Funakoshi, Y., Noguchi, K. and Shimizu, A.
Osaka J. Math.
60 (2023), 671–682

UNKNOTTABILITY OF SPATIAL GRAPHS BY REGION CROSSING CHANGES

Yukari FUNAKOSHI, Kenta NOGUCHI and Ayaka SHIMIZU

(Received October 26, 2020, revised July 19, 2022)

Abstract

A region crossing change is a local transformation on spatial graph diagrams switching the over/under relation at all the crossings on the boundary of a region. In this paper, we show that a spatial graph of a planar graph is unknottable by region crossing changes if and only if the spatial graph is non-Eulerian or is Eulerian and proper.

1. Introduction

A *knot* is an embedding of a circle to a three-sphere. A *link of r components* is an embedding of r circles to a three-sphere. A *spatial graph of r components* is an embedding of r connected graphs to a three-sphere. By regarding a circle to be a graph without vertices, we assume that knots and links belong to spatial graphs. Each spatial graph S is represented by a *diagram* on a two-sphere, a projection of S to a two-sphere with over/under information at each intersection, where each intersection is a double point of edges and called a *crossing*. It is well-known that two diagrams represent the same spatial graph if and only if one of them can be transformed into the other by a finite number of the Reidemeister moves RI to RV shown in Fig.1 ([8]). A *self-crossing* (resp. *non-self-crossing*) on a diagram is a crossing between edges of the same (resp. different) component. A *planar graph* is a graph which can be embedded to a two-sphere without creating crossings. A spatial graph S of a planar graph is *unknotted* if S has a diagram which has no crossings. A spatial graph S is *completely splitted* if S has a diagram which has no non-self-crossings. A diagram D of a spatial graph is *unknotted* (resp. *completely splitted*) if D represents an unknotted (resp. completely splitted) spatial graph. A graph G is *Eulerian* if the degree of every vertex of G is even. A spatial graph S is *Eulerian* if S is an embedding of an Eulerian graph. We assume that knots and links are Eulerian.

Studies of local transformations have a key role in knot theory and spatial graph theory to measure a complexity of a spatial graph or to consider the relation or classification of spatial graphs. For example, a *Delta move* is a local transformation on spatial graphs shown in Fig.2. It is shown in [10] that a Delta move is an unknotting operation for knots, i.e., we can unknot any knot by applying a finite number of Delta moves and Reidemeister moves on its diagram. On the other hand, a Delta move is not an unknotting operation for links and spatial graphs. Then the equivalent classes of links and spatial graphs on Delta moves are studied using and applying to other invariants, such as the Conway polynomial and the Wu

2020 Mathematics Subject Classification. Primary 57K10; Secondary 57M15.

invariant ([11, 13, 14, 16, 21]).

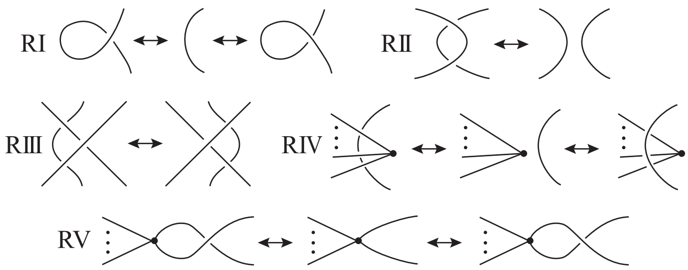

Fig. 1. Reidemeister moves.

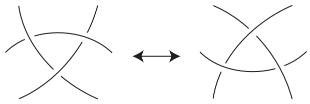

Fig. 2. A Delta move.

A Δ_{13}-*move* is a local transformation on spatial graphs shown in Fig.3 [12]. It is shown in [12] that a Δ_{13}-move is an unknotting operation for knots, and is not for links. For spatial graphs, it is shown in [15] that a spatial graph S of a planar graph can be unknotted by Δ_{13}-moves if and only if S is non-Eulerian or is Eulerian and proper. The definition of the properness is given in Section 3.

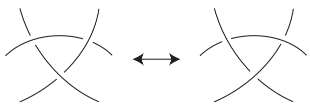

Fig. 3. A Δ_{13}-move.

