therefore it suffices to consider the case $H < M$. As an inductive assumption, suppose that $W_M^K = W_M^M$ for all $K \in \mathcal{U}_M(H)$. Then each connected component of $W_M^H \setminus W_M^{>H}$ ($W_M^{>H} = W_M^M$) is diffeomorphic to the circle. We can readily remove those undesired connected components of W_M^H by M -surgeries of isotropy type $(H)_M$ to obtain the property $W_M^H = W_M^M$. $\square$

In the following sections, we assume that $\mathbf{f}$ and F_M are adjusted by Lemma 9.2.

Proposition 9.3. *Let H be a solvable subgroup of G . Suppose the G -framed map $\mathbf{f} = (f, b)$ above satisfies the modified weak gap condition at H and the condition that*

($\mathcal{G}_1$) *X^K is $\mathbb{Z}$ -acyclic for all $K \in \mathcal{U}_G(H)$ such that $H \triangleleft K$ and K/H is a hyper-elementary group.*

Set $n_H = \dim V^H$ and let k_H be the integer satisfying $n_H = 2k_H$ or $2k_H + 1$. Suppose $n_H \geq 5$ (resp. $n_H = 3$) and X^H is $(k_H - 1)$ -connected. Then $([G/G] - \beta_G) \mathbf{f}$ is G -framed cobordant rel. ∂ to $\mathbf{f}' = (f', b')$, where $f' : (X', \partial X') \rightarrow (Y, \partial Y)$ and $b' : \tau_{X'} \rightarrow f'^* \tau_Y$, by G -surgeries of isotropy type $(H)_G$, such that X'^H is contractible (resp. $\mathbb{Z}$ -acyclic).

Here we remark that the equalities $X^L = \emptyset = X'^L$ and $\dim X^H = \dim Y^H = \dim X'^H$ hold for $L \in S(G)_{\text{non-sol}}$ and $H \in S(G)_{\text{sol}}$, respectively.

Proof. Note that X^H is 1-connected and $f^H : X^H \rightarrow Y^H$ is k_H -connected (resp. X^H is connected and $f_\#^H : \pi_1(X^H) \rightarrow \pi_1(Y^H)$ is surjective). Let $L(f^H; \mathbb{Z})$ be the surgery kernel. By the condition ($\mathcal{G}_1$) above, $L(f^H; \mathbb{Z})$ is stably free over $\mathbb{Z}[\overline{H}]$, where $\overline{H} = N_G(H)/H$. By Lemma 8.1, we obtain the obstruction $\sigma_{G,H}(\mathbf{f}; \mathbb{Z})$ in $\mathcal{L}_{V,H}(\mathbb{Z}[\overline{H}])$ to convert $\mathbf{f}$ so that $f^H : X^H \rightarrow Y^H$ would be a homotopy equivalence (resp. a $\mathbb{Z}$ -homology equivalence) by G -surgeries rel. ∂ of isotropy type $(H)_G$. Note the property

$$\sigma_{G,H}([G/G] - \beta_G) \mathbf{f}; \mathbb{Z}) = ([\overline{H}/\overline{H}] - \beta_G^H) \sigma_{G,H}(\mathbf{f}; \mathbb{Z}),$$

where β_G^H is the element $[X_1^H] - [X_2^H] \in \Omega(\overline{H})$ if $\beta = [X_1] - [X_2]$ for finite G -sets X_1 and X_2 . Recall the induction theory of equivariant-surgery-obstruction groups, see [10, 11], [2], [14, Corollary 1.4], and [25, Theorems 1.1 and 13.5]. If $H \trianglelefteq K \in S(G)$ and K/H is solvable, then K is solvable and

$$\text{res}_{K/H}^{\overline{H}}([G/G] - \beta_G)^H = (\text{res}_K^G([G/G] - \beta_G))^H = 0 \text{ in } \Omega(K/H).$$

It follows that

$$\text{res}_{K/H}^{\overline{H}}([G/G] - \beta_G)^H \sigma_{G,H}(\mathbf{f}; \mathbb{Z}) = 0$$

for all $K/H \in S(\overline{H})_{\text{sol}}$ and

$$([\overline{H}/\overline{H}] - \beta_G^H) \sigma_{G,H}(\mathbf{f}; \mathbb{Z}) = 0.$$

Therefore, $([G/G] - \beta_G) \mathbf{f}$ is G -framed cobordant rel. ∂ to $\mathbf{f}'$ stated in the proposition by G -surgeries of isotropy type $(H)_G$. $\square$

10. Simply organized families and G -surgeries

Let G be a nonsolvable group and V an $S(G)_{\text{nonsol}}$ -free real G -representation. Set

$$(10.1) \quad \mathcal{H}(G, V, 0) = \{H \in S(G)_{\text{sol}} \mid \dim V^H = 0\}.$$

