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A NEW COMPACTIFICATION OF TEICHMÜLLER SPACE

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Abstract

We construct a new compactification of Teichmüller space. We prove that this new compactification is finer than the Gardiner–Masur compactification of Teichmüller space and the action of the mapping class group on Teichmüller space extends continuously to this new compactification. We also construct some special points in the new boundary. The construction of the new compactification is based on the Hubbard-Masur theorem, which states that there is an one-to-one corresponding between holomorphic differentials and measured foliations.

1. Introduction

Let S be an oriented surface of genus g with n punctures. We assume that $3g-3+n>0$. Let $\mathcal{T}(S)$ be the Teichmüller space of S. Different parameterizations of $\mathcal{T}(S)$ give different compactifications of $\mathcal{T}(S)$. In particular, parameterizing $\mathcal{T}(S)$ by the extremal lengths of simple closed curves, Gardiner and Masur [5] constructed a compactification of $\mathcal{T}(S)$, which is called the Gardiner–Masur compactification of $\mathcal{T}(S)$ and is denoted by $\mathcal{T}^{GM}(S)$. The boundary $\mathcal{T}^{GM}(S)-\mathcal{T}(S)$ is called the Gardiner–Masur boundary of $\mathcal{T}(S)$ and is denoted by GM. Miyachi [11] proved that the action of the mapping class group $Mod(S)$ on $\mathcal{T}(S)$ extends continuously to $\mathcal{T}^{GM}(S)$. The structure of the Gardiner–Masur boundary GM is interesting and was widely studied (see [11], [12], [9], [16], etc.). Besides, it is also interseting to study other compactifications of $\mathcal{T}(S)$, such as the Thurston compactification, the Teichmüller compactification, the Bers compactification and so on (see [8], [2], [14], [1], etc.).

Hubbard and Masur [6] proved that there is an one-to-one corresponding between the space $Q(x)$ of holomorphic quadratic differentials on any $x\in\mathcal{T}(S)$ and the space $\mathcal{MF}$ of measured foliations on S. Based on this result, we give a new parameterization of $\mathcal{T}(S)$ and construct a new compactification of $\mathcal{T}(S)$ by this parameterization. We prove that this new compactification is finer than the Gardiner–Masur compactification and the action of $Mod(S)$ extends continuously to this new compactification. We also construct some special points in the boundary of this new compactification.

Before stating the main results, we need some notations. For any $x\in\mathcal{T}(S)$, sending $q\in Q(x)$ to its horizonzal foliation and vertical foliation, we have the horizontal foliation map $H_x : Q(x)\to\mathcal{MF}$ and the vertical foliation map $V_x : Q(x)\to\mathcal{MF}$ corresponding to x, respectively. By the result of [6], H_x and V_x are both homeomorphisms. Then $\tau_x = H_x\circ V_x^{-1} : \mathcal{MF}\to\mathcal{MF}$ is a homogeneous continuous map from $\mathcal{MF}$ to $\mathcal{MF}$. Let Ω be the

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space of homogeneous maps from $\mathcal{MF}$ to $\mathcal{MF}$. Endow Ω with the pointwise convergence topology. Let $P\Omega = \Omega - \{0\}/\mathbf{R}_+$ be the projective space under the action of multiplication by $\mathbf{R}_+$. Endow $P\Omega$ with the quotient topology. For any $\tau \in \Omega - \{0\}$, let $[\tau] \in P\Omega$ be the projective class of τ.

Note that for any $x \in \mathcal{T}(S)$, $\tau_x \in \Omega - \{0\}$ and $[\tau_x] \in P\Omega$. Sending $x \in \mathcal{T}(S)$ to $[\tau_x] \in P\Omega$, we have a map $\Phi : \mathcal{T}(S) \to P\Omega$. The main results of this paper are the following:

- The map $\Phi : \mathcal{T}(S) \to P\Omega$ is an embedding and the closure of $\Phi(\mathcal{T}(S))$ is compact (Theorem 3.2). Thus $\overline{\mathcal{T}}(S) = Cl(\Phi(\mathcal{T}(S)))$ is a new compactification of $\mathcal{T}(S)$ and $\partial\mathcal{T}(S) = Cl(\Phi(\mathcal{T}(S))) - \mathcal{T}(S)$ is a new boundary of $\mathcal{T}(S)$.
- The new compactification $\overline{\mathcal{T}}(S)$ is finer than the Gardiner–Masur compactification $\mathcal{T}^{GM}(S)$ (Theorem 4.1).
- The action of $Mod(S)$ on $\mathcal{T}(S)$ extends continuously to $\overline{\mathcal{T}}(S)$ (see Section 5).
- For any $F \in \mathcal{MF}$, $[i(F, \cdot)F : \mathcal{MF} \to \mathcal{MF}] \in \partial\mathcal{T}(S)$ (Theorem 5.10). In particular, $\partial\mathcal{T}(S) \neq \emptyset$.

Moreover, we may ask the following question:

QUESTION 1.1. Is the new compactification $\overline{\mathcal{T}}(S)$ strictly finer than the Gardiner–Masur compactification $\mathcal{T}^{GM}(S)$?

We will study Question 1.1 in coming future.

This paper is organized as follows.

Section 2 contains background materials on Teichmüller space, measured foliations and the Gardiner–Masur compactification. Section 3 is devoted to the construction of the new compactification $\overline{\mathcal{T}}(S)$. In Section 4, we study the relation between $\overline{\mathcal{T}}(S)$ and $\mathcal{T}^{GM}(S)$. Section 5 is devoted to the extended action of $Mod(S)$ on $\overline{\mathcal{T}}(S)$.

2. Preliminaries

2.1. Teichmüller space. A marked Riemann surface is a pair (X, f), where X is a Riemann surface of analytically finite type and $f : S \to X$ is an orientation-preserving homeomorphism. Note that a Riemann surface is called analytically finite if it is a closed Riemann surface minus a finite set. Two marked Riemann surfaces (X_1, f_1) and (X_2, f_2) are equivalent if there exists a conformal map $g : X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. The Teichmüller space $\mathcal{T}(S)$ is defined to be the set of equivalence classes of marked Riemann surfaces. For the sake of simplicity, we denote a marked Riemann surface (X, f) or its equivalence class by X, without explicit reference to the marking.

For any two points $x_1 = (X_1, f_1)$ and $x_2 = (X_2, f_2)$ in $\mathcal{T}(S)$, the Teichmüller distance between them is defined as

$$d_T(x_1, x_2) = \frac{1}{2} \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal mappings $f : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ is the maximal dilatation of f.

The mapping class group $Mod(S)$ of S is defined as the set of isotopy classes of orientation-preserving homeomorphisms of S. $Mod(S)$ acts on $\mathcal{T}(S)$: for any $g \in Mod(S)$ and $x = (X, f) \in \mathcal{T}(S)$, $g(x) = (X, f \circ g^{-1})$. And this action is isometric with respect to the

Teichmüller metric d_T: for any $f \in Mod(S)$ and $x, y \in \mathcal{T}(S)$, $d_T(f(x), f(y)) = d_T(x, y)$.

2.2. Measured foliations. Denote by $\mathcal{S}$ the set of homotopy classes of unoriented essential simple closed curves in S. Note that a simple closed curve is essential if it is not homotopic to a point or to a puncture. For any $\alpha, \beta \in \mathcal{S}$, denote their geometric intersection number by $i(\alpha, \beta)$.

Let $\mathbf{R}_{\geq 0} = \{x \in \mathbf{R} : x \geq 0\}$ and $\mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$. Denote by $\mathbf{R}_{\geq 0}^{\mathcal{S}}$ the set of all non-negative functions on $\mathcal{S}$, which is endowed with the topology of pointwise convergence.

Let $\mathbf{R}_+ \times \mathcal{S} = \{t \cdot \alpha : t > 0, \alpha \in \mathcal{S}\}$ be the set of weighted simple closed curves. It is known that

$$i_* : \mathbf{R}_+ \times \mathcal{S} \to \mathbf{R}_{\geq 0}^{\mathcal{S}}, \ t \cdot \alpha \mapsto t \cdot i(\alpha, \cdot)$$

is injective and induces a topology on $\mathbf{R}_+ \times \mathcal{S}$. Under this topology, i_* is an embedding.

The closure of $i_*(\mathbf{R}_+ \times \mathcal{S})$ in $\mathbf{R}_{\geq 0}^{\mathcal{S}}$ is called the space of measured foliations on S, which is denote by $\mathcal{MF}$. Naturally, $\mathbf{R}_+$ acts on $\mathbf{R}_{\geq 0}^{\mathcal{S}}$ by multiplication. Denote $\mathbf{R}_{\geq 0}^{\mathcal{S}} - \{0\}/\mathbf{R}_+$ by $P\mathbf{R}_{\geq 0}^{\mathcal{S}}$ and $\mathcal{MF} - \{0\}/\mathbf{R}_+$ by $\mathcal{PMF}$. $\mathcal{PMF}$ is called the space of projective measured foliations. For $F \in \mathcal{MF} - \{0\}$, denote $[F] \in \mathcal{PMF}$ to be the projective class of F. Note that $\mathcal{S}$ is embedded in $P\mathbf{R}_{\geq 0}^{\mathcal{S}}$, and the closure of $\mathcal{S}$ in $P\mathbf{R}_{\geq 0}^{\mathcal{S}}$ is $\mathcal{PMF}$. It is well known that $\mathcal{MF}$ is homeomorphic to $\mathbf{R}^{6g-6+2n}$ and $\mathcal{PMF}$ is homeomorphic to $S^{6g-7+2n}$ (see [4]).

Define the intersection number between weighted simple closed curves $t\alpha, s\beta \in \mathbf{R}_+ \times \mathcal{S}$ by the homogeneous formula $i(t\alpha, s\beta) = tsi(\alpha, \beta)$. Then the intersection number function i extends continuously to $i : \mathcal{MF} \times \mathcal{MF} \to \mathbf{R}_{\geq 0}$.

Any $F \in \mathcal{MF} - \{0\}$ is represented by a pair of singular foliation and a transverse measure μ in the sense that for any simple closed curve α,

$$i(F, \alpha) = \inf_{\alpha'} \int_{\alpha'} d\mu,$$

where the infimum runs over all simple closed curves α' homotopic to α.