Let D be a diagram of a spatial graph, and let R be a region of D. The *region crossing change at R* is defined as a local transformation on the spatial graph diagram D by changing the over/under information at all the crossings on the boundary of R. The following theorem is shown for knot diagrams [1].

Theorem 1.1 ([19])**.** *Any diagram of a knot can be unknotted by region crossing changes.*

Note that it had already been shown in [2] that any knot has a diagram which can be transformed into an unknotted diagram by a single "n-gon move", a kind of region crossing changes. The point of Theorem 1.1 is that we can unknot any fixed diagram of a knot by region crossing changes without applying Reidemeister moves. For links, the following is shown.

Theorem 1.2 ([3])**.** *Any diagram of a link L can be unknotted by region crossing changes if and only if L is proper.*

[1] An alternative proof of Theorem 1.1 is given in [5] using graph theory.

The point of Theorem 1.2 is that the unknottability of link diagrams by region crossing changes depends only on the properness of a link itself. The following theorem is shown for spatial graphs of a connected planar graph.

Theorem 1.3 ([7])**.** *Any diagram of a spatial graph of a connected planar graph can be unknotted by region crossing changes.*

Theorem 1.1 implies that a region crossing change is an unknotting operation for knot diagrams and Theorem 1.2 implies that it is not an unknotting operation for link diagrams. Again, the point of Theorems 1.1, 1.2 and 1.3 is that it does not depend on the choice of a diagram. In general, the unknottability by region crossing changes depends on the choice of a diagram of the spatial graph as pointed out in [20]. We define that a spatial graph S is *unknottable (resp. completely splittable) by region crossing changes* if S has a diagram which can be unknotted (resp. completely splitted) by region crossing changes, where applying Reidemeister moves is not allowed during region crossing changes. Note that any spatial graph of a planar graph is unknottable by (the classical) crossing changes. In this paper, we show the following theorems as a generalization of Theorems 1.1, 1.2 and 1.3.

Theorem 1.4. *A spatial graph S of a planar graph is unknottable by region crossing changes if and only if S is non-Eulerian or is Eulerian and proper.*

Theorem 1.5. *A spatial graph S is completely splittable by region crossing changes if and only if S is non-Eulerian or is Eulerian and proper.* [2]

The rest of the paper is organized as follows: In Section 2, we consider non-Eulerian spatial graphs. In Section 3, we consider Eulerian spatial graphs. In Section 4, we prove Theorems 1.4 and 1.5.

2. Non-Eulerian spatial graphs

In this section we consider non-Eulerian spatial graphs and show the following lemma:

Lemma 2.1. *Let S be a non-Eulerian spatial graph. Let D be a diagram of S, and let D' be a diagram which is obtained from D by some crossing changes. There exists a diagram E of S such that E can be transformed into a diagram representing the same spatial graph to D' by region crossing changes.*

See Fig.4 for example. For a spatial graph diagram D, a crossing c, a vertex v and a path P connecting c and v, we define the following transformation and denote it by cPv. Take an (over or under) arc α of c which does not belong to P. Stretch α along P to pass v as shown in Fig.5 (cf. [17]). Note that cPv is realized by Reidemeister moves. We call the stretched α the *spur* of cPv. Note that the over/under relation for the spur to all the edges around the vertices on P are the same to that for α to P. To prove Lemma 2.1, we need the following lemma.

---

[2] Any spatial graph of a non-Eulerian graph is completely splittable by region crossing changes. This means that the splitness by region crossing changes is *intrinsic* (see [6]) to non-Eulerian graphs. On the other hand, since it depends on the way of embedding, splitness by region crossing changes is not intrinsic to Eulerian graphs.

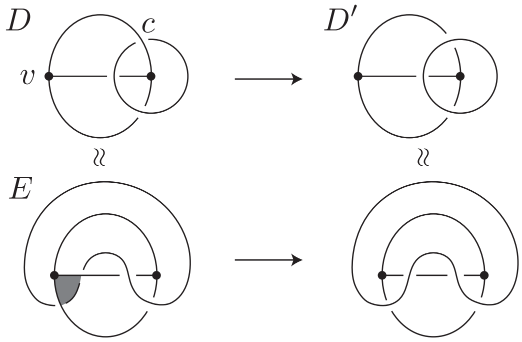

Fig.4. From the diagram D, we obtain D' by a crossing change at the crossing c. From the diagram E, representing the same spatial graph to D, we obtain a diagram representing the same spatial graph to D' by a region crossing change at the shaded region.