Let $\mathbf{f} = (f, b)$ and $\mathbf{F}_M = (F_M, B_M)$ be the G -framed map and the M -framed cobordisms, where $M \in \max(S(G)_{\text{sol}})$, obtained in Lemma 9.1. Let $Z = I \times Y$, $\partial_0 Z = \{0\} \times Y$, $\partial_1 Z = \{1\} \times Y$, and $\partial_{01} Z = I \times \partial Y$. We suppose that $\mathbf{f}$ and $\mathbf{F}_M$ are adjusted by Lemma 9.2. In this situation, for every $M \in \max(S(G)_{\text{sol}})$, W_M^M is diffeomorphic to $I \times Y^M$, X^M is diffeomorphic to Y^M , $f^M : X^M \rightarrow Y^M$ is homotopic rel. ∂ to a diffeomorphism, W_M^M is diffeomorphic to $I \times Y^M$, and $F_M^M : W_M^M \rightarrow I \times Y^M$ is homotopic rel. $\partial_1 W_M^M \cup \partial_{01} W_M^M$ to a diffeomorphism. In addition, we have $X^{=H} = \emptyset$ and $W_M^{=H} = \emptyset$ for all $H \in \mathcal{H}(G, V, 0) \setminus \max(S(G)_{\text{sol}})$ and $M \in \max(S(G)_{\text{sol}})$ such that $H \subset M$.

For a subset $\mathcal{H}$ of $S(G)$, let $X(\mathcal{H})$ denote the union of X^H , where H ranges over $\mathcal{H}$. Let $M \in \max(S(G)_{\text{sol}})$ and set $\mathcal{H}_M = \{M\} \cup \mathcal{H}(G, V, 0)$. Let $N_M(X(\mathcal{H}_M), X)$ be an M -regular neighborhood of $X(\mathcal{H}_M)$ in X . In this section we set $X^{(0)} = X$, $\mathbf{f}^{(0)} = \mathbf{f}$, $W_M^{(0)} = W_M$, $\mathbf{F}_M^{(0)} = \mathbf{F}_M$, for $M \in \max(S(G)_{\text{sol}})$, and $\mathbf{F}_G^{(0)} = I \times \mathbf{f}$. It is easy to obtain a product M -embedding $\Phi_M^{(0)} : I \times N_M(X(\mathcal{H}_M), X) \rightarrow W_M$ and an M -homotopy

$$\mathbb{H}_M^{(0)} : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ such that $\mathbb{H}_M^{(0)}|_{W_M \times \{0\}} = F_M$ and $\mathbb{H}_M^{(0)}|_{\text{Im}(\Phi_M^{(0)}) \times \{1\}}$ is a diffeomorphism to its image.

Now let $\mathcal{F}$ be a G -conjugation-invariant and upper-closed subset of $S(G)_{\text{sol}}$ and suppose $\mathcal{F}$ is G -simply organized with respect to $\rho_{\max} : \mathcal{F}^* \rightarrow \max(\mathcal{F})^*$, where $\max(\mathcal{F})^* = \mathcal{F}^* \cap \max(\mathcal{F})$. By Definition 7.1, the equality

$$(10.2) \quad X(\mathcal{U}_G(H)) = X(\mathcal{U}_M(H)) \cup X(\mathcal{Y}(G, M, H))$$

holds for $H \in \mathcal{F}^*$ and $M = \rho_{\max}(H)$, where $\mathcal{Y}(G, M, H)$ is the set of subgroups $K \in \mathcal{U}_G(H)$ such that $K \cap M = H$. Here we note that $X(\mathcal{U}_G(H)) = X^{>H}$ and $X(\mathcal{U}_M(H)) = (\text{res}_M^G X)^{>H}$.

Lemma 10.1. *Suppose $\mathcal{F}$ contains $\mathcal{F}^{(0)} = \max(S(G)_{\text{sol}}) \cup \mathcal{H}(G, V, 0)$. In addition suppose the next condition is fulfilled.*

$$(D1) \quad \dim V^K = 0 \text{ for all } H \in (\mathcal{F}^* \cap \text{Iso}(G, V \setminus \{0\})) \setminus \mathcal{F}^{(0)} \text{ and } K \in \mathcal{Y}(G, \rho_{\max}(H), H).$$

Then there are a G -framed map $\mathbf{f}'$ rel. ∂ , a G -framed cobordism $\mathbf{F}_G$ from $\mathbf{f}$ to $\mathbf{f}'$ rel. ∂ and $S(M)_{\text{nonsol}}$, and an M -framed cobordism $\mathbf{F}'_M$ from $\text{res}_M^G \mathbf{f}'$ to $\text{res}_M^G \text{id}_Y$ rel. ∂ for each $M \in \max(\mathcal{F})^$ having the following properties.*

- (1) X'^H is diffeomorphic to Y^H and $f'^H : X'^H \rightarrow Y^H$ is $N_G(H)$ -homotopic rel. ∂ to a diffeomorphism for all $H \in \mathcal{F}$.

(2) For each $M \in \max(\mathcal{F})^*$, there is an M -homotopy

$$\mathbb{H}'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W'_M \cup \partial_{01} W'_M$ such that $\mathbb{H}'_M|_{W'_M \times \{0\}}$ coincides with F'_M and $\mathbb{H}'_M|_{W'_M \times \{1\}}$ is a diffeomorphism to its image for every $H \in \mathcal{F}^*$ with $\rho_{\max}(H) = M$.