The action of $Mod(S)$ on $\mathcal{S}$ extends continuously to $\mathcal{MF}$. And its action preserves the intersection number: for any $f \in Mod(S)$ and $F, G \in \mathcal{MF}$, $i(f(F), f(G)) = i(F, G)$.

See [4] for more details on measured foliations.

2.3. Quadratic differentials. A holomorphic quadratic differential q on a Riemann surface X is a tensor of the form $q(z)dz^2$ such that $q(z)$ is holomorphic under the local coordinate $z = x + iy$ and is allowed to have simple poles at punctures of X. Denote $Q(X)$ to be the space of all holomorphic quadratic differentials on X. The 1-norm on $Q(X)$ is defined by

$$||q|| = \int_X |q|.$$

This norm induces a topology on $Q(X)$.

A quadratic differential q determines two measured foliations: the horizontal foliation $H(q)$ and the vertical foliation $V(q)$. The leaf of $H(q)$ is defined by y = constant and the measure on $H(q)$ is defined by $|dy|$. The leaf of $V(q)$ is defined by x = constant and the measure on $V(q)$ is defined by $|dx|$.

Denote by $Q(S)$ the bundle of holomorphic quadratic differentials over the Teichmüller space $\mathcal{T}(S)$ and by $p : Q(S) \to \mathcal{T}(S)$ the natural projection. Note that for any $x \in \mathcal{T}(S)$,

$p^{-1}(x) = Q(x)$.

In [6], Hubbard and Masur proved the following result.

Theorem 2.1. *The two maps*

$$H' : Q(S) \to \mathcal{T}(S) \times \mathcal{MF}, q \mapsto (p(q), H(q))$$

and

$$V' : Q(S) \to \mathcal{T}(S) \times \mathcal{MF}, q \mapsto (p(q), V(q))$$

are homeomorphisms.

In particular, $H_x : Q(x) \to \mathcal{MF}$ and $V_x : Q(x) \to \mathcal{MF}$ are both homeomorphisms, where H_x and V_x are the horizontal foliation map and the vertical foliation map corresponding to x, respectively.

The mapping class group $Mod(S)$ acts continuously on $Q(S)$ by push-forward: for any $q \in Q(S)$ and $f \in Mod(S)$, $f(q) = f_* q$. Conjugated through the homeomorphisms $H' : Q(S) \to \mathcal{T}(S) \times \mathcal{MF}$ or $V' : Q(S) \to \mathcal{T}(S) \times \mathcal{MF}$ in Theorem 2.1, this action is also described as follows: for any $f \in Mod(S)$ and $(x, F) \in \mathcal{T}(S) \times \mathcal{MF}$, $f(x, F) = (f(x), f(F))$.

2.4. Extremal length and the Gardiner–Masur compactification. A conformal metric on a Riemann surface X is a metric of the form $\sigma(z)|dz|$ in a local conformal coordinate of X, where $\sigma(z)$ is a non negative, Borel measurable function. The σ-area of X is defined by

$$Area_\sigma(X) = \int_X \sigma^2(z)|dz|^2,$$

and the σ-length of a simple closed curve α is defined by

$$L_\sigma(\alpha) = \inf_{\alpha'} \int_{\alpha'} \sigma(z)|dz|,$$

where the infimum runs over all simple closed curves α' homotopic to α.

For any $\alpha \in \mathcal{S}$, the extremal length of α on X is defined by

$$Ext(X, \alpha) = \sup_\sigma \frac{L_\sigma^2(\alpha)}{Area_\sigma(X)},$$

where the supremum runs over all conformal metrics σ with finite area on X.

For any $x = (X, f) \in \mathcal{T}(S)$ and $\alpha \in \mathcal{S}$, the extremal length of α on x is defined by

$$Ext(x, \alpha) = Ext(X, f(\alpha)).$$

The extremal lengths of simple closed curves extend continuously to the extremal lengths of measured foliations (see [8]) and the extremal lengths are related to quadratic differentials.

Theorem 2.2. *For any $x \in \mathcal{T}(S)$, there exists a continuous function*

$$Ext(x, \cdot) : \mathcal{MF} \to \boldsymbol{R}_{\geq 0},$$

such that

$$\mathcal{S} \ni \alpha \mapsto Ext(x, \alpha)$$

and

$$Ext(x, kF) = k^2 Ext(x, F)$$

for any $k > 0$ *and* $F \in \mathcal{MF}$.

What's more, for any $q \in Q(x)$, $Ext(x, H(q)) = Ext(x, V(q)) = \|q\|$.

In [8], Kerckhoff proved the following result (Kerckhoff's formula).

Theorem 2.3. *For any* $x, y \in \mathcal{T}(S)$, *we have*

$$d_T(x, y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{Ext(y, \alpha)}{Ext(x, \alpha)}.$$

Because of the density of $\mathbf{R}_+ \times \mathcal{S}$ in $\mathcal{MF}$ and the continuity of $Ext(x, \cdot)$,

$$d_T(x, y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{Ext(y, \alpha)}{Ext(x, \alpha)} = \frac{1}{2} \log \max_{F \in \mathcal{MF}} \frac{Ext(y, F)}{Ext(x, F)}.$$

In [10], Minsky proved the following result (Minsky's inequality).

Theorem 2.4. *For any* $F, G \in \mathcal{MF}$ *and* $x \in \mathcal{T}(S)$, *we have*

$$i(F, G)^2 \leq Ext(x, F) \cdot Ext(x, G),$$

and the equality holds if and only if there exists a quadratic differential q *on* x *such that* $[H(q)] = [F]$ *and* $[V(q)] = [G]$.

In [5], Gardiner and Masur constructed a compactification of Teichmüller space by the extremal lengths of simple closed curves. Define a map:

$$\widetilde{\Phi}_{GM} : \mathcal{T}(S) \to \mathbf{R}^{\mathcal{S}}_{\geq 0},$$
$$x \mapsto Ext^{\frac{1}{2}}(x, \cdot) : \mathcal{S} \to \mathbf{R}_{\geq 0}.$$

Let the map $\pi : \mathbf{R}^{\mathcal{S}}_{\geq 0} - \{0\} \to P\mathbf{R}^{\mathcal{S}}_{\geq 0}$ be the projective map. Then the map

$$\Phi_{GM} = \pi \circ \widetilde{\Phi}_{GM} : \mathcal{T}(S) \to P\mathbf{R}^{\mathcal{S}}_{\geq 0},$$
$$x \mapsto [Ext^{\frac{1}{2}}(x, \cdot)]$$

is an embedding and the closure of its image is compact. Thus we have a compactification of $\mathcal{T}(S)$ denoted by $\mathcal{T}^{GM}(S) = \mathcal{T}(S) \bigcup GM$. $\mathcal{T}^{GM}(S)$ is the Gardiner–Masur compactification of $\mathcal{T}(S)$ and GM is the Gardiner–Masur boundary of $\mathcal{T}(S)$.

Since $\mathcal{S}$ is dense in $\mathcal{PMF}$, it is natural to extend the domain from $\mathcal{S}$ to $\mathcal{MF}$. Precisely, Miyachi [11] proved

Proposition 2.5. *Fix a base point* $x_0 \in \mathcal{T}(S)$. *Let* x_n *be a sequence in* $\mathcal{T}(S)$. *Then the followings are equivalent:*

(1) x_n *converges to a boundary point* $p \in \mathcal{T}^{GM}(S)$.

(2) *There exists a continuous map* $\varepsilon_p : \mathcal{MF} \to \boldsymbol{R}_{\geq 0}$ *such that* $e^{-d_T(x_0, x_n)} Ext^{\frac{1}{2}}(x_n, \cdot)$ *converges uniformly to* ε_p *on any compact subset of* $\mathcal{MF}$.

(3) *There exists a continuous map* $\varepsilon'_p : \mathcal{MF} \to \boldsymbol{R}_{\geq 0}$ *such that* $\lim_{n \to \infty} e^{-d_T(x_0, x_n)} Ext^{\frac{1}{2}}(x_n, F) = \varepsilon'_p(F)$ *for any* $F \in \mathcal{MF}$.

Moreover, if one of above holds, then $\varepsilon_p = \varepsilon'_p$ *and* $p = [\varepsilon_p|_{\mathcal{S}}(\cdot) : \mathcal{S} \to \boldsymbol{R}_{\geq 0}]$.

Proof. Miyachi[11] proved that (1) is equivalent to (2); and if (1) or (2) holds, then $p = [\varepsilon_p|_{\mathcal{S}}(\cdot) : \mathcal{S} \to \mathbf{R}_{\geq 0}]$. Since the convergence on compact sets is stronger than pointwise convergence, (2) implies (3) and $\varepsilon_p = \varepsilon'_p$. Suppose (3) holds. Since $\mathcal{S} \subseteq \mathcal{MF}$, we have $\lim_{n\to\infty} e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, \alpha) = \varepsilon'_p(\alpha)$ for any $\alpha \in \mathcal{S}$, which implies that x_n converges to $[\varepsilon'_p]$ in $\mathcal{T}^{GM}(S)$. Thus (3) implies (1). □

Miyachi [11] also proved that the action of $Mod(S)$ on $\mathcal{T}(S)$ extends continuously to $\mathcal{T}^{GM}(S)$. Precisely, the action of $Mod(S)$ on $\mathcal{T}^{GM}(S)$ can be defined as follows. For any $p \in \mathcal{T}^{GM}(S)$, let $[\varepsilon_p(\cdot)] \in P\mathbf{R}^{\mathcal{S}}_{\geq 0}$ be the representative of p. Then for any $f \in Mod(S)$, the representative of $f(p)$ is $[\varepsilon_p \circ f^{-1}(\cdot)] \in P\mathbf{R}^{\mathcal{S}}_{\geq 0}$.

By the results in [5], identifying any $[F] \in \mathcal{PMF}$ with $[i(F, \cdot) : \mathcal{S} \to \mathbf{R}_{\geq 0}] \in P\mathbf{R}^{\mathcal{S}}_{\geq 0}$, $\mathcal{PMF}$ is a proper subset of GM. See [11], [12], [9] and [16] for more details on the Gardiner–Masur boundary.

2.5. Some basic lemmas. In this subsection, we introduce some basic results in general topology, which are necessary for our topic.