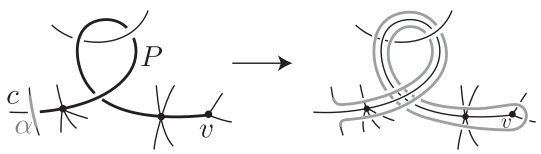

Fig.5. The transformation cPv. The spur is thickened on the right figure.

Lemma 2.2. *Let D be a spatial graph diagram which has a path P connecting a crossing c and a vertex v of odd degree, where P has no vertices of odd degree except v. Let D' be a diagram obtained from D by a crossing change at c. Let D'' be a diagram obtained from D by cPv. Then D'' can be transformed into a diagram representing the same spatial graph to D' by region crossing changes.*

Proof. Let $u_1, u_2, \dots, u_k = v$ be the vertices on P in order from the side of c along P. We locally consider edges not on P which are incident to a vertex u_i; say $e_1, e_2, \dots, e_l$, in cyclic order along the spur, as shown in Fig.6. We remark that l is an even number because each vertex u_i has locally even number of edges which intersect the spur. Take the regions in the spur along P between e_i and e_{i+1}, where we cancel the region which we encounter twice at a self-crossing of P. We call the set of the regions Q_i. Let $R(cPv)$ be the symmetric difference of $Q_1, Q_3, Q_5, \dots, Q_{l-1}$. By applying region crossing changes at all the regions in $R(cPv)$, the over/under relation around all the vertices in P is changed, and any other crossing is unchanged. Hence, we can shrink the spur back through the other side, and obtain D'. □

Proof of Lemma 2.1. Let S be a spatial graph that has a connected non-Eulerian graph G as a component, and let D be a diagram of S. Note that by the Handshaking Lemma, G has two or more even number of vertices of odd degree. Let $V_{\text{odd}}(G)$ be the set of all the vertices of odd degree of G. Let $B = \{b_1, b_2, \dots, b_k\}$ be a set of some crossings of D which are on G, where a crossing on G means a self-crossing or non-self-crossing which belongs to the diagram of G. Let $C = \{c_1, c_2, \dots, c_l\}$ be a set of some crossings of D which are not on G. Let D' be a diagram which is obtained from D by crossing changes at all the crossings in B and C. We show that we can retake a diagram E of S from D so that E can be transformed

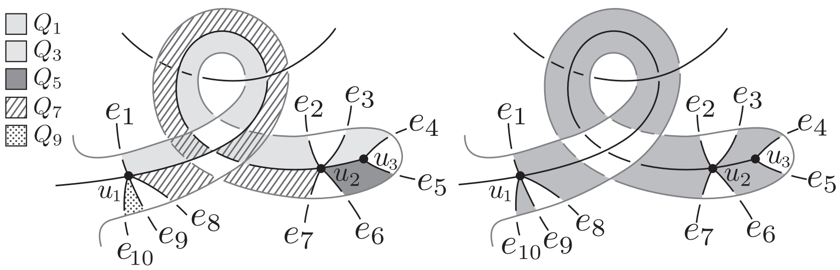

Fig. 6. Regions in Q_1, Q_3, Q_5, Q_7, Q_9 are patterned on the left. The regions of $R(cPv)$ are shaded on the right.

into a diagram representing the same spatial graph to D' by region crossing changes.
(a) Let $F_0 = D$.
(b) Take a path P_i on F_{i-1} which connects b_i and one of the vertices v_i in $V_{\mathrm{odd}}(G)$ so that P_i does not include any other vertices in $V_{\mathrm{odd}}(G)$, where v_i and v_j may be the same vertex for $i \neq j$. Apply $b_iP_iv_i$ and let F_i be the resulting diagram.
Repeat the procedure (b) from $i = 1$ to $i = k$, and let $F = F_k$.

For F, take the symmetric difference R^F of $R(b_iP_iv_i)$ for $i = 1, 2, \ldots, k$. Note that some regions in $R(b_iP_iv_i)$ may be divided by $b_jP_jv_j$ $(i < j)$. In that case, retake all the corresponding regions as $R(b_iP_iv_i)$. Thus, by Lemma 2.2, all the over/under relation around the vertices of P_i will be changed for every spur of $b_iP_iv_i$ if we apply region crossing changes at the regions of R^F.