Proof. We can write $\mathcal{F}$ in the form

$$\mathcal{F} = \mathcal{F}^{(0)} \amalg (H_1)_G \amalg (H_2)_G \amalg \cdots \amalg (H_m)_G,$$

where $H_i \in \mathcal{F}^*$ for $1 \leq i \leq m$, satisfying the condition that if $|H_i| > |H_j|$ then $i < j$. Set $M_i = \rho_{\max}(H_i)$. Let H_i be one of the subgroups above such that $(H_i)_G$ is a maximal conjugacy class in $\mathcal{F} \setminus \mathcal{F}^{(0)}$. For $H = H_i$ and $M = \rho_{\max}(H)$, since $X^{>H} \subset X(\mathcal{F}^{(0)})$, we will adopt a restriction of $\Phi_M^{(0)}$ as a product M -embedding $\Psi_i^{(i)} : I \times N_M(M \cdot X^{>H}, X) \rightarrow W_M$.

For $k = 1, \dots, m$, we inductively define $\mathcal{F}^{(k)}$ by $\mathcal{F}^{(k)} = \mathcal{F}^{(k-1)} \amalg (H_k)_G$. We prove the lemma by induction on $k = 1, \dots, m$. Recall that for integers i and j , we mean by $[i..j]$ the set of integers t such that $i \leq t \leq j$. Suppose that (for fixed k) we have obtained inductively,

- G -framed maps $f^{(i)}$ rel. ∂ , where $f^{(i)} : (X^{(i)}, \partial X^{(i)}) \rightarrow (Y, \partial Y)$,
- G -framed cobordisms $F_G^{(i)}$ rel. ∂ and $\mathcal{V}_G(H_i)$, from $f^{(i-1)}$ to $f^{(i)}$, where

$$F_G^{(i)} : (W_G^{(i)}, \partial_0 W_G^{(i)}, \partial_1 W_G^{(i)}, \partial_{01} W_G^{(i)}) \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$

- M -framed cobordisms $F_M^{(i)}$ rel. ∂ from $\text{res}_M^G f^{(i)}$ to $\text{res}_M^G id_Y$, where

$$F_M^{(i)} : (W_M^{(i)}, \partial_0 W_M^{(i)}, \partial_1 W_M^{(i)}, \partial_{01} W_M^{(i)}) \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$

such that $F_M^{(i)}$ is obtained by M -surgeries rel. $\partial_1 W_M^{(i-1)} \cup \partial_{01} W_M^{(i-1)}$ on $F_M^{(i-1)}$ of isotropy types $(K)_M$, where K runs over $\{L \cap M \mid L \in (H_i)_G\}$,

for $i \in [0..(k-1)]$ and $M \in \max(\mathcal{F})^*$,

- product M_j -embeddings $\Psi_j^{(i)} : I \times N_{M_j}(M_j \cdot (\text{res}_{M_j}^G X^{(i-1)})^{>H_j}, X^{(i-1)}) \rightarrow W_{M_j}^{(i-1)}$ such that $\Psi_j^{(i)} = \Psi_j^{(i-1)}$ whenever $j \leq i-1$,
- product M_j -embeddings $\Phi_j^{(i)} : I \times N_{M_j}(M_j \cdot (X^{(i)})^{H_j}, X^{(i)}) \rightarrow W_{M_j}^{(i)}$ such that $\Phi_j^{(i)} = \Phi_j^{(i-1)}$ whenever $j \leq i-1$ and that $\Psi_j^{(i)>H_j} = \bigcup_L \Phi_j^{(i)L}$, where L runs over $\mathcal{U}_{M_j}(H_j)$, and
- M_j -homotopies

$$\mathbb{H}_j^{(i)} : (W_{M_j}^{(i)}, \partial_0 W_{M_j}^{(i)}, \partial_1 W_{M_j}^{(i)}, \partial_{01} W_{M_j}^{(i)}) \times I \rightarrow (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W_{M_j}^{(i)} \cup \partial_{01} W_{M_j}^{(i)}$ such that $\mathbb{H}_j^{(i)}|_{W_{M_j}^{(i)} \times \{0\}}$ coincides with $F_{M_j}^{(i)}$ and $\mathbb{H}_j^{(i)}|_{\text{Im}(\Phi_j^{(i)}) \times \{1\}}$ is a diffeomorphism to its image,

for $i \in [1..(k-1)]$ and $j \in [1..i]$.

Note that $\dim V^{H_k} > 0$.

Case 1: $H_k \notin \text{Iso}(G, V)$. By (10.2), there is a subgroup $K \in \mathcal{U}_{M_k}(H_k)$ such that $\dim V^K = \dim V^{H_k} > 0$. It follows that $X^{>H_k} \subset X^K$ and $X^{H_k} = X^K \amalg X^{=H_k}$. By Lemma 6.2, we can obtain $f^{(k)}$, $F_{M_k}^{(k)}$, $\Phi_k^{(k)}$, and $\mathbb{H}_k^{(k)}$. Let $M \in \max(\mathcal{F})^* \setminus \{M_k\}$ and set $F_M^{(k)'} = F_G \cup_{f^{(k-1)}} F_M^{(k-1)}$. Note that $W_G^{H_j}$ is a product cobordism for each $j \in [1..(k-1)]$. Therefore, for $j \in [1..(k-1)]$, by deforming $F_{M_j}^{(k)'}$, we can obtain desired $F_{M_j}^{(k)}$, $\Psi_j^{(k)}$, $\Phi_j^{(k)}$, and $\mathbb{H}_j^{(k)}$, where $W_{M_j}^{(k)}$ is M_j -

homeomorphic to $W_G^{(k)} \cup_{X^{(k-1)}} W_{M_j}^{(k-1)}$. For $t \in [(k+1)..m]$, we adopt $F_{M_t}^{(k)'} as $F_{M_t}^{(k)}$.$