Let X be a set and Y be a topological space. Let Y^X be the set of all maps from X to Y. For any $x \in X$ and any open set U of Y, set $V_{x,U} = \{f \in Y^X : f(x) \in U\}$. Endow Y^X with the topology generated by $\{V_{x,U} : x \in X, U \text{ is an open set of } Y\}$. Under this topology, a sequence $f_n \in Y^X$ converges to f if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in X$. Thus this topology is called the pointwise convergence topology on Y^X. Note that if Y is a Hausdorff space, then Y^X is also a Hausdorff space. If Y is first-countable (any point $y \in Y$ has a countable basis of neighbourhoods) and X is a countable set, then Y^X is also first-countable. By the well-known Tykhonov theorem, we have

Lemma 2.6. *Let $\{M_x : x \in X\}$ be a family of compact subsets of Y. Then*

$$\{f \in Y^X : f(x) \in M_x \text{ for any } x \in X\}$$

is a compact subset of Y^X.

See [13] for more details on the pointwise convergence topology and Tykhonov theorem.

In particular, if $X = Y = \mathcal{MF}$, then we have a space $\mathcal{MF}^{\mathcal{MF}}$ with pointwise convergence topology. Since $\mathcal{MF}$ is a Hausdorff space, $\mathcal{MF}^{\mathcal{MF}}$ is also a Hausdorff space.

Let Ω be the set of all homogeneous maps from $\mathcal{MF}$ to $\mathcal{MF}$, where a map $f : \mathcal{MF} \to \mathcal{MF}$ is called homogeneous if for any $k \geq 0$ and $F \in \mathcal{MF}$, $f(kF) = kf(F)$. Note that $\Omega \subseteq \mathcal{MF}^{\mathcal{MF}}$. Then we have

Lemma 2.7. *Ω is closed in $\mathcal{MF}^{\mathcal{MF}}$.*

Proof. For any $k \geq 0$ and $F \in \mathcal{MF}$, set $U_{k,F} = \{f \in \mathcal{MF}^{\mathcal{MF}} : f(kF) \neq kf(F)\}$. Note that $U_{k,F}$ is open in $\mathcal{MF}^{\mathcal{MF}}$. Thus $\bigcup_{k\geq 0, F\in\mathcal{MF}} U_{k,F}$ is open in $\mathcal{MF}^{\mathcal{MF}}$. By the definition of Ω, we have $\Omega = \mathcal{MF}^{\mathcal{MF}} - \bigcup_{k\geq 0, F\in\mathcal{MF}} U_{k,F}$. Thus Ω is closed in $\mathcal{MF}^{\mathcal{MF}}$. □

We endow Ω with the subspace topology from the pointwise convergence topology on $\mathcal{MF}^{\mathcal{MF}}$ and call it the pointwise convergence topology on Ω. Note that $\mathbf{R}_+$ acts on Ω by multiplication. Let $P\Omega = \Omega - \{0\}/\mathbf{R}_+$ be the projective space. Let $pr : \Omega - \{0\} \to P\Omega$ be the natural projection. For any $f \in \Omega - \{0\}$, let $[f] = pr(f) \in P\Omega$ be its projective class. Naturally, we endow $P\Omega$ with the quotient topology from the pointwise convergence topology on Ω. Then we have

Lemma 2.8. *$P\Omega$ is a Hausdorff space.*

Proof. Firstly, taking a point x_0 in $\mathcal{T}(S)$, we claim that the map

$$\theta : \mathcal{MF} - \{0\} \to \mathcal{PMF} \times \mathbf{R}_+,\ F \mapsto ([F], Ext^{\frac{1}{2}}(x_0, F))$$

is a homeomorphism. Obviously, θ is continuous and bijective. Since $\mathcal{MF} \approx \mathbf{R}^{6g-6+2n}$ and $\mathcal{PMF} \approx S^{6g-7+2n}$, we have $\mathcal{MF} - \{0\} \approx \mathcal{PMF} \times \mathbf{R}_+ \approx \mathbf{R}^{6g-6+2n} - \{0\}$. By the invariance of domain, we know that θ is a homeomorphism.

Now we are ready to prove that $P\Omega$ is a Hausdorff space. For any $[f] \neq [g]$ in $P\Omega$, setting $l(f) = \{kf : k > 0\}$ and $l(g) = \{kg : k > 0\}$, we need to find two open subsets U, V of $\Omega - \{0\}$ such that $l(f) \subseteq U, l(g) \subseteq V, U \bigcap V = \emptyset$ and U, V are invariant under the action of $\mathbf{R}_+$, that is, $pr^{-1}(pr(U)) = U, pr^{-1}(pr(V)) = V$. There are three possible cases:
Case (1): for any $F \in \mathcal{MF}$, $[f(F)] = [g(F)]$ or $f(F) = g(F) = 0$. In this case, combining the fact that $f, g \neq 0 \in \Omega$ and $[f] \neq [g]$, we know that there are $F_1, F_2 \in \mathcal{MF}$ and $k_1, k_2 > 0$ such that $F_1 \neq F_2$, $k_1 \neq k_2$, $g(F_1) = k_1 f(F_1) \neq 0$ and $g(F_2) = k_2 f(F_2) \neq 0$. Consider a continuous map:

$$\theta_1 : \Omega \to \mathbf{R}^2_{\geq 0}, f \mapsto (Ext^{\frac{1}{2}}(x_0, f(F_1)), Ext^{\frac{1}{2}}(x_0, f(F_2))).$$

Set $l'(f) = \theta_1(l(f)), l'(g) = \theta_1(l(g))$. Then $l'(f)$ and $l'(g)$ are two disjoint straight lines in $\mathbf{R}^2_{\geq 0}$. Thus we can find two disjoint open subsets U', V' of $\mathbf{R}^2_{\geq 0} - \{0\}$ such that $l'(f) \subseteq U'$, $l'(g) \subseteq V'$ and U', V' are both invariant under the action of $\mathbf{R}_+$. Set $U = \theta_1^{-1}(U'), V = \theta_1^{-1}(V')$. Note that $U \bigcap V = \emptyset$, $l(f) \subseteq U, l(g) \subseteq V$ and U, V are both invariant under the action of $\mathbf{R}_+$. By the continuity of θ_1, U, V are both open subsets of $\Omega - \{0\}$.
Case(2): there exists $F_1 \in \mathcal{MF}$ such that $f(F_1) = 0, g(F_1) \neq 0$ or $g(F_1) = 0, f(F_1) \neq 0$. By symmetry, we may assume that $f(F_1) = 0, g(F_1) \neq 0$. In this case, using the fact that $f \neq 0 \in \Omega$, we know that there exists $F_2 \in \mathcal{MF}$ such that $f(F_2) \neq 0$. Then considering the same continuous map θ_1 as case (1) and using a similar argument, we also construct two desired open subsets U, V.
Case (3): there exists $F \in \mathcal{MF}$ such that $f(F), g(F) \neq 0$ and $[f(F)] \neq [g(F)]$. Since $\mathcal{PMF} \approx S^{6g-7+2n}$ is Hausdorff, there are two disjoint open subsets U', V' of $\mathcal{PMF}$ such that $[f(F)] \in U', [g(F)] \in V'$. Now we use the homeomorphism θ constructed above:

$$\theta : \mathcal{MF} - \{0\} \to \mathcal{PMF} \times \mathbf{R}_+,\ F \mapsto ([F], Ext^{\frac{1}{2}}(x_0, F)).$$

Set $U'' = \theta^{-1}(U' \times \mathbf{R}_+), V'' = \theta^{-1}(V' \times \mathbf{R}_+)$. Then U'' and V'' are disjoint open subsets of $\mathcal{MF} - \{0\}$ and are both invariant under the action of $\mathbf{R}_+$. Set $U = \{f \in \Omega : f(F) \in U''\}$ and $V = \{g \in \Omega : g(F) \in V''\}$. Then U, V are disjoint open subsets of $\Omega - \{0\}$ and are both invariant under the action of $\mathbf{R}_+$. Note that $l(f) \subseteq U, l(g) \subseteq V$. Thus U, V satisfy the desired properties. □

Moreover, we need the following result (see [13], Theorem 21.3)

Lemma 2.9. *Let X, Y be two topological spaces. Suppose that X is first-countable, that is, any point $x \in X$ has a countable basis of neighbourhoods. Let $f : X \to Y$ be a map from X to Y. Then the followings are equivalent:*

(1) *f is continuous;*
(2) *if a sequence $x_n \in X$ converges to $x \in X$, then $f(x_n)$ converges to $f(x)$.*

Besides, we need a result which extends Lemma 3.1 of [15]:

Lemma 2.10. *Let X, Y be two topological spaces. Let (Z, d) be a metric space. Let $H : X \times Y \to (Z, d)$ be a continuous map. Suppose a sequence $x_n \in X$ converges to $x \in X$. Then $H(x_n, \cdot)$ converges to $H(x, \cdot)$ uniformly on any compact subsets of Y. In particular, $H(x_n, \cdot)$ converges to $H(x, \cdot)$ pointwise, that is, $\lim_{n\to\infty} H(x_n, y) = H(x, y)$ for any $y \in Y$.*

Proof. The proof is similar to that of Lemma 3.1 of [15]. Let M be a compact subset of Y. By the continuity of $H : X \times Y \to (Z, d)$, for any $\epsilon > 0$ and $y \in Y$, there exists an open neighbourhood $U_y \subseteq X$ of x and an open neighbourhood $V_y \subseteq Y$ of y such that $d(H(x', y'), H(x, y)) < \epsilon$ for any $x' \in U_y$ and $y' \in V_y$. Note that M is covered by $\{V_y : y \in M\}$. By the compactness of M, there is a finite sub-covering $\{V_{y_1}, V_{y_2}, ..., V_{y_n}\}$. Set $U = \bigcap_{i=1}^n U_{y_i}$. Note that U is a neighbourhood of x such that $d(H(x', y), H(x, y)) < \epsilon$ for any $y \in M$ and $x' \in U$. Suppose $x_n \in X$ converges to x. Then there exists $N > 0$ such that $x_n \in U$ for any $n > N$. Thus $d(H(x_n, y), H(x, y)) < \epsilon$ for any $n > N$ and $y \in M$. This means that $H(x_n, \cdot)$ converges to $H(x, \cdot)$ uniformly on compact set M. In particular, since a single point set $\{y\} \subseteq Y$ is compact, we also have $\lim_{n\to\infty} H(x_n, y) = H(x, y)$ for any $y \in Y$. □

3. The definition of the new compactification

We introduce the new compactification of $\mathcal{T}(S)$ in this section. Recall that for any $x \in \mathcal{T}(S)$, the horizontal foliation map H_x and the vertical foliation map V_x are both homeomorphisms. Thus for any $x \in \mathcal{T}(S)$, $\tau_x : \mathcal{MF} \to \mathcal{MF}$, $F \mapsto H_x \circ V_x^{-1}(F)$ is a homeomorphism.