Then for F, take two vertices v_1 and v_2 from $V_{\mathrm{odd}}(G)$ such that v_1 and v_2 are connected by a path P which does not have any other vertices of odd degree.
(c) Let $E_0 = F$.
(d) Take an arc α in P, and stretch α to c_i going through the over side of other edges as shown in the middle of Fig.7. Four crossings are created and we call the crossings on the ends c_i^1 and c_i^2. Apply $c_i^1P_i^1v_1$ and $c_i^2P_i^2v_2$, where P_i^j is the path connecting c_i^j and v_j, and we remark that there are no vertices of odd degree in P_i^j except v_j. Let E_i be the resulting diagram. We call the region adjacent to c_i which is created by the stretch of α in the above procedure R_i. Note that if we apply region crossing changes at R_i, $R(c_i^1P_i^1v_1)$ and $R(c_i^2P_i^2v_2)$, then the over/under relation will be changed at the two spurs and R_i, and then the crossing change at c_i will be realized.

Repeat the procedure (d) from $i = 1$ to $i = l$, and let $E = E_l$.

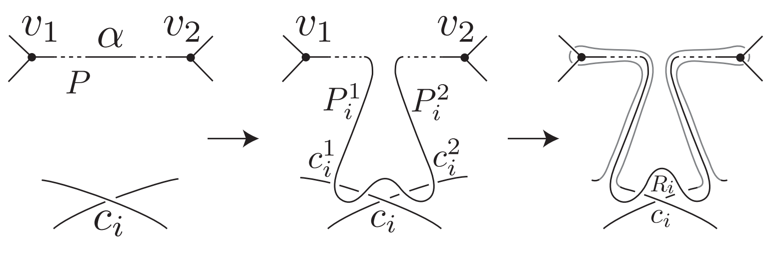

Fig. 7. The procedure (d).

For E, take the symmetric difference of R^F, R_i and $R(c_i^jP_i^jv_j)$ for all $i = 1, 2, \ldots, l$ and

$j = 1, 2$. Note that some regions in R^F, $R(c_i^j P_i^j v_j)$ and R_i may be divided by the procedure for c_h ($i < h$). In that case, retake all the corresponding regions. Thus, by applying region crossing changes and shrinking, we obtain D'. Note that the above transformations do not influence each other. □

3. Eulerian spatial graphs

In this section, we review the study of region crossing changes for links and consider Eulerian spatial graphs.

3.1. Linking number of links. Let $L = K_1 \cup K_2 \cup \cdots \cup K_r$ be an oriented link of r components $K_1, K_2, \ldots, K_r$. Let $D = D_1 \cup D_2 \cup \cdots \cup D_r$ be a diagram of L, where each D_l corresponds to K_l ($1 \leq l \leq r$). The *linking number* $lk(D_i, D_j)$ *between* D_i *and* D_j is the value of half the sum of the signs (see Fig.8) for all the crossings between D_i and D_j. The value of $lk(D_i, D_j)$ is an integer because the number of non-self-crossings of two components is an even number. It is well-known that the value $lk(D_i, D_j)$ does not depend on the choice of the diagram since it is unchanged over Reidemeister moves RI, RII and RIII (see, for example, [1]). Hence we can regard the value of $lk(D_i, D_j)$ as a link invariant and call it the *linking number* $lk(K_i, K_j)$ *between* K_i *and* K_j. A link L is *proper* if the value $\sum_{j \neq i} lk(K_i, K_j)$ is even for all $i \in \{1, 2, \ldots, r\}$ with an orientation. The properness does not depend on the choice of orientation because we have $lk(K_i, -K_j) = -lk(K_i, K_j)$, where $-K_j$ means K_j with orientation reversed. Since the number of crossings between a component K_i and the other components at the boundary of each region is an even number, the following holds:

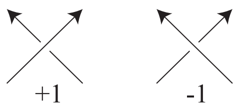

Fig. 8. The sign of a crossing.

Lemma 3.1 ([3, 4])**.** *The properness of a link is preserved over region crossing changes.*

The following lemma is shown in [3].

