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ENTROPY OF THE COMPOSITION OF TWO SPHERICAL TWISTS

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Abstract

Given a categorical dynamical system, *i.e.* a triangulated category together with an endofunctor, one can try to understand the complexity of the system by computing the entropy of the endofunctor. Computing the entropy of the composition of two endofunctors is hard, and in general the result doesn't have to be related to the entropy of the single pieces.

In this paper we compute the entropy of the composition of two spherical twists around spherical objects, showing that it depends on the dimension of the graded vector space of morphisms between them. As a consequence of these computations we produce new counterexamples to Kikuta–Takahashi's conjecture. In particular, we describe the first counterexamples in odd dimension and examples for the d-Calabi–Yau Ginzburg dg algebra associated to the A_2 quiver.

Contents

1. Introduction

In [5] the authors introduced the notion of a *categorical dynamical system*: a couple ($\mathscr{T}$, Φ) of a triangulated category together with an endofunctor $\Phi : \mathscr{T} \to \mathscr{T}$, and that of the *entropy* of an endofunctor: a function $h_t(\Phi) : \mathbb{R} \to [-\infty, +\infty)$.

Since their introduction, these ideas have received a lot of attention and many people have made contributions to the subject. However, computing explicit examples of entropies of endofunctors is a highly non-trivial task which has been accomplished only in a few cases, e.g. tensor product with lines bundles [5], spherical twists around spherical objects [21], and $\mathbb{P}$-twist around $\mathbb{P}$-objects [9].

Recently, in [17] the second author proved a theorem that relates the entropy of the twist around a spherical functor with that of (a shift of) the cotwist. Such result potentially allows

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one to estimate the entropy of any autoequivalence as it is known that any autoequivalence is the spherical twist around a spherical functor, see [22]. Moreover, as a fixed autoequivalence can be realised as a spherical twist in many different ways, one can try to make the computations easier by choosing a good representation as a spherical twist.

In [3] the first author described how to realise the composition of the twists around two spherical functors as a single twist, and therefore there is a natural candidate to which Kim's result can be applied in order to compute the entropy of the composition of two autoequivalences.

Even though these ideas seem to be profitable the general case is out of reach for the moment. For this reason we concentrate on the case of the composition of two spherical twists around spherical objects, which already shows interesting features.

A detailed statement would require us to consider various different cases and it goes beyond the scope of this introduction. Hence, we will content ourselves with an imprecise formulation.

Theorem 1.0.1. *If E_1 and E_2 are two d-spherical objects in a k-linear, proper, dg enhanced, triangulated category $\mathscr{T}$ with a split-generator and a Serre functor, and $V := \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$ satisfies $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$, then, we can explicitly compute or "precisely" bound $h_t(T_{E_2} \circ T_{E_1})$ when* $\dim V = 0, 1, 2$.

In contrast to the theory of entropy for dynamical systems from which it draws inspiration, the entropy of endofunctors naturally incorporates the dependence on a real variable $t \in \mathbb{R}$. When evaluating the entropy at 0, $h_0(\Phi)$, we speak of the *categorical entropy* of Φ.

Even though we are not able to compute the entropy of $T_{E_2} \circ T_{E_1}$ for all values of $t \in \mathbb{R}$, we are able to compute its categorical entropy. More precisely, we have

Theorem 1.0.2. *With the same notation and assumptions as in* Theorem 1.0.1*, we have*

$$h_0(T_{E_2} \circ T_{E_1}) = \begin{cases} 0 & \dim V = 0, 1, 2 \\ \log\left(\dfrac{(\dim V)^2 - 2 + \sqrt{(\dim V)^4 - 4(\dim V)^2}}{2}\right) > 0 & \dim V \geq 3 \end{cases}.$$

The content of the above two theorems is summed up in Theorem 4.0.1 and Theorem 4.0.4.

In [16] the authors proposed a conjecture that relates the categorical entropy of an autoequivalence with the spectral radius of the induced linear isomorphism on $K_{\mathrm{num}}(\mathscr{T})$. More precisely, if $\Phi : \mathscr{T} \to \mathscr{T}$ is an autoequivalence and $K_{\mathrm{num}}(\mathscr{T})$ is the numerical Grothendieck group of $\mathscr{T}$, then the conjecture says

$$h_0(\Phi) = \log \rho([\Phi]),$$

where $[\Phi] : K_{\mathrm{num}}(\mathscr{T}) \to K_{\mathrm{num}}(\mathscr{T})$ is the induced map and $\rho([\Phi]) = \max\{|\lambda| : \lambda \text{ eigenvalue of } [\Phi]\}$. In [15] the authors proved the lower bound $\geq$, but since then counterexamples have been found, [8], [21], [19].

Using Theorem 1.0.2 we are able to give a numerical condition that ensures when Kikuta–Takahashi's conjecture holds for the composition of two spherical twists, see Corollary 5.0.1. In particular, we are able to produce the first counterexamples to Kikuta–Takahashi in odd dimension (as hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$), see Example 5.0.2, and examples in the subgroup

$\langle T_{S_1}, T_{S_2}\rangle$ of $\mathrm{Aut}(\mathrm{D}(\Gamma_2^d))$, where Γ_2^d is the d-Calabi-Yau Ginzburg dg algebra and S_i's are the two spherical objects supported at the vertices, see Corollary 5.1.1.

The motivation of Fan's first counterexample to Kikuta–Takahashi's conjecture was to find a mirror counterpart of Thurston's construction of a map on a surface with positive topological entropy acting trivially on homology, [8]. The existence of a 4-dimensional example of such a map was shown recently in [14]. In Corollary 5.1.3, we give an interpretation of the A_2 Ginzburg dg algebra example in terms of symplectic geometry and see that certain compositions of Dehn twists give examples of such a map in even dimensions, see Remark 5.1.4.

2. Entropy of the spherical twist around a spherical functor

Let $\mathscr{T}$ be a k-linear triangulated category. In this paper, we study *categorical dynamical systems*, *i.e.* couples $(\mathscr{T}, \Phi)$ where $\Phi : \mathscr{T} \to \mathscr{T}$ is an exact endofunctor. To study the complexity of a categorical dynamical system, [5] introduced the notion of *categorical entropy*.

Definition 2.0.1. For $E, F \in \mathscr{T}$, the *categorical complexity* of F with respect to E is the function $\delta_t(E, F) : \mathbb{R} \to [0, \infty]$ given by

$$\delta_t(E,F) = \inf\left\{ \sum_{i=1}^{k} e^{n_i t} \;\middle|\; \begin{tikzcd}[column sep=small] 0 \arrow[rr] & & \ast \arrow[rr] \arrow[dl] & & \ast & \cdots & \ast \arrow[rr] & & F \oplus F' \arrow[dl] \\ & E[n_1] \arrow[ul, "+1"] & & E[n_2] \arrow[ul, "+1"] \arrow[from=ur] & & & & E[n_k] \arrow[ul, "+1"] & \end{tikzcd} \right\}$$

if $F \not\cong 0$, and $\delta_t(E, F) = 0$ if $F \cong 0$. Here the infimum is taken over all possible cone decompositions of objects of the form $F \oplus F'$ into $E[n_i]$'s.