Proposition 3.1. *For any $x \in \mathcal{T}(S)$, we have*
(1) *$\tau_x^2 = id_{\mathcal{MF}}$;*
(2) *τ_x is homogeneous: for any $F \in \mathcal{MF}$ and $k \geq 0$, $\tau_x(kF) = k\tau_x(F)$;*
(3) *for any $x \in \mathcal{T}(S)$ and $F \in \mathcal{MF}$, $Ext(x, F) = Ext(x, \tau_x(F)) = i(\tau_x(F), F)$.*

Proof. (1) comes from the fact that for any $q \in Q(x)$, $H(q) = V(-q)$. (2) comes from the homogeneity of $H(\cdot)$ and $V(\cdot)$. (3) comes from the the fact that $Ext(x, H(q)) = Ext(x, V(q)) = \|q\|$ (Theorem 2.2) and Minsky's inequality (Theorem 2.4). □

By Proposition 3.1, we have $\tau_x \in \Omega - \{0\}$ and $[\tau_x] \in P\Omega$ for any $x \in \mathcal{T}(S)$. Sending $x \in \mathcal{T}(S)$ to $[\tau_x] \in P\Omega$, we have a map

$$\Phi : \mathcal{T}(S) \to P\Omega, \ x \mapsto [\tau_x].$$

Then we have

Theorem 3.2. *Φ is an embedding and the closure of the image $\Phi(\mathcal{T}(S))$ is compact.*

Proof. Taking a base point $x_0 \in \mathcal{T}(S)$, we consider a map

$$\widetilde{\Phi} : \mathcal{T}(S) \to \Omega, \ x \mapsto e^{-2d_T(x_0,x)}\tau_x.$$

Now we proceed to prove the desired result for $\widetilde{\Phi}$: $\widetilde{\Phi}$ is an embedding and the closure of the image $\widetilde{\Phi}(\mathcal{T}(S))$ is compact.

Firstly, we prove the continuity of $\widetilde{\Phi}$. By Theorem 2.1, $\tau = H' \circ V'^{-1} : \mathcal{T}(S) \times \mathcal{MF} \to \mathcal{T}(S) \times \mathcal{MF},\ (x, F) \mapsto (x, \tau_x(F))$ is a homeomorphism, which implies that the map $(x, F) \mapsto \tau_x(F)$ is continuous. Then the map $(x, F) \mapsto e^{-2d_T(x_0,x)}\tau_x(F)$ is also continuous. Now suppose $x_n \in \mathcal{T}(S)$ converges to $x \in \mathcal{T}(S)$. By Lemma 2.10, we have $\lim_{n\to\infty} e^{-2d_T(x_0,x_n)}\tau_{x_n}(F) = e^{-2d_T(x_0,x)}\tau_x(F)$ for any $F \in \mathcal{MF}$. By Lemma 2.9 and the definition of pointwise convergence topology on Ω, this implies the continuity of $\widetilde{\Phi}$.

Secondly, we prove that $\widetilde{\Phi}$ is injective. Suppose that $\widetilde{\Phi}(x) = \widetilde{\Phi}(y)$ for some $x, y \in \mathcal{T}(S)$, that is, $e^{-2d_T(x_0,x)}\tau_x = e^{-2d_T(x_0,y)}\tau_y$. By Proposition 3.1(3), we know that for any $\alpha \in \mathcal{S}$,

$$e^{-2d_T(x_0,x)}Ext(x, \alpha) = e^{-2d_T(x_0,x)}i(\tau_x(\alpha), \alpha) = e^{-2d_T(x_0,y)}i(\tau_y(\alpha), \alpha) = e^{-2d_T(x_0,y)}Ext(y, \alpha),$$

which implies that $[Ext^{\frac{1}{2}}(x, \cdot)] = [Ext^{\frac{1}{2}}(y, \cdot)] \in P\mathbf{R}^{\mathcal{S}}_{\geq 0}$. Since $\Phi_{GM} : \mathcal{T}(S) \to P\mathbf{R}^{\mathcal{S}}_{\geq 0},\ x \mapsto [Ext^{\frac{1}{2}}(x, \cdot)]$ is an embedding, we have $x = y$. Thus $\widetilde{\Phi}$ is injective.

Thirdly, we prove the continuity of the inverse of $\widetilde{\Phi}$. Consider the space $\mathcal{MF}^{\mathcal{S}}$, which is the set of all maps from $\mathcal{S}$ to $\mathcal{MF}$. Endow $\mathcal{MF}^{\mathcal{S}}$ with the pointwise convergence topology. Since $\mathcal{S}$ is countable and $\mathcal{MF}$ is first-countable, we know that $\mathcal{MF}^{\mathcal{S}}$ is first-countable. Since $\mathcal{S} \subseteq \mathcal{MF}$, we have a natural continuous projection $I_1 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{S}}, f \mapsto f|_{\mathcal{S}}$. Now consider the map $I_2 : \mathcal{MF}^{\mathcal{S}} \to \mathbf{R}^{\mathcal{S}}_{\geq 0}, f \mapsto i(f(\cdot), \cdot)^{\frac{1}{2}}$. Suppose $f_n \in \mathcal{MF}^{\mathcal{S}}$ converges to $f \in \mathcal{MF}^{\mathcal{S}}$, that is, $\lim_{n\to\infty} f_n(\alpha) = f(\alpha)$ for any $\alpha \in \mathcal{S}$. Then by the continuity of $i(\cdot, \cdot)$, we have $\lim_{n\to\infty} i(f_n(\alpha), \alpha) = i(f(\alpha), \alpha)$ for any $\alpha \in \mathcal{S}$. By Lemma 2.9, this implies that $I_2 : \mathcal{MF}^{\mathcal{S}} \to \mathbf{R}^{\mathcal{S}}_{\geq 0}$ is continuous. Let $\pi : \mathbf{R}^{\mathcal{S}}_{\geq 0} - \{0\} \to P\mathbf{R}^{\mathcal{S}}_{\geq 0}$ be the natural projection from $\mathbf{R}^{\mathcal{S}}_{\geq 0} - \{0\}$ to $P\mathbf{R}^{\mathcal{S}}_{\geq 0}$. Considering the map $I_2 \circ I_1 \circ \widetilde{\Phi} : \mathcal{T}(S) \to \mathbf{R}^{\mathcal{S}}_{\geq 0}$, we know that its image $I_2 \circ I_1 \circ \widetilde{\Phi}(\mathcal{T}(S))$ does not contain $0 \in \mathbf{R}^{\mathcal{S}}_{\geq 0}$. Thus we have a map: $\pi \circ I_2 \circ I_1 \circ \widetilde{\Phi} : \mathcal{T}(S) \to P\mathbf{R}^{\mathcal{S}}_{\geq 0}$. And it is easy to verify that $\pi \circ I_2 \circ I_1 \circ \widetilde{\Phi} = \Phi_{GM}$, where Φ_{GM} is the Gardiner–Masur embedding. Then $\widetilde{\Phi}^{-1} = \Phi^{-1}_{GM} \circ \pi \circ I_2 \circ I_1$, which is continuous by the continuities of $I_1, I_2, \pi, \Phi^{-1}_{GM}$.

Therefore, $\widetilde{\Phi}$ is an embedding.

Now we prove the compactness of the closure of image $\widetilde{\Phi}(\mathcal{T}(S))$. By Proposition 3.1 (3) and Kerckhoff's formula (Theorem 2.3), for any $x \in \mathcal{T}(S)$ and $F \in \mathcal{MF}$,

$$\begin{aligned} Ext(x_0, \widetilde{\Phi}(x)(F)) &= Ext(x_0, e^{-2d_T(x_0,x)}\tau_x(F)) \\ &\leq e^{-4d_T(x_0,x)} \cdot e^{2d_T(x_0,x)}Ext(x, \tau_x(F)) \\ &= e^{-2d_T(x_0,x)}Ext(x, F) \leq Ext(x_0, F). \end{aligned}$$

Note that for any $M \geq 0$, $\{E \in \mathcal{MF} : Ext(x_0, E) \leq M\}$ is compact. Then by Lemma 2.6, the set

$$A = \{f \in \mathcal{MF}^{\mathcal{MF}} : Ext(x_0, f(F)) \leq Ext(x_0, F) \text{ for any } F \in \mathcal{MF}\}$$

is a compact subset of $\mathcal{MF}^{\mathcal{MF}}$. By Lemma 2.7, Ω is closed in $\mathcal{MF}^{\mathcal{MF}}$. Thus $A' = A \bigcap \Omega$ is a compact subset of Ω. Since $\widetilde{\Phi}(\mathcal{T}(S))$ is contained in compact set A', we know that the closure of $\widetilde{\Phi}(\mathcal{T}(S))$ is compact.

Besides, we prove that $0 \notin Cl(\widetilde{\Phi}(\mathcal{T}(S)))$, where $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$ is the closure of $\widetilde{\Phi}(\mathcal{T}(S))$ in Ω. Suppose $0 \in Cl(\widetilde{\Phi}(\mathcal{T}(S)))$. Considering the continuous projection $I_1 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{S}}, f \mapsto f|_{\mathcal{S}}$ constructed above, we have $0 \in Cl(I_1(\widetilde{\Phi}(\mathcal{T}(S)))$. Since $\mathcal{MF}^{\mathcal{S}}$ is first-countable, this means that there exists a sequence $x_n \in \mathcal{T}(S)$ such that for any $\alpha \in \mathcal{S}$,

$$\lim_{n\to\infty} e^{-2d_T(x_0,x_n)}\tau_{x_n}(\alpha) = 0.$$

By Proposition 3.1(3), this implies that for any $\alpha \in \mathcal{S}$,

$$\lim_{n\to\infty} e^{-d_T(x_0,x_n)}Ext^{\frac{1}{2}}(x_n,\alpha) = \lim_{n\to\infty} i(e^{-2d_T(x_0,x_n)}\tau_{x_n}(\alpha),\alpha)^{\frac{1}{2}} = 0.$$

By the compactness of $\mathcal{T}^{GM}(S)$, we may assume that x_n converges to some $p \in \mathcal{T}^{GM}(S)$. By Proposition 2.5, we have $\varepsilon_{x_n}(\cdot) = e^{-d_T(x_0,x_n)}Ext^{\frac{1}{2}}(x_n,\cdot) : \mathcal{S} \to \mathbf{R}_{\geq 0}$ converges pointwise to some $\varepsilon_p(\cdot) : \mathcal{S} \to \mathbf{R}_{\geq 0}$ such that $p = [\varepsilon_p]$. Thus the representative of p is 0, which is impossible.