Lemma 3.2 ([3])**.** *Let D be a diagram of a link. Take n knot components $D_1, D_2, \ldots, D_n$ such that D_i and D_j have crossings for $|i-j| = 1$ or $|i-j| = n-1$. Let C be a set of n crossings $c_1, c_2, \ldots, c_n$ where c_i is of D_i and D_{i+1} with $D_{n+1} = D_1$. Then the crossing changes at the n crossings in C can be realized by region crossing changes on D.*

In particular, the following lemma holds.

Lemma 3.3 ([3, 4])**.** *Let D be a diagram of a link. Crossing changes on D at any pair of crossings between two knot components, say D_i and D_j, can be realized by region crossing changes for any i and j.*

3.2. Linking number of Eulerian spatial graphs. In this subsection, we give the definition of the linking number for Eulerian spatial graphs, which is equivalent to the definition given in [15]. Let G be a graph with an orientation on each edge. For a vertex v, the *indegree* (resp. *outdegree*) of v is the number of incident edges to v whose orientation is incoming to (resp. outgoing from) v. Let S be an Eulerian spatial graph consisting of connected graphs $G_1, G_2, \ldots, G_r$. Give an orientation O_l to G_l so that the indegree equals the outdegree at each vertex of G_l. We call such orientation an *Eulerian orientation*. Note that we can take an Eulerian orientation for every Eulerian graph since it has an "Eulerian circuit".

Unless otherwise stated, an oriented Eulerian spatial graph means an Eulerian spatial graph with an Eulerian orientation in this paper. We define the linking number for oriented Eulerian spatial graphs. Let D be a diagram of an oriented Eulerian spatial graph $S = S_1 \cup S_2 \cup \cdots \cup S_r$. The *linking number* $lk(S_i, S_j)$ between S_i and S_j is the value of half the sum of the signs for all the crossings between S_i and S_j in D. The value of $lk(S_i, S_j)$ is an integer since we can confirm that the number of crossings between S_i and S_j is an even number by considering their Eulerian circuits and assuming them a link. The value $lk(S_i, S_j)$ is preserved over Reidemeister moves RI, RII and RIII as well as for links. For RIV, the value is also preserved because the number of positive crossings and negative crossings are the same around a vertex which is applied an RIV. For RV, the value is unchanged because there are no change for non-self crossings. Hence $lk(S_i, S_j)$ is an invariant for oriented Eulerian spatial graphs. Moreover, we have the following:

Lemma 3.4. *Regardless of the Eulerian orientation, the parity of $lk(S_i, S_j)$ is an invariant for Eulerian spatial graphs.*

Proof. Let $lk(S_i, S_j)$ be the linking number between S_i and S_j with Eulerian orientations O_i of S_i and O_j of S_j. Let $lk'(S_i, S_j)$ be that with Eulerian orientations $O_i{}'$ of S_i and O_j of S_j. Take the subgraph H of S_i by taking edges which have different orientations between O_i and $O_i{}'$. Then H is an Eulerian subgraph of S_i. Since H and S_j are Eulerian, the number of crossings between H and S_j is an even number. Hence the difference between $lk(S_i, S_j)$ and $lk'(S_i, S_j)$ is the half of some multiple of four, i.e., multiple of two. □

An Eulerian spatial graph S of r components is *proper* if $\sum_{j \neq i} lk(S_i, S_j)$ is even for all $i \in \{1, 2, \ldots, r\}$ with an Eulerian orientation. We have the following corollary as well as Lemma 3.1.

Corollary 3.5. *For diagrams of an Eulerian spatial graph, the properness is preserved over region crossing changes.*

Proof. Fix an Eulerian orientation. Since the number of crossings between S_i and the other components at the boundary of each region is an even number, the parity of $\sum_{i \neq j} lk(S_i, S_j)$ is unchanged by region crossing changes for each i. □

Next, we introduce the warping degree for spatial graph diagrams and consider the relation to the linking number for Eulerian spatial graph diagrams. Let $D = D_1 \cup D_2 \cup \cdots \cup D_r$ be a diagram of a spatial graph of r components with an order. A *warping crossing point between D_i and D_j* ($i < j$) is a crossing point such that D_j is over than D_i. The *warping degree $w(D_i, D_j)$ between D_i and D_j* ($i < j$) is the number of warping crossing points between D_i and D_j. Note that a diagram D with $w(D_i, D_j) = 0$ for all $i < j$ represents a completely splitted spatial graph. The following holds. (See [9] and [18] for links.)