An object $G \in \mathscr{T}$ is called a *split-generator* if the smallest full triangulated subcategory containing G and closed under taking direct summands coincides with $\mathscr{T}$ itself.

Definition 2.0.2. Let G be a split-generator of $\mathscr{T}$. The *categorical entropy* of an exact endofunctor $\Phi : \mathscr{T} \to \mathscr{T}$ is the function $h_t(\Phi) : \mathbb{R} \to [-\infty, \infty)$ given by

$$h_t(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, \Phi^n(G)).$$

Remark 2.0.3. The categorical entropy is well-defined, *i.e.* the limit exists in $[-\infty, \infty)$ and does not depend on the choice of a split-generator [5]. Moreover it can be also written as

$$h_t(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, \Phi^n(G'))$$

for any choice of split-generators G, G' of $\mathscr{T}$, see [13].

Let $\mathscr{D}, \mathscr{T}$ be k-linear triangulated categories with dg enhancements.

Definition 2.0.4. An exact functor $f : \mathscr{D} \to \mathscr{T}$ with right and left adjoint functors f^R, f^L is called a *spherical functor* if it satisfies the following conditions:

(1) The *twist functor* $T_f = \mathrm{cone}(ff^R \xrightarrow{\varepsilon} \mathrm{Id}_{\mathscr{T}})$ is an exact autoequivalence of $\mathscr{T}$, where

$\varepsilon : ff^R \to \mathrm{Id}_{\mathscr{T}}$ is the counit of the adjoint pair $f \dashv f^R$.

(2) The *cotwist functor* $C_f = \mathrm{cone}(\mathrm{Id}_{\mathscr{D}} \xrightarrow{\eta} f^R f)[-1]$ is an exact autoequivalence of $\mathscr{D}$, where $\eta : \mathrm{Id}_{\mathscr{D}} \to f^R f$ is the unit of the adjoint pair $f \dashv f^R$.

(3) $f^R \cong f^L T_f[-1]$.

(4) $f^R \cong C_f f^L[1]$.

Remark 2.0.5. The original definition of a spherical functor [2] requires that the isomorphisms in (3), (4) should come from some canonical natural transformations. However, in [1, §1.1], the author proved that, if there is any isomorphism, then the canonical one is also an isomorphism under the assumption that the cotwist functor is an autoequivalence.

In [17, Theorem 1.6, 1.7], the second author proved the following theorem which relates the entropy of the twist with that of the cotwist.

Theorem 2.0.6. *Let $f : \mathscr{D} \to \mathscr{T}$ be a spherical functor with right adjoint functor f^R.*

(1) *Assume that the essential image of f^R contains a split-generator of $\mathscr{D}$. Then*

$$h_t(C_f[2]) \le h_t(T_f) \le \begin{cases} 0 & \text{for every } t \text{ such that } h_t(C_f[2]) \le 0, \\ h_t(C_f[2]) & \text{for every } t \text{ such that } h_t(C_f[2]) \ge 0. \end{cases}$$

(2) *Assume that* $\mathrm{Ker}\, ff^R \neq 0$. *Then*

$$h_t(T_f) \ge 0.$$

Example 2.0.7. This theorem can be considered as a generalisation of the computations of the entropy of the spherical twist around a spherical object [21], and the $\mathbb{P}$-twist around a $\mathbb{P}$-object [9].

Indeed, if E is a d-spherical object ($d \ge 1$) in $\mathscr{T}$, the functor $f = - \otimes_k E : \mathrm{D^b}(k) \to \mathscr{T}$ is spherical and

$$T_f \cong T_E, \qquad C_f \cong [-1-d]$$

where T_E denotes the *spherical twist* around E. Since by [5, Theorem 2.6] in $\mathrm{D^b}(k)$ we have $h_t([m]) = mt$ for any $m \in \mathbb{Z}$, Theorem 2.0.6 implies that

$$(1-d)t \le h_t(T_E) \le \begin{cases} 0 & t \ge 0 \\ (1-d)t & t \le 0 \end{cases}.$$

Moreover, it also implies that if $E^\perp := \{F \in \mathscr{T} \mid \mathrm{Hom}^\bullet_{\mathscr{T}}(E, F) = 0\} \neq 0$, then $h_t(T_E) = 0$ for all $t \ge 0$. This is exactly the main result of [21].

The main result of [9] can be obtained similarly using a presentation of the $\mathbb{P}$-twist around a $\mathbb{P}$-object as a spherical twist [22].

In general, it is not easy to verify the technical conditions of the above theorem. However, the following lemma from [17] provides a useful sufficient condition for the assumption of part (1) of Theorem 2.0.6.

Lemma 2.0.8. *Let $f : \mathscr{D} \to \mathscr{T}$ be a spherical functor with a right adjoint functor f^R and G be a split-generator of $\mathscr{D}$. Assume that there is an integer $n > 0$ such that* $\mathrm{Hom}_{\mathscr{D}}(C_f^n(G), G) = 0$. *Then $f^R f(G \oplus C_f(G) \oplus \cdots \oplus C_f^{n-1}(G))$ is a split-generator of $\mathscr{D}$.*

Proof. Set $X_1 = f^R f(G)$. It sits in the distinguished triangle

$$G \xrightarrow{\eta_G} f^R f(G) \xrightarrow{\phi_1} C_f(G)[1] \to G[1] \tag{1}$$

defining the cotwist functor C_f. Then, we inductively define a sequence $\{X_n\}_{n=1}^{\infty}$ of objects of $\mathscr{D}$ by the commutative diagram

$$\begin{tikzcd}
X_{n-1}[-1] \arrow[r, "{\phi_{n-1}[-1]}"] \arrow[d, equal] & C_f^{n-1}(G) \arrow[r] \arrow[d, "\eta_{C_f^{n-1}(G)}"] & G \arrow[r] \arrow[d] & X_{n-1} \arrow[d, equal] \\
X_{n-1}[-1] \arrow[r] & f^R f C_f^{n-1}(G) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, "\phi_n"] & X_{n-1} \\
 & C_f^n(G)[1] \arrow[r, equal] \arrow[d] & C_f^n(G)[1] \arrow[d] & \\
 & C_f^{n-1}(G)[1] \arrow[r] & G[1] &
\end{tikzcd}$$

obtained by applying the octahedral axiom.