Let $pr : \Omega - \{0\} \to P\Omega$ be the natural projection. Since $0 \notin Cl(\widetilde{\Phi}(\mathcal{T}(S)))$, restricting pr to $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$, we have a map $pr : Cl(\widetilde{\Phi}(\mathcal{T}(S))) \to P\Omega$. We claim that this map is an embedding. By Lemma 2.8, $P\Omega$ is a Hausdorff space. By the fact that a continuous and injective map from a compact space to a Hausdorff space is an embedding, we only need to prove the injectivity of $pr : Cl(\widetilde{\Phi}(\mathcal{T}(S))) \to P\Omega$. Suppose that $[f] = [g]$ for some $f, g \in Cl(\widetilde{\Phi}(\mathcal{T}(S)))$, that is, $f = kg$ for some $k > 0$. By the continuity of $I_1 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{S}}, f \mapsto f|_{\mathcal{S}}$, we know that $I_1(Cl(\widetilde{\Phi}(\mathcal{T}(S)))) \subseteq Cl(I_1(\widetilde{\Phi}(\mathcal{T}(S)))) \subseteq \mathcal{MF}^{\mathcal{S}}$. Since $\mathcal{MF}^{\mathcal{S}}$ is first-countable, there are two sequences $x_n, y_n \in \mathcal{T}(S)$ such that

$$\lim_{n\to\infty} I_1(\widetilde{\Phi}(x_n)) = I_1(f),\ \lim_{n\to\infty} I_1(\widetilde{\Phi}(y_n)) = I_1(g).$$

This means that $\lim_{n\to\infty} e^{-2d_T(x_0,x_n)}\tau_{x_n}(\alpha) = f(\alpha)$ and $\lim_{n\to\infty} e^{-2d_T(x_0,y_n)}\tau_{y_n}(\alpha) = g(\alpha)$ for any $\alpha \in \mathcal{S}$. By the continuity of $I_2 : \mathcal{MF}^{\mathcal{S}} \to \mathbf{R}^{\mathcal{S}}_{\geq 0}, f \mapsto i(f(\cdot),\cdot)^{\frac{1}{2}}$, we have $\lim_{n\to\infty} e^{-d_T(x_0,x_n)}Ext^{\frac{1}{2}}(x_n,\alpha) = i(f(\alpha),\alpha)^{\frac{1}{2}}$ and $\lim_{n\to\infty} e^{-d_T(x_0,y_n)}Ext^{\frac{1}{2}}(y_n,\alpha) = i(g(\alpha),\alpha)^{\frac{1}{2}}$ for any $\alpha \in \mathcal{S}$. Since $f = kg$, by Lemma 2.5, we have x_n and y_n converges to a same point $[i(f(\alpha),\alpha)^{\frac{1}{2}}]$ in $\mathcal{T}^{GM}(S)$. Again by Lemma 2.5, we have $f = g$.

Now we are ready to prove the desired result: Φ is an embedding and the closure of the image $\Phi(\mathcal{T}(S))$ is compact. Since $pr : Cl(\widetilde{\Phi}(\mathcal{T}(S))) \to P\Omega$ is an embedding, pr is a homeomorphism from $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$ to its image $pr(Cl(\widetilde{\Phi}(\mathcal{T}(S))))$. Note that $\Phi = pr \circ \widetilde{\Phi} : \mathcal{T}(S) \to P\Omega$. Thus Φ is an embedding and $Cl(\Phi(\mathcal{T}(S))) = pr(Cl(\widetilde{\Phi}(\mathcal{T}(S))))$, which is compact. Moreover, $Cl(\Phi(\mathcal{T}(S)))$ and $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$ are equivalent compactifications of $\mathcal{T}(S)$. □

Hence, $Cl(\Phi(\mathcal{T}(S)))$ is a compactification of $\mathcal{T}(S)$ and $Cl(\Phi(\mathcal{T}(S))) - \Phi(\mathcal{T}(S))$ is a new boundary of $\mathcal{T}(S)$. For simplicity, we denote $Cl(\Phi(\mathcal{T}(S)))$ by $\overline{\mathcal{T}}(S)$ and denote $Cl(\Phi(\mathcal{T}(S))) - \Phi(\mathcal{T}(S))$ by $\partial\mathcal{T}(S)$.

By the proof of Theorem 3.2, we have

Proposition 3.3. *Fix a base point $x_0 \in \mathcal{T}(S)$. A sequence $x_n \in \mathcal{T}(S)$ converges to a point $p \in \partial\mathcal{T}(S)$ if and only if $e^{-2d_T(x_0,x_n)}\tau_{x_n}(\cdot)$ converges to some $\tau_p(\cdot) \in \Omega$ such that $p = [\tau_p]$.*

Besides, we have

Proposition 3.4. *In $\overline{\mathcal{T}}(S)$, $\mathcal{T}(S)$ is open and $\partial\mathcal{T}(S)$ is closed. Moreover, $\partial\mathcal{T}(S)$ is compact.*

Proof. We only need to prove the openness of $\mathcal{T}(S)$ in $\overline{\mathcal{T}}(S)$. For any $x \in \mathcal{T}(S)$, let $B(x,1) = \{y \in \mathcal{T}(S) : d_T(x,y) < 1\}$ be the unit open ball with center x and $\overline{B}(x,1) = \{y \in$

$\mathcal{T}(S) : d_T(x, y) \le 1\}$ be the unit closed ball with center x, where d_T is the Teichmüller metric. Since $B(x, 1)$ is open in $\mathcal{T}(S)$, there exists an open set U of $\overline{\mathcal{T}}(S)$ such that $U \bigcap \mathcal{T}(S) = B(x, 1)$. It is well-known that $\overline{B}(x, 1)$ is compact. Since $\overline{\mathcal{T}}(S)$ is Hausdorff, we know that $\overline{B}(x, 1)$ is closed in $\overline{\mathcal{T}}(S)$. Thus $\overline{\mathcal{T}}(S) - \overline{B}(x, 1)$ is open in $\overline{\mathcal{T}}(S)$. Thus $U \bigcap (\overline{\mathcal{T}}(S) - \overline{B}(x, 1)) = U - \overline{B}(x, 1)$ is an open set of $\overline{\mathcal{T}}(S)$. Since $U \bigcap \mathcal{T}(S) = B(x, 1)$, we have $U - \overline{B}(x, 1) \subseteq \overline{\mathcal{T}}(S) - \mathcal{T}(S) = \partial\mathcal{T}(S)$. We claim that $U - \overline{B}(x, 1) = \emptyset$. Otherwise, suppose $U - \overline{B}(x, 1) \neq \emptyset$. Then there exists a point $p \in \partial\mathcal{T}(S)$ with $U - \overline{B}(x, 1)$ as open neighbourhood. Since $\mathcal{T}(S)$ is dense in $\overline{\mathcal{T}}(S)$, there exists a point $y \in \mathcal{T}(S)$ such that $y \in U - \overline{B}(x, 1)$, which contradicts with $U - \overline{B}(x, 1) \subseteq \partial\mathcal{T}(S)$. Thus $U - \overline{B}(x, 1) = \emptyset$, which implies that $U = B(x, 1)$. Then $B(x, 1) \subseteq \mathcal{T}(S)$ is an open neighbourhood of x in $\overline{\mathcal{T}}(S)$, which implies that $\mathcal{T}(S)$ is open in $\overline{\mathcal{T}}(S)$. □

4. Finer than the Gardiner–Masur compactification

Let X be a locally compact Hausdorff space. Let $f_1 : X \to X_1$ and $f_2 : X \to X_2$ be two compactifications of X, that is, f_1, f_2 are embeddings; X_1, X_2 are compact; $f_1(X)$ is dense in X_1 and $f_2(X)$ is dense in X_2. We call compactification X_1 is finer than compactification X_2 if there exists a continuous map $F : X_1 \to X_2$ such that $f_2 = F \circ f_1$. When X_1 and X_2 are both Hausdorff, we know that if such a F exists, then it is surjective and unique.

Theorem 4.1. *As compactifications of $\mathcal{T}(S)$, $\overline{\mathcal{T}}(S)$ is finer than $\mathcal{T}^{GM}(S)$ through the surjective continuous map*

$$\Theta : \overline{\mathcal{T}}(S) \to \mathcal{T}^{GM}(S),\ [f] \mapsto [i(f|_S(\cdot), \cdot)^{\frac{1}{2}}].$$

Proof. Recall that the Gardiner–Masur embedding is defined as $\Phi_{GM} : \mathcal{T}(S) \to P\mathbf{R}^{\mathcal{S}}_{\ge 0}$, $x \mapsto [Ext^{\frac{1}{2}}(x, \cdot)]$ and the embedding inducing the new compactification $\overline{\mathcal{T}}(S)$ is defined as $\Phi : \mathcal{T}(S) \to P\Omega$, $x \mapsto [\tau_x]$. And $\mathcal{T}^{GM}(S) = Cl(\Phi_{GM}(\mathcal{T}(S))) \subseteq P\mathbf{R}^{\mathcal{S}}_{\ge 0}$ and $\overline{\mathcal{T}}(S) = Cl(\Phi(\mathcal{T}(S))) \subseteq P\Omega$.