Lemma 3.6. *Let $D = D_1 \cup D_2 \cup \cdots \cup D_r$ be a diagram of an oriented Eulerian spatial graph $S = S_1 \cup S_2 \cup \cdots \cup S_r$, where each D_l corresponds to S_l. Then $w(D_i, D_j) \equiv lk(S_i, S_j)$*

(mod 2) *holds for each $i < j$.*

Proof. If $w(D_i, D_j) = k$, apply crossing changes at all the warping crossing points between D_i and D_j, and let $D^0 = D_1^0 \cup D_2^0 \cup \cdots \cup D_r^0$ be the obtained diagram, where each D_l^0 corresponds to D_l. Since $w(D_i^0, D_j^0) = 0$, the two components represented by D_i^0 and D_j^0 are split and hence the linking number is zero. This implies that $lk(S_i, S_j) \equiv k \pmod 2$ because the value of the linking number is changed by one by each single crossing change. Hence $w(D_i, D_j) \equiv lk(S_i, S_j) \pmod 2$ holds. □

3.3. Vertex splittings. For a graph, a *vertex splitting at a vertex v into v' and v''* is the following transformation. Add two vertices v' and v'', reattach the edges incident to v to exactly one of v' or v'', and remove v (see Fig.9). We have the following:

Lemma 3.7. *Let G be a connected Eulerian graph, and let v be a vertex of G. Let e_1, e_2 and e_3 be edges of G which is incident to v. Let G_{12} (resp. G_{23}) be a graph obtained from G by a vertex splitting of v such that only e_1 and e_2 (resp. e_2 and e_3) are incident to v'. Either G_{12} or G_{23} is connected.*

Proof. Assume that G_{12} is not connected. Then G_{12} has a cycle H including e_1 and e_2 since G_{12} is also Eulerian. Then, there is a path connecting e_1 and e_2 in G_{23} which corresponds to $H - v'$ in G_{12}. This implies G_{23} is connected. □

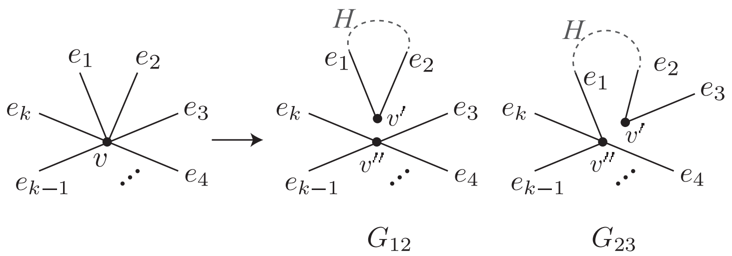

Fig.9. Vertex splittings.

By repeating vertex splittings to a connected Eulerian graph keeping connected, and ignoring vertices of degree two, we obtain a closed curve without vertices. In terms of spatial graphs, we obtain a knot from a connected Eulerian spatial graph. We show the following:

Lemma 3.8. *Any diagram D of a spatial graph of a connected Eulerian planar graph can be unknotted by region crossing changes.*

Proof. Let C be a set of crossings of D such that the crossing changes at all the crossings in C make D unknotted. Apply the vertex splittings with Lemma 3.7 to D to obtain a knot diagram D^k. Since any crossing change on any knot diagram can be realized by region crossing changes as shown in [19], D^k has a set R of regions such that the region crossing changes realize the crossing changes at all the crossings in C. Apply region crossing changes to D at the corresponding regions to R. □

Note that Lemma 3.8 is contained by Theorem 1.3. We have the following. (See Fig.10 for example.)

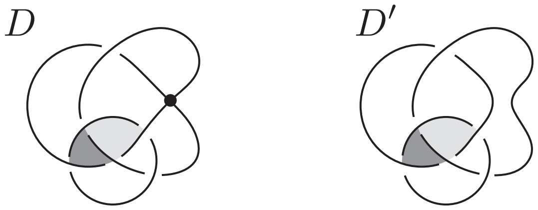

Fig. 10. The diagram D' is obtained from D by a vertex splitting. Both of D and D' are proper and can be completely splitted by region crossing changes.