The assumption implies that the exact triangle

$$G \to X_n \to C_f^n(G)[1] \to G[1]$$

splits, and therefore $X_n \cong G \oplus C_f^n(G)[1]$. The lemma follows since X_n is split-generated by $f^R f(G \oplus C_f(G) \oplus \cdots \oplus C_f^{n-1}(G))$ by construction. $\square$

Corollary 2.0.9. *Let $f : \mathscr{D} \to \mathscr{T}$ be a spherical functor with a right adjoint functor f^R. If there exists a split-generator G of $\mathscr{D}$ and an integer $n > 0$ such that $\operatorname{Hom}_{\mathscr{D}}(C_f^n(G), G) = 0$, then the essential image of $f^R f$ contains a split-generator of $\mathscr{D}$.*

In general, computing the entropy of an endofunctor is a very hard task, and we will try to tackle this question using Theorem 2.0.6. However, there is a case in which we can bound the entropy using its value at zero and some asymptotic behaviour.

Proposition 2.0.10 ([10, Theorem 2.1.7], [7, Proposition 6.13, 6.14])**.** *For any non-nilpotent endofunctor F of $\mathscr{T} = \mathrm{D}(T)^c$, T a smooth and proper dg algebra, the limits*

$$\lim_{t \to \pm\infty} \frac{h_t(F)}{t} = \tau^{\pm}(F)$$

are finite and we have the inequalities

$$\tau^+(F)t \le h_t(F) \le h_0(F) + \tau^+(F)t \qquad t \ge 0,$$

$$\tau^-(F)t \le h_t(F) \le h_0(F) + \tau^-(F)t \qquad t \le 0.$$

3. Upper triangular dg algebras and gluing

Let us consider two dg algebras A, B and an A-B bimodule V. From this data we can construct a new dg algebra $R := B \oplus A \oplus V$, where the grading and the differential are defined componentwise, and the multiplication is $(b, a, v) \cdot (b', a', v') = (bb', aa', vb' + av')$.

This new dg algebra is sometimes denoted

$$R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix},$$

and is called an upper triangular dg algebra.

In [3] the first author used such a dg algebra to represent the composition of two spherical twists around two spherical objects as the spherical twist around a single spherical functor.

Let us briefly recall this construction. Consider $\mathscr{T}$ a k-linear, proper, dg enhanced triangulated category with a split-generator and a Serre functor $\mathbb{S}_{\mathscr{T}}$. Take E_1 and E_2 two d-spherical objects in $\mathscr{T}$, *i.e.* they satisfy

$$\mathbb{S}_{\mathscr{T}} E_i \simeq E_i[d], \qquad \mathrm{Hom}^\bullet_{\mathscr{T}}(E_i, E_i) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{T}}(E_i, E_i[n])[-n] \simeq k[t]\big/t^2\ , \ \deg(t) = d,$$

where the second isomorphism is of graded algebras.

Then, we can construct the autoequivalence

$$T_i(F) := \mathrm{cone}\left(\mathrm{Hom}^\bullet_{\mathscr{T}}(E_i, F) \otimes E_i \to F\right)$$

called the *spherical twist* around E_i, see [23].

Consider dg lifts E_1', E_2' of the E_i's in a dg enhancement $\mathcal{D}$ of $\mathscr{T}$; then, the dg k-module $\mathrm{Hom}_{\mathcal{D}}(E_2', E_1')$ has associated graded module $H^\bullet(\mathrm{Hom}_{\mathcal{D}}(E_2', E_1'))$ isomorphic to $V := \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$. Define the upper triangular dg algebra $R' = k \oplus k \oplus \mathrm{Hom}_{\mathcal{D}}(E_2', E_1')$ and consider $E_2' \oplus E_1'$ as a left dg module over R'. Here the first (resp. second) copy of k acts on the right (resp. left) $\mathrm{Hom}_{\mathcal{D}}(E_2', E_1')$ via the identity of E_2' (resp. E_1'). Notice however that such an upper triangular dg algebra is formal because we can write an explicit quasi isomorphism $H^\bullet(R') \to R'$ by choosing representatives of the cohomology classes of $\mathrm{Hom}_{\mathcal{D}}(E_2', E_1')$. In particular, the dg enhancement of $\mathscr{T}$ doesn't matter in this particular construction, and we directly consider the graded algebra $R := k \oplus k \oplus V$.

With these remarks in mind, [3, Theorem 3.2.1] can be stated as follows[1]

Theorem 3.0.1. *The left R-module $E_2 \oplus E_1$ defines a spherical functor*

$$\mathrm{D}(R)^c \xrightarrow{f := -\overset{L}{\otimes}_R (E_2 \oplus E_1)} \mathscr{T}$$

whose twist is given by $T_f \simeq T_2 \circ T_1$ and whose cotwist is given by $C_f \simeq - \overset{L}{\otimes}_R R^*[-1-d]$.

REMARK 3.0.2. In [3] the cotwist was described for the dg algebra and not for its associated graded algebra. The description of the cotwist in the above formulation follows from the fact that if $A \to B$ is a quasi isomorphism of dg algebras, then the dual map $B^* \to A^*$ is quasi isomorphism of A-A dg bimodules.

In particular, as R is smooth and proper, we see that the cotwist gives Serre duality on $\mathrm{D}(R)^c$ up to a shift, see [24].

3.1. A distinguished triangle. Our aim is now to give sufficient conditions under which the assumption of part (1) of Theorem 2.0.6 is verified for the case of the composition of two spherical twists around spherical objects.

[1] Here, for a triangulated category $\mathscr{T}$ with arbitrary direct sums we denote $\mathscr{T}^c$ the subcategory of compact objects.

As a consequence of Theorem 3.0.1, we get the distinguished triangle of right R dg modules

$$R \to \mathrm{RHom}_{\mathscr{T}}(E_2 \oplus E_1, E_2 \oplus E_1) \to R^*[-d] \to R[1],$$

where $\mathrm{RHom}_{\mathscr{T}}(E_2 \oplus E_1, E_2 \oplus E_1)$ denotes the dg endomorphism algebra of (a dg lift of) $E_2 \oplus E_1$ in some dg enhancement of $\mathscr{T}$. This is triangle (1) for the spherical functor of Theorem 3.0.1.

By Lemma 2.0.8 we know that to satisfy the technical condition of Theorem 2.0.6 is enough to prove

$$0 = \mathrm{Hom}_{\mathrm{D}(R)}(R^*[-d], R[1]) \simeq H^{1+d}(R^!), \quad R^! = \mathrm{RHom}_{R\text{-}R}(R, R \otimes_k R).$$

Recall that $V = \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$. We have

Lemma 3.1.1. *If*

$$V^{1+d} = (V^*)^d = (V \otimes_k V^*)^d = (V^* \otimes_k V \otimes_k V)^d = 0,$$

then $H^{1+d}(R^!) = 0$.