Now we recall serval maps used in the proof of Theorem 3.2:

$$\widetilde{\Phi} : \mathcal{T}(S) \to \Omega,\ x \mapsto e^{-2d_T(x_0,x)}\tau_x,$$

$$I_1 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{S}}, f \mapsto f|_S,$$

$$I_2 : \mathcal{MF}^{\mathcal{S}} \to \mathbf{R}^{\mathcal{S}}_{\ge 0}, f \mapsto i(f(\cdot), \cdot)^{\frac{1}{2}},$$

$$\pi : \mathbf{R}^{\mathcal{S}}_{\ge 0} - \{0\} \to P\mathbf{R}^{\mathcal{S}}_{\ge 0}.$$

And in the proof of Theorem 3.2, we identify the compactification $\overline{\mathcal{T}}(S) = Cl(\Phi(\mathcal{T}(S)))$ with another compactification $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$.

Note that $\Phi_{GM} = \pi \circ I_2 \circ I_1 \circ \widetilde{\Phi} : \mathcal{T}(S) \to P\mathbf{R}^{\mathcal{S}}_{\ge 0}$. Thus

$$\mathcal{T}^{GM}(S) = Cl(\Phi_{GM}(\mathcal{T}(S))) \supseteq \pi \circ I_2 \circ I_1(Cl(\widetilde{\Phi}(\mathcal{T}(S))))$$

and

$$\pi \circ I_2 \circ I_1 : Cl(\widetilde{\Phi}(\mathcal{T}(S))) \to \mathcal{T}^{GM}(S),\ f \mapsto [i(f|_S(\cdot), \cdot)^{\frac{1}{2}}]$$

is a continuous map. This means that $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$ is finer than $\mathcal{T}^{GM}(S)$. Note that $\overline{\mathcal{T}}(S) = Cl(\Phi(\mathcal{T}(S)))$ is equivalent to $Cl(\widetilde{\Phi}(\mathcal{T}(S)))$ through the natural projection

$$pr : Cl(\widetilde{\Phi}(\mathcal{T}(S))) \to Cl(\Phi(\mathcal{T}(S))) = \overline{\mathcal{T}}(S),\ f \mapsto [f].$$

Thus as compactifications of $\mathcal{T}(S)$, $\overline{\mathcal{T}}(S)$ is finer than $\mathcal{T}^{GM}(S)$ through the continuous map

$$\Theta : \overline{\mathcal{T}}(S) \to \mathcal{T}^{GM}(S),\ [f] \mapsto [i(f|_S(\cdot), \cdot)^{\frac{1}{2}}].$$

Moreover, since $\overline{\mathcal{T}}(S)$ and $\mathcal{T}^{GM}(S)$ are both Hausdorff, Θ is surjective. □

Recall that $GM = \mathcal{T}^{GM}(S) - \mathcal{T}(S)$ is the Gardiner–Masur boundary and $\partial\mathcal{T}(S) = \overline{\mathcal{T}}(S) - \mathcal{T}(S)$ is the new boundary. Then we have

Proposition 4.2. $\Theta(\partial\mathcal{T}(S)) = GM$.

Proof. Since Θ is surjective and $\Theta(\mathcal{T}(S)) = \mathcal{T}(S)$, we only need to prove that $\Theta(\partial\mathcal{T}(S)) \subseteq GM$. Otherwise, suppose there exists some $p \in \partial\mathcal{T}(S)$ such that $\Theta(p) = x \in \mathcal{T}(S)$. Let $B(x,1) = \{y \in \mathcal{T}(S) : d_T(x,y) < 1\}$ and $\overline{B}(x,1) = \{y \in \mathcal{T}(S) : d_T(x,y) \leq 1\}$, where d_T is the Teichmüller metric. Note that $B(x,1)$ is an open set of $\mathcal{T}^{GM}(S)$. By $\Theta(p) = x$ and the continuity of Θ, we know that $\Theta^{-1}(B(x,1))$ is an open neighbourhood of p in $\overline{\mathcal{T}}(S)$. Since $\Theta|_{\mathcal{T}(S)} = id : \mathcal{T}(S) \to \mathcal{T}(S)$, we have $\Theta^{-1}(B(x,1)) \bigcap \mathcal{T}(S) = B(x,1)$. Note that $\overline{B}(x,1)$ is compact, which is closed in $\overline{\mathcal{T}}(S)$. Thus $\Theta^{-1}(B(x,1)) - \overline{B}(x,1)$ is an open set of $\overline{\mathcal{T}}(S)$. Then $\Theta^{-1}(B(x,1)) - \overline{B}(x,1)$ is an open neighbourhood of p in $\overline{\mathcal{T}}(S)$, which is disjoint with $\mathcal{T}(S)$. But this contradicts the denseness of $\mathcal{T}(S)$ in $\overline{\mathcal{T}}(S)$. □

Now we consider a special subset of $\partial\mathcal{T}(S)$:

$$\partial'\mathcal{T}(S) = \{p \in \partial\mathcal{T}(S) : \text{there exists a sequence } x_n \in \mathcal{T}(S) \text{ such that } \lim_{n\to\infty} x_n = p\}.$$

We don't know whether $\overline{\mathcal{T}}(S)$ is first-countable. If $\overline{\mathcal{T}}(S)$ is first-countable, then $\partial\mathcal{T}(S) = \partial'\mathcal{T}(S)$. $\partial'\mathcal{T}(S)$ is related to the Gardiner–Masur boundary GM through Θ:

Proposition 4.3. *For any* $[\tau] \in \partial'\mathcal{T}(S)$*, let* $[\varepsilon_p(\cdot)] = \Theta([\tau]) = [i(\tau(\cdot), \cdot)^{\frac{1}{2}}] \in GM$.
(1) *There exists a constant* $C > 0$ *such that for any* $F, G \in \mathcal{MF}$*,* $i(\tau(F), G) \leq C\varepsilon_p(F) \cdot \varepsilon_p(G)$.
(2) *For any* $F \in \mathcal{MF}$*,* $\tau(F) = 0$ *if and only if* $\varepsilon_p(F) = 0$.
(3) *For any* $F, G \in \mathcal{MF}$*,* $\tau(G) = 0$ *or* $\varepsilon_p(G) = 0$ *implies* $i(\tau(F), G) = 0$.
(4) $\tau^2 = 0$.

Proof. Fix a base point $x_0 \in \mathcal{T}(S)$. Since $[\tau] \in \partial'\mathcal{T}(S)$, we take a sequence $x_n \in \mathcal{T}(S)$ such that $\lim_{n\to\infty} x_n = [\tau]$ in $\overline{\mathcal{T}}(S)$. By the continuity of $\Theta : \overline{\mathcal{T}}(S) \to \mathcal{T}^{GM}(S)$, we have $\lim_{n\to\infty} x_n = [\varepsilon_p]$ in $\mathcal{T}^{GM}(S)$. By Proposition 3.3, $e^{-2d_T(x_0,x_n)}\tau_{x_n}(\cdot)$ converges pointwise to $C_1\tau(\cdot)$ for some constant $C_1 > 0$. By Proposition 2.5, $e^{-d_T(x_0,x_n)}Ext^{\frac{1}{2}}(x_n, \cdot)$ converges uniformly to $C_2\varepsilon_p(\cdot)$ on any compact set of $\mathcal{MF}$ for some constant $C_2 > 0$.
(1) By Minsky's inequality (Theorem 2.4) and Proposition 3.1 (3), for any $F, G \in \mathcal{MF}$,

$$\begin{aligned} i(\tau(F), G) &= C_1^{-1} \lim_{n\to\infty} e^{-2d_T(x_0,x_n)} i(\tau_{x_n}(F), G) \\ &\leq C_1^{-1} \lim_{n\to\infty} e^{-2d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, \tau_{x_n}(F)) \cdot Ext^{\frac{1}{2}}(x_n, G)) \end{aligned}$$

$$= C_1^{-1} \lim_{n\to\infty} (e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, F)) \cdot (e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, G)) = C_1^{-1} C_2^2 \varepsilon_p(F) \cdot \varepsilon_p(G).$$

Set $C = C_1^{-1} C_2^2$. We get the desired result.
(2) By the fact that $[\varepsilon_p(\cdot)] = [i(\tau(\cdot), \cdot)^{\frac{1}{2}}]$, $\tau(F) = 0$ implies $\varepsilon_p(F) = 0$. By (1), $\varepsilon_p(F) = 0$ implies that $i(\tau(F), G) = 0$ for any $G \in \mathcal{MF}$, which is equivalent to $\tau(F) = 0$.
(3) This result comes from (1) and (2).
(4) For any $F \in \mathcal{MF}$, we have $\lim_{n\to\infty} e^{-2d_T(x_0,x_n)} \tau_{x_n}(F) = C_1 \tau(F)$. Since $\mathcal{MF} \approx \mathbf{R}^{6g-6+n}$, there exists a compact neighbourhood A of $C_1\tau(F)$ such that $e^{-2d_T(x_0,x_n)}\tau_{x_n}(F) \in A$ for any $n > 0$. Note that $e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, \cdot)$ converges to $C_2\varepsilon_p(\cdot)$ uniformly on compact set A. Therefore, for any $\epsilon > 0$, there exists $N_1 > 0$ such that for any $n > N_1$,

$$|e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, e^{-2d_T(x_0,x_n)}\tau_{x_n}(F)) - C_2\varepsilon_p(e^{-2d_T(x_0,x_n)}\tau_{x_n}(F))| < \frac{\epsilon}{2}.$$

By the continuity of ε_p, there exists $N_2 > 0$ such that for any $n > N_2$,

$$|C_2\varepsilon_p(e^{-2d_T(x_0,x_n)}\tau_{x_n}(F)) - C_1C_2\varepsilon_p(\tau(F))| < \frac{\epsilon}{2}.$$

Thus for any $n > \max\{N_1, N_2\}$,

$$|e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, e^{-2d_T(x_0,x_n)}\tau_{x_n}(F)) - C_1C_2\varepsilon_p(\tau(F))| < \epsilon,$$

which implies $\lim_{n\to\infty} e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, e^{-2d_T(x_0,x_n)}\tau_{x_n}(F)) = C_1C_2\varepsilon_p(\tau(F))$.