Lemma 3.9. *Let D be a diagram of an Eulerian spatial graph, and let D' be a link diagram obtained from D by vertex splittings on each component. The following* (i) *to* (iv) *are equivalent:*
(i) *D can be completely splitted by region crossing changes.*
(ii) *D' can be completely splitted by region crossing changes.*
(iii) *D' is proper.*
(iv) *D is proper.*

Proof. (i) $\Rightarrow$ (iv): The contraposition holds by Corollary 3.5.
(ii) $\Leftrightarrow$ (iii): By Theorem 1.2.
(iii) $\Leftrightarrow$ (iv): Give an orientation to D', and give the same orientation to each edge of D. Then the orientation of D is Eulerian, and we can see that the properness is the same for D and D'. Note that even if D' has extra crossings created by the vertex splittings, there are no influences because they are self-crossings.
(ii) $\Rightarrow$ (i): Let R be a set of regions of D' such that the region crossing changes at the regions in R make D' completely splitted. Let $D'' = D''_1 \cup D''_2 \cup \cdots \cup D''_r$ be the resulting diagram. Since the linking number is zero for each pair of components, the value of the warping degree $w(D''_i, D''_j)$ is even by Lemma 3.6. By Lemma 3.3, we can realize pairwise crossing changes at all the warping crossing points between D''_i and D''_j by region crossing changes at some regions, say R_{ij}. Apply region crossing changes to D at the corresponding regions to the symmetric difference of R and R_{ij} for all $i < j$. Then we obtain a diagram with warping degree zero for any pair of components. Hence, D can also be completely splitted by region crossing changes. □

4. Proof of the main theorems

In this section, we prove Theorems 1.4 and 1.5. For non-Eulerian spatial graphs, we have the following theorem by Lemma 2.1.

Theorem 4.1. *Every non-Eulerian spatial graph is completely splittable by region crossing changes.*

Proof. Let $S = S_1 \cup S_2 \cup \cdots \cup S_r$ be a non-Eulerian spatial graph of r components. Let D be a diagram of S. Take a set C of all the non-self-crossings between S_i and S_j such that S_j is over than S_i for all $i < j$. By Lemma 2.1, D can be transformed into a suitable diagram to change all the crossings in C by region crossing changes. □

Similarly, we have the following theorem.

Theorem 4.2. *Every spatial graph of a non-Eulerian planar graph is unknottable by region crossing changes.*

Proof. Let D be a diagram of a spatial graph S of a non-Eulerian planar graph. Since S is an embedding of a planar graph, we can transform D into an unknotted diagram by some crossing changes. By Lemma 2.1, D can be transformed into the appropriate diagram to realize such crossing changes by region crossing changes. □

For Eulerian spatial graphs, the following theorem follows from Corollary 3.5 and Lemma 3.9.

Theorem 4.3. *An Eulerian spatial graph S is completely splittable by region crossing changes if and only if S is proper.*

Proof. Let S be an Eulerian spatial graph. If S is proper, then S is completely splittable by region crossing changes by Lemma 3.9. If S is not proper, any diagram of S is not proper, and furthermore any diagram which is obtained from a diagram of S by region crossing changes is not proper by Corollary 3.5. Then S is not completely splittable by region crossing changes by Lemma 3.9. □

The following theorem also follows.

Theorem 4.4. *A spatial graph S of an Eulerian planar graph is unknottable by region crossing changes if and only if S is proper.*

Proof. Let $D = D_1 \cup D_2 \cup \cdots \cup D_r$ be a diagram of Eulerian planar graph S of r components. If S is proper, then D has a set R_0 of regions which makes D completely splitted by region crossing changes by Lemma 3.9. Also, D has a set R_i of regions which makes D_i unknotted by region crossing changes by Lemma 3.8. Hence, the symmetric difference R of $R_0, R_1, R_2, \ldots, R_r$ makes D unknotted by region crossing changes. We remark that some reducible crossings of D_i may have different results of region crossing changes between R_i and R, where a *reducible crossing* is a crossing such that the same region meets diagonally at the crossing. There is no problem in that case because the crossing information at reducible crossings does not matter for unknottedness.

If S is not proper, then S is not completely splittable by Corollary 3.5 and Lemma 3.9 and hence is not unknottable by region crossing changes. □

We prove Theorems 1.4 and 1.5.