Case 2: $H_k \in \text{Iso}(G, V)$. In this case we have $\dim V^K < \dim V^{H_k}$ for all $K \in \mathcal{U}_G(H_k)$. By performing G -surgeries of isotropy type $(H_k)_G$ on $f^{(k-1)}$ (resp. M_k -surgeries of isotropy type $(H_k)_{M_k}$ on $F_{M_k}^{(k-1)}$), we can assume without any loss of generality that $X^{(k-1)H_k}$ (resp. $W_{M_k}^{(k-1)}$) is connected. We can obtain an M_k -product embedding $\Psi_k^{(k)} : I \times N_{M_k}(M_k \cdot (\text{res}_{M_k}^G X^{(k-1)})^{>H_k}, \text{res}_{M_k}^G X^{(k-1)}) \rightarrow W_{M_k}^{(i-1)}$ from $\Phi_k^{(0)}$ and $\Phi_k^{(j)}$, where j runs over the set

$$J_k = \{j \in [1..(k-1)] \mid \rho_{\max}(H_j) = M_k\}.$$

Recall the condition that $\dim V^K = 0$ for $K \in \mathcal{Y}(G, M_k, H_k)$ is fulfilled. By Lemma 6.4, we can obtain $f^{(k)}$, $F_{M_k}^{(k)}$, $\Phi_k^{(k)}$, and $\mathbb{H}_k^{(k)}$. Moreover we can obtain $F_M^{(k)}$ for $M \in \max(\mathcal{F})^* \setminus \{M_k\}$, and $\Psi_j^{(k)}$, $\Phi_j^{(k)}$, and $\mathbb{H}_j^{(k)}$ for $j \in [1..(k-1)]$ quite similarly to Case 1.

Putting Cases 1 and 2 together, we set $f' = f^{(m)}$,

$$F_G = F_G^{(1)} \cup_{f^{(1)}} F_G^{(2)} \cup_{f^{(2)}} \cdots \cup_{f^{(m-1)}} F_G^{(m)},$$

and $F'_M = F_M^{(m)}$ and $\mathbb{H}'_M = \mathbb{H}_j^{(m)}$, where $M = M_j$. Then the conclusion of Lemma 10.1 follows. $\square$

11. Construction theorems of one-fixed-point actions on spheres

In the present section, let G be a nonsolvable group, let $\mathcal{F}$ and $\mathcal{H}$ be G -conjugation-invariant and upper-closed subsets of $S(G)_{\text{sol}}$ such that $\mathcal{F}$ is G -simply organized with respect to $\rho_{\max} : \mathcal{F}^* \rightarrow \max(\mathcal{F})^*$, where $\max(\mathcal{F})^* = \mathcal{F}^* \cap \max(\mathcal{F})$, and

$$(11.1) \quad \max(S(G)_{\text{sol}}) \cup \mathcal{H}(G, V, 0) \subset \mathcal{F} \subset \mathcal{H},$$

let β_G be the element of $\Omega(G)$ defined in (3.1), and let V be an $S(G)_{\text{nonsol}}$ -free real G -representation. Suppose V is ample for β_G and satisfy the condition (D1) in Lemma 10.1. Let f and F_M be a G -framed map and M -framed cobordisms, where $M \in \max(S(G)_{\text{sol}})^*$, obtained in Lemma 9.1. In this section we suppose that f and F_M are adjusted by Lemmas 9.2 and 10.1.

Theorem 11.1. *Further suppose V satisfies*

(D2) $\dim V^H = 3$ or $\dim V^H \geq 5$ for $H \in \mathcal{H} \setminus \mathcal{F}$, and

(D3) *the modified weak gap condition at H , for $H \in \mathcal{H} \setminus \mathcal{F}$.*

Then there exists a G -framed map $f' = (f', b')$, where $f' : (X', \partial X') \rightarrow (Y, \partial Y)$, satisfying the following conditions.

- (1) f' is G -framed cobordant rel. ∂ and $S(G)_{\text{nonsol}}$ to f_m , where $f_i = ([G/G] - \beta_G)f_{i-1}$ ($i \in [1..m]$) and $f_0 = f$, for some $m \in \mathbb{N}$. Therefore X'^G is the empty set.
- (2) $f'^H : X'^H \rightarrow Y^H$ is $N_G(H)$ -homotopic rel. ∂ to a diffeomorphism for $H \in \mathcal{F}$.
- (3) $f'^H : X'^H \rightarrow Y^H$ is a homotopy equivalence rel. ∂ for $H \in \mathcal{H}$ with $\dim V^H \neq 3$.
- (4) $f'^H : X'^H \rightarrow Y^H$ is a $\mathbb{Z}$ -homology equivalence rel. ∂ for $H \in \mathcal{H}$ with $\dim V^H = 3$.

Proof. Inductively applying Proposition 9.3 to $H \in \mathcal{H} \setminus \mathcal{F}$, we obtain the theorem. $\square$

Theorem 11.2. *In the situation of Theorem 11.1, suppose $\mathcal{H} = S(G)_{\text{sol}}$ and $\dim V > 5$. Then there exists a one-fixed-point G -action on the standard sphere S such that $T_{x_0}(S) \cong V$*

as real G -representations, where x_0 is the G -fixed point of S .