Proof. For clarity let us denote k_1 the copy of k acting on R via id_{E_1} and k_2 the one acting via id_{E_2}. Then, by the definition of R we have the distinguished triangle of R-R bimodules (see e.g. the proof of [18, Proposition 3.11])

$$R \otimes_{k_1} V \otimes_{k_2} R \to R \otimes_{k_1} R \oplus R \otimes_{k_2} R \to R \to R \otimes_{k_1} V \otimes_{k_2} R[1].$$

Applying the functor $\mathrm{RHom}_{R\text{-}R}(-, R \otimes_k R)$, we get the distinguished triangle

$$R^! \to \begin{array}{c} \mathrm{RHom}_{R\text{-}R}(R \otimes_{k_1} R, R \otimes_k R) \\ \oplus \\ \mathrm{RHom}_{R\text{-}R}(R \otimes_{k_2} R, R \otimes_k R) \end{array} \to \mathrm{RHom}_{R\text{-}R}(R \otimes_{k_1} V \otimes_{k_2} R, R \otimes_k R) \to R^![1].$$

Now notice that[2]

$${}_{k_1}R \simeq k_1 \oplus V \quad {}_{k_2}R \simeq k_2 \quad R_{k_1} \simeq k_1 \quad R_{k_2} \simeq V \oplus k_2.$$

Using these isomorphisms of bimodules we can simplify the above distinguished triangle and get

$$R^! \to k \oplus k \oplus V \oplus V \to V^* \oplus V^* \otimes V \oplus V^* \otimes V \oplus V^* \otimes V \otimes V \to R^![1].$$

Then, the statement follows from taking the long exact sequence induced by the above distinguished triangle. □

We now wish to show that all of the conditions of Lemma 3.1.1 can be achieved if V satisfies[3] $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$.

Definition 3.1.2. For a graded vector space V we denote $\max V$ (resp. $\min V$) the maximum (resp. minimum) degree of a non-zero element of V.

[2]Here the subscript means that we are restricting the action via the inclusion $k_i \hookrightarrow R$.

[3]Notice that V is bounded by construction as $\mathscr{T}$ is proper.

Notice that if $\max V \leq d$, then $V^{1+d} = 0$; if $\max V < -d$, then $(V^*)^d = 0$; and if $2\max V - \min V < d$, then $(V^* \otimes_k V \otimes_k V)^d = 0$. Furthermore, notice that if we exchange E_1 with $E_1[n]$ the spherical twist does not change: $T_{E_1[n]} \simeq T_1$, but the degrees in which $V_n = \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1[n])$ lives do. More precisely, we have

$$\max V_n = \max V - n \qquad 2\max V_n - \min V_n = 2\max V - \min V - n.$$

In particular, if we take $n >> 0$ the three inequalities above can always be achieved, and the only remaining vanishing required by Lemma 3.1.1 is $(V^* \otimes_k V)^d = \mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$.

Hence we get

Lemma 3.1.3. *Let E_1 and E_2 be two d-spherical objects in $\mathscr{T}$ a k-linear, proper, dg enhanced triangulated category with a split-generator and a Serre functor. Set $V := \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$.*

If $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$, *then, up to replacing E_1 with $E_1[n]$ and V with $V[n]$ for $n >> 0$, the assumption of part* (1) *of* Theorem 2.0.6 *are satisfied for the spherical functor*

$$\mathrm{D}(R)^c \xrightarrow{f:=-\overset{L}{\otimes}_R(E_2 \oplus E_1)} \mathscr{T}.$$

Let us remark that we do not know whether the condition $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$ is really needed or whether it can be removed by a more thorough study of the map $R^*[-d] \to R[1]$.

REMARK 3.1.4. In principle what we did in this section can be done for any upper triangular dg algebra, and hence one could try to find sufficient conditions under which the hypothesis of part (1) of Theorem 2.0.6 is satisfied for any couple of spherical functors.

Unfortunately, the problem is that the terms involved are now RHom's between dg bimodules over the dg algebras A and B from which the upper triangular dg algebra R is constructed. Hence, homs can go in any direction regardless of the cohomological bounds we impose.

However, it is worthy to point out that in the case of the dg algebra arising from [3, Theorem 4.1.2] for the composition of many spherical twists around spherical objects it is still possible to give sufficient conditions based on cohomological bounds (because we can bring all the RHom's back at the vertices of the dg algebra).

3.2. Categorical entropy of the Serre functor. Theorem 3.0.1 and Lemma 3.1.1 tell us that if $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$, $V = \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$, then the entropy of $T_2 \circ T_1$ can be computed by computing the entropy of the Serre functor for $\mathrm{D}(R)^c$, $R = k \oplus k \oplus V$ (up to shift V, but we will not care about this because shifting V will not affect the final result, as it ought to be).

Even though our motivation for computing the entropy of the Serre functor of $\mathrm{D}(R)^c$ is computing the entropy of $T_2 \circ T_1$, the results of this section apply for any upper triangular dg algebra of the form $k \oplus k \oplus W$ where W is a graded vector space with $2 \leq \dim W < \infty$. Hence, in the following A will denote any such upper triangular dg algebra.

We know by [24] that the Serre functor for $\mathrm{D}(A)^c$ is given by $\mathbb{S}_A := - \overset{L}{\otimes}_A A^*$, so the only thing we have to do is to compute $h_t(\mathbb{S}_A)$.

Unfortunately, this is not an easy task for a general $t \in \mathbb{R}$. However, using the results of [6], we will be able to compute the categorical entropy of $\mathbb{S}_A$, *i.e.* $h_0(\mathbb{S}_A)$ (and the entropy itself in the case $\dim W = 2$).

REMARK 3.2.1. In [5, pag. 32] the authors state the value of the entropy of the Serre functor for the derived category of the Kronecker quiver with $m \geq 3$ arrows. Our computations will recover that value when W lives only in degree 0, and they will show that the grading on W does not affect $h_0(\mathbb{S}_A)$.

As our category is of the form $\mathrm{D}(A)^c$, by [5, Theorem 2.6] we know that computing $h_0(\mathbb{S}_A)$ amounts to computing

$$\lim_{m\to+\infty} \frac{1}{m} \log\left(\sum_{n\in\mathbb{Z}} \dim H^n\left((A^*)^{\otimes_A m}\right)\right).$$

Thanks to[4] [6, Lemma 8.2], we know that

$$\sum_{n\in\mathbb{Z}} \dim H^n\left((A^*)^{\otimes_A m}\right) = d_{2m-2} + d_{2m-3} + d_{2m-1} + d_{2m-2},$$

where d_m satisfies the relations

$$\begin{aligned} d_{m+2} + d_m &= d_{m+1} \cdot \dim W \quad \forall\, m \geq -1 \\ d_1 &= \dim W \\ d_0 &= 1 \\ d_{-1} &= 0. \end{aligned} \tag{2}$$

Set $N = \dim W$. To solve this recurrence relation we use the characteristic equation

$$N\sigma^{-1} - \sigma^{-2} = 1 \iff \sigma_\pm = \frac{N \pm \sqrt{N^2-4}}{2}.$$

We see that we have to distinguish between two cases.