Proof of Theorem 1.5. It follows from Theorems 4.1 and 4.3. □

Proof of Theorem 1.4. It follows from Theorems 4.2 and 4.4. □

ACKNOWLEDGEMENTS. The authors are very grateful to Ryo Nikkuni for helpful comments. They are also very grateful to the anonymous referee for careful reading and suggestions. The second author's work was partially supported by JSPS KAKENHI Grant Number 17K14239.

References

[1] C.C. Adams: The Knot Book, An Elementary Introduction to the Mathematical Theory of Knots, W. H. Freeman, New York, 1994.

[2] H. Aida: *Unknotting operations for polygonal type*, Tokyo J. Math. **15** (1992), 111–121.

[3] Z. Cheng: *When is region crossing change an unknotting operation?*, Math. Proc. Cambridge Philos. Soc. **155** (2013), 257–269.

[4] Z. Cheng and H. Gao: *On region crossing change and incidence matrix*, Sci. China Math. **55** (2012), 1487–1495.

[5] O. Dasbach and H. Russell: *Equivalence of edge bicolored graphs on surfaces*, Electron. J. Combin. **25** (2018), Paper 1.59, 15 pp.

[6] E. Flapan, T. Mattman, B. Mellor, R. Naimi and R. Nikkuni: *Recent developments in spatial graph theory*; in Knots, links, spatial graphs, and algebraic invariants, Contemp. Math. **689** (2017), Amer. Math. Soc., Providence, RI, 81–102.

[7] K. Hayano, A. Shimizu and R. Shinjo: *Region crossing change on spatial-graph diagrams*, J. Knot Theory Ramifications **24** (2015), 1550045, 12 pp.

[8] L.H. Kauffman: *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. **311** (1989), 697–710.

[9] A. Kawauchi: Lectures on Knot Theory (in Japanese), Kyoritsu shuppan, Tokyo, 2007.

[10] H. Murakami and Y. Nakanishi: *On a certain move generating link-homology*, Math. Ann. **284** (1989), 75–89.

[11] T. Motohashi and K. Taniyama: *Delta unknotting operation and vertex homotopy of spatial graphs in R^3*; in Proceedings of Knots '96 Tokyo, World Scientific Publ. River Edge, NJ, 1997, 185–200.

[12] Y. Nakanishi: *Replacements of the Conway third identity*, Tokyo J. Math. **14** (1991), 197–203.

[13] Y. Nakanishi: *Delta link homotopy for two component links*, Topology Appl. **121** (2002), 169–182.

[14] Y. Nakanishi and Y. Ohyama: *Delta link homotopy for two component links. III*, J. Math. Soc. Japan **55** (2003), 641–654.

[15] Y. Ohyama: *Local moves on a graph in R^3*, J. Knot Theory Ramifications **5** (1996), 265–277.

[16] M. Okada: *Delta-unknotting operation and the second coefficient of the Conway polynomial*, J. Math. Soc. Japan **42** (1990), 713–717.

[17] M. Ozawa: *Edge number of knots and links*, arXiv:0705.4348.

[18] A. Shimizu: *The warping degree of a link diagram*, Osaka J. Math. **48** (2011), 209–231.

[19] A. Shimizu: *Region crossing change is an unknotting operation*, J. Math. Soc. Japan **66** (2014), 693–708.

[20] A. Shimizu and R. Takahashi: *Region crossing change on spatial theta-curves*, J. Knot Theory Ramifications **29** (2020), 2050028, 11 pp.

[21] K. Taniyama: *Homology classification of spatial embeddings of a graph*, Topology Appl. **65** (1995), 205–228.

Yukari Funakoshi
Faculty of Education
Gifu Shotoku Gakuen University
1–1 Takakuwa-nishi, Yanaizu-shi
Gifu, 501–6194
Japan
e-mail: yukarifunakoshi@gifu.shotoku.ac.jp

Kenta Noguchi
Department of Information Sciences
Tokyo University of Science
2641 Yamazaki, Noda, Chiba, 278–8510
Japan
e-mail: noguchi@rs.tus.ac.jp

Ayaka Shimizu
Department of General Education
National Institute of Technology (KOSEN)
Gunma College
580 Toriba, Maebashi-shi, Gunma, 371–8530
Japan
e-mail: shimizu@gunma-ct.ac.jp