Proof. Let X' be the G -manifold obtained in Theorem 11.1 and set $\Sigma = D(V) \cup_{\partial} X'$. It is clear that Σ is a homotopy sphere with exactly one G -fixed point, say x_0 , and $T_{x_0}(\Sigma) \cong_G V$. Let S be the G -connected sum $([G/G] - \beta_G)\Sigma$ with respect to the expression (3.2) of β_G . Then S is the standard sphere with exactly one G -fixed point, cf. [16, Proposition 1.3]. $\square$

Let $\tilde{G}$ be an extension of G by a finite solvable group N , i.e. we have the exact sequence

$$E \longrightarrow N \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow E.$$

A subgroup $\tilde{H}$ of $\tilde{G}$ is solvable if and only if $\pi(\tilde{H})$ is solvable. It follows that

$$\beta_{\tilde{G}} = \pi^* \beta_G \text{ and } S(\tilde{G})_{\text{sol}} = \pi^{-1}(S(G)_{\text{sol}}).$$

Let $\tilde{U}$ be a free real $\tilde{G}$ -representation and set

$$\tilde{V} = \tilde{U} \oplus \pi^* V.$$

Let $\tilde{Y}$ be the unit disk of $\tilde{V}$. There are a $\tilde{G}$ -framed map $\tilde{f} = (\tilde{f}, \tilde{b})$ rel. ∂ , where $\tilde{f} : (\tilde{X}, \partial\tilde{X}) \rightarrow (\tilde{Y}, \partial\tilde{Y})$, $\tilde{b} : \tau_{\tilde{X}} \rightarrow \tilde{f}^* \tau_{\tilde{Y}}$, $\tau_{\tilde{X}} = \varepsilon_{\tilde{X}}(\mathbb{R}) \oplus T(\tilde{X}) \oplus \varepsilon_{\tilde{X}}(\mathbb{R}^\ell)$, and $\tau_{\tilde{Y}} = \varepsilon_{\tilde{Y}}(\mathbb{R}) \oplus T(\tilde{Y}) \oplus \varepsilon_{\tilde{Y}}(\mathbb{R}^\ell)$, and $\tilde{M}$ -framed cobordisms $\tilde{F}_{\tilde{M}} = (\tilde{F}_{\tilde{M}}, \tilde{B}_{\tilde{M}})$, where $M \in \max(S(G)_{\text{sol}})$, $\tilde{M} = \pi^{-1}(M)$, $\tilde{F}_{\tilde{M}} : \tilde{W}_{\tilde{M}} \rightarrow I \times \tilde{Y}$, and

$$\tilde{B}_{\tilde{M}} : T(\tilde{W}_{\tilde{M}}) \oplus \varepsilon_{\tilde{W}_{\tilde{M}}}(\mathbb{R}^\ell) \rightarrow \tilde{F}_{\tilde{M}}^*(T(I \times \tilde{Y})) \oplus \varepsilon_{\tilde{W}_{\tilde{M}}}(\mathbb{R}^\ell)$$

such that

$$\tilde{f}^N = f \text{ and } \tilde{F}_{\tilde{M}}^N = F_M.$$

Theorem 11.3. *In the situation of Theorem 11.1, suppose $\mathcal{H} = S(G)_{\text{sol}}$. Let $\tilde{G}$ and $\tilde{U}$ be as above. Suppose the condition that*

$$(D4) \dim \tilde{U} > \dim V \text{ and } \dim \tilde{U} + \dim V > 5$$

is fulfilled. Then there exists a one-fixed-point $\tilde{G}$ -action on the standard sphere $\tilde{S}$ such that $T_{x_0}(\tilde{S}) \cong \tilde{U} \oplus \pi^ V$ as real $\tilde{G}$ -representations, where x_0 is the $\tilde{G}$ -fixed point of $\tilde{S}$.*

Proof. Let f' be the G -framed map rel. ∂ stated in Theorem 11.1. There is a $\tilde{G}$ -framed map $\tilde{f}' = (\tilde{f}', \tilde{b}')$ rel. ∂ , where $\tilde{f}' : (\tilde{X}', \partial\tilde{X}') \rightarrow (\tilde{Y}, \partial\tilde{Y})$, such that $\tilde{f}'^N = f'$. Then $\tilde{f}'^K$ is a $\mathbb{Z}$ -homology equivalence for every $K \in S(\tilde{G})_{\text{sol}} \setminus \{E\}$. By the condition (D4), $\tilde{X}'$ satisfies the gap condition at E , because

$$2 \dim \tilde{X}'^H = 2 \dim \tilde{U}^H + 2 \dim V^{\pi(H)} = 2 \dim V^{\pi(H)} \leq 2 \dim V < \dim \tilde{U} + \dim V = \dim \tilde{X}'$$