By Proposition 3.1 (3),

$$\begin{aligned}
&\lim_{n\to\infty} e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, e^{-2d_T(x_0,x_n)}\tau_{x_n}(F)) \\
&= \lim_{n\to\infty} e^{-d_T(x_0,x_n)} [e^{-2d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, \tau_{x_n}(F))] \\
&= \lim_{n\to\infty} e^{-2d_T(x_0,x_n)} [e^{-d_T(x_0,x_n)} Ext^{\frac{1}{2}}(x_n, F)] \\
&= 0 \cdot C_2\varepsilon_p(F) \\
&= 0,
\end{aligned}$$

which implies $\varepsilon_p(\tau(F)) = 0$. By (2), we have $\tau^2(F) = \tau(\tau(F)) = 0$. □

Remark 4.4. (1) If we replace $\partial'\mathcal{T}(S)$ by the whole boundary $\partial\mathcal{T}(S)$, then the proof of Proposition 4.3 is not effective. Because the proof is based on the fact that there exists a sequence $x_n \in \mathcal{T}(S)$ such that x_n converges to $[\tau]$ in $\overline{\mathcal{T}}(S)$. We conjecture that the results of Proposition 4.3 are also true for any points in the whole boundary $\partial\mathcal{T}(S)$.
(2) We may compare Proposition 4.3 (4) with Proposition 3.1 (1): $\tau_x^2 = id_{\mathcal{MF}}$ for any $x \in \mathcal{T}(S)$; while $\tau^2 = 0$ for any $[\tau] \in \partial'\mathcal{T}(S)$.

5. The extended action of the mapping class group

Recall that the mapping class group $Mod(S)$ is the set of isotopy classes of orientation-preserving homeomorphisms of S. And $Mod(S)$ acts naturally on several spaces, such as $\mathcal{T}(S)$, $\mathcal{MF}$ and $Q(S)$. Since the action of $Mod(S)$ on $\mathcal{T}(S)$ extends continuously to $\mathcal{T}^{GM}(S)$, it is also natural to extend to $\overline{\mathcal{T}}(S)$. For this, we need a lemma:

Lemma 5.1. *For any $x \in \mathcal{T}(S)$ and $f \in Mod(S)$, $\tau_{f(x)} = f \circ \tau_x \circ f^{-1}$.*

Proof. By the action of $Mod(S)$ on $Q(S)$ described at the end of Subsection 2.3, we know that for any $q \in Q(x)$, $H_{f(x)}(f(q)) = f(H_x(q))$ and $V_{f(x)}(f(q)) = f(V_x(q))$. Thus $H_{f(x)} = f \circ H_x \circ f^{-1}$ and $V_{f(x)} = f \circ V_x \circ f^{-1}$, which implies that $\tau_{f(x)} = H_{f(x)} \circ V_{f(x)}^{-1} = f \circ H_x \circ f^{-1} \circ (f \circ V_x \circ f^{-1})^{-1} = f \circ H_x \circ V_x^{-1} \circ f^{-1} = f \circ \tau_x \circ f^{-1}$. □

Lemma 5.1 inspires us to define the action of $Mod(S)$ on $\overline{\mathcal{T}}(S)$ as follows:

Lemma 5.2. *For any $f \in Mod(S)$, the map*

$$F : \overline{\mathcal{T}}(S) \to \overline{\mathcal{T}}(S),\ [\tau] \mapsto [f \circ \tau \circ f^{-1}]$$

is a well-defined homeomorphism.

Proof. Fix $f \in Mod(S)$. Firstly, we consider the following map:

$$F_1 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{MF}},\ g \mapsto f \circ g.$$

Note that the pointwise convergence topology on $\mathcal{MF}^{\mathcal{MF}}$ is generated by

$$\{V_{x,U} : x \in \mathcal{MF}, U \text{ is a open set of } \mathcal{MF}\},$$

where $V_{x,U} = \{g \in \mathcal{MF}^{\mathcal{MF}} : g(x) \in U\}$. For any $V_{x,U}$,

$$F_1^{-1}(V_{x,U}) = \{g \in \mathcal{MF}^{\mathcal{MF}} : f \circ g(x) \in U\} = \{g \in \mathcal{MF}^{\mathcal{MF}} : g(x) \in f^{-1}(U)\}.$$

Since f acts continuously on $\mathcal{MF}$, we know that $f^{-1}(U)$ is an open set of $\mathcal{MF}$. Then $F_1^{-1}(V_{x,U}) = V_{x,f^{-1}(U)}$ is an open set of $\mathcal{MF}^{\mathcal{MF}}$. Thus F_1 is continuous. Replacing f by f^{-1} and using the same argument, we know that F_1 is a homeomorphism.

Secondly, we consider another map:

$$F_2 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{MF}},\ g \mapsto g \circ f^{-1}.$$

For any $V_{x,U}$,

$$F_2^{-1}(V_{x,U}) = \{g \in \mathcal{MF}^{\mathcal{MF}} : g \circ f^{-1}(x) \in U\} = V_{f^{-1}(x),U},$$

which is an open set of $\mathcal{MF}^{\mathcal{MF}}$. Thus F_2 is continuous. Replacing f by f^{-1} and using the same argument, we know that F_2 is a homeomorphism.

Thus the map

$$F' = F_1 \circ F_2 : \mathcal{MF}^{\mathcal{MF}} \to \mathcal{MF}^{\mathcal{MF}},\ g \mapsto f \circ g \circ f^{-1}$$

is a homeomorphism. Since $f : \mathcal{MF} \to \mathcal{MF}$ is homogeneous, we have $F'(\Omega) = \Omega$. Note that $F'(kg) = kF'(g)$ for any $k \geq 0$ and $g \in \mathcal{MF}^{\mathcal{MF}}$. Therefore, F' induces a homeomorphism $F : P\Omega \to P\Omega$ such that $pr \circ F' = F \circ pr$, where $pr : \Omega - \{0\} \to P\Omega$ is the natural projection. Note that $\mathcal{T}(S)$ is embedded in $P\Omega$ and its closure is $\overline{\mathcal{T}}(S)$. By Lemma 5.1, if we regard $\mathcal{T}(S)$ as a subset of $P\Omega$, then $F(\mathcal{T}(S)) = \mathcal{T}(S)$. Since F is a homeomorphism, we have $F(\overline{\mathcal{T}}(S)) = \overline{\mathcal{T}}(S)$. Thus $F : \overline{\mathcal{T}}(S) \to \overline{\mathcal{T}}(S)$ is a well-defined homeomorphism. □

By Lemma 5.1 and Lemma 5.2, the action of $Mod(S)$ on $\mathcal{T}(S)$ extends continuously to $\overline{\mathcal{T}}(S)$: for any $f \in Mod(S)$ and $[\tau] \in \overline{\mathcal{T}}(S)$, $f([\tau]) = [f \circ \tau \circ f^{-1}]$.

$\Theta : \overline{\mathcal{T}}(S) \to \mathcal{T}^{GM}(S)$ is $Mod(S)$-covariant:

Theorem 5.3. *For any $f \in Mod(S)$, $f \circ \Theta = \Theta \circ f : \overline{\mathcal{T}}(S) \to \mathcal{T}^{GM}(S)$.*

Proof. By the definitions of the actions of $Mod(S)$ on $\mathcal{T}^{GM}(S)$ and $\overline{\mathcal{T}}(S)$, we know that for any $[\tau] \in \overline{\mathcal{T}}(S)$ and $f \in Mod(S)$,

$$\begin{aligned} f \circ \Theta([\tau]) &= f([i(\tau|_S(\cdot), \cdot)^{\frac{1}{2}}]) = [i(\tau|_S \circ f^{-1}(\cdot), f^{-1}(\cdot))^{\frac{1}{2}}] = [i((f \circ \tau \circ f^{-1})|_S(\cdot), \cdot)^{\frac{1}{2}}] \\ &= \Theta([f \circ \tau \circ f^{-1}]) = \Theta \circ f([\tau]). \end{aligned}$$

□

Next we construct boundary points in $\partial\mathcal{T}(S)$ based on the action of $Mod(S)$. We need a lemma:

Lemma 5.4. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $Mod(S)$ and $\{t_n\}_{n=1}^{\infty}$ be a positive sequence. Then the followings are equivalent:*
(1) $t_n f_n$ converges pointwise to some $g_1 \neq 0$ on $\mathcal{MF}$;
(2) $t_n f_n$ converges uniformly to some $g_2 \neq 0$ on any compact subsets of $\mathcal{MF}$;
(3) $t_n f_n^{-1}$ converges pointwise to some $g_3 \neq 0$ on $\mathcal{MF}$;
(4) $t_n f_n^{-1}$ converges uniformly to some $g_4 \neq 0$ on any compact subsets of $\mathcal{MF}$.

Moreover, if one of them holds, then g_1, g_2, g_3, g_4 are all continuous maps from $\mathcal{MF}$ to $\mathcal{MF}$ and $g_1 = g_2$, $g_3 = g_4$.

Proof. Firstly, we recall two basic results: (a) under the uniform convergence on compact sets, the limit of a sequence of continuous maps is still continuous; (b) the uniform convergence on compact sets is stronger than the pointwise convergence. By (a) and (b), we have (2) implies (1) and if (2) holds, then $g_2 = g_1$ are both continuous; (4) implies (3) and if (4) holds, then $g_4 = g_3$ are both continuous. By the fact that $(f_n^{-1})^{-1} = f_n$, to prove the four statements above are equivalent, we only need to prove that (1) implies (4).

To do this, we need a coordinate for $\mathcal{MF}$. By the results in [4], there are finite simple closed curves $\alpha_1, \alpha_2, ..., \alpha_N$ filling up the surface S such that the map $\varphi : \mathcal{MF} \to \mathbf{R}^N, F \mapsto (i(\alpha_i, F))_{i=1}^N$ is an embedding. Note that $\alpha_1, \alpha_2, ..., \alpha_N$ fill up the surface S if and only if $\sum_{i=1}^N i(\alpha_i, F) > 0$ for any $F \in \mathcal{MF} - \{0\}$. Now we claim that $\varphi(\mathcal{MF})$ is closed in $\mathbf{R}^N$. Suppose $\lim_{n\to\infty} \varphi(F_n) = (a_1, a_2, ..., a_N)$ in $\mathbf{R}^N$ for some sequence $F_n \in \mathcal{MF}$. Then we have $\lim_{n\to\infty} \sum_{i=1}^N i(\alpha_i, F_n) = \sum_{i=1}^N a_i$, which implies that $\sum_{i=1}^N i(\alpha_i, F_n) \leq M$ for some $M > 0$. By the result in [3], since $\alpha_1, \alpha_2, ..., \alpha_N$ fill up the surface S, $\{F \in \mathcal{MF} : \sum_{i=1}^N i(\alpha_i, F) \leq M\}$ is compact. Thus there exists a subsequence F_{n_k} such that $\lim_{k\to\infty} F_{n_k} = F_0$ for some $F_0 \in \mathcal{MF}$. Then $(a_1, a_2, ..., a_N) = \lim_{k\to\infty} \varphi(F_{n_k}) = \varphi(F_0) \in \varphi(\mathcal{MF})$. Thus $\varphi(\mathcal{MF})$ is closed in $\mathbf{R}^N$.