If $N = 2$ the the solution to the recurrence equation is given by

$$d_m = m + 1.$$

If $N \geq 3$ then the solution is given by

$$d_m = \alpha\sigma_-^m + \beta\sigma_+^m, \quad \alpha = \frac{1}{2} - \frac{N}{2\sqrt{N^2-4}}, \quad \beta = \frac{1}{2} + \frac{N}{2\sqrt{N^2-4}}.$$

Hence we get

Lemma 3.2.2. *We have*

$$h_0(\mathbb{S}_A) = \begin{cases} 0 & \dim W = 2 \\ \log\left(\dfrac{(\dim W)^2 - 2 + \sqrt{(\dim W)^4 - 4(\dim W)^2}}{2}\right) > 0 & \dim W \geq 3 \end{cases}.$$

Proof. Notice that by the recurrence relations (2) we have

$$\sum_{n\in\mathbb{Z}} \dim H^n\left((A^*)^{\otimes_A m}\right) = (2 + \dim W) d_{2m-2}.$$

---

[4]What we denote d_m is $\dim \psi_m(W)$ in *ibidem*.

Hence we have

$$h_0(\mathbb{S}_A) = \lim_{m\to+\infty} \frac{1}{m} \log(d_{2m-2}).$$

If $N = 2$ we have

$$h_0(\mathbb{S}_A) = \lim_{m\to+\infty} \frac{1}{m} \log(2m-1) = 0.$$

If $N \geq 3$ have

$$\begin{aligned} h_0(\mathbb{S}_A) &= \lim_{m\to+\infty} \frac{1}{m} \log(\alpha\sigma_-^{2m-2} + \beta\sigma_+^{2m-2}) \\ &= \lim_{m\to+\infty} \frac{1}{m}(2m-2)\log(\sigma_+) = \log(\sigma_+^2), \end{aligned}$$

where in the second line we used that $\sigma_+ > \sigma_-$ and $\beta \neq 0$. □

Proposition 3.2.3. *Set* $w = \max W - \min W$*. If* $\dim W = 2$*, we have*

$$h_t(\mathbb{S}_A) = \begin{cases} (1-w)t & t \leq 0 \\ (1+w)t & t \geq 0 \end{cases}.$$

Proof. This follows from Lemma 3.2.2, [6, Proposition 8.4], [7, Proposition 6.13], and Proposition 2.0.10. □

4. Composition of two spherical twists around spherical objects

Now that we have introduced all the pieces that we need, we can move on to compute the categorical entropy of the composition of two spherical twists around spherical objects, and in some cases the entropy itself.

Let us recall the setting. We have $\mathscr{T}$ a k-linear, proper, dg enhanced triangulated category with a split-generator and a Serre functor. Moreover, we have two d-spherical objects E_1, $E_2 \in \mathscr{T}$, and we want to compute the entropy of $T_2 \circ T_1$, where $T_i = T_{E_i}$.

By Theorem 3.0.1 we know that for $f = - \overset{L}{\otimes}_R (E_2 \oplus E_1)\colon \mathrm{D}(R)^c \to \mathscr{T}$, where $R = k \oplus k \oplus \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$, we have $T_2 \circ T_1 \simeq T_f$, $C_f \simeq \mathbb{S}_R[-1-d]$.

Moreover, by Lemma 3.1.3 we know that if $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$, $V = \mathrm{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$, then we can compute the entropy of $T_2 \circ T_1$ using Theorem 2.0.6.

Finally, by Lemma 3.2.2 we know the exact value $h_0(\mathbb{S}_R)$ when $\dim V \geq 2$.

Let us put together all these pieces to get the following results.

Theorem 4.0.1. *Assume* $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$*. Then the categorical entropy of* $T_2 \circ T_1$ *is given by*

$$h_0(T_2 \circ T_1) = \begin{cases} 0 & \dim V = 0, 1, 2 \\ \log\left(\dfrac{(\dim V)^2 - 2 + \sqrt{(\dim V)^4 - 4(\dim V)^2}}{2}\right) > 0 & \dim V \geq 3 \end{cases}.$$

Moreover, if $\dim V = 0$ *we have*

$$h_t(T_2 \circ T_1) = \begin{cases} (1-d)t & t \leq 0 \\ \leq 0 & \text{otherwise} \end{cases},$$

while if $\dim V = 1$ *we have*

$$h_t(T_2 \circ T_1) = \begin{cases} \left(\frac{4}{3} - d\right)t & \forall t : \left(\frac{4}{3} - d\right)t \geq 0 \\ \leq 0 & \text{otherwise} \end{cases}.$$

In all cases, if $E_1^{\perp} \cap E_2^{\perp} \neq 0$*, then* $h_t(T_2 \circ T_1) \geq 0$.

Proof. When $\dim V \geq 2$, the first statement is a rephrasing of Lemma 3.2.2 taking into account $C_f = \mathbb{S}_R[-1-d]$.

If $\dim V = 0$ we have $R = k \oplus k$ and $\mathbb{S}_R = \mathrm{id} \oplus \mathrm{id}$ on $\mathrm{D}(R)^c \simeq \mathrm{D}(k)^c \oplus \mathrm{D}(k)^c$. Hence, using [5, Theorem 2.6] to compute $h_t(\mathbb{S}_R)$, we have

$$h_t(T_2 \circ T_1) = \underbrace{h_t(\mathbb{S}_R)}_{=0} + (1-d)t = (1-d)t \quad \forall t : (1-d)t \geq 0$$

and $h_t(T_2 \circ T_1) \leq 0$ otherwise.

If $\dim V = 1$ we can always shift V so that R is the path algebra of the Dynkin quiver A_2. Hence, $\mathrm{D}(R)^c$ is fractional Calabi–Yau of dimension $1/3$, see [11], [4]. For any $d \geq 1$ Lemma 3.1.3 applies (without need of further shifting V), and therefore, using once again [5, Theorem 2.6], we get

$$h_t(T_2 \circ T_1) = h_t(\mathbb{S}_R) + (1-d)t = \left(\frac{4}{3} - d\right)t \quad \forall t : \left(\frac{4}{3} - d\right)t \geq 0$$

and $h_t(T_2 \circ T_1) \leq 0$ otherwise.

The statement about the case in which the common orthogonal is not zero follows from Theorem 2.0.6. □

REMARK 4.0.2. When $\dim V = 0$ the twists T_2 and T_1 commute with each other. In this case the result we obtained can also be proved using the same strategy used in [21, Theorem 3.1].

REMARK 4.0.3. It was noticed in [20, Theorem 3.1] and [19, Remark 3.5] that the composition of many spherical twists can have positive categorical entropy, but the value of the entropy was not computed. The above theorem gives the precise value of the entropy of the composition of two spherical twists and tells us when it is positive.