for $H \in S(\tilde{G}) \setminus \{E\}$ such that $\tilde{X}'^H \neq \emptyset$. Without any loss of generality, we can suppose $\tilde{f}'$ is connected up to the middle dimension. We have the $\tilde{G}$ -surgery obstruction $\sigma_{\tilde{G}, E}(\tilde{f}')$ of isotropy type $(E)_{\tilde{G}}$ in $\mathcal{L}_{\tilde{V}, E}(\mathbb{Z}[\tilde{G}])$. Recall Proposition 9.3. Performing $\tilde{G}$ -surgeries rel. ∂ of isotropy type $(E)_{\tilde{G}}$ on $([\tilde{G}/\tilde{G}] - \beta_{\tilde{G}})\tilde{f}'$, we can obtain a $\tilde{G}$ -framed map $\tilde{f}'' = (\tilde{f}'', \tilde{b}'')$, where $\tilde{f}'' : (\tilde{X}'', \partial\tilde{X}'') \rightarrow (\tilde{Y}, \partial\tilde{Y})$, such that $\tilde{X}''^L = \emptyset$ for all $L \in S(\tilde{G})_{\text{non sol}}$ and $\tilde{f}''$ is a homotopy equivalence. Then $\tilde{\Sigma} = D(\tilde{V}) \cup_{\partial} \tilde{X}''$ is a homotopy sphere with exactly one $\tilde{G}$ -fixed point, say x_0 . We have $T_{x_0}(\tilde{\Sigma}) \cong \tilde{V}$ as real $\tilde{G}$ -representations. Let $\tilde{S}$ be the $\tilde{G}$ -connected sum $([\tilde{G}/\tilde{G}] - \beta_{\tilde{G}})\tilde{\Sigma}$ with respect to the expression of $\beta_{\tilde{G}}$ induced from the expression (3.2) of

β_G . Then $\widetilde{S}$ is the standard sphere with exactly one $\widetilde{G}$ -fixed point, cf. [16, Proposition 1.3].

□

12. Proof of Theorem 1.3

In this section we prove Theorem 1.3 on a case-by-case basis. Before the proof, we recall that the condition (D1) in Lemma 10.1 (concerning the primitive gap condition for $(G, \rho_{\max}(H), H)$) will be requested for $H \in \mathcal{F} \setminus (\max(\mathcal{S}(G)_{\text{sol}}) \cup \mathcal{H}(G, V, 0))$, and that the conditions (D2) and (D3) in Theorem 11.1 (concerning the modified weak gap condition at H) will be requested for $H \in \mathcal{S}(G)_{\text{sol}} \setminus \mathcal{F}$. We will give Figures 3–7 to help readers follow the arguments. In the diagrams, we adopt the following conventions.

- (1) For a subgroup H , $H^{(m)}$ indicates $\dim V^H = m$.
- (2) For subgroups H and K of G , an arrow (resp. a dotted arrow) from $H^{(m_1)}$ to $K^{(m_2)}$ indicates $\rho_{\max}(H) = K$ and $H \triangleleft K$ (resp. $\rho_{\max}(H) = K$ and $H \not\triangleleft K$).

Proof in Case $n = 6$ (i). Here $G = A_5$ and V has the form $V = V_3 \oplus V'_3$ for irreducible real G -representations V_3 and V'_3 of dimension 3. The element β_G has the form (3.3). The fixed-point-set dimensions of V for A_5 are as in Figure 3.

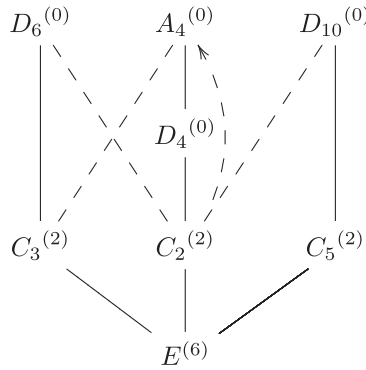


Fig.3.

By Proposition 3.3 (1), V is ample for β_G . Let $\mathcal{F} = \mathcal{S}(G)_{\text{sol}} \setminus \{E\}$ and $\mathcal{H} = \mathcal{S}(G)_{\text{sol}}$. By Proposition 7.1, $\mathcal{F}$ is G -simply organized. By Proposition 7.2, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G -action on S^6 . □

Proof in Case $n = 6$ (ii). Here $G = S_5$ and V is an irreducible real G -representation of dimension 6. The element β_G has the form (3.5). The fixed-point-set dimensions of V for S_5 are as in Figure 4.

By Proposition 3.3 (2), V is ample for β_G . Let $\mathcal{F} = \mathcal{S}(G)_{\text{sol}} \setminus (\{E\} \cup (\mathbb{C}_2)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\text{sol}}$. By Proposition 7.4, $\mathcal{F}$ is G -simply organized. By Proposition 7.5, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G -action on S^6 . □

Proof in Case $n = 6$ (iii). Here $G = A_5 \times Z$, where $|Z| = 2$, and V has the form $V = V^Z \oplus V_Z$ such that V^Z and V_Z are irreducible real G -representations of dimension 3. The