Now suppose that (1) holds, that is, $t_n f_n$ converges pointwise to some $g_1 \neq 0$ on $\mathcal{MF}$. Then for $i = 1, 2, ..., N$, $\lim_{n\to\infty} t_n f_n(\alpha_i) = g_1(\alpha_i)$. Since $i(\cdot, \cdot)$ is continuous, by Lemma 2.10, we have $i(t_n f_n(\alpha_i), \cdot)$ converges uniformly to $i(g_1(\alpha_i), \cdot)$ on any compact subset A of $\mathcal{MF}$. Thus $(i(t_n f_n(\alpha_i), \cdot))_{i=1}^N$ converges uniformly to $(i(g_1(\alpha_i), \cdot))_{i=1}^N$ on A.

Note that $\varphi \circ (t_n f_n^{-1}(\cdot)) = (i(\alpha_i, t_n f_n^{-1}(\cdot)))_{i=1}^N = (i(t_n f_n(\alpha_i), \cdot))_{i=1}^N$. Thus $\varphi \circ (t_n f_n^{-1}(\cdot))$ converges uniformly to $(i(g_1(\alpha_i), \cdot))_{i=1}^N$ on A. Since $\varphi(\mathcal{MF})$ is closed in $\mathbf{R}^N$, we know that

$$(i(g_1(\alpha_i), F))_{i=1}^N = \lim_{n\to\infty} (i(t_n f_n(\alpha_i), F))_{i=1}^N = \lim_{n\to\infty} (i(\alpha_i, t_n f_n^{-1}(F)))_{i=1}^N \in \varphi(\mathcal{MF})$$

for any $F \in \mathcal{MF}$. Thus we have a well-defined map $g_4 = \varphi^{-1}(i(g_1(\alpha_i), \cdot))_{i=1}^N : \mathcal{MF} \to \mathcal{MF}$.

Since φ is an embedding and $\varphi \circ (t_n f_n^{-1}(\cdot))$ converges uniformly to $\big(i(g_1(\alpha_i), \cdot)\big)_{i=1}^N$ on A, we know that $t_n f_n^{-1}$ converges uniformly to g_4 on A. Thus $t_n f_n^{-1}$ converges uniformly to g_4 on any compact subsets of $\mathcal{MF}$.

Now we need to prove that $g_1 \neq 0$ implies $g_4 \neq 0$. Suppose $g_4 = 0$. Then for any $F \in \mathcal{MF}$, we have

$$\varphi(g_1(F)) = \lim_{n\to\infty} \big(i(\alpha_i, t_n f_n(F))\big)_{i=1}^N = \lim_{n\to\infty} \big(i(t_n f_n^{-1}(\alpha_i), F)\big)_{i=1}^N = \big(i(g_4(\alpha_i), F)\big)_{i=1}^N = 0,$$

which implies that $g_1 = 0$. □

By Lemma 5.4, we have

Proposition 5.5. *Let $\{f_n\}_{n=1}^\infty$ be a sequence in Mod(S) and $x \in \mathcal{T}(S)$. Suppose that there exists a positive sequence $\{t_n\}_{n=1}^\infty$ such that $t_n f_n$ converges pointwise to some $f_0 \neq 0$ on $\mathcal{MF}$. Then $\lim_{n\to\infty} f_n(x) = [f_0 \circ \tau_x \circ f_0']$ in $\overline{\mathcal{T}}(S)$ for some continuous map $0 \neq f_0' : \mathcal{MF} \to \mathcal{MF}$.*

Proof. By Lemma 5.4, f_0 is continuous; $t_n f_n^{-1}$ converges pointwise to some continuous map $f_0' \neq 0$; and $t_n f_n$ converges uniformly to f_0 on any compact subset of $\mathcal{MF}$. Since τ_x is continuous, we have $\tau_x \circ (t_n f_n^{-1})$ converges pointwise to $\tau_x \circ f_0'$. Since f_0 is continuous, we have $f_0 \circ \tau_x \circ (t_n f_n^{-1})$ converges pointwise to $f_0 \circ \tau_x \circ f_0'$.

Since $\mathcal{MF}$ is homeomorphic to $\mathbf{R}^{6g-6+2n}$, we can choose a metric d on $\mathcal{MF}$. For any $F \in \mathcal{MF}$, since $\lim_{n\to\infty} \tau_x \circ (t_n f_n^{-1})(F) = \tau_x \circ f_0'(F)$, $\{\tau_x \circ (t_n f_n^{-1})(F)\}_{n=1}^\infty \subseteq A$ for some compact set $A \subseteq \mathcal{MF}$.

Since $\lim_{n\to\infty} f_0 \circ \tau_x \circ (t_n f_n^{-1})(F) = f_0 \circ \tau_x \circ f_0'(F)$, we know that for any $\varepsilon > 0$, there exists $N_1 > 0$ such that for any $n > N_1$, $d(f_0 \circ \tau_x \circ (t_n f_n^{-1})(F), f_0 \circ \tau_x \circ f_0'(F)) < \frac{\varepsilon}{2}$.

Since $t_n f_n$ converges uniformly to f_0 on compact set A, there exists $N_2 > 0$ such that for any $n > N_2$ and $E \in A$, $d(t_n f_n(E), f_0(E)) < \frac{\varepsilon}{2}$. Thus for any $n > N_2$, $d\big((t_n f_n) \circ \tau_x \circ (t_n f_n^{-1})(F), f_0 \circ \tau_x \circ (t_n f_n^{-1})(F)\big) < \frac{\varepsilon}{2}$.

By the triangle inequality, for any $n > \max\{N_1.N_2\}$,

$$d\big((t_n f_n) \circ \tau_x \circ (t_n f_n^{-1})(F), f_0 \circ \tau_x \circ f_0'(F)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $(t_n f_n) \circ \tau_x \circ (t_n f_n^{-1})$ converges pointwise to $f_0 \circ \tau_x \circ f_0'$ on $\mathcal{MF}$, which implies that $\lim_{n\to\infty} f_n(x) = [f_0 \circ \tau_x \circ f_0']$ in $\overline{\mathcal{T}}(S)$. □

We need two results (see [7]).

Proposition 5.6. *Let $f = T_{\alpha_1}^{n_1} \circ T_{\alpha_2}^{n_2} \circ \cdots \circ T_{\alpha_k}^{n_k}$, where $\alpha_1, ..., \alpha_k$ are pairwise disjoint simple closed curves, T_{α_i} is the Dehn Twist of α_i and $n_i \in \mathbf{Z}$ $(i = 1, 2, ..., k)$. Then for any $F \in \mathcal{MF}$, we have*

$$\lim_{n\to\pm\infty} \frac{f^n(F)}{|n|} = \sum_{i=1}^k |n_i| i(\alpha_i, F)\alpha_i.$$

Proposition 5.7. *Let $f \in Mod(S)$ be a Pseudo-Anosov element such that $f(F^s) = \lambda^{-1}F^s$, $f(F^u) = \lambda F^u$ with $\lambda > 1$, $F^s, F^u \in \mathcal{MF}$ and $i(F^s, F^u) = 1$. Then for any $F \in \mathcal{MF}$, we have*

$$\lim_{n\to\infty} \frac{f^n(F)}{\lambda^n} = i(F^s, F)F^u, \ \lim_{n\to\infty} \frac{f^{-n}(F)}{\lambda^n} = i(F^u, F)F^s.$$

By Proposition 5.5, Proposition 5.6 and Proposition 5.7, we have

Proposition 5.8. (1) *With the assumption of* Proposition 5.6, *for any* $x \in \mathcal{T}(S)$,

$$\lim_{n\to\pm\infty} f^n(x) = [f_0 \circ \tau_x \circ f_0] \in \partial'\mathcal{T}(S),$$

where $f_0 = \sum_{i=1}^{k} |n_i| i(\alpha_i, \cdot)\alpha_i$.
(2) *With the assumption of* Proposition 5.7, *for any* $x \in \mathcal{T}(S)$,

$$\lim_{n\to\infty} f^n(x) = [i(F^u, \cdot)F^u] \in \partial'\mathcal{T}(S), \quad \lim_{n\to\infty} f^{-n}(x) = [i(F^s, \cdot)F^s] \in \partial'\mathcal{T}(S).$$

Remark 5.9. By Proposition 5.8, we know that $\partial'\mathcal{T}(S) \neq \emptyset$.

Besides, through Proposition 5.8, we construct some special boundary points in $\partial\mathcal{T}(S)$.

Theorem 5.10. *For any* $F \in \mathcal{MF} - \{0\}$, $[i(F, \cdot)F] \in \partial\mathcal{T}(S)$.

Proof. By Proposition 5.8(1), $[i(\alpha, \cdot)\alpha] \in \partial'\mathcal{T}(S) \subseteq \partial\mathcal{T}(S)$ for any $\alpha \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $\mathcal{PMF}$, for any $F \in \mathcal{MF} - \{0\}$, there are a positive sequence $\{t_n\}_{n=1}^{\infty}$ and a sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ such that $\lim_{n\to\infty} t_n\alpha_n = F$. Thus the sequence $[i(\alpha_n, \cdot)\alpha_n] \in \partial\mathcal{T}(S)$ converges to $[i(F, \cdot)F]$. By the closeness of $\partial\mathcal{T}(S)$, we have $[i(F, \cdot)F] \in \partial\mathcal{T}(S)$. □

Remark 5.11. For any $x \in \mathcal{T}(S)$, τ_x is continuous. Inspired by this, we conjecture that any boundary point $[\tau] \in \partial\mathcal{T}(S)$ is continuous. There are two evidences supporting this conjecture: the special boundary points constructed in Proposition 5.8 and Theorem 5.10 are continuous; $\Theta([\tau]) = [i(\tau(\cdot), \cdot)] \in GM$ is continuous.

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