Theorem 4.0.4. *Assume* $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$ *and set* $w = \max V - \min V$. *If* $\dim V = 2$*, then we have the following:*

(1) *if* $d + w \geq 2$ *and* $d - w > 2$*, then*

$$h_t(T_2 \circ T_1) = \begin{cases} (2-(d+w))t & t \leq 0 \\ \leq 0 & t \geq 0 \end{cases};$$

(2) *if* $d + w \geq 2$ *and* $d - w \leq 2$*, then*

$$h_t(T_2 \circ T_1) = \begin{cases} (2-(d+w))t & t \leq 0 \\ (2-(d-w))t & t \geq 0 \end{cases};$$

(3) *if* $d + w < 2$ *and* $d - w > 2$*, then*

$$h_t(T_2 \circ T_1) \leq 0 \quad \forall t \in \mathbb{R};$$

(4) *if $d + w < 2$ and $d - w \leq 2$, then*

$$h_t(T_2 \circ T_1) = \begin{cases} \leq 0 & t \leq 0 \\ (2 - (d - w))t & t \geq 0 \end{cases}.$$

In all cases, if $E_1^\perp \cap E_2^\perp \neq 0$, then $h_t(T_2 \circ T_1) \geq 0$.

Proof. The assumptions, together with Theorem 2.0.6, Lemma 3.1.3, and Proposition 3.2.3, imply that[5]

$$h_t(T_2 \circ T_1) = h_t(\mathbb{S}_R) + (1 - d)t = \begin{cases} (2 - (d + w))t & t \leq 0 \\ (2 - (d - w))t & t \geq 0 \end{cases}$$

as long as the right hand side is bigger than or equal to 0, and $h_t(T_2 \circ T_1) \leq 0$ otherwise. The statement of the theorem then follows by a case by case argument. □

5. Counterexamples to Kikuta–Takahashi

In this section, using Theorem 4.0.1, we will produce new counterexamples to Kikuta–Takahashi's conjecture [16]. In particular, we will produce the first counterexamples in odd dimension.

Let $\mathscr{T}$ be a k-linear, proper, dg enhanced triangulated category with a Serre functor and a split-generator, and let $K(\mathscr{T})$ be its Grothendieck group. The *Euler form* $\chi\colon K(\mathscr{T}) \times K(\mathscr{T}) \to \mathbb{Z}$ is defined by

$$\chi([E], [F]) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \operatorname{Hom}_{\mathscr{T}}(E, F[i]).$$

We define the *numerical Grothendieck group* $K_{\mathrm{num}}(\mathscr{T})$ as[6]

$$K_{\mathrm{num}}(\mathscr{T}) = K(\mathscr{T}) / \langle [E] \in K(\mathscr{T}) \,|\, \chi([E], -) = 0 \rangle.$$

Note that the induced Euler form $\chi : K_{\mathrm{num}}(\mathscr{T}) \times K_{\mathrm{num}}(\mathscr{T}) \to \mathbb{Z}$ is non-degenerate. In this section, we only consider triangulated categories whose numerical Grothendieck groups are of finite rank.

Corollary 5.0.1. *Let $E_1, E_2 \in \mathscr{T}$ be d-spherical objects and $V = \operatorname{Hom}^\bullet_{\mathscr{T}}(E_2, E_1)$. Suppose $[E_1], [E_2]$ are non-zero and linearly independent in $K_{\mathrm{num}}(\mathscr{T})$, that $\operatorname{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$, and that $\chi([E_2], [E_1]) \neq \pm 2$ if d is even, $\chi([E_2], [E_1]) \neq 0$ if d is odd. If $\dim V = 0, 1, 2$, then*

$$h_0(T_2 \circ T_1) = \log \rho([T_2 \circ T_1]) = 0,$$

and if $\dim V \geq 3$, then

$$h_0(T_2 \circ T_1) \geq \log \rho([T_2 \circ T_1]),$$

[5]Here R depends on how much we shift E_1, but the entropy of the Serre functor does not, so we drop the dependence on n.

[6]Notice that the existence of a Serre functor implies that the right and left radical of χ agree, so there is no ambiguity in the definition of $K_{\mathrm{num}}(\mathscr{T})$.

where the equality holds if and only if $\chi([E_2],[E_1]) = \pm \dim V$.

Proof. First of all, notice that as $[E_1],[E_2]$ are assumed to be non-zero and linearly independent the subspace $W := \mathrm{Span}([E_1],[E_2])$ is two dimensional. Moreover, the fact that $[E_1],[E_2]$ are d-spherical implies $W^{\perp} = {}^{\perp}W$.

The assumptions on $\chi([E_2],[E_1])$ imply that the restriction of χ to W is non-degenerate. Hence, we get a basis of $K_{\mathrm{num}}(\mathscr{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ by taking $[E_1]$,$[E_2]$ and a basis of $W^{\perp}$.

By definition, we have

$$[T_i](v) = v - \chi([E_i],v)[E_i] \qquad \forall\, v \in K_{\mathrm{num}}(\mathscr{T}).$$

Denote $\lambda = \chi([E_2],[E_1])$. Then, with respect to the previously chosen basis,

$$[T_2 \circ T_1] = \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

where

$$A = (-1)^{1-d} \begin{pmatrix} 1 & \lambda \\ -\lambda & 1-\lambda^2 \end{pmatrix}.$$

The eigenvalues are

$$\frac{\lambda^2 - 2 \pm \sqrt{\lambda^4 - 4\lambda^2}}{2},$$

and the logarithm of the spectral radius is[7]

$$\log \rho([T_2 \circ T_1]) = \log \left| \frac{\lambda^2 - 2 + \sqrt{\lambda^4 - 4\lambda^2}}{2} \right|.$$

By Theorem 4.0.1, we have

$$h_0(T_2 \circ T_1) = \log \left| \frac{(\dim V)^2 - 2 + \sqrt{(\dim V)^4 - 4(\dim V)^2}}{2} \right|.$$

This shows the statement for $\dim V = 0,1,2$. As the function

$$x \mapsto \log \left(\frac{x - 2 + \sqrt{x^2 - 4x}}{2} \right)$$

is injective on $x \geq 4$, we also get the statement for $\dim V \geq 3$. □

Example 5.0.2. Consider $\mathbb{P}^n \times \mathbb{P}^m$ with either

(1) $n \geq 3$, n odd, $m \geq 2$, m even;
(2) $n, m \geq 2$, n, m even.

Take X to be the zero locus of a general section of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(n+1, m+1)$. Then, from the exact sequence

$$\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(-n-1, -m-1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m} \to \mathcal{O}_X$$

we see that X is a true Calabi–Yau manifold of dimension $m + n - 1$. In particular, line

[7]Notice that here we are taking the absolute value of (possibly) a complex number.

bundles on X are $d := m + n - 1$ spherical objects.

Consider $\mathcal{L} = \mathcal{O}_X(n+1, 0)$. Then, from the above exact sequence we see that

$$R\Gamma(\mathcal{L}) \simeq k^N \oplus k[-m+1] \qquad N = \binom{2n+1}{n+1}$$

In particular, if we set $V = \mathrm{Hom}^\bullet_{\mathrm{D^b}(X)}(\mathcal{O}_X, \mathcal{L})$, we have: $\dim V = N + 1$, and $\lambda = N - 1 > 2$ in case (1) (m is even and $n \geq 3$), $\lambda = N - 1 > 0$ in case (2) (m is even and $n \geq 2$). Moreover, we have $\max V - \min V = m - 1 < d$, and therefore $\mathrm{Hom}_{\mathrm{D^b}(k)}(V, V[d]) = 0$.