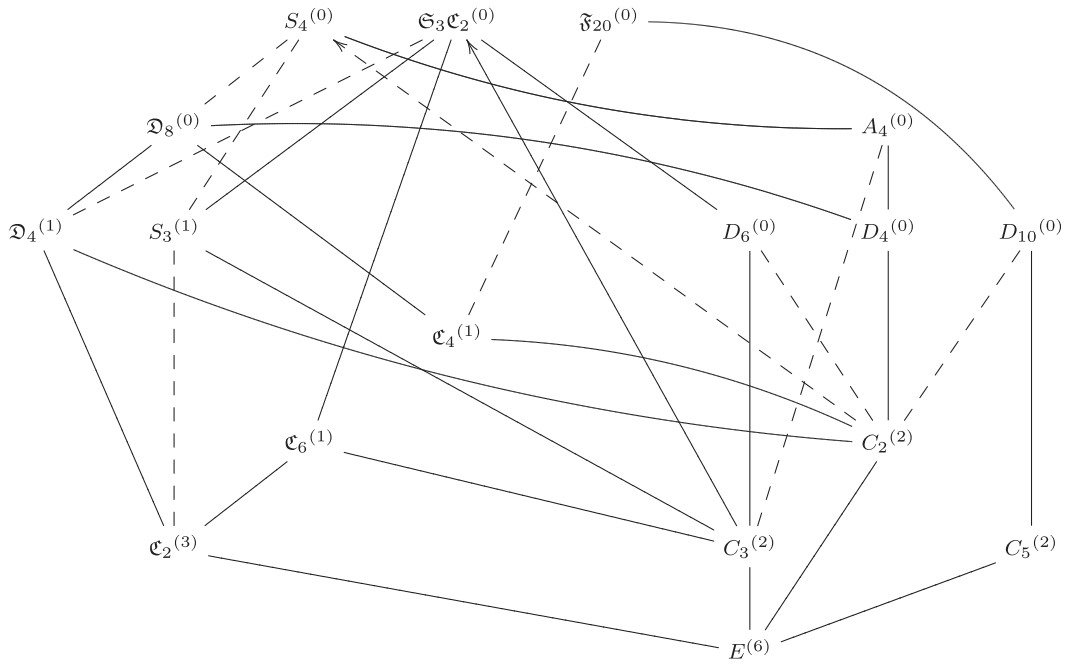


Fig.4.

element β_G has the form $\beta_G = \pi^*\beta_L$, where $L = A_5$ and $\pi : G \rightarrow L$ is an epimorphism. The fixed-point-set dimensions of V for $A_5 \times Z$ are as in Figure 5.

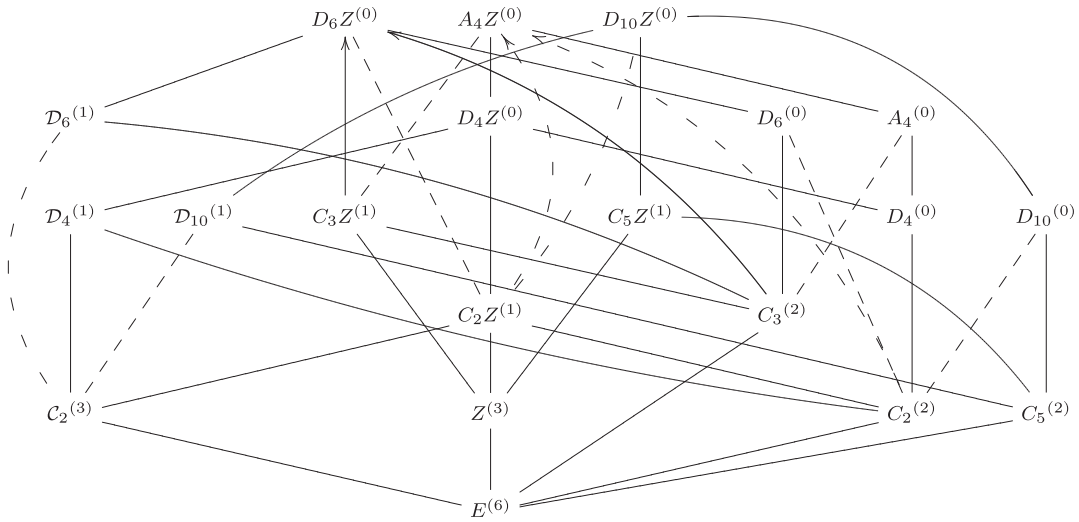


Fig.5.

By Proposition 3.3 (3), V is ample for β_G . Let $\mathcal{F} = S(G)_{\text{sol}} \setminus (\{E, Z\} \cup (C_2)_G)$ and $\mathcal{H} = S(G)_{\text{sol}}$. By Proposition 7.6, $\mathcal{F}$ is G -simply organized. By Proposition 7.7, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G -action on S^6 . $\square$

Proof in Case $n = 7$ (iv). Here $G = A_5$ and V has the form $V = V_3 \oplus V_4$, where V_3 and V_4 are irreducible real G -representations of dimension 3 and 4, respectively. The fixed-point-set dimensions of V for A_5 are as in Figure 6.

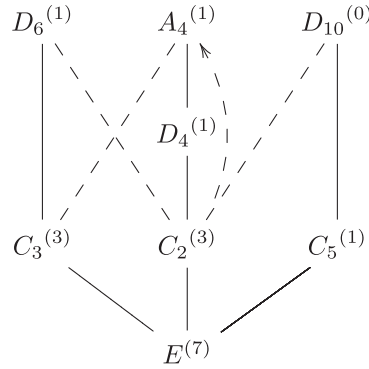


Fig.6.

By Proposition 3.3 (1), V is ample for β_G . Let $\mathcal{F} = S(G)_{\text{sol}} \setminus (\{E\} \cup (C_2)_G \cup (C_3)_G)$ and $\mathcal{H} = S(G)_{\text{sol}}$. By Proposition 7.1, $\mathcal{F}$ is G -simply organized. By Proposition 7.3, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G -action on S^7 . $\square$

Proof in Case $n = 7$ (v). Here $G = A_5 \times Z$, where $|Z| = 2$, and V has the form $V = V^Z \oplus V_Z$ such that V^Z and V_Z are irreducible real G -representations of dimension 3 and 4, respectively. The fixed-point-set dimensions of V for $A_5 \times Z$ are as in Figure 7.

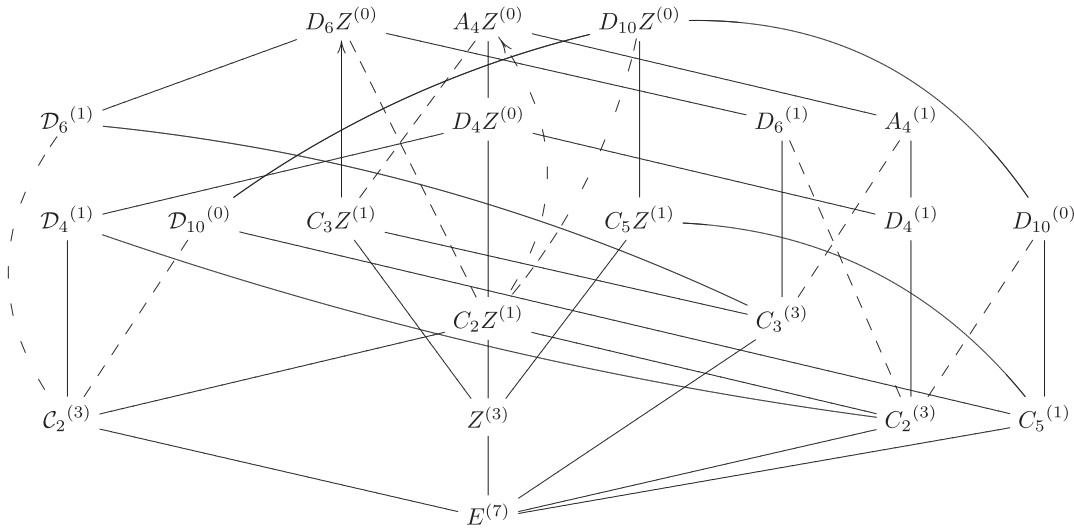


Fig.7.

By Proposition 3.3 (3), V is ample for β_G . Let $\mathcal{F} = S(G)_{\text{sol}} \setminus (\{E, Z\} \cup (C_2)_G \cup (C_2)_G \cup (C_3)_G)$ and $\mathcal{H} = S(G)_{\text{sol}}$. By Proposition 7.6, $\mathcal{F}$ is G -simply organized. By Proposition 7.8, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also

fulfilled. Therefore Theorem 11.2 gives a desired one-fixed-point G -action on S^7 . $\square$

Proof in Case $n = 3 + 4k$ (vi). Changing notation, let $\tilde{G} = \mathrm{SL}(2, 5) \times Z_m$ and $G = A_5$. Let $\pi : \tilde{G} \rightarrow G$ be an epimorphism. Changing notation, let V be an irreducible real G -representation of dimension 3, let $\tilde{U}$ be a free real $\tilde{G}$ -representation of dimension $4k$, and set $\tilde{V} = \tilde{U} \oplus \pi^*V$. The kernel N of π is $Z \times Z_m$, where $Z = \mathrm{Center}(\mathrm{SL}(2, 5))$. The element $\beta_{\tilde{G}}$ has the form $\beta_{\tilde{G}} = \pi^*\beta_G$. By Proposition 3.3 (1), V is ample for β_G . Let $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \setminus \{E\}$ and $\mathcal{H} = \mathcal{S}(G)_{\mathrm{sol}}$. By Proposition 7.1, $\mathcal{F}$ is G -simply organized. By Proposition 7.2, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.3 gives a desired one-fixed-point $\tilde{G}$ -action on S^{3+4k} . $\square$

Proof in Case $n = 6 + 8k$ (vi). Changing notation, let $\tilde{G} = \mathrm{TL}(2, 5) \times Z_m$ and $G = S_5$. Let $\pi : \tilde{G} \rightarrow G$ be an epimorphism. Changing notation, let V be an irreducible real G -representation of dimension 6, let $\tilde{U}$ be a free real $\tilde{G}$ -representation of dimension $8k$, and set $\tilde{V} = \tilde{U} \oplus \pi^*V$. The kernel N of π is $Z \times Z_m$, where $Z = \mathrm{Center}(\mathrm{TL}(2, 5))$. The element $\beta_{\tilde{G}}$ has the form $\beta_{\tilde{G}} = \pi^*\beta_G$. By Proposition 3.3 (2), V is ample for β_G . Let $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \setminus (\{E\} \cup (\mathbb{C}_2)_G)$ and $\mathcal{H} = \mathcal{S}(G)_{\mathrm{sol}}$. By Proposition 7.4, $\mathcal{F}$ is G -simply organized. By Proposition 7.5, V satisfies (D1) in Lemma 10.1 and (D2), (D3) in Theorem 11.1. The condition (11.1) is also fulfilled. Therefore Theorem 11.3 gives a desired one-fixed-point $\tilde{G}$ -action on S^{6+8k} . $\square$

We remark that the real G -representation V in Theorem 1.3 is faithful and therefore the G -action on V is effective. Since $T_{x_0}(S) \cong V$, the G -action on S obtained in Theorem 1.3 is effective.

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