As the line bundles $\mathcal{O}_X$ and $\mathcal{L}$ have linearly independent classes in $K_{\mathrm{num}}(\mathrm{D^b}(X))$,[8] Corollary 5.0.1 applies. In particular, we get

$$\log\left(\rho(T_{\mathcal{O}_X} \circ T_{\mathcal{L}})\right) < h_0\left(T_{\mathcal{O}_X} \circ T_{\mathcal{L}}\right),$$

thus contradicting Kikuta–Takahashi's conjecture.

5.1. A_2 **Ginzburg dg algebra.** The *d-Calabi–Yau Ginzburg dg algebra* Γ_2^d associated to the A_2 quiver is defined as follows. First, as a graded algebra, it is the path algebra of the graded quiver with two vertices $\{1, 2\}$ and four arrows: $a : 1 \to 2$ in degree 0, $a^* : 2 \to 1$ in degree $2-d$ and $t_i : i \to i$ $(i = 1, 2)$ in degree $1-d$. The differential is given by $da = da^* = 0$, $dt_1 = aa^*$ and $dt_2 = -a^*a$.

Let $\mathscr{D}_2^d$ be the derived category of dg Γ_2^d-modules with finite dimensional cohomology. It is known that $\mathscr{D}_2^d$ is a d-Calabi–Yau category and the simple modules S_1, S_2 are spherical objects such that $V = \mathrm{Hom}^\bullet_{\mathscr{D}_2^d}(S_2, S_1) = \mathbb{C}[1-d]$. Denote by T_1, T_2 the spherical twists around them; we obtain a braid group action via $\mathrm{Br}_3 = \langle \sigma_1, \sigma_2 \rangle \ni \sigma_i \mapsto T_i$. We call an object a *reachable spherical object* if it is isomorphic to an object σS_i for some $\sigma \in \mathrm{Br}_3$ and $i = 1, 2$. For two reachable spherical objects E_1, E_2, the Poincaré polynomial of $\mathrm{Hom}^\bullet_{\mathscr{D}_2^d}(E_2, E_1)$, *i.e.*

$$p(E_2, E_1) = \sum_{n \in \mathbb{Z}} \dim \mathrm{Hom}_{\mathscr{D}_2^d}(E_2, E_1[n]) q^n \tag{3}$$

coincides with a weighted intersection number of some arcs on the disk with 3 marked points, [12]. Let us recall the precise statement.

Let (D, Δ) be the unit disk D with 3 marked points $\Delta = \{p_1, p_2, p_3\} \subset D$. A *closed arc* in (D, Δ) is an embedding $c : [0, 1] \to D$ such that $c^{-1}(\Delta) = \{0, 1\}$. Define $P = \mathbb{P}(T(D \setminus \Delta))$ to be the real projectivisation of the tangent bundle of $D \setminus \Delta$. By considering an oriented trivialization of D, we can identify P with $\mathbb{RP}^1 \times (D \setminus \Delta)$. For each p_i, take a small loop λ_i winding p_i positively once. Then $[\mathrm{pt} \times \lambda_i]$ and $[\mathbb{RP}^1 \times \mathrm{pt}]$ form a basis of $H_1(P, \mathbb{Z})$. Define $\alpha \in H^1(P, \mathbb{Z}^2)$ by $\alpha([\mathrm{pt} \times \lambda_i]) = (-2, 1)$ and $\alpha([\mathbb{RP}^1 \times \mathrm{pt}]) = (1, 0)$. Let $\tilde{P}$ be the covering space with covering group $\mathbb{Z}$ determined by α. A *bigraded closed arc* $(c, \tilde{c})$ (or $\tilde{c}$ for short) in (D, Δ) is a closed arc c in (D, Δ) together with a lift $\tilde{c} : (0, 1) \to \tilde{P}$ of the section $s_c : (0, 1) \to P$ given by $s_c(t) = T_{c(t)}c$.

Let $\tilde{c}_0, \tilde{c}_1$ be bigraded closed arcs having minimal intersection in the sense that they intersect transversely and do not bound a disk. We shall define a bigrading of an intersection point $z \in c_0 \cap c_1$. Take a small loop l around z and an arc $a : [0, 1] \to l \subset D$ which moves clockwise along l and $a^{-1}(c_i) = \{i\}$ for $i = 0, 1$. Let us also take a path $\pi : [0, 1] \to P$ such that $\pi(t) \in P_{a(t)}$ for all t, $\pi(i) = T_{a(i)}c_i$ for $i = 0, 1$ and $\pi(t) \neq T_{a(t)}l$ for all t. Let

[8] E.g. their Euler pairing is non-zero.

$\tilde{\pi} : [0,1] \to \tilde{P}$ be the lift of π with $\tilde{\pi}(0) = \tilde{c}_0(a(0))$. Then we have $\tilde{c}_1(a(1)) = (\mu_1, \mu_2) \cdot \tilde{\pi}(1)$ for a unique $(\mu_1, \mu_2) \in \mathbb{Z}^2$ which acts as a covering transformation. In this case, we denote $(\mu_1(z), \mu_2(z)) = (\mu_1, \mu_2)$ and define the *bigraded intersection number* of $\tilde{c}_0$ and $\tilde{c}_1$ to be

$$I(\tilde{c}_0, \tilde{c}_1) = (1 + q_1^{-1} q_2) \sum_{z \in (c_0 \cap c_1) \setminus \Delta} q_1^{\mu_1(z)} q_2^{\mu_2(z)} + \sum_{z \in c_0 \cap c_1 \cap \Delta} q_1^{\mu_1(z)} q_2^{\mu_2(z)} \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}].$$

It was shown in [12] that the behavior of reachable spherical objects of $\mathscr{D}_2^d$ can be read off from the topology of bigraded closed arcs in (D, Δ). More precisely, there are some bigraded closed arcs $\tilde{b}_1, \tilde{b}_2$ and a braid group action $\mathrm{Br}_3 = \langle \sigma_1, \sigma_2 \rangle \ni \sigma_i \mapsto t_i$, where t_i is the half twist around b_i, on the set of isotopy classes of (admissible) bigraded curves satisfying

$$p(\sigma S_i, \tau S_j) = I(\sigma \tilde{b}_i, \tau \tilde{b}_j)|_{q_1 = q, q_2 = q^d} \tag{4}$$ [9]

for any $\sigma, \tau \in \mathrm{Br}_3$ and $i, j = 1, 2$.

Corollary 5.1.1. *Let E_1, E_2 be reachable spherical objects which are non-isomorphic to each other up to shift. Then self extensions of $V = \mathrm{Hom}^\bullet_{\mathscr{D}_2^d}(E_2, E_1)$ have degree of the form $k(d-1)$ for some $k \geq 0$. In particular, we have*

$$h_0(T_2 \circ T_1) = \log \left| \frac{(\dim V)^2 - 2 + \sqrt{(\dim V)^4 - 4(\dim V)^2}}{2} \right|.$$

Moreover,

$$h_0(T_2 \circ T_1) = \log \rho([T_2 \circ T_1])$$

holds if and only if $\dim V = 1, 2$ *or* $\dim V \geq 3$ *and d is odd.*

Remark 5.1.2. Notice that in the above corollary the case $\dim V = 0$ does not appear. This is because the dimension of V is at least one in the above setup. Indeed, each E_i corresponds to some closed arc b_i on the disk with 3 marked points, and the dimension of V can be computed via (3) and (4) setting $q = 1$. Since b_1 and b_2 share at least one marked point, the dimension of V is at least 1.

Proof. For simplicity, we assume that the marked points on the disk are $p_1 = (-\frac{1}{2}, 0)$, $p_2 = (0,0)$, $p_3 = (\frac{1}{2}, 0)$. Without loss of generality, we can assume that $E_2 = S_2$ and a bigraded closed arc $\tilde{b}_2$ corresponding to it is the straight arc connecting p_2 and p_3. Let $\tilde{c}$ be a bigraded closed arc corresponding to E_1. By twisting around b_2, we can assume c has p_2 as one of its end points. Denote the intersection points of b_2 and c by $z_i = (a_i, 0)$ where $0 = a_1 < a_2 < \cdots < a_n \leq \frac{1}{2}$.

To prove the first claim, it is enough to prove that

$$\delta_i = \mu_1(z_{i+1}) + d\mu_2(z_{i+1}) - (\mu_1(z_i) + d\mu_2(z_i)) = k(d-1)$$

for some $k \in \mathbb{Z}$ and for all i. Here, we shall only show it for the case $i = 1$ as the other cases can be shown similarly. Up to twisting around b_2,[10] we have four possibilities in that case which are depicted in Figure 1. For each of four cases, δ_1 is $d-1, 2(d-1), -2(d-1)$ and $-3(d-1)$ respectively.

[9] We only need the existence of such bigraded closed arcs, for their explicit description see [12].

[10] Note that twisting around b_2 doesn't change δ_i.

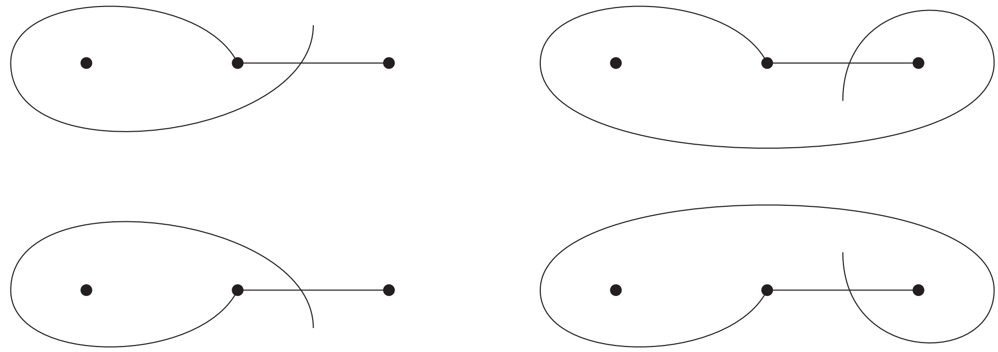

Fig. 1. δ_1 is $d-1$ (top left), $2(d-1)$ (top right), $-2(d-1)$ (bottom left) and $-3(d-1)$ (bottom right) respectively.

The second claim can be seen by noticing that $\chi([E_2],[E_1]) = \dim V$ when d is odd while $|\chi([E_2],[E_1])| = 1, 2$ when d is even.[11] □

This example has the following symplecto-geometric interpretation. Let X_2^d be the Milnor fiber of A_2-singularity of dimension $2d > 2$ and L_1, L_2 be the vanishing cycles (equipped with suitable grading structures). It is known that L_1, L_2 split-generate the (split-closed) derived Fukaya category $D^\pi\mathcal{F}(X_2^d)$. Since S_1, S_2 also split-generate the finite-dimensional derived category $\mathscr{D}_2^d$ and the graded algebra

$$\bigoplus_{i,j=1}^{2} \mathrm{Hom}^\bullet_{D^\pi\mathcal{F}(X_2^d)}(L_i, L_j) \cong \bigoplus_{i,j=1}^{2} \mathrm{Hom}^\bullet_{\mathscr{D}_2^d}(S_i, S_j)$$

is intrinsically formal (see [23, Lemma 4.21]), we have an exact equivalence

$$D^\pi\mathcal{F}(X_2^d) \simeq \mathscr{D}_2^d.$$

In particular, under this equivalence, L_i corresponds to S_i and the Dehn twist τ_i around L_i corresponds to the spherical twist T_i around S_i. Therefore, Corollary 5.1.1 can be stated in terms of symplectic geometry.

Corollary 5.1.3. *Let L_1, L_2 be reachable Lagrangian spheres in X_2^d. Then we have*

$$h_0(\tau_2 \circ \tau_1) = \log \left| \frac{m^2 - 2 + \sqrt{m^4 - 4m^2}}{2} \right|$$

where $m = \dim HF^\bullet(L_2, L_1)$.

REMARK 5.1.4. Let d be even and $\tilde{b}_1, \tilde{b}_2$ be the bigraded closed arcs corresponding to L_1, L_2 respectively. Suppose $\tilde{b}_1$ and $\tilde{b}_2$ share only one end point. Then, $p(L_1, L_2) = I(\tilde{b}_1, \tilde{b}_2)|_{q_1=q, q_2=q^d}$ implies that $\lambda = \chi(L_1, L_2) = \pm 1$. Thus, by the Picard–Lefschetz formula, $(\tau_2 \circ \tau_1)^3$ acts on $H_d(X_2^d, \mathbb{Z}) = \langle [L_1], [L_2] \rangle$ as

[11] When d is even, the absolute value of $\chi([E_2],[E_1])$ is exactly the number of common end points of the closed arcs corresponding to E_1, E_2.

$$\begin{pmatrix} -1 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. it is in the symplectic Torelli group of X_2^d. As we have seen, the categorical entropy of $\tau_2 \circ \tau_1$ (and also $(\tau_2 \circ \tau_1)^3$) is positive whenever $\dim HF^\bullet(L_2, L_1) \geq 3$. Therefore, in such a case, $(\tau_2 \circ \tau_1)^3$ gives a higher-dimensional counterexample to Kikuta–Takahashi's conjecture coming from an element in the symplectic Torelli group having positive categorical entropy. This answers a question in [14, Problem 1.2] about the existence of such an autoequivalence for higher-dimensions.

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